

ON A GENERAL FORM OF THE SECOND MAIN THEOREM

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ABSTRACT. We give a proof of a general form of the Second Main Theorem for holomorphic curves with a good error term. Two applications of this general form are also provided.

1. INTRODUCTION

In this paper, we give a proof of a general form of the Second Main Theorem for holomorphic curves in Nevanlinna theory (Theorem 2.3) by using the techniques of Ye [19] to get a good error term. This theorem was first proved by Vojta [15], but with a less precise error term. We then explain why this general form is more useful than the usual form of the Second Main Theorem by giving two applications. The first application is a simple proof of the Second Main Theorem with moving targets, originally obtained by Ru and Stoll [8]. We indicate that the general form of the SMT with fixed hyperplanes (Theorem 2.1) implies the Second Main Theorem with moving targets. The second application is to establish a Second Main Theorem for linearly degenerate holomorphic mappings with a good error term. We note that all the proofs in this paper work also when the domain is \mathbf{C}^m instead of \mathbf{C} .

2. A GENERAL FORM OF THE SECOND MAIN THEOREM WITH A GOOD ERROR TERM

Motivated by Schmidt's subspace theorem in number theory, Vojta [15] formulated the following general form of the Second Main Theorem for holomorphic curves. Here we use the standard notation in Nevanlinna theory.

Theorem 2.1 (A general form of the SMT). *Let $f = [f_0 : \cdots : f_n] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Let H_1, \dots, H_q be arbitrary hyperplanes in $\mathbf{P}^n(\mathbf{C})$. Then, for every $\epsilon > 0$,*

$$\int_0^{2\pi} \max_K \sum_{j \in K} \lambda_{H_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq (n+1+\epsilon)T_f(r),$$

where \leq means the inequality holds for all r outside of a set E with finite Lebesgue measure, the maximum is taken over all subsets K of $\{1, \dots, q\}$ such that the linear forms $H_j, j \in K$, are linearly independent, and $\lambda_H(f(re^{i\theta}))$ is defined as follows:

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for each hyperplane $H = \{[x_0 : \cdots : x_n] \in \mathbf{P}^n(\mathbf{C}) \mid a_0x_0 + \cdots + a_nx_n = 0\}$, denote $\|H\| = \max_{0 \leq j \leq n} |a_j|$ and $\|f(z)\| = \max_{0 \leq j \leq n} |f_j(z)|$; then

$$(2.1) \quad \lambda_H(f(re^{i\theta})) = \log \frac{\|f(re^{i\theta})\| \|H\|}{H(f(re^{i\theta}))}.$$

We note that the hyperplanes H_1, \dots, H_q are not assumed to lie in general position in the above theorem. By the product to sum estimate, it implies the following standard form of the Second Main Theorem for holomorphic curves.

Theorem 2.2 (SMT). *Let H_1, \dots, H_q be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position. Let $f : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Then, for any $\epsilon > 0$,*

$$\sum_{j=1}^q m_f(H_j, r) \leq (n+1+\epsilon)T_f(r),$$

where \leq means the inequality holds for all r outside of a set E with finite Lebesgue measure, and the proximity function $m_f(H, r)$ is defined as

$$m_f(H, r) = \int_0^{2\pi} \lambda_H(f(re^{i\theta})) \frac{d\theta}{2\pi}.$$

In this paper, we examine the error term which appeared in Theorem 2.1, and obtain the following result:

Theorem 2.3 (A general form of the SMT with a good error term). *Let $f = [f_0 : \cdots : f_n] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Let H_1, \dots, H_q be arbitrary hyperplanes in $\mathbf{P}^n(\mathbf{C})$. Let ψ and ϕ be increasing functions in \mathbf{R}^+ with*

$$\int_e^\infty \frac{dr}{r\psi(r)} < \infty, \text{ and } \int_e^\infty \frac{dr}{\phi(r)} = \infty.$$

Then the inequality

$$\begin{aligned} & \int_0^{2\pi} \max_K \sum_{k \in K} \lambda_{H_k}(f(re^{i\theta})) \frac{d\theta}{2\pi} + N(R_f, r) \\ & \leq (n+1)T_f(r) + \frac{n(n+1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1) \end{aligned}$$

holds for all r outside a set E with $\int_E dr/\phi(r) < \infty$. Here the maximum is taken over all subsets K of $\{1, \dots, q\}$ such that the linear forms H_j , $j \in K$, are linearly independent, and $N(R_f, r)$ is the ramification term.

The error term problem was first raised by S. Lang, motivated by its analogy with number theory. Since then, a number of results have been obtained; see Lang [4], Lang-Cherry [2], Wong [17], Wong-Stoll [18] and Ye [19]. Our proof here basically repeats Ye's argument (cf. [19]).

Recall a lemma in [19],

Lemma 2.4 (Lemma 6 of [19]). *Let g be a non-constant meromorphic function. For arbitrary α with $0 < \alpha < 1/2$, there exists a constant C, C_1, C_2 such that for*

any $r < \rho < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{g^{(l)}(re^{i\theta})}{g(re^{i\theta})} \right|^\alpha d\theta \leq C \left(\frac{\rho}{r(\rho-r)} \right)^{l\alpha} [C_1 T_g(\rho) + C_2 \log \frac{R}{R-\rho} T_g(\rho)]^{l\alpha}.$$

Remark. Thanks to William Cherry, (4.14) [19] is incorrect, thus, the above lemma is slightly different from Lemma 6 in [19]. To derive this lemma from Lemma 6 in [19], only (4.13) is used, which remains correct.

