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ON THE NUMBER OF GEODESIC SEGMENTS CONNECTING TWO POINTS ON MANIFOLDS OF NON-POSITIVE CURVATURE

PAUL HORJA

Abstract. We prove that on a complete Riemannian manifold M of dimension n with sectional curvature $K_M < 0$, two points which realize a local maximum for the distance function (considered as a function of two arguments) are connected by at least $2n+1$ geodesic segments. A simpler version of the argument shows that if one of the points is fixed and $K_M \leq 0$ then the two points are connected by at least $n+1$ geodesic segments. The proof uses mainly the convexity properties of the distance function for metrics of negative curvature.

1. Introduction

Let M be a compact Riemannian manifold of dimension $n \geq 2$, with a metric of negative sectional curvature ($K_M < 0$). In this paper we prove that on such manifolds there exist pairs of points connected by at least $2n+1$ geometrically distinct geodesic segments (i.e. length minimizing). A class of points which provide examples in this class are the points situated at distance equal to the diameter of the manifold. A simplified version of the method allows us to show that in the case of non-positive curvature ($K_M \leq 0$) for any point there exist another point and $n+1$ geometrically distinct geodesic segments connecting them. Actually the assumptions on compactness and curvature can be relaxed, and this will be explained later in the paper. The essential ingredient in the proofs is the basic metric property that the distance function of spaces of non-positive and negative curvature is convex and, in a sense that will be explained in the paper, even almost strictly convex. The results can also be seen as estimates for the "order" of the points in the cut locus for manifolds of non-positive curvature.

In the case of positive curvature the situation changes: for an ellipsoid in $\mathbb{R}^3$ with axes of different lengths, the points at maximal distance are connected by two geodesic segments, but for the sphere by infinitely many geodesic segments. For the flat torus obtained as a quotient of $\mathbb{R}^2$ by a lattice not generated by two orthogonal vectors, the maximal "order" of the points in the cut locus is 3. Interesting is the situation for convex polyhedra in $\mathbb{R}^3$, which is intermediate between the cases of negative and positive curvature. For a large class of them, namely for those admitting two points at maximal intrinsic distance which are not vertices, the result remains true, i.e. there are at least 5 geodesic segments connecting the two points. Moreover, the convex polyhedra in the above class with two points at maximal

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intrinsic distance connected by exactly 5 geodesic segments form a dense set in the class. The results concerning the polyhedra are not treated in this paper.

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2. Preliminaries

We recall some definitions. Complete explanations can be found in [2] and [5].

A function $f : I \to \mathbb{R}, I$ being an interval in $\mathbb{R}$, is said to be *convex* if for $a \neq b, a, b \in I$ and $t \in (0,1)$ we have the inequality

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).$$

The function is called *strictly convex* if the inequality is strict. If N is a Riemannian manifold of dimension $n \geq 2$, consider a geodesic segment parametrized proportionally to arc length $\gamma : [0,1] \to N, \gamma(0) = p_0, \gamma(1) = p_1$. Denote by $[p_0, p_1]$ the set $\gamma([0,1])$. A subset $V \subset N$ is called *convex*, if for every $p, q \in V$ there is a unique geodesic segment from p to q and this is contained in V. A function $f : V \to \mathbb{R}$ is called *(strictly) convex* if for every nontrivial geodesic $\gamma : [0,1] \to V$ parametrized proportionally to arc length the function $f \circ \gamma$ is (strictly) convex.

We now introduce a notion useful in what follows. For an open nonempty convex set $V \subset N$ and for a point $p \in V$, a convex function $f : V \to \mathbb{R}$ is called *almost strictly convex* at p, if there exists a line $l_p^f \subset \mathrm{T}_pV$ (passing through the origin) such that for every geodesic $\gamma : [0,1] \to V$ parametrized proportionally to arc length with $\gamma(0) = p$, $\dot\gamma(0) \notin l_p^f$, the function $f \circ \gamma$ is not constant. If f is almost strictly convex for all $p \in V$, we say that f is almost strictly convex on V. Of course, every strictly convex function is almost strictly convex.

Remark 2.1. If M_1 and M_2 are Riemannian manifolds, then the product Riemannian metric on $M_1 \times M_2$ is given by the action of the two metrics on the product tangent space. So a curve

$$\gamma : [0,1] \to M_1 \times M_2, \quad \gamma(t) = (\gamma_1(t), \gamma_2(t)),$$

is a geodesic parametrized proportionally to arc length in $M_1 \times M_2$ if and only if $\gamma_i : [0,1] \to M_i, i = 1,2$, are also geodesics parametrized proportionally to arc length. It follows that if $U_i \subset M_i$ are nonempty convex sets $(i = 1,2)$, then $U_1 \times U_2$ is also a convex set in $M_1 \times M_2$.

