POISSON TRANSFORMS ON VECTOR BUNDLES

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Abstract. Let $G$ be a connected real semisimple Lie group with finite center, and $K$ a maximal compact subgroup of $G$. Let $(\tau, V)$ be an irreducible unitary representation of $K$, and $G \times_K V$ the associated vector bundle. In the algebra of invariant differential operators on $G \times_K V$ the center of the universal enveloping algebra of Lie($G$) induces a certain commutative subalgebra $Z_\tau$. We are able to determine the characters of $Z_\tau$. Given such a character we define a Poisson transform from certain principal series representations to the corresponding space of joint eigensections. We prove that for most of the characters this map is a bijection, generalizing a famous conjecture by Helgason which corresponds to $\tau$ the trivial representation.

Introduction

Let $G$ be a connected real semisimple Lie group with finite center, and $K$ a maximal compact subgroup of $G$. Then $G/K$ is a Riemannian symmetric space of noncompact type. We fix an Iwasawa decomposition $G = KAN$. Let $M$ be the centralizer of $A$ in $K$. Let $g$ and $a$ be the Lie algebras of $G$ and $A$, respectively, and $\Sigma(g, a)$ the root system for $g, a$. Let $\Sigma^+(g, a)$ be the positive roots in $\Sigma(g, a)$ for the ordering given by $N$. Let $D(G/K)$ be the algebra of invariant differential operators on $G/K$. It is well-known that the characters of $D(G/K)$ are parametrized by $\lambda \in a_+^\ast$, the complex dual space of $a$. Let $E_\lambda(G/K)$ denote the space of joint eigenfunctions corresponding to $\lambda$. We write $g = k(g) \exp(g)n(g)$ for each $g \in G$ according to $G = KAN$. For each $\phi \in C^\infty(K/M)$ we define $P_\lambda \phi \in C^\infty(G/K)$ by

$$P_\lambda \phi(g) = \int_K \phi(k) e^{-\langle\lambda + \rho, H(g^{-1}k) \rangle} dk.$$ 

Here $\rho$ is the half sum of $\Sigma^+(g, a)$ (including multiplicities). It turns out that $P_\lambda \phi \in E_\lambda(G/K)$. One can easily extend the definition of $P_\lambda$ to the space $D'(K/M)$ (resp. $A'(K/M)$) of distributions (resp. analytic functionals) on $K/M$. In this paragraph we fix $\lambda \in a_+^\ast$ such that $2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle$ is not in $-\mathbb{N} - \{0\}$, for each $\alpha \in \Sigma^+(g, a)$. It is proved by Helgason [Helg2] that $P_\lambda$ defines a bijection from $C^\infty(K/M)_{K\text{-finite}}$ onto $E_\lambda(G(K)/K_{K\text{-finite}})$. He also proves in the rank one case $P_\lambda$ is a bijection from $A'(K/M)$ onto $E_\lambda(G(K)/K)$. He then conjectured this should be true for the higher rank case. The conjecture was eventually proved by six Japanese mathematicians in [KKMOOT]. It should be mentioned that a representation theoretic proof by Schmid, starting from the $K$-finite result, is indicated in [Sch]. Lewis, then a student of Helgason, made the following observation: Let $E_\lambda(G/K)$ be the subspace of $E_\lambda(G/K)$ where each element increases at most exponentially (see §2 for definition);

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then $P_\lambda$ maps $D'(K/M)$ into $\mathcal{E}_\Lambda^\infty(G/K)$. He was able to prove in the rank one case $P_\lambda$ is a bijection from $D'(K/M)$ onto $\mathcal{E}_\Lambda^\infty(G/K)$. See [Lew]. This result has been generalized to the higher rank case by Oshima and Sekiguchi [OS]. There is an alternative and independent proof by Wallach [Wall1]. By refining Wallach’s idea van den Ban and Schlichtkrull have a third proof in [BS]. They define $\mathcal{E}_\Lambda^\infty(G/K)$ as the subspace of $\mathcal{E}_\Lambda(G/K)$ where each element and its derivatives increase at most exponentially (uniformly). Then they prove $P_\lambda$ is a bijection from $C^\infty(K/M)$ onto $\mathcal{E}_\Lambda^\infty(G/K)$. The bijectivity of $P_\lambda$ from $D'(K/M)$ to $\mathcal{E}_\Lambda^\infty(G/K)$ follows easily.

Let $(\tau, V)$ be an irreducible unitary representation of $K$. Let $G \times_K V$ be the associated vector bundle over $G/K$. The space of smooth sections of this vector bundle can be identified with

$$C^\infty \text{Ind}_K^G(\tau) = \{ f \in C^\infty(G, V) | f(gk) = \tau(k^{-1})f(g), \forall g \in G, \forall k \in K \}.$$ 

Let $D_\tau$ denote the algebra of invariant differential operators on $C^\infty \text{Ind}_K^G(\tau)$. Notice when $(\tau, V)$ is the trivial representation we go back to the previous case. In the case where $\dim V = 1$, $D_\tau$ is commutative and its characters can be parameterized by $\lambda \in \mathfrak{a}_C^\ast$. In [Shim] Shimeno is able to characterize the joint eigenspace of $D_\tau$ in terms of a Poisson transform for most of $\lambda$. Gaillard’s results about the eigenforms of the Laplacian on hyperbolic spaces are illuminating. They show considerable variety even for a simple space. See [Ga] for details. van der Ven [Ven] considers vector-valued Poisson transforms in the rank one case, extending Gaillard’s results.

His emphasis, however, is on the singular eigenvalues. Minemura [Min] studies the properties of $D_\tau$ and obtains a result on the dimension of the spherical eigensections.

One of the difficulties people run into when trying to generalize the classical results is the complexity of $D_\tau$, in particular its noncommutativity. The remedy used was either a condition on $\tau$ or a condition on $(G/K)$. We put a mild condition on $g$ (see beginning of §4) but no restriction on $\tau$. We replace $D_\tau$ with a subalgebra $Z_\tau$ coming from $\mathcal{Z}(g)$, the center of the universal enveloping algebra of $\mathfrak{g}_C$. Then we are able to determine the characters of $Z_\tau$. It turns out they are given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}_C^\ast$, and $\Lambda$ is given by the infinitesimal character of an irreducible representation of $M$ contained in $\tau$ (see Proposition 1.11).