The following is a standard lemma in Nevanlinna theory; for reference, see [3].

Lemma 2.5. *Let ψ, ϕ be defined in Theorem 2.3 and T a non-decreasing function on \mathbf{R}^+ . Then there is a constant $C > 1$ such that*

$$T\left(r + \frac{\phi(r)}{\psi(T(r))}\right) \leq CT(r),$$

for all large r outside a set E with $\int_E dr/\phi(r) < \infty$.

We now prove Theorem 2.3.

Proof. Denote by $K \subset \{1, \dots, q\}$ such that linear forms $H_k, k \in K$, are linearly a set of indices independent. Without loss of generality, we may assume that $q \geq n + 1$ and $\#K = n + 1$. Let T be the set of all injective maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, q\}$ such that $H_{\mu(0)}, \dots, H_{\mu(n)}$ are linearly independent. Then

$$\begin{aligned} (2.2) \quad & \int_0^{2\pi} \max_K \sum_{k \in K} \lambda_{H_k}(f(re^{i\theta})) \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \max_{\mu \in T} \sum_{j=0}^n \log \left(\frac{\max_{0 \leq l \leq n} |f_l(re^{i\theta})|}{|H_{\mu(j)}(f(re^{i\theta}))|} \right) \frac{d\theta}{2\pi} + O(1) \\ &= \int_0^{2\pi} \log \left\{ \max_{\mu \in T} \left(\left(\max_{0 \leq l \leq n} |f_l(re^{i\theta})| \right)^{n+1} / \prod_{j=0}^n |H_{\mu(j)}(f(re^{i\theta}))| \right) \right\} \frac{d\theta}{2\pi} + O(1) \\ &\leq \int_0^{2\pi} \log \left\{ \sum_{\mu \in T} \left(\left(\max_{0 \leq l \leq n} |f_l(re^{i\theta})| \right)^{n+1} / \prod_{j=0}^n |H_{\mu(j)}(f(re^{i\theta}))| \right) \right\} \frac{d\theta}{2\pi} + O(1) \\ &= \int_0^{2\pi} \log \left\{ \sum_{\mu \in T} \frac{|W(H_{\mu(0)}(f), \dots, H_{\mu(n)}(f))|}{|H_{\mu(0)}(f)H_{\mu(1)}(f) \cdots H_{\mu(n)}(f)|} (re^{i\theta}) \right\} \frac{d\theta}{2\pi} \\ &+ \int_0^{2\pi} \log \left\{ \left(\max_{0 \leq l \leq n} |f_l(re^{i\theta})| \right)^{n+1} / |W(f_0, \dots, f_n)|(re^{i\theta}) \right\} \frac{d\theta}{2\pi} + O(1). \end{aligned}$$

In the above, we used the property of Wronskians that

$$|W(f_0, \dots, f_n)| = |W(H_{\mu(0)}(f), \dots, H_{\mu(n)}(f))| \cdot C,$$

where C is a constant. We estimate the first term on the right-hand side of (2.2). Denote

$$g_{\mu(l)}(z) = \frac{H_{\mu(l)}(f)}{H_{\mu(0)}(f)}(z), \quad 0 \leq l \leq n.$$

From the concavity of the logarithm, the Hölder inequality, Lemma 2.4 and the inequality $(\sum_{i,j} a_{ij})^\alpha \leq C \sum_{i,j} a_{ij}^\alpha$, for $a_{ij} \geq 0$ and any $\alpha > 0$, we have, for $\alpha n(n+1) < 1$,

(2.3)