Consider now a simply connected complete Riemannian manifold N of dimension $n \geq 2$, with all the sectional curvatures bounded above by $\chi \leq 0$ (shortly, of nonpositive curvature). We will say that N has negative curvature if $\chi < 0$.

We introduce some notations. For two distinct points $p_0, p_1 \in N$ consider the geodesic segment $\gamma : [0,1] \to N$ parametrized proportionally to arc length, such that $\gamma(0) = p_0, \gamma(1) = p_1$, and for t $(0 \leq t \leq 1)$ denote by $(1-t)p_0 + tp_1$ the point $\gamma(t)$. Suppose that $\mathbb{H}_\chi$ is the 2-dimensional space of constant curvature $\chi \leq 0$ (i.e. the Euclidean plane for $\chi = 0$, and the hyperbolic plane with constant curvature χ for $\chi < 0$). For a triple of points $[p_1, p_2, p_3]$ in N the corresponding triple of points in $\mathbb{H}_\chi$ is denoted by $[p_1^*, p_2^*, p_3^*]$, and has the property:

$$d(p_i, p_j) = d(p_i^*, p_j^*), \qquad 1 \leq i < j \leq 3,$$

where d^* is the distance function on $\mathbb{H}_\chi$. For a point $q \in [p_1, p_2]$, the corresponding point $q^* \in [p_1^*, p_2^*]$ has the property $d(p_1, q) = d^*(p_1^*, q^*)$.

It is known then that N has non-positive (negative, respectively) curvature in the metric sense, too (see for example [5], page 52). This means that for every triple of points $[p_1, p_2, p_3]$ in N and the points $q \in [p_1, p_2], r \in [p_1, p_3]$ the corresponding triple $[p_1^*, p_2^*, p_3^*]$ and the points $q^* \in [p_1^*, p_2^*], r^* \in [p_1^*, p_3^*]$ in $\mathbb{H}_\chi$ have the property: $d(q, r) \leq d^*(q^*, r^*)$.

Remark 2.2. If $\mathbb{H}^n$ is the n-dimensional hyperbolic space, then the distance function $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ has the following property:

$$d(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}) < \frac{d(p_1, p_2) + d(q_1, q_2)}{2}$$

where p_1, p_2, q_1, q_2 are points in $\mathbb{H}^n$ which are not on the same maximal geodesic (see [3], page 37).

It is elementary that a continuous mid-convex function is convex. It is also easy to see that strict inequality for the middle of a segment implies that the inequality is strict for every interior point of the segment. It follows that, for t $(0 < t < 1)$, we have

$$d(tp_1 + (1-t)q_1, tp_2 + (1-t)q_2) \leq td(p_1, p_2) + (1-t)d(q_1, q_2).$$

Of course, when the four points are on the same geodesic, the inequality becomes equality. We need only the case $n = 2$, and this can be also verified by direct computation using the cosine formula in the hyperbolic plane. In fact, the following is true.

Lemma 2.3. *Let N be a simply connected complete Riemannian manifold of negative curvature as above, and U_1 and U_2 two nonempty convex sets in N, such that $U_1 \cap U_2 = \emptyset$. Then the restriction of the distance function $d : U_1 \times U_2 \to \mathbb{R}$ is almost strictly convex.*

Proof. From Remark 2.1, it follows that the set $U_1 \times U_2$ is a nonempty convex subset of the product manifold $N \times N$, so it makes sense to talk about the convexity of the function $d : U_1 \times U_2 \to \mathbb{R}$. Take the points $p_1, q_1 \in U_1, p_2, q_2 \in U_2$, $p_i \neq q_i, i = 1, 2$. Consider t, $0 < t < 1$; we shall prove that

$$d(tp_1 + (1-t)q_1, tp_2 + (1-t)q_2) \leq td(p_1, p_2) + (1-t)d(q_1, q_2),$$

with equality only when p_1, q_1, p_2, q_2 are on the same geodesic.