Let $V$ be the representation space of $\tau$, and

$$V = \bigoplus_{\sigma \in \hat{M}} V(\sigma)$$

the isotypic decomposition of $V$ into $M$-isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$. Write

$$V(\Lambda) = \bigoplus_{\sigma \in \tau, \Lambda_\sigma = \Lambda} V(\sigma).$$

Here $\Lambda_\sigma$ is given by the infinitesimal character of $\sigma$. Let $\tau(\Lambda)$ be the restriction of $\tau$ to $M$ with representation space $V(\Lambda)$. We define a Poisson transform (see §1 for definition)

$$P_\lambda : C^\infty \text{Ind}_{MAN}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \to \mathcal{E}_\Lambda^\infty \text{Ind}_K^G(\tau)$$

by

$$P_\lambda \phi(g) = \int_K \tau(k)\phi(gk)dk.$$
Let $\Sigma(\mathfrak{g})$ be the algebra of differential operators on $G$, with the left translations by elements of $G$. Let $\text{Ind}_{\mathfrak{k},\mathfrak{a},\mathfrak{n}}^G(\tau)\otimes (-\lambda)\otimes 1$ is the space defined by
$$\{\phi \in C^\infty(G,\tau)\otimes (\mathfrak{g}) \mid \phi(g \mathfrak{m} \mathfrak{a} \mathfrak{n}) = a^{\lambda - \rho}(m^{-1})\phi(g)\},$$
and $\mathcal{E}_{\lambda-\tau}^G$ the subspace of the total eigenspace where each element and its derivatives increase at most exponentially (uniformly). Let $C(\lambda)$ be the generalized Harish-Chandra $C$-function corresponding to $\tau$ (Proposition 2.3), $C_0(\lambda)$ the restriction of $C(\lambda)$ to $V(\Lambda)$, and $\Sigma(\mathfrak{g},\mathfrak{h}_C)$ as defined after Remark 1.5.

**Theorem.** Let $\lambda - \Lambda \in \mathfrak{h}_C^*$, satisfying the following conditions:

1. $2\langle \lambda - \Lambda, \alpha \rangle/\langle \alpha, \alpha \rangle \notin \mathbb{Z}$, $\forall \alpha \in \Sigma(\mathfrak{g},\mathfrak{h}_C)$, with $\alpha|\alpha \neq 0$;
2. $2\langle \lambda, \beta \rangle/\langle \beta, \beta \rangle \notin -\mathbb{Z}$, $\forall \beta \in \Sigma(\mathfrak{g},\mathfrak{a})$;
3. $\det C_0(\lambda) \neq 0$.

Then $P_\lambda$ is a bijection.

This generalizes the result of van den and Ban and Schlichtkrull mentioned above which corresponds to $\lambda$ the trivial representation.

We have similar results about distributions and $K$-finite sections, generalizing the above-mentioned results for $\tau$ trivial.

The main idea in the proof is to generalize the theory of asymptotic expansions developed in [Ban] and [BS]. By invoking Casselman’s deep result [Ca] on globalization of Harish-Chandra modules, one might simplify our argument somehow. But we prefer a self-contained account. Besides, we think the theory of asymptotic expansions developed here is of interest on its own.

The paper is organized as follows: in Section 1 we study the invariant differential operators on $G \times K$. In Section 2 we introduce some function spaces on $G$. In Section 3 we state some results on the asymptotic expansion of an eigensection. In Section 4 we study the algebraic structure of a $(\mathfrak{g},K)$-module. In Sections 5 and 6 we prove the results stated in Section 3. In Section 7 we study the leading terms of the asymptotic expansion. In Section 8 we give an inversion formula to the Poisson transform. In Sections 9 and 10 we extend the Poisson transform to vector-valued distributions.

1. **Notations and preliminaries**

Let $G$ be a connected real semisimple Lie group with finite center and $K$ a maximal compact subgroup of $G$. Then $G/K$ is a Riemannian symmetric space. We fix an Iwasawa decomposition $G = KAN$, and let $M$ be the centralizer of $A$ in $K$. $M'$ the normalizer of $A$ in $K$, $W = M'/M$ the Weyl group. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, and $\mathfrak{m}$ be the corresponding Lie algebras of $G, K, A, N$, and $M$, respectively, and $U(\mathfrak{g}), U(\mathfrak{k}), U(\mathfrak{a}), U(\mathfrak{n})$, and $U(\mathfrak{m})$ the corresponding universal enveloping algebras of the complexified Lie algebras. Let $\Sigma(\mathfrak{g},\mathfrak{a})$ be the restricted root system for $(\mathfrak{g},\mathfrak{a})$, and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots for the ordering of $\Sigma(\mathfrak{g},\mathfrak{a})$ given by $N$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. If $g \in G$ we write $g = k(g) \exp H(g)n(g)$ according to $G = KAN$.

Fix once and for all an irreducible unitary representation $(\tau, V)$ of $K$. Denote by $G \times K V$ the associated vector bundle. Then the space of its smooth sections may be identified with the following space:

$$C^\infty \text{Ind}_{\mathfrak{k},\mathfrak{a},\mathfrak{n}}^G(\tau) = \{f \in C^\infty(G,\tau) \mid f(gk) = \tau(k)^{-1}f(g), \forall g \in G, \forall k \in K\}.$$

Let $D_{\tau}$ denote the algebra of differential operators on $C^\infty \text{Ind}_{\mathfrak{k},\mathfrak{a},\mathfrak{n}}^G(\tau)$ that commute with the left translations by elements of $G$. The remainder of this section will be
devoted to the study of this algebra. First for each \( f \in C^\infty(G, V) \) and \( X \in \mathfrak{g} \) we define \( L_X f \) and \( R_X f \) as follows:

\[
L_X f(g) = \left. \left( \frac{d}{dt} f(\exp(-tX)g) \right) \right|_{t=0},
\]

\[
R_X f(g) = \left. \left( \frac{d}{dt} f(g \exp tX) \right) \right|_{t=0}, \quad \forall g \in G.
\]

Then \( L \) and \( R \) define two representations of \( \mathfrak{g} \) which we extend to representations of \( U(\mathfrak{g}) \). Let \( \text{End}(V) \) denote the space of linear maps from \( V \) to itself. Then \( U(\mathfrak{g}) \otimes \text{End}(V) \) is an associative algebra with the natural multiplication. Let \( I(\tau) \) be the left ideal of \( U(\mathfrak{g}) \otimes \text{End}(V) \) generated by \( \{ X \otimes 1 + 1 \otimes \tau(X) | X \in \mathfrak{t} \} \).