$$\begin{aligned}
& \int_0^{2\pi} \log \left\{ \sum_{\mu \in T} \frac{|W(H_{\mu(0)}(f), \dots, H_{\mu(n)}(f))|}{|H_{\mu(0)}(f)H_{\mu(1)}(f) \cdots H_{\mu(n)}(f)|} (re^{i\theta}) \right\} \frac{d\theta}{2\pi} \\
& \leq \frac{1}{\alpha} \int_0^{2\pi} \log \sum_{\mu \in T} \left(\frac{|W(H_{\mu(0)}(f), \dots, H_{\mu(n)}(f))|}{|H_{\mu(0)}(f)H_{\mu(1)}(f) \cdots H_{\mu(n)}(f)|} (re^{i\theta}) \right)^\alpha \frac{d\theta}{2\pi} + O(1) \\
& = \frac{1}{\alpha} \int_0^{2\pi} \log \sum_{\mu \in T} \left(\frac{|W(1, g_{\mu(1)}, \dots, g_{\mu(n)})|}{|g_{\mu(1)} \cdots g_{\mu(n)}|} (re^{i\theta}) \right)^\alpha \frac{d\theta}{2\pi} + O(1) \\
& \leq \frac{1}{\alpha} \int_0^{2\pi} \log \left\{ \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq n(n+1)/2} \left| \frac{g_{\mu(1)}^{(i_1)} \cdots g_{\mu(n)}^{(i_n)}}{g_{\mu(1)} \cdots g_{\mu(n)}} \right|^\alpha (re^{i\theta}) \right\} \frac{d\theta}{2\pi} + O(1) \\
& \leq \frac{1}{\alpha} \log \int_0^{2\pi} \left\{ \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq n(n+1)/2} \left| \frac{g_{\mu(1)}^{(i_1)} \cdots g_{\mu(n)}^{(i_n)}}{g_{\mu(1)} \cdots g_{\mu(n)}} \right|^\alpha (re^{i\theta}) \right\} \frac{d\theta}{2\pi} + O(1) \\
& \leq \frac{1}{\alpha} \log \left\{ \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq n(n+1)/2} \prod_{l=1}^n \left(\int_0^{2\pi} \left| \frac{g_{\mu(l)}^{(i_l)}}{g_{\mu(l)}} \right|^{\alpha(n+1)} (re^{i\theta}) \frac{d\theta}{2\pi} \right)^{1/(n+1)} \right\} + O(1) \\
& \leq \frac{1}{\alpha} \log \left\{ \sum_{\mu \in T} \sum_{i_1 + \dots + i_n \leq n(n+1)/2} \prod_{l=1}^n \left(\left(\frac{\rho}{r(\rho-r)} \right)^{i_l \alpha} [C_1 T_{g_{\mu(l)}}(\rho) \right. \right. \\
& \qquad \qquad \qquad \left. \left. + C_2 \log \frac{R}{\rho(R-\rho)} T_{g_{\mu(l)}}(\rho) \right]^{i_l \alpha} \right) \right\} + O(1) \\
& \leq \frac{n(n+1)}{2} \log \left\{ \frac{\rho}{r(\rho-r)} \sum_{\mu \in T} [C_1 T_f(\rho) + C_2 \log \frac{R}{\rho(R-\rho)} T_f(\rho)] \right\} + O(1).
\end{aligned}$$

Taking

$$R = r + \frac{\phi(r)}{\psi(T_f(r))} \text{ and } \rho = \frac{R+r}{2} = r + \frac{\phi(r)}{2\psi(T_f(r))},$$

and applying Lemma 2.5 gives

$$\begin{aligned}
T_f(\rho) & \leq T_f(R) \leq CT_f(r), \\
\frac{R}{\rho(R-\rho)} & \leq \frac{2\Psi(T_f(r))}{\phi(R)}.
\end{aligned}$$

Recall that Ψ is the function such that $\int \frac{dx}{x\Psi(x)} < \infty$, and the smaller Ψ , the better. So WLOG, we assume that $\Psi(x) \leq x$. Thus we have

$$\frac{R}{\rho(R-\rho)} \leq CT_f(r).$$

Thus, (2.3) becomes

$$(2.4) \quad \int_0^{2\pi} \log \left\{ \sum_{\mu \in T} \frac{W(H_{\mu(0)}(f), \dots, H_{\mu(n)}(f))}{|H_{\mu(0)}(f)H_{\mu(1)}(f) \cdots H_{\mu(n)}(f)|} (re^{i\theta}) \right\} \frac{d\theta}{2\pi} \\ \leq \frac{n(n+1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1).$$

Now

$$(2.5) \quad \int_0^{2\pi} \log \left\{ \left(\max_{0 \leq l \leq n} |f_l(re^{i\theta})| \right)^{n+1} / |W(f_0, \dots, f_n)(re^{i\theta})| \right\} \frac{d\theta}{2\pi} \\ = \int_0^{2\pi} \log \left(\max_{0 \leq l \leq n} |f_l(re^{i\theta})| \right)^{n+1} \frac{d\theta}{2\pi} + \int_0^{2\pi} \log \frac{1}{|W(f_0, \dots, f_n)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ = (n+1)T_f(r) - N_W(0, r) \\ = (n+1)T_f(r) - N(R_f, r).$$

Combining (2.2), (2.4) and (2.5), we conclude the proof. □

3. APPLICATION I: A SIMPLE PROOF OF THE SECOND MAIN THEOREM FOR MOVING TARGETS

The purpose of this section is to show how the general form of SMT with fixed targets (Theorem 2.1) implies the SMT with moving targets which was originally proved by Ru-Stoll ([8]) in 1991. We note that the usual form of the SMT with fixed targets doesn't directly imply the SMT with moving targets, but the general form does! This is one of the motivations for writing this paper. This observation is also used in another paper, joint with Vojta ([10]), to prove Schmidt's subspace theorem with moving targets in number theory.

Theorem 3.1. *Let $f = [f_0 : \dots : f_n] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$ be a holomorphic map. Let \mathcal{G} be an arbitrary finite set of q holomorphic maps $H_j = [a_{j,0} : \dots : a_{j,n}] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})^*$. Let $\mathcal{R}_{\mathcal{G}}$ be the smallest field which contains \mathbf{C} and all $a_{j\mu}/a_{j\nu}$ with $a_{j\nu} \neq 0$. If f is non-degenerate over $\mathcal{R}_{\mathcal{G}}$, meaning that f_0, \dots, f_n are linearly independent over $\mathcal{R}_{\mathcal{G}}$, then for every $\epsilon > 0$, we have*

$$\int_0^{2\pi} \max_K \sum_{k \in K} \lambda_{H_k(re^{i\theta})}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq (n+1+\epsilon)T_f(r) + O\left(\max_{1 \leq j \leq q} T_{H_j}(r)\right),$$

where $\leq \cdot$ means the inequality holds except for those r contained in a set $E \subset \mathbf{R}_{>0}$ of finite Lebesgue measure, and where \max_K is taken over all subsets $K \subset \{1, \dots, q\}$ such that $H_k(z)$, for $k \in K$, are linearly independent for some (and hence for almost all) $z \in \mathbf{C}$.