Suppose they are not, and denote by r_1 the point $tp_1 + (1-t)q_1$ and by r_2 the point $tp_2 + (1-t)q_2$. Consider also the point $r \in [p_1, q_2]$, $r = tp_1 + (1-t)q_2$. Using the comparison triangle $[p_1^*, q_1^*, q_2^*]$ and the corresponding points $r_1^*, r^* \in \mathbb{H}_\chi$, we obtain

$$d(r_1, r) \leq d^*(r_1^*, r^*) < (1-t)d^*(q_1^*, q_2^*) = (1-t)d(q_1, q_2),$$

where we have assumed that $q_2 \notin [p_1, q_1]$ (possible, because p_1, q_1, p_2, q_2 are not all on the same geodesic), and we have used Remark 2.2. In a similar way, we have

$$d(r, r_2) \leq td(p_1, p_2),$$

and then, by addition,

$$d(r_1, r_2) \leq d(r_1, r) + d(r, r_2) < td(p_1, p_2) + (1-t)d(q_1, q_2).$$

When the four points are on the same geodesic the inequality becomes equality. Consider now a geodesic $\gamma : [-\delta, 1+\delta] \to N$, $\gamma(0) = p_1, \gamma(1) = p_2, \delta > 0$; from the inequality proved above it can be inferred that the function $d : U_1 \times U_2 \to \mathbb{R}$ is convex. Furthermore, for a geodesic $\tilde{\gamma} : [0,1] \to N \times N$ such that $\tilde{\gamma}(0) = (p_1, p_2)$, it follows that the function $d \circ \tilde{\gamma} : [0,1] \to \mathbb{R}$ is strictly convex if and only if $\dot{\tilde{\gamma}}(0)$, the tangent vector at p, is not in the 2-plane generated in $\mathrm{T}_{(p_1,p_2)}(N \times N)$ by the vectors $(\dot{\gamma}(0), 0)$ and $(0, \dot{\gamma}(1))$. It's not difficult to see that the function $d \circ \tilde{\gamma} : [0,1] \to \mathbb{R}$ is constant if and only if the tangent vector $\dot{\tilde{\gamma}}(0)$ is on the line given by the vector $(\dot{\gamma}(0), \dot{\gamma}(1))$ in the tangent space to the product manifold $N \times N$ at (p_1, p_2). So the restriction of the distance function to $U_1 \times U_2$ is almost strictly convex, and the direction of the line $l^d_{(p_1,p_2)}$ (see the definition of an almost strictly convex function) is given by the vector $(\dot{\gamma}(0), \dot{\gamma}(1))$.

Note. As Professor Jianguo Cao pointed out to me, Lemma 2.3 is a consequence of Proposition 1 from the paper of Schoen and Yau [7] (for similar results, see also [4]). Our notion of almost strictly convexity for a function $f : V \to \mathbb{R}$ corresponds to the fact that the rank of the Hessian of f at the considered point is at least $\dim V - 1$.

For the case of non-positive curvature we will use the following statement:

Lemma 2.4. *Let N be a simply connected complete Riemannian manifold of non-positive curvature, $p \in N$ a fixed point, and U_2 a nonempty convex set in N such that $p \notin U_2$. Then the function $f : U_2 \to \mathbb{R}$ defined by $f(q) = d(q,p)$ is almost strictly convex. Moreover, for any point $q \in U_2$ and any geodesic $\gamma : [0,1] \to N$, $\gamma(0) = q$, the function $f \circ \gamma$ is not constant (i.e. the direction l^f_q can be chosen arbitrarily).*

Proof. In this case, the restriction of the distance function is considered with one argument fixed. Using the same notations as in the previous lemma, but keeping $p_1 = q_1 = p$, and by comparing this time with triangles in the Euclidean plane, we have

$$d(p, r_2) \le t d(p, p_2) + (1-t) d(p, q_2),$$

with equality if and only if the three points are on the same geodesic. This shows that f is almost strictly convex. Moreover, at a closer look, for a point $q \in U_2$, even for the geodesic $\gamma : [0, 1+\delta] \to N$, $\gamma(0) = p, \gamma(1) = q, \delta > 0$, the function $f \circ \gamma$ (where the composition makes sense) is not constant. This proves the last assertion in Lemma 2.4. Notice though that, in this situation, the function f falls short of being strictly convex—the strict inequality does not hold in the direction $\dot{\gamma}(1)$, but holds in all the others.