**Proposition 1.1.** With the above notations, we have

\[
U(\mathfrak{g}) \otimes \text{End}(V) = (U(\mathfrak{a}) \otimes \text{End}(V)) \oplus (nU(\mathfrak{g}) \otimes \text{End}(V) + I(\tau)).
\]

**Proof.** It suffices to show the left-hand side is contained in the right-hand side. Suppose \( u \otimes T \in U(\mathfrak{g}) \otimes \text{End}(V) \). By Poincaré-Birkhoff-Witt we can assume \( u = u_1 u_2 u_3 \), where \( u_1 \in U(\mathfrak{n}), \ u_2 \in U(\mathfrak{a}), \) and \( u_3 \in U(\mathfrak{t}) \). If \( u_1 \in nU(\mathfrak{n}) \) then \( u \otimes T \in nU(\mathfrak{g}) \otimes \text{End}(V) \). So we can assume \( u = u_2 u_3 \), where \( u_2 \in U(\mathfrak{a}), \) and \( u_3 \in U(\mathfrak{t}) \). Let \( u_3 = X_1 \cdots X_j \), for \( X_1, \ldots, X_j \in \mathfrak{t} \). It is easy to show \( u_2 u_3 \otimes T \in (U(\mathfrak{a}) \otimes \text{End}(V)) + I(\tau) \) by induction on \( j \).

Define a \( K \)-action on \( U(\mathfrak{g}) \otimes \text{End}(V) \) by

\[
k(X \otimes T) = \text{Ad}(k)X \otimes \tau(k)T \tau(k)^{-1},
\]

for each \( k \in K \). Let \((U(\mathfrak{g}) \otimes \text{End}(V))^K \) be the fixed elements.

**Proposition 1.2.** Let \( \Gamma_1 : U(\mathfrak{g}) \otimes \text{End}(V) \to U(\mathfrak{a}) \otimes \text{End}(V) \) be the projection map according to the decomposition in Proposition 1.1. Then \( \Gamma_1 \) is a homomorphism from \((U(\mathfrak{g}) \otimes \text{End}(V))^K \) into \( U(\mathfrak{a}) \otimes \text{End}_M(V) \), where

\[
\text{End}_M(V) = \{ T \in \text{End}(V) | \tau(m)T = T \tau(m), \forall m \in M \}.
\]

**Proof.** Since \( M \) preserves \( \mathfrak{n} \), it is easy to see \( \Gamma_1 \) maps \((U(\mathfrak{g}) \otimes \text{End}(V))^K \) into \( U(\mathfrak{a}) \otimes \text{End}_M(V) \). We now check \( \Gamma_1 \) is a homomorphism.

Suppose \( D_1, D_2 \in (U(\mathfrak{g}) \otimes \text{End}(V))^K \). Then

\[
D_1 - \Gamma_1(D_1) \in nU(\mathfrak{g}) \otimes \text{End}(V) + I(\tau).
\]

Hence

\[
D_1 D_2 - \Gamma_1(D_1) D_2 \in nU(\mathfrak{g}) \otimes \text{End}(V) + I(\tau) D_2.
\]

Assume \( D_2 = \sum u_i \otimes T_i \), for \( u_i \in U(\mathfrak{g}), \) and \( T_i \in \text{End}(V) \). Then for any \( X \in \mathfrak{t}, \)

\[
(X \otimes 1 + 1 \otimes \tau(X)) D_2 = \sum (X u_i \otimes T_i + u_i \otimes \tau(X) T_i)
= \sum (\text{ad}(X) u_i \otimes T_i + u_i \otimes [\tau(X), T_i])
+ \sum (u_i X \otimes T_i + u_i \otimes T_i \tau(X)).
\]

Then first summation is zero since \( D_2 \in (U(\mathfrak{g}) \otimes \text{End}(V))^K \). The second one is just \( D_2(X \otimes 1 + 1 \otimes \tau(X)) \). So we have proved \( I(\tau) D_2 \subset I(\tau) \). Hence

\[
D_1 D_2 - \Gamma_1(D_1) D_2 \in nU(\mathfrak{g}) \otimes \text{End}(V) + I(\tau).
\]
Therefore, 

\[ D_2 - \Gamma_1(D_2) \in nU(g) \otimes \text{End}(V) + I(\tau), \]

and

\[ \Gamma_1(D_1)(nU(g) \otimes \text{End}(V) + I(\tau)) \subset nU(g) \otimes \text{End}(V) + I(\tau). \]

Therefore

\[ D_1D_2 - \Gamma_1(D_1)\Gamma_1(D_2) \in nU(g) \otimes \text{End}(V) + I(\tau). \]

This proves \( \Gamma_1(D_1D_2) = \Gamma_1(D_1)\Gamma_1(D_2). \)

For \( D = \sum u_i \otimes T_i \in U(g) \otimes \text{End}(V), \) and \( f \in C^\infty(G, V), \) we define

\[ \mu_1(D)f = \sum T_i R_{u_i}f. \]

It is not difficult to show for \( D \in (U(g) \otimes \text{End}(V))^K \) and \( f \in C^\infty \text{Ind}_K^G(\tau), \mu_1(D)f \)
remains in \( C^\infty \text{Ind}_K^G(\tau). \) So \( \mu_1(D) \in D_\tau. \) In fact \( \mu_1 \) is a surjective homomorphism
from \( (U(g) \otimes \text{End}(V))^K \) onto \( D_\tau. \)

We define \( \mu(D) = \mu_1(D \otimes 1), \) for each \( D \in U(g)^K. \) By a theorem of Burnside which asserts that \( \tau(U(t)) = \text{End}(V), \) one can prove \( \mu \) is a surjective homomorphism from \( U(g)^K \) onto \( D_\tau, \) using the surjectivity of \( \mu_1. \)

For each \( \lambda \in a_+^\ast, \) we introduce an important function \( \Psi_\lambda \) on \( G \) with values in \( \text{End}(V) \) as follows:

\[ \Psi_\lambda(nak) = a^{\lambda + \rho_\tau}(k)^{-1}, \]

for \( n \in N, a \in A, \) and \( k \in K. \) Here \( \rho \) is the half sum of the positive roots for \( (g, a). \)
Notice that for each \( v \in V, \) the function: \( g \to \Psi_\lambda(g \cdot v) \) belongs to \( C^\infty \text{Ind}_K^G(\tau). \)

