

## RESTRICTION OF STABLE BUNDLES IN CHARACTERISTIC $p$

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ABSTRACT. Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $X$  be a nonsingular projective variety defined over  $k$  and  $H$  an ample line bundle on  $X$ . We shall prove that there exists an explicit number  $m_0$  such that if  $E$  is a  $\mu$ -stable vector bundle of rank at most three, then the restriction  $E|_D$  is  $\mu$ -stable for all  $m \geq m_0$  and all smooth irreducible divisors  $D \in |mH|$ . This result has implications to the geometry of the moduli space of  $\mu$ -stable bundles on a surface or a projective space.

### INTRODUCTION

Let  $X$  be a nonsingular projective variety of dimension  $n$  defined over an algebraically closed field  $k$ , and let  $E$  be a vector bundle of rank  $r$  on  $X$ . The problem of whether the  $\mu$ -(semi)stability of  $E$  with respect to an ample divisor  $H$  is preserved under restriction to a divisor  $D \in |mH|$  has been studied intensively in recent years. When the characteristic of  $k$  is arbitrary, a fundamental theorem of Mehta-Ramanathan states that the answer is positive for a general member  $D$  and sufficiently large  $m$  ([M-R1], [M-R2]). However, most of the important results have been proved under the assumption that the characteristic is zero. For example, H. Flenner's theorem on the effective estimate of  $m$  has no known analogue in positive characteristic ([F]).

In this paper we would like to contribute to the restriction problem when  $\text{char } k = p > 0$ . Under the assumption that  $r = 2$  or  $3$ , we shall give an explicit number  $m_0$  such that for all  $m \geq m_0$ , every  $\mu$ -stable bundle  $E$  with fixed Chern classes remains  $\mu$ -stable when restricted to arbitrary smooth irreducible divisors  $D \in |mH|$ . As a consequence, we obtain the restriction morphism between moduli spaces of  $\mu$ -stable bundles

$$j_D : M_X(r, c_i) \rightarrow M_D(r, c_{i|D}),$$

where  $M_X(r, c_i)$  denotes the moduli space of  $\mu$ -stable bundles  $E$  of rank  $r$  ( $r = 2, 3$ ) with Chern classes  $c_i(E) = c_i$ . We remark that a restriction theorem with no genericity assumption on  $D$  has been obtained by F. Bogomolov if  $n = 2$  and  $\text{char } k = 0$  ([B1], [B2]). Recently, A. Moriwaki has generalized the result to higher dimensional varieties ([Mo3]).

The lower bound  $m_0$  mentioned above can be expressed as a function of  $\delta_H(E)$  and  $\mu_H(X)$ , where  $\delta_H(E) = \{2rc_2(E) - (r-1)c_1(E)^2\} \cdot H^{n-2}$  and  $\mu_H(X)$  denotes the minimal slope of the Harder-Narasimhan filtration of the tangent bundle of  $X$ . The appearance of the number  $\mu_H(X)$  comes from our use of an analogue of

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Bogomolov-Gieseker inequality, which has been proved for rank two bundles in [N1]. The inequality is described as follows. If  $E$  is a  $\mu$ -semistable torsion-free sheaf of rank  $r = 2$  or  $3$ , we have  $\delta_H(E) \geq 0$  if  $\mu_H(X) \geq 0$ , and otherwise

$$\delta_H(E) \geq -\alpha \frac{\mu_H(X)^2}{p^2 H^n},$$

where  $\alpha = 1$  if  $r = 2$ ,  $\alpha = 9$  if  $r = 3$ . Inequalities of different types have been obtained for rank two bundles on surfaces ([S-B]) and non-uniruled varieties ([Mo2]).

It is natural to ask whether the results in the present paper generalize to bundles of rank greater than three. This problem is closely related to the boundedness of the family of  $\mu$ -semistable sheaves in positive characteristic, which is at present unknown except when  $r \leq 3$  or  $n = 2$ . Roughly speaking, the restriction theorem, the Bogomolov-Gieseker inequality and the boundedness may be considered as almost equivalent properties of  $\mu$ -semistable sheaves.

In the first section we will prove analogues of the Bogomolov-Gieseker inequality. The restriction theorem will be proved in section two. This result will be applied in the last section to the study of moduli spaces of  $\mu$ -stable bundles on surfaces and projective spaces.

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## 1. THE BOGOMOLOV-GIESEKER INEQUALITY

Throughout this paper all varieties are defined over an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $X$  be a nonsingular projective variety of dimension  $n \geq 2$  over  $k$  and  $E$  a rank  $r$  torsion-free sheaf on  $X$ . For an ample line bundle  $H$  on  $X$ , we define the slope of  $E$  by

$$\mu_H(E) = \frac{c_1(E) \cdot H^{n-1}}{r}$$

and let

$$\delta_H(E) := \{2rc_2(E) - (r-1)c_1(E)^2\} \cdot H^{n-2}.$$

Even if  $E$  is  $\mu$ -semistable, the Bogomolov-Gieseker inequality  $\delta_H(E) \geq 0$  does not hold in general. Nevertheless, one expects that some lower bound for  $\delta_H(E)$ , which depends on  $X$  and  $H$ , may exist. In fact, the boundedness of semistable sheaves implies the existence of a lower bound. For rank two  $\mu$ -semistable bundles, an effective lower bound has been given in [N1]. The purpose of this section is to extend the result to  $\mu$ -semistable torsion-free sheaves of rank at most three.