We will need also the following elementary fact:

Lemma 2.5. *Consider in $\mathbb{R}^n$ $(n \ge 2)$ for every i, $1 \le i \le k, k \le n$, an $(n-1)$-dimensional linear subspace H_i, and denote by $\overline{S}_i$ one of the closed half-spaces determined by H_i. If $\bigcup_{i=1}^{k} \overline{S}_i = \mathbb{R}^n$, then $\bigcap_{i=1}^{k} H_i$ is a linear subspace of dimension at least $(n-k+1)$.*

Proof. The lemma is clearly true if there exist i, j $(1 \le i \le k)$ such that $H_i = H_j$. Suppose that all the hyperplanes are mutually distinct.

We use induction relative to n. For $n = 2, 3$ the lemma is true. Suppose it's true for n. Then in $\mathbb{R}^{n+1}$, take $S'_i = S_i \cap H_1$, for every i, $2 \leq i \leq k$. Since $\bigcup_{i=1}^{k} \overline{S}_i = \mathbb{R}^n$, and $H_1 \neq H_i$, for all i, $2 \leq i \leq k$, it follows that $\bigcup_{i=2}^{k} \overline{S}'_i = H_1$. Using induction, we have

$$\dim(\bigcap_{i=2}^{k} H_i) \geq n - (k-1) + 1 = (n+1) - k + 1,$$

so

$$\dim(\bigcap_{i=1}^{k} H_i) \geq (n+1) - k + 1,$$

which ends the proof of the lemma. □

3. Main results

The main tool in proving the theorems will be the following:

Proposition 3.1. *Suppose N is a Riemannian manifold of dimension $m \geq 2$, not necessarily of negative curvature, V an open convex set in N, $p \in V$. Assume that there exists a natural number $k \geq 1$ such that for every i, $1 \leq i \leq k$, there is an almost strictly convex function $f_i : V \to \mathbb{R}$, $f_i(p) = 0$, with the property:*
(i) there exists $\epsilon > 0$, such that for every point $q \in B(p, \epsilon) \subset V$,

$$\min_{1 \leq i \leq k} f_i(q) \leq 0.$$

Then $k \geq m$.
Moreover, if the following condition holds also:
(ii) for every $i_1, i_2, \ldots, i_m$ $(1 \leq i_1 < i_2 < \cdots < i_m \leq k)$,

$$l_p^{f_{i_1}} \cap l_p^{f_{i_2}} \cap \cdots \cap l_p^{f_{i_m}} = \{0\},$$

then $k \geq m + 1$.

Proof. We can suppose that $\overline{B(p, \epsilon)} \subset V$ and $B(p, \epsilon)$ is convex (make ϵ smaller, if necessary). For every i, $1 \leq i \leq k$, consider the open set

$$A_i = \{q \mid q \in B(p, \epsilon), f_i(q) < 0\}$$

and the closed set

$$B_i = \{q \mid q \in \overline{B(p, \epsilon)}, f_i(q) \leq 0\}.$$

Of course, $\overline{A_i} \subset B_i$, and since the functions f_i are convex, it follows that A_i and B_i are convex sets. For every i, $1 \leq i \leq k$, consider the geodesic $\gamma_i : [-\epsilon, \epsilon] \to \overline{B(p, \epsilon)}$, parametrized by arc length, such that $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) \in l_p^i$. Denote by C_i the set $\gamma_i([-\epsilon, \epsilon])$. We make the following remark:

$$B_i \setminus \overline{A_i} \subset C_i \tag{3.2}$$

Indeed, if $B_i \setminus \overline{A_i} = \emptyset$, there is nothing to prove. Suppose that $B_i \setminus \overline{A_i} \neq \emptyset$. For every point $q \in B_i \setminus \overline{A_i}$, we have that $f_i(q) = 0$.

Consider first the case $A_i = \emptyset$. If B_i consists of the single point p, the property is clear. For another point $q \in B_i$, we have that

$$f_i(q) = f_i(p) = 0, \quad [p,q] \subset B_i, \qquad f_i \geq 0 \quad \text{on } \overline{B(p,\epsilon)},$$

because $A_i = \emptyset$. The convexity of the function f_i implies then that for every point $r \in [p,q]$, $f_i(r) = 0$, so f_i is constant along the segment $[p,q]$, and this gives $B_i \subset C_i$.