**Proposition 1.3.** For each \( D \in U(g)^K, \) and \( v \in V, \)

\[ \mu(D)(\Psi_\lambda \cdot v) = \Psi_\lambda \cdot (\Gamma_1(D \otimes 1)(\lambda + \rho)v). \]

**Proof.** Since both sides are left \( N \)-invariant and behave in the same way under the right \( K \)-action, it is sufficient to show they are equal when restricted to \( A. \) By definition

\[ D \otimes 1 = D_1 + \Gamma_1(D \otimes 1) + D_2, \]

where \( D_1 \in nU(g) \otimes \text{End}(V), \) and \( D_2 \in I(\tau). \) It is easy to see that

\[ \mu_1(D_1)(\Psi_\lambda \cdot v)|A = 0, \]

and

\[ \mu_1(D_2)(\Psi_\lambda \cdot v) = 0. \]

So

\[ \mu(D)(\Psi_\lambda \cdot v)|A = a^{\lambda + \rho_\tau}(D \otimes 1)(\lambda + \rho)v. \]

**Corollary 1.4.** There exists a homomorphism \( \Gamma' : D_\tau \to U(a) \otimes \text{End}_M(V). \) Moreover, for each \( D \in U(a)^K, \Gamma'(\mu(D)) = \Gamma_1(D \otimes 1). \)

**Remark 1.5.** It has been proved in Section 3 in [Min] that \( \Gamma' \) is injective, using results from [Lep].
Let $\Sigma$ be the Cartan involution of $\mathfrak{g}$, let $\gamma$ be the identity map of $\mathfrak{h}$, and let $\tau$ be the Harish-Chandra homomorphism. For each irreducible representation $(\sigma, V_\sigma)$ of $M$, we get a Lie algebra representation of $\mathfrak{m}$ by differentiation. We denote the representation by $d\sigma$. In general this is not irreducible. Fortunately it is a multiple of an irreducible representation of $\mathfrak{m}$. This fact can be seen in the following way. Let $M_0$ be the identity component of $M$. By structure theory (see 1.1.3.8 in [War]) one can find $Z(A)$, a finite subgroup of $M$ where each element commutes with every element of $M_0$. Choose an irreducible representation $(\sigma, V_\sigma)$ of $M_0$ in $(\sigma, V_\sigma)$. For each $z \in Z(A)$, $(\sigma, \sigma(z) V_\sigma)$ gives an irreducible representation of $M_0$ in $(\sigma, V_\sigma)$, which is equivalent to $(\sigma, V_\sigma)$. Since $\sigma$ is irreducible, $V_\sigma = \sum_{z \in Z(A)} \sigma(z) V_\sigma$. So by Schur’s lemma the center $Z(\mathfrak{m})$ of $U(\mathfrak{m})$ acts on $V_\sigma$ by scalars. The action is determined by $\Lambda_\sigma \in -\mathfrak{h}^*$ as follows: For each $Z$ in $Z(M)$, $d\sigma(Z) = \gamma(Z)(\Lambda_\sigma) I_{V_\sigma}$, where $\gamma$ is the Harish-Chandra homomorphism for $(\mathfrak{m}, \mathfrak{t})$, and $I_{V_\sigma}$ the identity map of $V_\sigma$. We choose $\Lambda_\sigma$ the highest weight of $\sigma$ plus $\rho_0$.

Let $\Gamma: D_\tau \to U(\mathfrak{a}) \otimes \text{End}_M(V)$ be defined by
\[
\Gamma(D)(\lambda) = \Gamma'(D)(\lambda + \rho).
\]

**Theorem 1.6.** For each $Z \in Z(\mathfrak{g})$, and $\lambda \in \mathfrak{a}^*_C$,
\[
\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma) I_{V(\sigma)}.
\]

We give a proof below using a well-known proposition about $Z(\mathfrak{g})$. A more self-contained proof is in [Wall]. First for the proof and later use we recall the definition of Poisson transforms.

Let $(\delta, V_\delta)$ be a finite dimensional representation of $B = MAN$, the maximal parabolic subgroup of $G$. Let $C^\infty \text{Ind}^G_B(\delta)$ be the space defined by
\[
\{ \phi \in C^\infty(G, V_\delta) | \phi(gman) = a^{-\delta-1}(man) \phi(g), \forall g \in G, \forall man \in B \}.
\]
Let \( C^\infty \text{Ind}^G_B(\delta) \) be endowed with the topology from \( C^\infty(G, V_\delta) \). We will specify the topology on \( C^\infty \text{Ind}^G_K(\tau) \) in the next section.

**Definition 1.7.** A Poisson transform is a continuous, linear, \( G \)-equivariant map from \( C^\infty \text{Ind}^G_B(\delta) \) into \( C^\infty \text{Ind}^G_K(\tau) \).

Given \( T \in \text{Hom}_M(V_\delta, V) \), and \( \phi \in C^\infty \text{Ind}^G_B(\delta) \), we write

\[
P_T \phi(g) = \int_K \tau(k)T(\phi(gk))dk.
\]

One can easily check \( P_T \) is a Poisson transform.

**Proposition 1.8.** The map \( T \rightarrow P_T \) is a bijection from \( \text{Hom}_M(V_\delta, V) \) onto the space of Poisson transforms.

This result appears in [Ven]. We include a proof for completeness. Suppose \( P \) is a Poisson transform from \( C^\infty \text{Ind}^G_B(\delta) \) into \( C^\infty \text{Ind}^G_K(\tau) \). Define the Poisson kernel \( p \in [C^\infty \text{Ind}^G_B(\delta)]' \otimes V \), the strong topological dual of \( C^\infty \text{Ind}^G_B(\delta) \) tensored by \( V \), by

\[
\langle p, \phi \rangle = (P\phi)(e),
\]

for each \( \phi \in C^\infty \text{Ind}^G_B(\delta) \). By the \( G \)-equivariance of \( P \) the Poisson kernel completely determines \( P \) by

\[
P\phi(x) = \langle p, L_{x^{-1}}\phi \rangle,
\]

for any \( \phi \in C^\infty \text{Ind}^G_B(\delta) \). Here \( L_{x^{-1}}\phi(g) = \phi(xg) \).