The moving hyperplanes H_1, \dots, H_q are said to be *in general position* if $H_1(z), \dots, H_q(z)$ are in general position for some (and hence for almost all) $z \in \mathbf{C}$. By the "product to sum estimate", the above theorem implies the SMT with moving targets [8]. For the sake of completeness, we recall the "product to sum estimate" lemma.

Let \mathcal{G} be a finite set of $q \geq n + 1$ holomorphic maps $H_j = [a_{j,0} : \cdots : a_{j,n}] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})^*$, $1 \leq j \leq q$, such that H_1, \dots, H_q are in general position. Define

$$\Gamma(\mathcal{G})(z) = \min \left\{ \frac{\|\tilde{H}_{\mu(0)}(z) \wedge \cdots \wedge \tilde{H}_{\mu(n)}(z)\|}{\|\tilde{H}_{\mu(0)}(z)\| \cdots \|\tilde{H}_{\mu(n)}(z)\|} \mid \mu : \mathbf{Z}[0, n] \rightarrow \{1, \dots, q\} \right\},$$

for all $z \in \mathbf{C}$, where $\tilde{H}_j = (a_{j,0}, \dots, a_{j,n})$. By the general position assumption,

$$D = \{z \in \mathbf{C} \mid \Gamma(\mathcal{G})(z) = 0\}$$

is a closed set of isolated points.

Lemma 3.2 (product to sum estimate; cf. Theorem 6.2 of [8]). *Let $f = [f_0 : \cdots : f_n] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$ be a holomorphic map. Let \mathcal{G} be a finite set of holomorphic maps $H_j = [a_{j,0} : \cdots : a_{j,n}] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})^*$, $1 \leq j \leq q$, such that H_1, \dots, H_q are in general position. Take a point $z \in \mathbf{C} - D$ such that $H(f(z)) \neq 0$ for all $H \in \mathcal{G}$. Then there exist $i(z, 0), \dots, i(z, n)$ among $1, \dots, q$ such that*

$$\prod_{H \in \mathcal{G}} \frac{\|f(z)\| \|H(z)\|}{|H(f(z))|} \leq \left(\frac{2^{3n+3}}{\Gamma(\mathcal{G})(z)} \right)^{q-n-1} \prod_{l=0}^n \frac{\|f(z)\| \|H_{i(z,l)}(z)\|}{|H_{i(z,l)}(f(z))|}.$$

Proof. Given any $z \in \mathbf{C} - D$, we have fixed hyperplanes $\{H(z) \mid H \in \mathcal{G}\}$, and they are in general position. It is well-known (cf. Theorem 6.1 of [8]) that for the number $a = (1/2)^{3n+3} \Gamma(\mathcal{G})(z)$ we have $\#I(f(z), a, \mathcal{G}) \leq n$, where

$$I(f(z), a, \mathcal{G}) = \left\{ j \mid \frac{|H_j(f(z))|}{\|f(z)\| \|H_j(z)\|} \leq a, 1 \leq j \leq q \right\}.$$

Choose $i(z, 0), \dots, i(z, n) \in \{1, \dots, q\}$ such that $I(f(z), a, \mathcal{G}) \subset \{i(z, 0), \dots, i(z, n)\}$; then

$$\begin{aligned} \prod_{H \in \mathcal{G}} \frac{\|f(z)\| \|H(z)\|}{|H(f(z))|} &\leq \left(\frac{1}{a} \right)^{q-n-1} \prod_{l=0}^n \frac{\|f(z)\| \|H_{i(z,l)}(z)\|}{|H_{i(z,l)}(f(z))|} \\ &= \left(\frac{2^{3n+3}}{\Gamma(\mathcal{G})(z)} \right)^{q-n-1} \prod_{l=0}^n \frac{\|f(z)\| \|H_{i(z,l)}(z)\|}{|H_{i(z,l)}(f(z))|}. \end{aligned}$$

□

It is now clear that Theorem 3.1, together with Lemma 3.2, implies the SMT with moving targets.

Theorem 3.3 (SMT with moving targets; cf. [8]). *Under the same assumptions in Theorem 3.1, if, in addition, we assume that the hyperplanes $H_j = [a_{j,0} : \cdots : a_{j,n}] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})^*$, $1 \leq j \leq q$, are in general position, then for every $\epsilon > 0$, we have*

$$\sum_{i=1}^q m_f(H_i, r) \leq (n + 1 + \epsilon) T_f(r) + O \left(\max_{1 \leq j \leq q} T_{H_j}(r) \right),$$

where \leq means the inequality holds for all r outside of a subset $E \subset \mathbf{R}_{>0}$ with finite Lebesgue measure.

We now prove Theorem 3.1.