Suppose next that there exists a point $p' \in A_i$. Then $f_i(p') < 0$, and, in fact, from the convexity of the function f_i it follows that $f_i(p'') < 0$, for every $p'' \in [p,p'], p'' \neq p$. So $p'' \in A_i$, which implies that $p \in \overline{A_i}$. For a point $q \in B_i \setminus \overline{A_i}$, we can find $\delta > 0$ such that $B(q,\delta) \cap A_i = \emptyset$; this means $f_i(r) \geq 0$, for every $r \in B(q,\delta)$. On the other hand, the convexity of f_i implies that for every point $r' \in [p,q] \cap B(q,\delta), r' \neq q$, we have that $f_i(r') \leq 0$ (because $f_i(p) = f_i(q) = 0$). In conclusion $f_i(r') = 0$, and f_i is constant along the segment $[p,q]$. So $[p,q] \subset C_i$, which ends the proof of the relation (3.2).

Condition (i) in the proposition says in fact that

$$\bigcup_{i=1}^{k} B_i = \overline{B(p,\epsilon)}.$$

Combining this with (3.2), we obtain

$$\bigcup_{i=1}^{k} (\overline{A_i} \cup C_i) = \overline{B(p,\epsilon)},$$

so

$$\overline{B(p,\epsilon)} \setminus \bigcup_{i=1}^{k} \overline{A_i} \subset \bigcup_{i=1}^{k} C_i.$$

But the difference of the two sets in the first part of the inclusion is an open set in $\overline{B(p,\epsilon)}$, and since $m = \dim N \geq 2$, it is clear that the union of the geodesics C_i cannot cover this open set, unless the open set is empty,

$$\overline{B(p,\epsilon)} \setminus \bigcup_{i=1}^{k} \overline{A_i} = \emptyset.$$

Consider then k_0 such that $A_i \neq \emptyset$ for all i, $1 \leq i \leq k_0$, and $A_j = \emptyset$ for all j, $k_0 \leq j \leq k$. We will prove that in fact $k_0 \geq m$ (respectively $k_0 \geq m+1$). Clearly

$$\bigcup_{i=1}^{k_0} \overline{A_i} = \overline{B(p,\epsilon)}.$$

For the non-empty convex set A_i, with $p \in \partial A_i$, we can apply proposition 4.9.2 from [6]: there exists a support hyperplane $H_p^i \subset \mathrm{T}_p N$ so that all the tangent vectors at p to the geodesic segments connecting p with points in the set A_i are in the same open half-space, denoted by S_p^i, $1 \leq i \leq k_0$, determined by H_p^i in $\mathrm{T}_p N$.

The sets $\overline{A_i}$ can cover the closed ball $\overline{B(p,\epsilon)}$ if and only if the corresponding closed half-spaces cover the tangent space at p (otherwise we would have a direction which is in none of the closed half-spaces, so a geodesic segment which, at least locally, is not contained in any of the closed sets $\overline{A_i}$). So

$$\bigcup_{i=1}^{k_0} \overline{S_p^i} = \mathrm{T}_p N,$$

and then Lemma 2.5 implies that

$$\dim(\bigcap_{i=1}^{k_0} H_p^i) \geq m - k_0 + 1.$$

Suppose now that $k_0 \leq m-1$; then

$$\dim(\bigcap_{i=1}^{k_0} H_p^i) \geq m - (m-1) + 1 = 2.$$

Define the set $U = B(p,\epsilon) \cap \exp_p(\bigcap_{i=1}^{k_0} H_p^i)$. Take a point $q \in U \setminus \{p\}$. Then there exists $i_0, 1 \leq i_0 \leq k_0$, such that $q \in \overline{A_{i_0}}$. But from the definition of the set U we have that $q \notin A_{i_0}$, so it follows that $q \in \overline{A_{i_0}} \setminus A_{i_0}$, or in other words $f_{i_0}(q) = 0$. Using the convexity of the function and the fact that $f_{i_0}(p) = 0$, we get as above that $q \in C_{i_0}$. The consequence of this argument is that $U \subset \bigcup_{i=1}^{k} C_i$. But this is impossible, because U is a submanifold of dimension at least 2, which cannot be covered by finitely many 1-dimensional submanifolds. This ends the proof of the first part of the proposition.

Consider now the case $k_0 = m$ and suppose that the condition (ii) holds. Then

$$\dim(\bigcap_{i=1}^{k_0} H_p^i) \geq 1.$$

Consider a vector $v \in \bigcap_{i=1}^{k_0} H_p^i$, $0 < ||v|| < \epsilon$, and define the point $q = \exp_p v$. It is not possible that $q \in \bigcup_{i=1}^{k_0} A_i$, because this would mean that $v \in S_p^i$ for some i, which is not true. On the other hand, if there is no i such that $q \in \overline{A_i}$, it would follow that there exists a neighborhood W of q, $W \subset B(p,\epsilon)$, such that

$$W \cap (\bigcap_{i=1}^{k_0} \overline{A_i}) = \emptyset,$$

which is impossible, too.