By Section 9 there is a \( K \)-equivariant isomorphism between \( (C^\infty \text{Ind}^G_B(\delta))' \) and \( C^{-\infty} \text{Ind}^K_M(\delta|M) \), where \( C^{-\infty} \text{Ind}^K_M(\delta|M) \) denotes the space of vector-valued distributions \( f: C^\infty(K, \mathbb{C}) \rightarrow V_\delta^* \), such that

\[
R_m f = \delta(m)^{-1} f,
\]

for any \( m \in M \). Here \( \delta \) is the dual representation of \( \delta|M \). And \( R_m f(\phi) = f(R_{m^{-1}} \phi) \), with \( (R_{m^{-1}} \phi)(k) = \phi(km^{-1}) \). So

\[
p \in C^{-\infty} \text{Ind}^K_M(\delta|M) \otimes V.
\]

However, for \( \phi \in C^\infty \text{Ind}^G_B(\delta) \),

\[
\langle p, L_k \phi \rangle = P(L_k \phi)(e) = P\phi(k^{-1}) = \tau(k)(P\phi(e)) = \tau(k)(\langle p, \phi \rangle).
\]

Hence \( p \in (C^{-\infty} \text{Ind}^K_M(\delta|M) \otimes V)^K \). Let \( \pi \) be the representation of \( K \) in \( V_\delta^* \otimes V \) defined by \( \pi(k)(v \otimes w) = v \otimes \tau(k)w \), for \( v \in V_\delta^* \), and \( w \in V \). Then \( p \in C^{-\infty}(K, V_\delta^* \otimes V) \), and \( L_k p = \pi(k^{-1})p \). By Lemma 9.3 \( p \) must be smooth. Its transformation properties imply that \( p \) is determined by \( p(e) \), which belongs to \( (V_\delta^* \otimes V)_M \cong \text{Hom}_M(V_\delta, V) \).

**Proof of Proposition 1.8.** From the definition of \( P_T \), it is immediate that the Poisson kernel of \( P_T \) evaluated at the identity is \( T \). This shows the map \( T \rightarrow P_T \) is injective. On the other hand, let \( P \) be a Poisson transform, and let \( p \) be its Poisson
kernel. Then
\[ P\phi(x) = \langle p, L_{x^{-1}} \phi \rangle \]
\[ = \int_K \langle p(k), \phi(xk) \rangle dk \]
\[ = \int_K \tau(k)p(e)\phi(xk) dk. \]

This proves \( P = P_{p(e)} \), whence the surjectivity. \( \square \)

Lemma 1.9.

\[ \int_K F(k)(g^{-1}k) dk = \int_K F(k)e^{-2\rho H(gk)} dk. \]

Let \( \sigma \) be a finite dimensional representation of \( M \) and \( \lambda \in \mathfrak{a}_c^\vee \). Then \( \sigma \otimes (-\lambda) \otimes 1 \) defines a representation of \( B \) by \( \text{man} \rightarrow a^{-\lambda}\sigma(m) \).

Corollary 1.10. For each \( \phi \in C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1) \), \( T \in \text{Hom}_M(V_\sigma, V) \),

\[ P_T \phi(g) = \int_K \Psi_\lambda(k^{-1}g)T\phi(k) dk. \]

Proof.

\[ P_T \phi(g) = \int_K \tau(k)T\phi(gk) dk \]
\[ = \int_K \tau(k)T\phi(k(gk) \exp H(gk)n(gk)) dk \]
\[ = \int_K e^{(\lambda-\rho)H(gk)}\tau(k)T\phi(k(gk)) dk. \]

By Lemma 1.9,

\[ \int_K e^{(\lambda-\rho)H(gk)}\tau(k)T\phi(k(gk)) dk \]
\[ = \int_K e^{(\lambda+\rho)H(gk(g^{-1}k))}\tau(k(g^{-1}k))T\phi(k(gk(g^{-1}k))) dk \]
\[ = \int_K e^{(-\lambda+\rho)H(g^{-1}k)}\tau(k(g^{-1}k))T\phi(k) dk \]
\[ = \int_K \Psi_\lambda(k^{-1}g)T\phi(k) dk. \] \( \square \)

Proof of Theorem 1.6. Let \( \delta \) be the restriction of \( \tau \) to \( M \) with \( V(\sigma) \) as the representation space. It is well-known that \( L_Z\phi = \gamma(Z)(\Lambda_\sigma - \lambda)\phi \) for each \( Z \in Z(\mathfrak{g}) \), and \( \phi \in C^\infty \text{Ind}_B^G(\delta \otimes (-\lambda) \otimes 1) \). See [Vogan]. Let * denote adjoint on \( U(\mathfrak{g}) \). By Corollary 5.31 on p. 324 in [Helg1],

\[ R_Z P_T \phi = L_Z P_T \phi = P_{T L_Z} \phi = P_T (\gamma(Z^*)(\Lambda_\sigma - \lambda)\phi) = P_{\gamma(Z^*)}^\tau(\Lambda_\sigma - \lambda)P_T \phi = P_{\gamma(Z)(-\Lambda_\sigma + \lambda)}T\phi. \]

On the other hand, by Proposition 1.3 and Corollary 1.10,

\[ R_Z P_T \phi = P_{T(\mu(Z))^\tau} \phi. \]

So

\[ P_{\gamma(Z)(-\Lambda_\sigma + \lambda)}T = P_{T(\mu(Z))^\tau}. \]
By Proposition 1.8 we conclude
\[ \Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma)|_{V(\sigma)}. \]

By definition a character of \( Z_\tau \) is a homomorphism from \( Z_\tau \) to \( \mathbb{C} \).

**Proposition 1.11.** A character \( \chi \) of \( Z_\tau \) is given by \( \lambda - \Lambda_\sigma \), where \( \lambda \in a_\tau^\circ \), and \( \sigma \in \tau \). More specifically, \( \chi(\mu(Z)) = \gamma(Z)(\lambda - \Lambda_\sigma) \), for each \( Z \in Z(\mathfrak{g}) \).