Proof. Without loss of generality, we can assume $q \geq n + 1$, and that at least $n + 1$ of the hyperplanes are linearly independent. Let T be the set of all maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, q\}$ such that $H_{\mu(0)}(z), \dots, H_{\mu(n)}(z)$ are linearly independent for some (thus for almost all) $z \in \mathbf{C}$. Without loss of generality we assume that (f_0, \dots, f_n) is a reduced representation for f , meaning that f_j are entire and without common zeros. For each $1 \leq j \leq q$, choose \bar{j} so that $a_{j,\bar{j}} \neq 0$, and define

$$(3.1) \quad \zeta_{j,l}(z) = a_{j,l}(z)/a_{j,\bar{j}}(z), \quad j = 1, \dots, q; \quad l = 0, \dots, n.$$

Let $\mathcal{L}(s)$ be the vector space generated over \mathbf{C} by

$$\left\{ \zeta_{1,0}^{n_{1,0}} \cdots \zeta_{q,0}^{n_{q,0}} \cdots \zeta_{1,n}^{n_{1,n}} \cdots \zeta_{q,n}^{n_{q,n}} \mid n_{i,j} \in \mathbf{N}, \sum_{i=1}^q \sum_{j=0}^n n_{i,j} = s \right\}.$$

We have $\mathcal{L}(s) \subset \mathcal{L}(s + 1)$. Let $\{b_1, \dots, b_{\ell(s+1)}\}$ be a basis of $\mathcal{L}(s + 1)$ such that $\{b_1, \dots, b_{\ell(s)}\}$ is a basis of $\mathcal{L}(s)$, where $\ell(s) = \dim \mathcal{L}(s)$. Let $F = \mathbf{P}(\tilde{F}) : \mathbf{C} \rightarrow \mathbf{P}(\mathbf{C}^{(n+1)\ell(s+1)})$ be the holomorphic map defined by

$$(3.2) \quad \begin{aligned} \tilde{F}(z) &= (b_1(z)f_0(z), \dots, b_{\ell(s+1)}(z)f_0(z), b_1(z)f_1(z), \dots, b_{\ell(s+1)}(z)f_n(z)) \\ &\in \mathbf{C}^{(n+1)\ell(s+1)}. \end{aligned}$$

Because f is linearly nondegenerate over $\mathcal{R}_{\mathcal{G}}$, $F(\mathbf{C})$ is not contained in any hyperplanes in $\mathbf{P}(\mathbf{C}^{(n+1)\ell(s+1)})$. We will apply the general form of the SMT (Theorem 2.1) to F . The next step is to construct, for each $\mu \in T$, fixed hyperplanes $\{\hat{H}_{l,j}(\mu) \mid l = 0, \dots, n; j = 1, \dots, \ell(s)\}$ in $\mathbf{P}(\mathbf{C}^{(n+1)\ell(s+1)})$ such that $\lambda_{\hat{H}_{l,j}(\mu)}(F(z)) \sim \lambda_{H_{\mu(l)}(z)}(f(z))$. For $j = 1, \dots, q$, let h_j be the meromorphic function defined by

$$(3.3) \quad h_j(z) = \zeta_{j,0} + \sum_{l=1}^n \zeta_{j,l}(z) \frac{f_l(z)}{f_0(z)} \quad (j = 1, \dots, q).$$

The $b_j h_{\mu(l)} (j = 1, \dots, \ell(s), l = 0, \dots, n)$ can be written as \mathbf{C} -linear combinations of $b_r, 1 \leq r \leq \ell(s + 1)$, and the products $b_{\alpha}(f_{\beta}/f_0)$ for $\alpha = 1, \dots, \ell(s + 1)$ and $\beta = 0, \dots, n$. In other words there is an $(n + 1)\ell(s) \times ((n + 1)\ell(s + 1))$ matrix $C(\mu)$ with entries in \mathbf{C} such that

$$(3.4) \quad \begin{pmatrix} b_1 h_{\mu(0)} \\ \vdots \\ b_{\ell(s)} h_{\mu(0)} \\ \vdots \\ b_1 h_{\mu(n)} \\ \vdots \\ b_{\ell(s)} h_{\mu(n)} \end{pmatrix} = C(\mu) \begin{pmatrix} b_1 \\ \vdots \\ b_{\ell(s+1)} \\ b_1 (f_1/f_0) \\ \vdots \\ b_{\ell(s+1)} (f_1/f_0) \\ \vdots \\ b_1 (f_n/f_0) \\ \vdots \\ b_{\ell(s+1)} (f_n/f_0) \end{pmatrix}.$$

For $l = 0, \dots, n, j = 1, \dots, \ell(s)$, let $\hat{H}_{l,j}(\mu)$ be the hyperplane in $\mathbf{P}(\mathbf{C}^{(n+1)\ell(s+1)})$ defined by the corresponding row in $C(\mu)$; i.e., if $c_{ij}(\mu)$ denote the elements of

$C(\mu)$, then

$$\begin{aligned}
 \hat{H}_{l,j}(\mu) &= \{ [y_{1,0} : \cdots : y_{\ell(s),0} : y_{1,1} : \cdots : y_{\ell(s+1),1} : \cdots : y_{1,n} : \cdots : y_{\ell(s+1),n}] \\
 (3.5) \quad &\in \mathbf{P}(\mathbf{C}^{(n+1)\ell(s+1)}) \left| \begin{array}{l} c_{l\ell(s)+j,1}(\mu)y_{1,0} + \cdots + c_{l\ell(s)+j,\ell(s+1)}(\mu)y_{\ell(s),0} \\ + c_{l\ell(s)+j,\ell(s+1)+1}(\mu)y_{1,1} + \cdots + c_{l\ell(s)+j,2\ell(s+1)}(\mu)y_{\ell(s+1),1} \\ + \cdots + c_{l\ell(s)+j,(n+1)\ell(s+1)}(\mu)y_{\ell(s+1),n} = 0 \end{array} \right\}.
 \end{aligned}$$