The argument shows that there exists i_0 such that $q \in \overline{A_{i_0}} \setminus A_{i_0}$. The fact that v is the tangent vector at p to the geodesic segment $[p,q]$ and the convexity of the function f_{i_0} and of the set $\overline{A_{i_0}}$ imply that $[p,q] \subset \overline{A_{i_0}} \setminus A_{i_0}$ (no interior point of $[p,q]$ can be in A_{i_0}, and f_{i_0} is constant along this geodesic segment). This means that $[p,q] \subset C_{i_0}$, so in fact $v \in l_p^{f_{i_0}}$. If k_1 is the number of A_i's with the property that $q \in \overline{A_i}$, condition (ii) gives that $k_1 \leq m-1$. The first part of the proposition can be applied now, and a contradiction is obtained. This ends the proof of the proposition.

Remark 3.3. Notice that, when conditions (i) and (ii) are satisfied, the argument from the last part of the proof shows that there exists a neighborhood W of p such that for $q \in W, q \neq p$, we have

$$\min_{1 \le i \le k} f_i(q) < 0.$$

In other words, the function $\min_{1 \le i \le k} f_i$ has a strict local maximum at p.

We are now in the position to prove the main results.

Theorem 3.4. *Let M be a complete Riemannian manifold of dimension $n \ge 2$ and negative curvature (all the sectional curvatures bounded above by $\chi < 0$). If $(p_1, p_2) \in M \times M$ is a local maximum for the distance function on M, $d : M \times M \to \mathbb{R}$, then the points p_1 and p_2 are connected by at least $2n+1$ distinct geodesic segments.*

Proof. Consider the universal covering space $\widetilde{M}$ and the covering map $\pi : \widetilde{M} \to M$. Denote by $\tilde{d}$ the distance function on $\widetilde{M}$, and by d_0 the distance between the points p_1 and p_2. Take a point $\tilde{p}_1 \in \widetilde{M}$ so that $\pi(\tilde{p}_1) = p_1$.

The closed ball $\overline{B(\tilde{p}_1, d_0)}$ is compact in $\widetilde{M}$. From the discreteness of the fiber $\pi^{-1}(p_2)$, it follows that p_1 and p_2 are connected in M by just finitely many geodesic segments

$$\gamma_i : [0,1] \to M, \quad 1 \le i \le k, \qquad \gamma_i(0) = p_1, \quad \gamma_i(1) = p_2,$$

parametrized proportionally to arc length. Consider their liftings to the universal cover

$$\tilde{\gamma}_i : [0,1] \to \widetilde{M}, \quad 1 \le i \le k, \qquad \tilde{\gamma}_i(0) = \tilde{p}_1, \quad \tilde{\gamma}_i(1) = \tilde{p}_2^i \in \pi^{-1}(p_2),$$

The discreteness of the fiber implies also that it is possible to find $\epsilon_0 > 0$ so that $\tilde{d}(\tilde{p}_1, \tilde{p}) > d_0 + \epsilon_0$, for every $\tilde{p} \in \pi^{-1}(p_2) \setminus \{\tilde{p}_2^1, \ldots, \tilde{p}_2^k\}$. Define the sets

$$U_1 = B(p_1, \frac{\epsilon_0}{4}), \qquad U_2 = B(p_2, \frac{\epsilon_0}{4}),$$

$$\widetilde{U}_1 = B(\tilde{p}_1, \frac{\epsilon_0}{4}), \qquad \widetilde{U}_2^i = B(\tilde{p}_2^i, \frac{\epsilon_0}{4}), \quad 1 \le i \le k.$$

We can suppose that ϵ_0 is small enough so that $U_1 \cap U_2 = \emptyset$, $\widetilde{U}_2^i \cap \widetilde{U}_2^j = \emptyset$ for $i \neq j$, and the restrictions $\pi_1 : \widetilde{U}_1 \to U_1$ and $\pi_{2,i} : \widetilde{U}_2^i \to U_2$ of the covering map are isometries. The negative curvature and the convexity of the distance function imply that the balls $U_1, U_2, \widetilde{U}_1, \widetilde{U}_2^i$ are convex.