**Lemma 1.12.** Let \( S \) be the common zeros of \( p_1, \ldots, p_m \) in \( S(\mathfrak{h}_C) \). Assume in addition \( S \) is \( \tilde{W} \)-invariant, \( \tilde{W} \) denoting the Weyl group for \( (\mathfrak{g}_C, \mathfrak{h}_C) \). Then one can find \( q_1, \ldots, q_n \) in \( I(\mathfrak{h}_C) \), the \( \tilde{W} \)-invariants in \( S(\mathfrak{h}_C) \), such that \( S \) is the common zeros of \( q_1, \ldots, q_n \).

Proof. Write \( R_i(X) = \prod_{s \in \tilde{W}} (X - p_i^s) \), and \( w = |\tilde{W}| \). Then
\[ R_i(X) = X^w + p_1 X^{w-1} + \cdots + p_n. \]

It is easy to see we can use \( p_{ij} \) as our \( q_1, \ldots, q_n \).

**Proof of Proposition 1.11.** Let \( A = \mu \circ \gamma^{-1} : I(\mathfrak{h}_C) \to Z_\tau \). By Theorem 1.6 \( \ker(A) = \{ \rho \in I(\mathfrak{h}_C) | p(-\Lambda_\sigma + a_\tau^\circ) = 0, \text{ for all } \sigma \in \tau \} \). Here we use Remark 1.5 which asserts that \( \Gamma \) is injective. Suppose \( \chi : Z_\tau \to \mathbb{C} \) is a character of \( Z_\tau \). Then there exists \( \mu \in h_\tau^\circ \), such that \( \chi \circ A = \chi_\mu \), where \( \chi_\mu \) is the homomorphism defined by evaluation at \( \mu \). Obviously \( p(\mu) = 0 \), for all \( p \in \ker(A) \). Let
\[ S = \bigcup_{\sigma \in \tau, w \in \tilde{W}} w(-\Lambda_\sigma + a_\tau^\circ) \subset h_\tau^\circ. \]

Obviously one can find \( p_1, \ldots, p_m \) in \( S(\mathfrak{h}_C) \) such that \( S \) is the common zeros of \( p_1, \ldots, p_m \). Then by Lemma 1.12 we can find \( q_1, \ldots, q_n \) in \( I(\mathfrak{h}_C) \) such that \( S \) is the common zeros of \( q_1, \ldots, q_n \). This shows \( q_1, \ldots, q_n \) are in \( \ker(A) \). So \( q_1(\mu) = \cdots = q_n(\mu) \). Therefore \( \mu \in S \), i.e. \( \mu = w(\lambda - \Lambda_\sigma) \) for some \( \lambda \in a_\tau^\circ, \sigma \in \tau \), and \( w \in \tilde{W} \).

For \( s \in M' \), define \( s.(X \otimes T) = \text{Ad}(s)X \otimes \tau(s)T\tau(s^{-1}) \), for \( X \in U(\mathfrak{a}) \), and \( T \in \text{End}(V) \). The next proposition is about a property of the generalized Harish-Chandra homomorphism. It is a weak version of a conjecture by Lepowsky.

**Proposition 1.13.** For each \( s \in M'/M \), \( s.\Gamma(D) = \Gamma(D) \), for each \( D \in Z_\tau \).

For the proof of this result we need more facts about Weyl groups. Let \( \tilde{W}_1 \subset \tilde{W} \) be the subgroup where every element stabilizes \( \mathfrak{a} \). It is well-known there is a surjective homomorphism \( \tilde{W}_1 \to M'/M \). The kernel \( \tilde{W}_0 \) is the Weyl group for \( (\mathfrak{m}, \mathfrak{t}) \).

**Lemma 1.14.** For each \( s \in M'/M \), choose \( w(s) \) in \( \tilde{W}_1 \) in the preimage of \( s \) under the homomorphism above. Then \( A_{s\tau} = w(s)A_\tau \).

Proof (by Vogan). Take a maximal torus \( T \) of \( M_0 \). \( sTs^{-1} \) is another maximal torus. So there is \( m \in M_0 \), such that \( msTs^{-1}m^{-1} = T \). To avoid cumbersome notations we assume \( sTs^{-1} = T \). It is easy to see that \( \text{Ad}(s)^* \), the transpose of \( \text{Ad}(s) \), preserves \( \Sigma(\mathfrak{m}, \mathfrak{t}) \). We can also assume \( \text{Ad}(s)^* \) preserves \( \Sigma^+(\mathfrak{m}, \mathfrak{t}) \). For \( Z \in Z(\mathfrak{m}) \),
\[ Z - \gamma'(Z) \in m^{-1}U(\mathfrak{m}). \]
Hence
\[ \text{Ad}(s)Z - \text{Ad}(s)\gamma'(Z) \in m^{-1}U(m). \]

So
\[ \sigma^*(Z) = \sigma(\text{Ad}(s)Z) = \text{Ad}(s)\gamma'(Z)(\Lambda_\sigma - \rho_0) \]
\[ = \gamma'(Z)(\text{Ad}(s)^*\Lambda_\sigma - \rho_0) = \gamma(Z)(\text{Ad}(s)^*\Lambda_\sigma). \]

Hence
\[ \Lambda_{\sigma^*} = \text{Ad}(s)^*\Lambda_\sigma = w(s)\Lambda_\sigma. \quad \Box \]

**Proof of Proposition 1.13.** Take \( Z \in \mathcal{Z}(\mathfrak{g}) \) such that \( D = \mu(Z) \). Then for each \( \lambda \in \mathfrak{a}_c^* \), and \( s \in M' \),
\[ s.\Gamma(D)(\lambda)|V(\sigma) = s.\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\text{Ad}(s)^*\lambda - \Lambda_{\sigma^*})I_{V(\sigma)}. \]

By Lemma 1.14, \( \Lambda_{\sigma^*} = w(s)\Lambda_\sigma \). So
\[ s.\Gamma(D)(\lambda)|V(\sigma) = \gamma(Z)(\text{Ad}(s)^*\lambda - w(s)\Lambda_\sigma)I_{V(\sigma)} \]
\[ = \gamma(Z)(\lambda - \Lambda_\sigma)I_{V(\sigma)} \]
\[ = \Gamma(\mu(Z))(\lambda)|V(\sigma) \]
\[ = \Gamma(D)(\lambda)|V(\sigma). \quad \Box \]

Now let \( \mathfrak{n} = \mathfrak{n}_0 \). Similarly as in Proposition 1.1 we get
\[ U(\mathfrak{g}) \otimes \text{End}(V) = U(\mathfrak{a}) \otimes \text{End}(V) \oplus (\mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau)) \]
Then we define \( \widetilde{\Gamma}_1 : U(\mathfrak{g}) \otimes \text{End}(V) \to U(\mathfrak{a}) \otimes \text{End}(V) \) as the projection according to this decomposition.