A torsion-free sheaf  $E$  is said to be *p-semistable* with respect to  $H$  if, for all  $m \geq 0$ , the  $m$ -th iterated Frobenius pull-back  $(F^m)^*E$  is  $\mu$ -semistable with respect to  $H$ . The following result has been proved for vector bundles in [Mo1, Theorem 1], and its generalization to torsion-free sheaves poses no significant problems.

**Proposition (1.1).** *Let  $X$  be a nonsingular projective variety of dimension  $n \geq 2$ . Let  $E$  be a torsion-free sheaf of rank  $r$  on  $X$  which is  $p$ -semistable with respect to  $H$ . Then  $\delta_H(E) \geq 0$  if  $r \leq 3$ .*

Let  $T_X$  denote the tangent bundle of  $X$  and let

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{s-1} \subset T_s = T_X$$

be the Harder-Narasimhan filtration (H-N filtration for short) with respect to  $H$ . We define

$$\mu_H(X) := \mu_H(T_X/T_{s-1}).$$

**Proposition (1.2).** *Assume that  $\mu_H(X) \geq 0$ , and let  $E$  be a  $\mu$ -semistable torsion-free sheaf of rank  $r$ . Then  $\delta_H(E) \geq 0$  if  $r \leq 3$ .*

Proof. It suffices to show that under the assumption  $\mu_H(X) \geq 0$ , every  $\mu$ -semistable sheaf  $E$  is  $p$ -semistable. Assume that  $E$  is not  $p$ -semistable, and let  $m$  be the smallest integer such that  $(F^m)^*E$  is unstable. Let

$$0 = E_0 \subset E_1 \subset \dots \subset E_{s-1} \subset E_s = (F^m)^*E$$

be the H-N filtration of  $(F^m)^*E$  and let  $G_i = E_i/E_{i-1}$ . By arguing as in the proof of [N1, Lemma 1], for  $0 < i < s$  we obtain a non-trivial  $\mathcal{O}_X$ -homomorphism

$$f_i : T_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(E_1, (F^m)^*E/E_1).$$

It follows that we can find  $0 < i < j \leq s$  and a non-zero map

$$f : T_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(G_i, G_j).$$

Since at least one of  $G_i$  and  $G_j$  is of rank one, the latter sheaf is  $\mu$ -semistable. Hence we obtain  $\mu_H(X) \leq \mu_H(G_j) - \mu_H(G_i) < 0$ , which is a contradiction. This completes the proof. □

*Remark.* The condition  $\mu_H(X) \geq 0$  is satisfied for varieties with nef tangent bundle  $T_X$  (that is, the tautological bundle  $\mathcal{O}(1)$  on  $\mathbb{P}(T_X)$  is nef). Another example is a Fano  $n$ -fold  $X$  such that  $T_X$  is  $\mu$ -semistable with respect to  $H = -K_X$ : we have  $\mu_H(X) = (-K_X)^n/n > 0$  in this case. If  $X$  is a del Pezzo surface, the  $\mu$ -semistability of  $T_X$  has been proved ([Fa]).

**Corollary (1.3).** *Let  $X$  be a Fano threefold. If the tangent bundle of  $X$  is  $\mu$ -semistable with respect to  $-K_X$ , then  $c_1(X)^3 \leq 72$ .*

Proof. Since we have  $\mu_H(X) = \mu_H(T_X) = (-K_X)^3/3 > 0$ , Proposition (1.2) yields

$$\{3c_2(X) - c_1(X)^2\} \cdot c_1(X) \geq 0.$$

We have  $H^3(X, \mathcal{O}_X) \cong H^0(X, K_X) = 0$ . Further, by [N1, Corollary 3]

$$H^2(X, \mathcal{O}_X) \cong H^1(X, K_X) = 0.$$

Hence  $\chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X) \leq 1$  and, by the Riemann-Roch formula,

$$1 \geq \frac{1}{24}c_1(X)c_2(X).$$

Therefore we obtain  $c_1(X)^3 \leq 72$ , as desired. □

In view of Proposition (1.1), the following can be proved as in [N1, Theorem 1].

**Proposition (1.4).** *Let  $X$  be a nonsingular projective variety of dimension  $n \geq 2$ . Let  $E$  be a rank two torsion-free sheaf which is  $\mu$ -semistable with respect to  $H$ . Then*

1. *If  $\mu_H(X) \geq 0$ , then  $\delta_H(E) \geq 0$ .*
2. *If  $\mu_H(X) < 0$ , then*

$$\delta_H(E) \geq -\frac{\mu_H(X)^2}{p^2 H^n}.$$

The rest of this section is devoted to proving an inequality of the above type for rank three  $\mu$ -semistable sheaves. The following lemma is elementary but very useful.