For every i, $1 \le i \le k$, consider the restriction of the distance function $\tilde{d}$ on $\widetilde{M}$, $\tilde{d}_i : \widetilde{U}_1 \times \widetilde{U}_2^i \to \mathbb{R}$, and the map $\pi_i : \widetilde{U}_1 \times \widetilde{U}_2^i \to U_1 \times U_2$ defined by $\pi_i = (\pi_1, \pi_{2,i})$. We have that π_1 and $\pi_{2,i}$ are isometries, so π_i is also an isometry. Define the function $f_i : U_1 \times U_2 \to \mathbb{R}$ by the relation $f_i = \tilde{d}_i \circ \pi_i^{-1}$. In fact, for $q_1 \in U_1, q_2 \in U_2$, it's clear that $f_i(q_1, q_2) = \tilde{d}(\tilde{q}_1, \tilde{q}_2^i)$, where $\tilde{q}_1 = \pi_1^{-1}(q_1)$ and $\tilde{q}_2^i = \pi_{2,i}^{-1}(q_2)$. The claim is that

$$d(q_1, q_2) = \min_{1 \le i \le k} f_i(q_1, q_2), \quad \text{for} \quad q_1 \in U_1, q_2 \in U_2. \tag{3.5}$$

For, notice that $d(q_1, q_2) = \min_{\tilde{q}_2 \in \pi^{-1}(q_2)} \tilde{d}(\tilde{q}_1, \tilde{q}_2)$. But

$$\tilde{d}(\tilde{q}_1, \tilde{q}_2^i) \le \tilde{d}(\tilde{q}_1, \tilde{p}_1) + \tilde{d}(\tilde{p}_1, \tilde{p}_2^i) + \tilde{d}(\tilde{p}_2^i, \tilde{q}_2^i) < \frac{\epsilon}{4} + d_0 + \frac{\epsilon}{4} = d_0 + \frac{\epsilon}{2}.$$

If $\tilde{q}_2 \in \pi^{-1}(q_2) \setminus \{\tilde{q}_2^1, \ldots, \tilde{q}_2^k\}$, then

$$\tilde{d}(\tilde{q}_1, \tilde{q}_2) \geq \tilde{d}(\tilde{p}_1, \tilde{p}) - \tilde{d}(\tilde{p}, \tilde{q}_2) - \tilde{d}(\tilde{q}_1, \tilde{p}_1) > d_0 + \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{4} = d_0 + \frac{\epsilon}{2},$$

where $\tilde{p}$ is the point of the fiber $\pi^{-1}(p_2)$ which is closest to $\tilde{q}_2$. The relation (3.5) is proved.

We have that $\widetilde{U}_2^i \cap \widetilde{U}_1 = \emptyset$, because $U_1 \cap U_2 = \emptyset$. By applying Lemma 2.3, we obtain that the functions $\tilde{d}_i$ are almost strictly convex, and that the lines $l_{\tilde{p}_i}^{\tilde{d}_i}$, $\tilde{p}_i = (\tilde{p}_1, \tilde{p}_2^i)$, are generated by the vectors $(\dot{\tilde{\gamma}}_i(0), \dot{\tilde{\gamma}}_i(1))$. From the equality

$$f_i \circ \gamma_i = (f_i \circ \pi_i) \circ (\pi_i^{-1} \circ \gamma_i) = \tilde{d}_i \circ \tilde{\gamma}_i,$$

and using the fact that π_i are isometries, it follows that the functions f_i are almost strictly convex with $l_p^{f_i} = (\pi_i)_*(l_{\tilde{p}_i}^{\tilde{d}_i})$, $p = (p_1, p_2)$. In order to apply Proposition 3.1 for the functions f_i, we have to prove that the conditions (i) and (ii) are satisfied.