**Corollary 1.15.** For each \( Z \in \mathcal{Z}(\mathfrak{g}) \), and \( \lambda \in \mathfrak{a}_c^* \),
\[ \widetilde{\Gamma}_1(Z \otimes 1)(\lambda) = \Gamma(\mu(Z))(\lambda + \rho). \]

**Proof.** Take \( s \in M' \) such that \( \text{Ad}(s)^*\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \Sigma^-(\mathfrak{g}, \mathfrak{a}) \). By definition
\[ Z \otimes 1 - \Gamma_1(Z \otimes 1) \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau). \]

Hence
\[ s.(Z \otimes 1) - s.\Gamma_1(Z \otimes 1) \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau). \]

So
\[ \widetilde{\Gamma}_1(Z \otimes 1) = s.\Gamma_1(Z \otimes 1). \]

Hence
\[ \widetilde{\Gamma}_1(Z \otimes 1)(\lambda) = \tau(s)\Gamma_1(Z \otimes 1)(\text{Ad}(s)^*\lambda)\tau(s^{-1}) \]
\[ = \tau(s)\Gamma(\mu(Z))(\text{Ad}(s)^*\lambda - \rho)\tau(s^{-1}) \]
\[ = \tau(s)\Gamma(\mu(Z))(\text{Ad}(s)^*(\lambda + \rho))\tau(s^{-1}) \]
\[ = \Gamma(\mu(Z))(\lambda + \rho). \quad \Box \]
2. Some function spaces on $G$

In this section we introduce a certain growth condition on a function on $G$ with values in $V$. It turns out the condition is satisfied by $P_T\phi$ for any $\phi \in C^\infty\text{Ind}^G_B(\delta)$, where $\delta$ is a certain finite dimensional representation of $B$.

For each $g \in G$, we denote by $\|g\|$ the operator norm of $\text{Ad}(g)$ on $\mathfrak{g}$, which is equipped with the inner product $\langle X,Y \rangle_\Theta = -K(X,\Theta Y)$. Here $K$ is the Killing form on $\mathfrak{g}$.

**Lemma 2.1.** (i) $\|g\| = \|\Theta g\| = \|g^{-1}\| \geq 1$;
(ii) $\|g_1g_2\| \leq \|g_1\| \|g_2\|$;
(iii) if $g = k_1ak_2$ with $k_1, k_2 \in K$, $a \in A$, then
$$\|g\| = \exp \left( \max_{\alpha \in \Sigma(\mathfrak{g},a)} |\alpha(\log a)| \right);$$
(iv) there are constants $C_1, C_2 > 0$, such that if $x = \exp X$ with $X \in \mathfrak{p}$, then $e^{C_1|X|} \leq \|x\| \leq e^{C_2|X|}$. Here $\mathfrak{p}$ is the $-1$ eigenspace of $\Theta$, and $|X| = \sqrt{\langle X,X \rangle_\Theta}$;
(v) $\|a\| \leq \|an\|$, for $a \in A$, and $n \in \mathbb{N}$.

**Proof.** See [BS].

For any function $f : G \to V$ and $r \in \mathbb{R}$, we write
$$\|f\|_r = \sup_{g \in G} \|g\|^{-r}|f(g)|.$$ We say $f$ increases at most exponentially if $\|f\|_r < \infty$, for some $r \in \mathbb{R}$. Let $C^r_r(G,V)$ denote the Banach space of continuous functions $f$ on $G$ with values in $V$ with $\|f\|_r \leq \infty$.

**Example 2.2.** Let $\lambda \in \mathfrak{a}_C^\circ$, and $\sigma$ a finite dimensional representation of $M$. Let $r(\lambda) = C_1^{-1}|\text{Re} \lambda - \rho|$, where $C_1$ is the constant in Lemma 2.1 (iv). Then for each $\phi \in C^\infty\text{Ind}^G_B(\sigma \otimes (-\lambda) \otimes 1)$, and $T \in \text{Hom}_M(V_\sigma, V)$, $P_T\phi$ belongs to $C^r_r(\lambda)(G,V)$. This is in [BS] when $\tau$ is trivial and $\tau$ in general does not offer additional difficulties.

Write
$$C^\infty_r(G,V) = \{ f \in C^\infty(G,V) | L_uf \in C^r_r(G,V), \forall u \in U(\mathfrak{g}) \}.$$ We endow $C^r_r(G,V)$ with its standard topology: Let $X_1, \ldots, X_p$ be a basis of $\mathfrak{g}$, and $X^I = X^{i_1} \cdots X^{i_p} \in U(\mathfrak{g})$ for $I = (i_1, \ldots, i_p) \in \mathbb{N}^p$. For $q \in \mathbb{N}$ and $f \in C^q(G,V)$, a $q$ times continuously differentiable function from $G$ to $V$, we define
$$\|f\|_{q,r} = \sum_{|I| \leq q} \|L_{X^I}f\|_r.$$ Endowed with this norm the space
$$C^q_r(G,V) = \{ f \in C^q(G,V) | \|f\|_{q,r} < \infty \}$$
is a Banach space. Obviously $C^q_r \subset C^{q'}_r$ if $q' \leq q$, $C^\infty_r(G,V) = \bigcap_q C^q_r(G,V)$. The topology on $C^\infty_r(G,V)$ is given by the family of norms $\|f\|_{q,r}$, $q \in \mathbb{N}$, on $C^\infty_r(G,V)$. We now consider for each $q \in \mathbb{N}$ the action of $L$ and $R$ on $C^\infty_r(G,V)$. Recall for $g,x \in G$, and $f \in C^q(G,V)$, $L_xf(g) = f(x^{-1}g)$, and $R_xf(g) = f(gx)$. Obviously $L_x$ leaves $C^q_r(G,V)$ invariant. In fact $\|L_xf\|_{q,r} \leq C\|x\|^{q+r}\|f\|_{q,r}$, for each $f \in C^q_r(G,V)$, and $x \in G$. Here $C$ and $s$ are constants. On the other hand,
\[ |R_x f|_{q,r} \leq \|x\|^\tau \|f\|_{q,r}. \] From Example 2.2, we see \( P_T \) maps \( C^\infty \text{Ind}^G_H(\sigma \otimes (-\lambda) \otimes 1) \) into \( C^\infty_r(G,V) \) continuously.