**Lemma (1.5).** *Let  $E$  be a torsion-free sheaf of rank  $r$  on  $X$ . Assume that  $E$  has a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that  $G_i = E_i/E_{i-1}$  is torsion-free of rank  $r_i$  for each  $i$ . We put

$$\alpha_i = \frac{r(\mu_H(G_i) - \mu_H(E))}{H^n}.$$

Then

$$\delta_H(E) \geq \sum_{i=1}^s \left\{ \frac{r}{r_i} \delta_H(G_i) - \frac{r_i}{r} \alpha_i^2 H^n \right\}.$$

*Proof.* We have

$$\begin{aligned} 2c_2(E) \cdot H^{n-2} &= \left\{ 2 \sum_{i=1}^s c_2(G_i) + 2 \sum_{i < j} c_1(G_i)c_1(G_j) \right\} \cdot H^{n-2} \\ &= \left\{ 2 \sum_{i=1}^s c_2(G_i) + c_1(E)^2 - \sum_{i=1}^s c_1(G_i)^2 \right\} \cdot H^{n-2} \\ &\geq \sum_{i=1}^s \left( \frac{\delta_H(G_i)}{r_i} - \frac{c_1(G_i)^2 \cdot H^{n-2}}{r_i} \right) + c_1(E)^2 \cdot H^{n-2}. \end{aligned}$$

On the other hand, since for each  $i$  we have

$$\left\{ \frac{r}{r_i} c_1(G_i) - c_1(E) - \alpha_i H \right\} \cdot H^{n-1} = 0,$$

the Hodge index theorem yields

$$\left\{ \frac{r}{r_i} c_1(G_i) - c_1(E) - \alpha_i H \right\}^2 \cdot H^{n-2} \leq 0.$$

Hence

$$-\frac{c_1(G_i)^2 \cdot H^{n-2}}{r_i} \geq \left\{ -\frac{2}{r} c_1(G_i) \cdot c_1(E) + \frac{r_i}{r^2} c_1(E)^2 \right\} \cdot H^{n-2} - \frac{r_i}{r^2} \alpha_i^2 H^n.$$

Since  $\sum_{i=1}^s c_1(G_i) = c_1(E)$  and  $\sum_{i=1}^s r_i = r$ , after summing up these inequalities for  $1 \leq i \leq s$ , we obtain

$$\sum_{i=1}^s -\frac{c_1(G_i)^2 \cdot H^{n-2}}{r_i} \geq -\frac{c_1(E)^2 \cdot H^{n-2}}{r} - \sum_{i=1}^s \frac{r_i}{r^2} \alpha_i^2 H^n.$$

Combining this with the inequality for  $2c_2(E) \cdot H^{n-2}$ , we obtain the claimed bound for  $\delta_H(E)$ . □

Assume that  $\mu_H(X) < 0$ . Let  $E$  be a  $\mu$ -semistable torsion-free sheaf of rank three such that  $F^*E$  is not  $\mu$ -semistable. Let

$$0 = E_0 \subset E_1 \subset \dots \subset E_{s-1} \subset E_s = F^*E$$

be the H-N filtration ( $s = 2$  or  $3$ ). Let  $G_i = E_i/E_{i-1}$  and  $r_i = \text{rk } G_i$ . There are three types of  $(r_i)$ :  $(2, 1)$ ,  $(1, 2)$ ,  $(1, 1, 1)$ . We have the following estimates for  $\alpha_i$ .

**Lemma (1.6).**  $\alpha_1 > 0$ , and

$$\begin{aligned} \frac{2\mu_H(X)}{H^n} &\leq \alpha_2 < \alpha_1 \leq -\frac{\mu_H(X)}{H^n} && \text{in case of type } (2, 1), \\ \frac{\mu_H(X)}{H^n} &\leq \alpha_2 < \alpha_1 \leq -\frac{2\mu_H(X)}{H^n} && \text{in case of type } (1, 2), \\ \frac{3\mu_H(X)}{H^n} &\leq \alpha_3 < \alpha_2 < \alpha_1 \leq -\frac{3\mu_H(X)}{H^n} && \text{in case of type } (1, 1, 1). \end{aligned}$$

*Proof.* The inequality  $\alpha_1 > 0$  is clear. As in the proof of Proposition (1.2), for each  $i = 1, 2$ , we have a non-trivial  $\mathcal{O}_X$ -homomorphism

$$f_i : T_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(E_i, F^*E/E_i) =: H_i.$$

Since the slope of the maximal destabilizing subsheaf of  $H_i$  is  $\mu_H(G_{i+1}) - \mu_H(G_i)$ , we see that  $\mu_H(X) \leq \mu_H(G_{i+1}) - \mu_H(G_i)$  for  $i = 1, 2$ . Then it is easy to deduce the following estimates for  $\mu_H(E_1)$ :

$$\begin{aligned} \mu_H(E_1) &\leq \mu_H(F^*E) - \frac{2}{3}\mu_H(X) && \text{in case of type } (2, 1), \\ \mu_H(E_1) &\leq \mu_H(F^*E) - \frac{1}{3}\mu_H(X) && \text{in case of type } (1, 2), \\ \mu_H(E_1) &\leq \mu_H(F^*E) - \mu_H(X) && \text{in case of type } (1, 1, 1). \end{aligned}$$

The claim follows immediately from the above inequalities. □

**Theorem (1.7).** *Let  $X$  be a nonsingular projective variety of dimension  $n \geq 2$  with an ample line bundle  $H$ . Let  $E$  be a  $\mu$ -semistable torsion-free sheaf of rank three on  $X$ .*