The fact that $p = (p_1, p_2) \in M \times M$ is a local maximum for the restriction to $U_1 \times U_2$ of the distance function on $d : M \times M \to \mathbb{R}$, together with the equality (3.5), provide exactly the condition (i) for the functions f_i. This implies that $k \geq 2n \geq 4$. Next, we prove that, in this particular case, condition (ii) holds in the stronger form:

(ii') for every i_1, i_2, i_3 $(1 \leq i_1 \leq i_2 \leq i_3 \leq k)$,

$$l_p^{f_{i_1}} \cap l_p^{f_{i_2}} \cap l_p^{f_{i_3}} = \{0\}$$

For, suppose that there exists a non-zero vector $v = (v_1, v_2)$, $v \in l_p^{f_{i_1}} \cap l_p^{f_{i_2}} \cap l_p^{f_{i_3}}$. But the direction of the line $l_p^{f_i}$, $i \in \{i_1, i_2, i_3\}$, is given by the vector $(\dot{\gamma}_i(0), \dot{\gamma}_i(1))$. Since $v \neq 0$, one of its components is non-zero, too. Suppose $v_1 \neq 0$. It follows that $v_1 = \pm\dot{\gamma}_i(0)$, for all $i \in \{i_1, i_2, i_3\}$ (the geodesic segments joining p_1 and p_2 are parametrized proportionally to arc length, and they have the same length). But this implies that at least two of the considered geodesic segments have the same tangent vectors at p_1, which is impossible. This contradiction shows that the condition (ii') holds for the functions f_i. Proposition 3.1 can be applied in this case, which ends the proof of the theorem.

Remark 3.6. If we translate Remark 3.3 into the language of Theorem 3.4, it follows that the pairs of points for which the distance function has a local maximum are isolated in the product topology.

Theorem 3.7. *Let M be a complete Riemannian manifold of dimension $n \geq 2$ and non-positive curvature (all the sectional curvatures bounded above by $\chi \leq 0$), p_1 a fixed point in M. If $p_2 \in M$ is a local maximum for the function $f : M \to \mathbb{R}$, defined by $f(q) = d(q, p_1)$ (d is the distance function on M), then the points p_1 and p_2 are connected by at least $n+1$ distinct geodesic segments.*

Proof. We will use the notations from the proof of the previous theorem. The universal covering space $\widetilde{M}$ is again diffeomorphic to $\mathbb{R}^n$, and using the same construction we define the functions $f_i : U_2 \to \mathbb{R}$ by the relation $f_i(q_2) = \tilde{d}(\tilde{p}_1, \tilde{q}_2^i)$. The same argument as above gives that $f(q_2) = d(p_1, q_2) = \min_{1 \leq i \leq k} f_i(q_2)$, for $q_2 \in U_2$. Notice that in this case, Lemma 2.4 implies that the functions f_i are almost strictly convex, but also that the "bad" directions $l_{p_2}^{f_i}$ can be chosen arbitrarily. This means that condition (ii) from Proposition 3.1 is satisfied easily with a general choice of the directions $l_{p_2}^{f_i}$. Proposition 3.1 implies $k \geq n+1$, which ends the proof of the theorem. □

Remark 3.8. As one can see, the proofs of Theorems 3.4 and 3.7. are "semi-global". The conditions on curvature can be relaxed so that this is bounded above by the corresponding χ on the union of the metric balls of radii $d_0/2$ centered at p_1 and p_2, where $d_0 = d(p_1, p_2)$. In particular, the conclusion of Theorem 3.4 holds for the case when all the sectional curvatures are strictly negative, but with the upper bound $\chi = 0$.

Corollary 3.9. *On every compact Riemannian manifold of negative curvature, there are pairs of points connected by at least* $2n+1$ *distinct geodesic segments (for example, the points at maximal distance).*

Corollary 3.10. *On every compact Riemannian manifold of non-positive curvature, for any given point* $p_1 \in M$*, there exist a point* $p_2 \in M$ *(for example the point at maximal distance from* p_1*) and at least* $n+1$ *distinct geodesic segments connecting the two points.*

Remark 3.11. The flat torus example given in the introduction shows that Corollary 3.10 is the best one can expect in general for the case of non-positive curvature. But the flatness is a very restrictive situation. It would be interesting to find conditions under which the result can be improved (for example, if we set as a hypothesis that the two points which realize the diameter of the manifold are isolated in the product topology).

Remark 3.12. As one can see, the definitions and the proofs, except the proof of Proposition 3.1, are essentially metric. It is likely that the results are true without the differentiability hypothesis for spaces of non-positive curvature (see [1], [5]) which are n-dimensional topological manifolds. What one will need is the generalization of the tangent space for metric spaces, the so-called space of directions. The notion is discussed in [1], but the details of the proof of the analog of Proposition 3.1 seem to be more difficult. A case which probably can be studied directly is the case of polyhedra of non-positive curvature, where the same methods as above should work.

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Department of Mathematics, Duke University, Durham, North Carolina 27708-0320
E-mail address: horja@math.duke.edu