Since $H_{\mu(0)}(z), \dots, H_{\mu(n)}(z)$ are linearly independent for some z and f_0, \dots, f_n are linearly independent over $\mathcal{R}_{\mathcal{G}}$, $h_{\mu(0)}, \dots, h_{\mu(n)}$ are linearly independent over $\mathcal{R}_{\mathcal{G}}$. Thus, by the choice of $b_1, \dots, b_{\ell(s)}$, $(b_j h_{\mu(l)})_{j=1, \dots, \ell(s); l=0, \dots, n}$ are linearly independent over \mathbf{C} . Hence, $\hat{H}_{l,j}(\mu), l = 0, \dots, n, j = 1, \dots, \ell(s)$, are linearly independent for each $\mu \in T$. Applying Theorem 2.1 for F , with the hyperplanes $\{\hat{H}_{l,j}(\mu) | l = 0, \dots, n; j = 1, \dots, \ell(s)\}$, yields

$$(3.6) \quad \int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^n \sum_{j=1}^{\ell(s)} \lambda_{\hat{H}_{l,j}(\mu)}(F(re^{i\theta})) \frac{d\theta}{2\pi} \leq ((n+1)\ell(s+1) + \epsilon/2) T_F(r).$$

We now compare $T_f(r)$ and $T_F(r)$. In fact, for each $z \in \mathbf{C}$, not in the set of the poles of $b_1, \dots, b_{\ell(s+1)}$, we have

$$\begin{aligned}
 \|\tilde{F}(z)\| &= \max(|b_1(z)f_0(z)|, \dots, |b_{\ell(s+1)}(z)f_0(z)|, |b_1(z)f_1(z)|, \dots, |b_{\ell(s+1)}(z)f_n(z)|) \\
 &\quad + O(1) \\
 &= \max(|f_0(z)|, \dots, |f_n(z)|) \cdot \max(|b_1(z)|, \dots, |b_{\ell(s+1)}(z)|) \\
 &= \|f\| \cdot \max(|b_1(z)|, \dots, |b_{\ell(s+1)}(z)|) + O(1).
 \end{aligned}$$

By first main theorem

$$\begin{aligned}
 T_F(r) &= \int_0^{2\pi} \log \frac{\|\tilde{F}\|}{|b_1 f_0|}(re^{i\theta}) d\theta + N(r, [b_1 f_0 = 0]) \\
 &\quad - N(r, [1/b_1 = 0]) + O(1) \\
 &= \int_0^{2\pi} \log \frac{\|f\|}{|f_0|}(re^{i\theta}) d\theta + N(r, [f_0 = 0]) + O\left(\max_{1 \leq q} T_{H_i}(r)\right).
 \end{aligned}$$

Therefore

$$(3.7) \quad T_F(r) = T_f(r) + O\left(\max_{1 \leq q} T_{H_i}(r)\right).$$

Next we compare $\lambda_{H_{\mu(l)}(z)}(f(z))$ and $\lambda_{\hat{H}_{l,j}(\mu)}(F(z))$, for each $\mu \in T$. By (2.1), (3.1), (3.2), (3.3), (3.4), and (3.5)

$$\begin{aligned} &\lambda_{\hat{H}_{l,j}(\mu)}(F(z)) \\ &= -\log \frac{|c_{l\ell(s)+j,1}(\mu)b_1(z)f_0(z) + \cdots + c_{l\ell(s)+j,(n+1)\ell(s+1)}(\mu)b_{\ell(s+1)}(z)f_n(z)|}{\max_{0 \leq r \leq n} |f_r(z)| \cdot \max_{1 \leq \alpha \leq \ell(s+1)} |b_\alpha(z)| \cdot \max_{1 \leq r \leq (n+1)\ell(s+1)} |c_{l\ell(s)+j,r}(\mu)|} \\ &= -\log \frac{|b_j(z)h_{\mu(l)}(z)| \cdot |f_0(z)|}{\max_{0 \leq r \leq n} |f_r(z)| \cdot \max_{1 \leq \alpha \leq \ell(s+1)} |b_\alpha(z)| \cdot \max_{1 \leq r \leq (n+1)\ell(s+1)} |c_{l\ell(s)+j,r}(\mu)|} \\ &= -\log \frac{|b_j(z)| \cdot |\zeta_{\mu(l),0}(z)f_0(z) + \cdots + \zeta_{\mu(l),n}(z)f_n(z)|}{\max_{0 \leq r \leq n} |f_r(z)| \cdot \max_{1 \leq \alpha \leq \ell(s+1)} |b_\alpha(z)| \cdot \max_{1 \leq r \leq (n+1)\ell(s+1)} |c_{l\ell(s)+j,r}(\mu)|} \\ &= \lambda_{H_{\mu(l)}(z)}(f(z)) - \log \frac{\max_{0 \leq r \leq n} |a_{\mu(l),r}(z)|}{|a_{\mu(l),\bar{\mu}(l)}(z)|} - \log \frac{|b_j(z)|}{\max_{1 \leq \alpha \leq \ell(s+1)} |b_\alpha(z)|} + O(1), \end{aligned}$$