1. *If  $\mu_H(X) \geq 0$ , then  $\delta_H(E) \geq 0$ .*
2. *If  $\mu_H(X) < 0$ , then*

$$\delta_H(E) > -\frac{9\mu_H(X)^2}{p^2H^n}.$$

*Proof.* (1) follows from Proposition (1.2). Assume  $\mu_H(X) < 0$ . Let  $m$  denote the smallest integer such that  $(F^m)^*E$  is not  $\mu$ -semistable. Let

$$0 = E_0 \subset E_1 \subset \dots \subset E_{s-1} \subset E_s = (F^m)^*E$$

be the H-N filtration of  $(F^m)^*E$  and put  $G_i = E_i/E_{i-1}$ . We will give the lower bounds for  $\delta_H(E)$  according to the three types of  $r_i = \text{rk } G_i$ . In case of type  $(2, 1)$ , we have  $\delta_H(G_1) \geq -\mu_H(X)^2/(p^2H^n)$  by Proposition (1.4) and  $\delta_H(G_2) \geq 0$ . Applying Lemmas (1.5) and (1.6) to  $(F^{m-1})^*E$ , we obtain

$$\delta_H((F^m)^*E) = p^{2m}\delta_H(E) > -\left(2 + \frac{3}{2p^2}\right)\frac{\mu_H(X)^2}{H^n}.$$

Hence

$$\delta_H(E) > - \left( 2 + \frac{3}{2p^2} \right) \frac{\mu_H(X)^2}{p^2 H^n}.$$

Similarly, in case of type (1, 2)

$$\delta_H(E) > - \left( 4 + \frac{3}{2p^2} \right) \frac{\mu_H(X)^2}{p^2 H^n}.$$

Finally, in case of type (1, 1, 1) we obtain

$$\delta_H(E) > - \frac{9\mu_H(X)^2}{p^2 H^n}.$$

Comparing the above bounds for  $\delta_H(E)$ , we get the theorem.  $\square$

**Corollary (1.8).** *Assume that  $X$  is a threefold with the ample canonical bundle  $K_X$ . If the tangent bundle of  $X$  is  $\mu$ -semistable with respect to  $K_X$ , then the following inequality holds:*

$$\{3c_2(X) - c_1(X)^2\} \cdot (-c_1(X)) \geq 0.$$

*Proof.* Applying Theorem (1.7) to  $E = T_X$  and  $H = K_X$ , we obtain

$$\{3c_2(X) - c_1(X)^2\} \cdot (-c_1(X)) > -\frac{3}{2p^2} > -1.$$

The claim is clear, since the left-hand side is an integer.  $\square$

## 2. THE RESTRICTION THEOREM

Let  $X$  be a nonsingular projective variety of dimension  $n \geq 2$  and let  $H$  be an ample line bundle on  $X$ . In this section we will prove the restriction theorem mentioned in the introduction. We will treat the two cases  $r = 2$  and  $r = 3$  separately. Let  $\delta = \delta_H(E)$ ,  $\mu = \mu_H(X)$  and  $h = H^n$ .

**Theorem (2.1).** *Let  $E$  be a rank two vector bundle on  $X$  which is  $\mu$ -stable with respect to  $H$ . Let  $D \subset X$  be a smooth irreducible divisor with  $D \in |mH|$ . Then  $E|_D$  is  $\mu$ -stable with respect to  $H|_D$  under either one of the following conditions:*

1.  $\mu \geq 0$  and  $m > \frac{1}{2} \left( \delta + \frac{1}{h} \right)$ ;
2.  $\mu < 0$  and

$$m > \max \left\{ \sqrt{\frac{1}{h} \left( \delta + \frac{\mu^2}{p^2 h} \right)}, \frac{1}{2} \left( \delta + \frac{1}{h} \right) \right\}.$$

*Proof.* Assume that  $E|_D$  is not  $\mu$ -stable with respect to  $H|_D$ . Then there exist a rank one torsion-free sheaf  $Q$  on  $D$  satisfying  $\mu_{H|_D}(Q) \leq \mu_{H|_D}(E|_D)$  and a surjection  $E|_D \rightarrow Q$ . Let  $E'$  be the rank two torsion-free sheaf on  $X$  defined by the exact sequence

$$0 \rightarrow E' \rightarrow E \rightarrow Q \rightarrow 0.$$

Then we have  $c_1(E') = c_1(E) - D$  and  $c_2(E') = c_2(E) - c_1(E) \cdot D + c_1(Q) \cdot H|_D$ . Hence

$$\delta_H(E') \leq \delta_H(E) - m^2 H^n.$$

By Proposition (1.4), it follows that under assumption (1) or (2),  $E'$  is unstable with respect to  $H$ . Consider the H-N filtration of  $E'$ :

$$0 \rightarrow G_1 \rightarrow E' \rightarrow G_2 \rightarrow 0,$$

where the  $G_i$  are rank one torsion-free sheaves. For  $i = 1, 2$ , let

$$\alpha_i = \frac{2(\mu_H(G_i) - \mu_H(E'))}{H^n}.$$

Since  $G_1$  is a subsheaf of  $E$  which is  $\mu$ -stable, we have

$$0 < \alpha_1 \leq m - \frac{1}{H^n},$$

and we have  $\alpha_2 = -\alpha_1$ . Applying Lemma (1.5) to  $E'$ , we have  $\delta_H(E') \geq -\alpha_1^2 H^n \geq -(m - 1/(H^n))^2 H^n$ . It follows that

$$m \leq \frac{1}{2} \left( \delta_H(E) + \frac{1}{H^n} \right).$$

This contradicts the assumption on  $m$ . Hence the theorem is proved. □

**Theorem (2.2).** *Let  $X, H$  be as before and let  $E$  be a rank three vector bundle on  $X$  which is  $\mu$ -stable with respect to  $H$ . Then for every smooth irreducible  $D \in |mH|$ ,  $E|_D$  is  $\mu$ -stable under one of the following conditions:*

1.  $\mu \geq 0$  and

$$m > \max \left\{ \sqrt{\frac{\delta}{2h}}, \sqrt{\frac{2}{3h} \left( \delta + \frac{2}{h} \right)}, \frac{1}{12} \left( \delta + \frac{1}{2h} \right), \frac{1}{2} \left( \frac{\delta}{2} + \frac{1}{h} \right) \right\};$$

2.  $\mu < 0$  and

$$m > \max \left\{ \sqrt{\frac{1}{2h} \left( \delta + \frac{9\mu^2}{p^2h} \right)}, \sqrt{\frac{2}{3h} \left( \delta + \frac{2}{h} + \frac{\mu^2}{p^2h} \right)}, \frac{1}{4} \left( \frac{\delta}{3} + \frac{\mu^2}{2p^2h} \right), \frac{1}{2} \left( \frac{\delta}{2} + \frac{1}{2h} \right) \right\}.$$

*Proof.* The proof is similar to the rank two case. Assume that  $E|_D$  is not  $\mu$ -stable and let  $Q$  be its destabilizing torsion-free quotient. First we treat the case when  $Q$  is of rank one. Let  $E'$  be the kernel of the surjection  $E|_D \rightarrow Q$ . As before, we have

$$\delta_H(E') \leq \delta_H(E) - 2m^2 H^n.$$

By Theorem (1.7),  $E'$  is unstable if  $m$  satisfies the inequality

$$m > \max \left\{ \sqrt{\frac{\delta_H(E)}{2H^n}}, \sqrt{\frac{1}{2H^n} \left( \delta_H(E) + \frac{9\mu_H(X)^2}{p^2H^n} \right)} \right\}.$$

We shall give upper bounds for  $m$  according to the types of the H-N filtration of  $E'$ . For  $1 \leq i \leq 3$ , define

$$\alpha_i = \frac{3(\mu_H(G_i) - \mu_H(E'))}{H^n}.$$

Assume that we are in case of type (2, 1). Then we have

$$0 < \alpha_1 \leq m - \frac{1}{2H^n}$$

since  $G_1$  is a subsheaf of  $E$ . Using Proposition (1.4) and Lemma (1.5), we obtain

$$m \leq \begin{cases} \frac{1}{12} \left( \delta_H(E) + \frac{1}{2H^n} \right) & \text{if } \mu_H(X) \geq 0, \\ \frac{1}{4} \left( \delta_H(E) + \frac{\mu_H(X)^2}{2p^2H^n} \right) & \text{if } \mu_H(X) < 0. \end{cases}$$

Similarly, in case of type (1, 2) we have  $0 < \alpha_1 \leq m - 1/(H^n)$ . Hence

$$m \leq \begin{cases} \sqrt{\frac{2}{3H^n} \left( \delta_H(E) + \frac{2}{H^n} \right) - \frac{1}{H^n}} & \text{if } \mu_H(X) \geq 0, \\ \sqrt{\frac{2}{3H^n} \left( \delta_H(E) + \frac{2}{H^n} + \frac{\mu_H(X)^2}{p^2H^n} \right) - \frac{1}{H^n}} & \text{if } \mu_H(X) < 0. \end{cases}$$

Assume that we are in case of type (1, 1, 1). We have  $0 < \alpha_1 \leq m - 1/(H^n)$ , since

$$\mu_H(G_1) \leq \mu_H(E') + \frac{mH^n - 1}{3}.$$

Hence

$$\begin{aligned} \mu_H(G_2) &= 2\mu_H(E_2) - \mu_H(G_1) \\ &> 2\mu_H(E') - \left( \mu_H(E') + \frac{mH^n - 1}{3} \right) \\ &= \mu_H(E') - \frac{mH^n - 1}{3}. \end{aligned}$$

It follows that  $\alpha_2 > -(m - 1/(H^n))$ . Finally, we have

$$\begin{aligned} \mu_H(G_3) &= 3\mu_H(E') - \mu_H(G_2) - \mu_H(G_3) \\ &> 3\mu_H(E') - 2 \left( \mu_H(E') + \frac{mH^n - 1}{3} \right) \\ &= \mu_H(E') - \frac{2(mH^n - 1)}{3}. \end{aligned}$$