where the above equality holds for those $z \in \mathbf{C}$ such that $\|F(z), \hat{H}_{l,j}(\mu)\| \neq 0, \|f(z), H_{\mu(l)}(z)\| \neq 0, f_0(z) \neq 0$, and such that z is not in the union of the sets of the zeros and poles of $\zeta_{j,l}$ and $b_r, 1 \leq j \leq q, 0 \leq l \leq n, 1 \leq r \leq \ell(s+1)$. Combining this with (3.6) and (3.7) gives

$$\begin{aligned} &\ell(s) \int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^n \lambda_{H_{\mu(l)}(re^{i\theta})}(f(re^{i\theta})) \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^n \sum_{j=1}^{\ell(s)} \lambda_{\hat{H}_{l,j}(\mu)}(F(re^{i\theta})) \frac{d\theta}{2\pi} + O\left(\max_{1 \leq i \leq q} T_{H_i}(r)\right) \\ &\leq \cdot ((n+1)\ell(s+1) + \epsilon/2) T_f(r) + O\left(\max_{1 \leq j \leq q} T_{H_j}(r)\right). \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^n \lambda_{H_{\mu(l)}(re^{i\theta})}(f(re^{i\theta})) \frac{d\theta}{2\pi} \\ &\leq \cdot ((1+n)\ell(s+1)/\ell(s) + \epsilon/2\ell(s)) T_f(r) + O\left(\max_{1 \leq j \leq q} T_{H_j}(r)\right). \end{aligned}$$

By Steinmetz [12],

$$\liminf_{s \rightarrow \infty} \frac{\ell(s+1)}{\ell(s)} = 1.$$

Thus, for the given $\epsilon > 0$ there exists an $s \in \mathbf{Z}_{>0}$ such that

$$\ell(s+1)/\ell(s) < 1 + \epsilon/2(n+1).$$

Choosing such an s concludes our proof. □

4. APPLICATION 2: THE SMT FOR LINEARLY DEGENERATE HOLOMORPHIC MAPPINGS WITH A GOOD ERROR TERM

In this section we will derive the Second Main Theorem with a good error term for the holomorphic curves whose image is contained in some k -dimensional subspace

of $\mathbf{P}^n(\mathbf{C})$. By using Nochka weights, we reduce the problem to Theorem 2.3, and so we regard this section as the second application of the general form of the Second Main Theorem (Theorem 2.3).

Theorem 4.1. *Let $f = [f_0 : \cdots : f_n] : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$ be a holomorphic map whose image is contained in some k -dimensional subspace but not in any subspace of dimension lower than k . Let $H_j = [a_{j,0} : \cdots : a_{j,n}]$, $1 \leq j \leq q$, be hyperplanes in general position. Let ψ and ϕ be increasing functions in \mathbf{R}^+ with*

$$\int_e^\infty \frac{dr}{r\psi(r)} < \infty, \text{ and } \int_e^\infty \frac{dr}{\phi(r)} = \infty.$$

Then, we have

$$\begin{aligned} & \sum_{i=1}^q m_{f,H_i}(r) + \binom{n+1}{k+1} N(R_f, r) \\ & \leq (2n - k + 1)T_f(r) + \frac{k(2n - k + 1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1), \end{aligned}$$

for all large r outside a set E with $\int_E dr/\phi(r) < \infty$.

Proof. Without loss of generality, we may assume that $f(\mathbf{C}) \subset \mathbf{P}^k(\mathbf{C})$. So $f : \mathbf{C} \rightarrow \mathbf{P}^k(\mathbf{C})$ is a non-degenerate holomorphic map. We also assume that $q \geq 2n - k + 1$. Denote $\hat{H}_j = H_j \cap \mathbf{P}^k(\mathbf{C})$. Then $\hat{H}_1, \dots, \hat{H}_q$ are in n -subgeneral position, i.e. for every set $P \subset \{1, \dots, q\}$ with $\#P = n + 1$ there are $\mu(0), \dots, \mu(k) \in P$ such that $\hat{H}_{\mu(0)}, \dots, \hat{H}_{\mu(k)}$ are linearly independent.

Recall the the following lemma about Nochka weights; for details, see [5], [11].

Lemma 4.2 (cf. [11]). *Let $\mathcal{B} = \{b_j \in \mathbf{P}^k(\mathbf{C}^*), 1 \leq j \leq q\}$ be a set of hyperplanes in n -subgeneral position with $2n - k + 1 \leq q$. For each $\emptyset \neq P \subset \{1, \dots, q\}$ let $L(P)$ be the linear space generated by $\{\tilde{b}_j | j \in P\}$, where \tilde{b}_j is a reduced representation of b_j . Then there exist a function $\omega : \{1, \dots, q\} \rightarrow \mathbf{R}(0, 1]$, called a Nochka weight, and a real number $\theta \geq 1$, called a Nochka constant, satisfying the following properties:*