Hence we obtain  $\alpha_3 > -2(m - 1/(H^n))$ . From these bounds for  $\alpha_i$ , we obtain  $\alpha_i^2 \leq (m - 1/(H^n))^2$  for  $i = 1, 2$ , and  $\alpha_3^2 < 4(m - 1/(H^n))^2$ . Thus, by Lemma (1.5),

$$\delta(E') > -2 \left( m - \frac{1}{H^n} \right)^2 H^n.$$

Therefore we obtain

$$m \leq \frac{1}{2} \left( \frac{\delta_H(E)}{2} + \frac{1}{H^n} \right).$$

Putting these altogether, we conclude that  $E|_D$  has no destabilizing rank one quotient sheaves. If  $Q$  is a sheaf of rank two, then let  $\tilde{Q} := \Lambda^2 Q / \text{tors}$  denote the quotient of the exterior product  $\Lambda^2 Q$  by its torsion subsheaf. Then  $\tilde{Q}$  is a rank one torsion-free sheaf, and there is a surjection

$$\bigwedge^2 E|_D \rightarrow \tilde{Q}.$$

Since we have  $\Lambda^2 E \cong E^\vee(c_1(E))$ ,  $\Lambda^2 E$  is  $\mu$ -stable and  $\delta_H(\Lambda^2 E) = \delta_H(E)$ . Further, it is easy to see that  $\tilde{Q}$  is a destabilizing quotient of  $\Lambda^2 E|_D$ . Thus we are reduced to the case  $Q$  is of rank one. This completes the proof of the theorem.  $\square$

3. APPLICATION TO THE MODULI SPACE

In this section we consider some applications of the restriction theorem to the study of moduli spaces of  $\mu$ -stable bundles.

Let  $S$  be a nonsingular projective surface with an ample line bundle  $H$ . For  $L \in \text{Pic}(S)$ , we denote by  $M = M_S(r, L, c_2)$  the moduli space of vector bundles of rank  $r$  on  $S$  with  $\det(E) \cong L$ ,  $c_2(E) = c_2$  which are  $\mu$ -stable with respect to  $H$ .  $M_S(r, L, c_2)$  has a natural compactification  $\overline{M}_S(r, L, c_2)$ , the moduli of Gieseker semistable torsion-free sheaves. We recall that  $\overline{M}_S(r, L, c_2)$  is constructed as the geometric invariant theory quotient of a certain Quot scheme  $\text{Quot}^{ss}$  ([G]). Let  $\pi : \text{Quot}^{ss} \rightarrow \overline{M}_S(r, L, c_2)$  be the quotient morphism.

We fix a tautological sheaf  $\mathcal{F}$  on  $S \times \text{Quot}^{ss}$ . For a smooth curve  $C \subset S$  of genus  $g(C)$ , we define  $\mathcal{F}^C = \mathcal{F}|_{C \times \text{Quot}^{ss}}$ . Let  $p_Q : C \times \text{Quot}^{ss} \rightarrow \text{Quot}^{ss}$  and  $p_C : C \times \text{Quot}^{ss} \rightarrow C$  be the natural projections. The proof of the following lemma is exactly the same as that of [N2, Proposition 1.3], where the result is stated in the case  $\text{char } k = 0$ .

**Lemma (3.1).** *Assume that  $c_1(L) \cdot C = rd$  for some integer  $d$ , and let  $A$  be a line bundle of degree  $-d + g(C) - 1$  on  $C$ . Then there exists a line bundle  $\text{Det}_{\mathcal{F}}(C)$  on  $\overline{M}_S(r, L, c_2)$  such that*

$$\pi^* \text{Det}_{\mathcal{F}}(C) \cong \det(p_Q)_!(\mathcal{F}^C \otimes p_C^* A)^\vee.$$

Let  $\overline{M}_C(r, L|_C)$  denote the moduli of semistable rank  $r$  vector bundles  $F$  on  $C$  with  $\det F \cong L|_C$ . Then we have

**Proposition (3.2).** *If  $r = 2$  or  $3$ , then there exists an explicit number  $m_0$  such that for all  $m \geq m_0$  and every smooth curve  $C \in |rmH|$ , we obtain the restriction morphism of moduli spaces*

$$j_C : M_S(r, L, c_2) \rightarrow \overline{M}_C(r, L|_C)$$

which satisfies  $\text{Det}_{\mathcal{F}}(C)|_M \cong j_C^* \mathcal{L}_C$  for an ample line bundle  $\mathcal{L}_C$  on  $\overline{M}_C(r, L|_C)$ . Furthermore, if  $m$  is sufficiently large,  $j_C$  is an injective immersion (namely, an injective morphism with injective differential map).

*Proof.* We obtain the number  $m_0$  immediately from Theorems (2.1) and (2.2). The existence of an ample line bundle  $\mathcal{L}_C$  with the required property follows from the description of  $\text{Pic } \overline{M}_C(r, L|_C)$  given in [D-N]. Thus the first claim is proved.

For the second claim, note that since  $M_S(r, \mathcal{L}, c_2)$  is of finite type, if  $m$  is sufficiently large we have  $H^q(S, \mathcal{H}om(E, F)(-C)) = 0$  for  $q \leq 1$  and every  $E, F \in M$  by Serre’s vanishing theorem. It follows that

$$H^0(S, \mathcal{H}om(E, F)) \cong H^0(C, \mathcal{H}om(E|_C, F|_C))$$

and the natural map  $H^1(S, \mathcal{H}om(E, F)) \rightarrow H^1(C, \mathcal{H}om(E|_C, F|_C))$  is injective. Therefore we deduce that  $j_C$  is an injective immersion.  $\square$

Next we study  $\mu$ -stable bundles on the projective space  $\mathbb{P}^n$  by restricting them to smooth complete intersections. Restrictions to hyperplanes have already been considered in [E] in the case  $\text{char } k > 0$ . The following approach to moduli spaces is similar to [P], which treats the case  $\text{char } k = 0$ .