- (i) *If $j \in \{1, \dots, q\}$, then $0 < \omega(j)\theta \leq 1$.*
- (ii) *$q - 2n + k - 1 = \theta(\sum_{j=1}^q \omega(j) - k - 1)$.*
- (iii) *If $\emptyset \neq P \subset \{1, \dots, q\}$ with $\#P \leq n + 1$, then $\sum_{j \in P} \omega(j) \leq \dim L(P)$.*
- (iv) *$1 \leq (n + 1)/(k + 1) \leq \theta \leq (2n - k + 1)/(k + 1)$.*
- (v) *Given E_1, \dots, E_q , real numbers ≥ 1 , and given any $Y \subset \{1, \dots, q\}$ with $0 < \#Y \leq n + 1$, there exists a subset M of Y with $\#M = \dim L(Y)$ such that $\{\tilde{b}_j\}_{j \in M}$ is a basis for $L(Y)$ and*

$$\prod_{j \in Y} E_j^{\omega(j)} \leq \prod_{j \in M} E_j.$$

Similarly to the proof of Lemma 3.2, since H_1, \dots, H_q are in general position, for the number $a = (1/2)^{3n+3}\Gamma(\mathcal{G})$ we have $\#I(f(z), a, \mathcal{G}) \leq n$, where

$$I(f(z), a, \mathcal{G}) = \left\{ j \mid \frac{|H_j(f(z))|}{\|f(z)\| \|H_j\|} \leq a, 1 \leq j \leq q \right\}.$$

Choose $i(z, 0), \dots, i(z, n) \in \{1, \dots, q\}$ such that $I(f(z), a, \mathcal{G}) \subset \{i(z, 0), \dots, i(z, n)\}$, and then

$$(4.1) \quad \prod_{j=1}^q \left(\frac{\|f(z)\| \|H_j\|}{|H_j(f(z))|} \right)^{\omega(j)} \leq \left(\frac{1}{a} \right)^{\sum_{j \neq i(z,l)} \omega(j)} \prod_{l=0}^n \left(\frac{\|f(z)\| \|H_{i(z,l)}\|}{|H_{i(z,l)}(f(z))|} \right)^{\omega(i(z,l))},$$

where $\omega(j)$ is the Nochka weight corresponding to \hat{H}_j . Applying Lemma 4.2 with $E_l = e^{\lambda_{\hat{H}_{i(z,l)}}(f(z))}$, $0 \leq l \leq n$, there is a subset M of $Y = \{i(z, 0), \dots, i(z, n)\}$ with $\#M = k + 1$ such that $\{\hat{H}_{i(z,j)} | i(z, j) \in M\}$ is linearly independent, and

$$\prod_{l=0}^n e^{\omega(i(z,l)) \lambda_{\hat{H}_{i(z,l)}}(f(z))} \leq \prod_{i(z,j) \in M} e^{\lambda_{\hat{H}_{i(z,j)}}(f(z))}.$$

Thus

$$\sum_{l=0}^n \omega(i(z,l)) \lambda_{\hat{H}_{i(z,l)}}(f(z)) \leq \sum_{i(z,j) \in M} \lambda_{\hat{H}_{i(z,j)}}(f(z)) \leq \max_{\gamma \in \Gamma} \sum_{l=0}^k \lambda_{\hat{H}_{\gamma(l)}}(f(z)),$$

where Γ is the set of all maps $\gamma : \{0, \dots, k\} \rightarrow \{1, \dots, q\}$ such that $\hat{H}_{\gamma(0)}, \dots, \hat{H}_{\gamma(k)}$ are linearly independent. Combining with (4.1) gives

$$\sum_{j=1}^q \omega(j) m_f(H_j, r) \leq \int_0^{2\pi} \max_{\gamma \in \Gamma} \sum_{l=0}^k \lambda_{\hat{H}_{\gamma(l)}}(f(re^{i\theta})) \frac{d\theta}{2\pi} + O(1).$$

Applying Theorem 2.3 yields

$$\begin{aligned} & \int_0^{2\pi} \max_{\gamma \in \Gamma} \sum_{l=0}^k \lambda_{\hat{H}_{\gamma(l)}}(f(re^{i\theta})) \frac{d\theta}{2\pi} \\ & \leq \cdot (k + 1)T_f(r) - N(R_f, r) + \frac{k(k + 1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{j=1}^q \omega(j) m_f(H_j, r) & \leq \cdot (k + 1)T_f(r) - N(R_f, r) \\ & \quad + \frac{k(k + 1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1). \end{aligned}$$

Combining this with Lemma 4.2, and recalling that $m_f(H_j, r) \leq T_f(r) + O(1)$, we have

$$\begin{aligned}
\sum_{j=1}^q m_f(H_j, r) &= \sum_{j=1}^q (1 - \theta\omega(j))m_f(H_j, r) + \sum_{j=1}^q \theta\omega(j)m_f(H_j, r) \\
&\leq \sum_{j=1}^q (1 - \theta\omega(j))m_f(H_j, r) + \theta(k+1)T_f(r) - \theta N(R_f, r) \\
&\quad + \theta \frac{k(k+1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1) \\
&\leq \sum_{j=1}^q (1 - \theta\omega(j))T_f(r) + \theta(k+1)T_f(r) - \left(\frac{n+1}{k+1}\right) N(R_f, r) \\
&\quad + \frac{(2n-k+1)k}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1) \\
&= \left\{ q - \theta \left(\sum_{1 \leq j \leq q} \omega(j) - k - 1 \right) \right\} T_f(r) - \left(\frac{n+1}{k+1}\right) N(R_f, r) \\
&\quad + \frac{(2n-k+1)k}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1) \\
&= (2n-k+1)T_f(r) - \left(\frac{n+1}{k+1}\right) N(R_f, r) \\
&\quad + \frac{(2n-k+1)k}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1).
\end{aligned}$$

□

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