For given integers  $r \geq 2$  and  $c_i$  ( $2 \leq i \leq r$ ), let  $M_{\mathbb{P}^n}(r, 0, c_i)$  be the moduli space of rank  $r$   $\mu$ -stable vector bundle  $E$  on  $\mathbb{P}^n$  with  $c_1(E) = 0$  and  $c_i(E) = c_i \in A^i(\mathbb{P}^n) \cong \mathbb{Z}$ , where  $A^i$  denotes the Chow group of codimension  $i$  cycles. Similarly,

for a smooth subvariety  $X \subset \mathbb{P}^n$ , we denote by  $M_X(r, 0, c_i)$  the moduli space of rank  $r$  bundles  $E$  on  $X$  which are  $\mu$ -stable with respect to  $H = \mathcal{O}_X(1)$ , with  $c_1(E) = 0$ ,  $c_i(E) = c_i H^i \in A^i(X)$ . We notice that if  $r \leq 3$ , these moduli spaces are quasi-projective schemes ([Ma1], [Ma2]).

**Lemma (3.3).** *Let  $X \subset \mathbb{P}^n$  be a smooth complete intersection of type  $(d_1, \dots, d_m)$  with  $\sum_{j=1}^m d_j \geq n + 1$ . If  $\dim(X) \geq 3$  or  $\dim(X) = 2$  and  $\text{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$ , then  $T_X$  is  $\mu$ -stable with respect to  $\mathcal{O}_X(1)$ .*

*Proof.* It suffices to prove the  $\mu$ -stability of the cotangent bundle  $\Omega_X^1$ . Assume that  $\Omega_X^1$  is not  $\mu$ -stable, and let  $F \subset \Omega_X^1$  be a destabilizing subsheaf of rank  $r$  with  $0 < r < n - m$ . We may assume  $F$  to be reflexive by considering its double dual; hence we obtain an invertible subsheaf  $\det F \subset \Omega_X^r$ . Since  $\text{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$  if  $\dim(X) \geq 3$ , we have  $\det F = \mathcal{O}_X(l)$  for some integer  $l$ . Thus  $H^0(X, \Omega_X^r(-l)) \neq 0$  and, moreover, the destabilizing condition yields

$$l \geq \frac{r(\sum_{j=1}^m d_j - n - 1)}{n - m} \geq 0.$$

However, this is a contradiction since we have  $H^0(X, \Omega_X^r(-l)) = 0$  for  $l \geq 0$  by [D, Théorème 1.5]. This completes the proof.  $\square$

**Theorem (3.4).** *Let  $n \geq 3$ , let  $c_i$  be integers and let  $r = 2$  or  $3$ . Let  $X \subset \mathbb{P}^n$  be a smooth complete intersection of type  $(d_1, \dots, d_m)$  with  $\dim(X) \geq 2$ . There exists an integer  $d_0$  depending only on  $n, m, c_2$  and  $p$  such that if  $d_j > d_0$ , then for all  $E \in M_{\mathbb{P}^n}(r, 0, c_i)$ ,  $E|_X$  is  $\mu$ -stable with respect to  $\mathcal{O}_X(1)$ . Furthermore, for sufficiently large  $d_j$  and  $d = \prod_{j=1}^m d_j$ , the restriction morphism*

$$j_X : M_{\mathbb{P}^n}(r, 0, c_i) \rightarrow M_X(r, 0, dc_i)$$

*defines an open immersion.*

*Proof.* We shall consider only the rank two case, since the proof in the rank three case is entirely similar. First we choose an integer  $d_1$  such that

$$d_1 > \max\{2c_2, n + 1\}$$

and let  $X \subset \mathbb{P}^n$  be a smooth hypersurface of degree  $d_1$ . By Theorem (2.1), every  $E \in M_{\mathbb{P}^n}(2, 0, c_2)$  restricts to a  $\mu$ -stable bundle  $E|_X$  with  $c_1(E|_X) = 0$ ,  $c_2(E|_X) = c_2 d_1 H^2$ . Since the tangent bundle of  $X$  is  $\mu$ -stable by Lemma (3.3), we have

$$\mu_H(X) = \mu_H(T_X) = \frac{(n + 1 - d_1)d_1}{n - 1} < 0.$$

Let  $d_2$  an integer such that

$$d_2 > \max \left\{ \sqrt{4c_2 d_1 + \frac{(n + 1 - d_1)^2}{p^2(n - 1)^2}}, 2c_2 d_1^2 + \frac{1}{2d_1} \right\}.$$

Applying again Theorem (2.1) to  $E|_X$ , we see that for every smooth  $Y \in |d_2 H|$ ,  $E|_Y$  is  $\mu$ -stable. Repeating this process, we get the first assertion.

Let  $\mathcal{I}_X$  denote the ideal sheaf of  $X$ . We observe that if  $\dim(X) \geq 2$  and  $d_j$  are sufficiently large, then we have  $H^q(\mathbb{P}^n, \mathcal{H}om(E, F) \otimes \mathcal{I}_X) = 0$  for  $q \leq 2$  and for all  $E, F \in M_{\mathbb{P}^n}(2, 0, c_2)$ . Indeed, since  $M_{\mathbb{P}^n}(2, 0, c_2)$  is of finite type, we can use Serre's

vanishing theorem and the Koszul resolution of  $\mathcal{I}_X$  to prove the claim. Therefore the cohomology sequences induced by the exact sequence

$$0 \rightarrow \mathcal{H}om(E, F) \otimes \mathcal{I}_X \rightarrow \mathcal{H}om(E, F) \rightarrow \mathcal{H}om(E|_X, F|_X) \rightarrow 0$$

yield the isomorphisms  $H^q(\mathbb{P}^n, \mathcal{H}om(E, F)) \cong H^q(X, \mathcal{H}om(E|_X, F|_X))$  for  $q \leq 1$ . It follows that the restriction morphism defines an open immersion.  $\square$

For the restriction to complete intersection curves, we have the following result.

**Theorem (3.5).** *Let  $n \geq 3$ ,  $c_i$  be integers and  $r = 2$  or  $3$ . For sufficiently large  $d_j$  and a general smooth complete intersection curve  $C \subset \mathbb{P}^n$  of type  $(d_1, \dots, d_{n-1})$ , every  $E \in M_{\mathbb{P}^n}(r, 0, c_i)$  restricts to a  $\mu$ -stable bundle on  $C$  and the restriction morphism defines an injective immersion of moduli spaces*

$$j_C : M_{\mathbb{P}^n}(r, 0, c_i) \rightarrow M_C(r, 0).$$

*Proof.* For sufficiently large  $d_j$ , a general complete intersection surface  $S \subset \mathbb{P}^n$  of type  $(d_1, \dots, d_{n-2})$  satisfies  $\text{Pic}(S) \cong \mathbb{Z}[\mathcal{O}_S(1)]$  (cf. [M-R1, 2.2. Proposition]), and we have  $H^q(\mathbb{P}^n, \mathcal{H}om(E, F) \otimes \mathcal{I}_C) = 0$  for  $q \leq 1$ . Hence the theorem follows as before.  $\square$

#### REFERENCES

- [B1] F.Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. of the USSR, Izvestija **13** (1979), 499–555. MR **80j**:14014
- [B2] F.Bogomolov, *Stability of vector bundles on surfaces and curves*, Einstein metrics and Yang-Mills connections, Lecture Notes Pure Appl. Math., vol. 145 Marcel Dekker, New York, 1993, 35–49. MR **94i**:14021
- [D] P.Deligne, *Cohomologie des intersections complètes*, SGA7, exp.XI., Lecture Notes in Math., 340, Springer, 1973, 39–61. MR **50**:7135
- [D-N] J.-M.Drezet, M.S.Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbe algébriques*, Invent.Math. **97** (1989), 53–94. MR **90d**:14008
- [E] L.Ein, *Stable vector bundles on projective spaces in char  $p > 0$* , Math. Ann. **254** (1980), 53–72. MR **81d**:14010
- [F] H.Flenner, *Restriction of semistable bundles on projective varieties*, Comment.Math.Helvetici **59** (1984), 635–650. MR **86m**:14014
- [Fa] R.Fahlaoui, *Stabilité du fibré tangent des surfaces de del Pezzo*, Math. Ann. **283** (1989), 171–176. MR **89k**:14063
- [G] D.Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann.Math. **106** (1977), 45–60. MR **81h**:14014
- [Ma1] M.Maruyama, *Moduli of stable sheaves I*, J.Math. Kyoto Univ. **17** (1977), 91–126. MR **56**:8567
- [Ma2] M.Maruyama, *On boundedness of families of torsion free sheaves*, J.Math. Kyoto Univ. **21** (1981), 673–701. MR **83a**:14019
- [Mo1] A.Moriwaki, *A note on Bogomolov-Gieseker's inequality in positive characteristic*, Duke Math.J **64** (1991), 361–375. MR **92m**:14054
- [Mo2] A.Moriwaki, *Frobenius pull-back of vector bundles of rank 2 over non-uniruled varieties*, Math. Ann. **296** (1993), 441–451. MR **94j**:14039
- [Mo3] A.Moriwaki, *Arithmetic Bogomolov-Gieseker's inequality*, Amer.J.Math. **117** (1995), 1325–1347. MR **96i**:14022
- [M-R1] V.B.Mehta, A.Ramanathan, *Semistable sheaves on projective varieties and their restriction to curves*, Math. Ann. **258** (1982), 213–224. MR **83f**:14013
- [M-R2] V.B.Mehta, A.Ramanathan, *Restriction of stable sheaves and representations of the fundamental group*, Invent.Math. **77** (1984), 163–172. MR **85m**:14026
- [N1] T.Nakashima, *Bogomolov-Gieseker inequality and cohomology vanishing in characteristic  $p$* , Proc.Amer.Math.Soc. **123** (1995), 3609–3613. MR **96b**:14058
- [N2] T.Nakashima, *Singularity of the moduli space of stable bundles on surfaces*, Compositio Math. **100** (1996), 125–130. MR **97d**:14018

- [P] R.Paoletti, *Seshadri constants, gonality of space curves, and restriction of stable bundles*, J.Diff.Geom. **40** (1994), 475–504. MR **95k**:14046
- [S-B] N.I.Shepherd-Barron, *Unstable vector bundles and linear systems on surfaces in positive characteristic*, Invent.Math. **106** (1991), 243–262. MR **92h**:14027

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