Recall from Proposition 1.11 a character of \( Z_{r} \) is given by \( \lambda - \Lambda \), where \( \lambda \in a^*_\mathbb{C} \), and \( \Lambda \) is the infinitesimal character of an irreducible representation of \( M \) in \( \tau \). Let \( \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) \) denote the corresponding eigenspace of \( Z_r \). Let

\[ \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) = \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) \cap C^\infty_r(G,V), \]

\[ \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) = \bigcup_{r \in \mathbb{R}} \mathcal{E}_{\lambda - \Lambda,r} \text{Ind}^G_K(\tau). \]

Our goal is to describe \( \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) \) in terms of the Poisson transform, at least for “generic” \( \lambda - \Lambda \). The following result due to Harish-Chandra is very important to us. See [Wall2].

**Proposition 2.3.** Let \( \mathcal{N} = \Theta N \). Then \( C(\lambda) \) defined by

\[ C(\lambda) = \int_\mathcal{N} \tau(k(\pi))e^{-(\lambda + \rho)H(\pi)}d\pi \]

is holomorphic on \( \{ \lambda \in a^*_\mathbb{C} | \text{Re}(\lambda, \alpha) > 0, \text{ for each } \alpha \in \Sigma^+(g,a) \} \). Moreover there exists a meromorphic continuation to \( a^*_\mathbb{C} \).

**Proposition 2.4.** Let \( \lambda \in a^*_\mathbb{C} \) such that \( \text{Re}(\lambda, \alpha) > 0, \text{ for } \alpha \in \Sigma^+(g,a) \). Then

\[ \lim_{t \to \infty} e^{-(\lambda + \rho)(H)} P_T \phi(g \exp tH) = C(\lambda)T \phi, \]

for each \( H \in a^+, T \in \text{Hom}_M(V_\sigma, V) \), and \( \phi \in C^\infty \text{Ind}^G_H(\sigma \otimes (-\lambda) \otimes 1) \). Here \( a^+ = \{ X \in a | \alpha(X) > 0, \forall \alpha \in \Sigma^+(g,a) \} \).

**Proof.** First we observe \( k \to \tau(k)T \phi(g \exp tHk) \) is a function on \( K/M \). By Theorem 5.20 in Chapter I in [Helg1],

\[ P_T \phi(g \exp tH) = \int_\mathcal{N} \tau(k(\pi))T \phi(g \exp tHk(\pi))e^{-2\rho H(\pi)}d\pi = \int_\mathcal{N} e^{-H(\pi)} T \phi(g \exp tH)\pi)d\pi = e^{-(\lambda + \rho)H} \int_\mathcal{N} e^{-2H(\pi)}T \phi(ga_t \pi a_t^{-1})d\pi. \]

Here \( a_t = \exp tH \). So

\[ e^{-(\lambda + \rho)H} P_T \phi(g \exp tH) = \int_\mathcal{N} e^{-H(\pi)} T \phi(ga_t \pi a_t^{-1})d\pi \]

since \( a_t \pi a_t^{-1} \to e \), as \( t \to \infty \). Formally we have

\[ P_T \phi(g \exp tH) \to C(\lambda)T \phi(g), \]

as \( t \to \infty \). To justify the exchange of two limits we use an argument due to Helgason. Let \( \lambda = \xi + \sqrt{-1} \eta, \) for \( \xi, \eta \in a^* \). Our assumption on \( \lambda \) amounts to \( A_\xi \in a^* \), where \( A_\xi \) is given by \( \langle \mu, A_\xi \rangle = K(\xi, \mu) \), for each \( \mu \in a^* \). It was proved by Harish-Chandra that

\[ B(H, H(\pi)) \geq 0, \quad B(H, H(\pi) - H(a_t \pi a_t^{-1})) \geq 0, \]

for each \( H \in a^+ \). Thus if we choose \( \varepsilon \) such that \( 0 < \varepsilon < 1, A_\rho - \varepsilon A_\xi \in a^+ \), and put

\[ C = \sup_{\pi_t} |\tau(k(\pi))T \phi(gk(a_t \pi a_t^{-1}))| < \infty, \]

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then
\[ |e^{-(\lambda + \rho)H(\pi)} \tau(k(\pi))T\phi(ga_i\pi a_i^{-1})| \]
\[ = |e^{-(\lambda + \rho)H(\pi)}e^{(\lambda - \rho)H(a_i\pi a_i^{-1})} \tau(k(\pi))T\phi(ga_i\pi a_i^{-1})| \]
\[ \leq Ce^{-(\xi + \rho)H(\pi)}e^{(\xi - \rho)H(a_i\pi a_i^{-1})} \]
\[ \leq Ce^{-(\xi + \rho)H(\pi)}e^{(\xi - \varepsilon \xi)H(a_i\pi a_i^{-1})} \]
\[ \leq Ce^{-(\xi - \rho)H(\pi)}. \]

This being integrable over \( \pi \) justifies letting \( t \to \infty \) under the integral sign and proves Proposition 2.4. \( \square \)

3. ASYMPTOTICS

By a formal expansion at a point \( H_0 \in a^+ \), we mean a formal sum
\[ \sum_{\xi \in X} p_{\xi}(H, t)e^{t\xi(H)}, \]
where \( X \) is a subset of \( a^+_C \) such that the subset \( X(N) \) given by
\[ X(N) = \{ \xi \in X | \text{Re}\xi(H_0) \geq N \} \]
is a finite set for each \( N \in \mathbb{R} \), where \( p_{\xi} \) is a continuous function defined in a neighborhood of \( \{H_0\} \times \mathbb{R} \) and polynomial in the last variable.

Let \( f \) be a function \( a^+ \to V \). If \( N \in \mathbb{R} \) we say the formal sum is asymptotic to \( f \) of order \( N \) at \( H_0 \), if there exist a neighborhood of \( H_0 \) in \( a^+ \), say \( U \), and constants \( \varepsilon \geq 0, C \geq 0 \), such that
\[ \left| f(tH) - \sum_{\xi \in X(N)} p_{\xi}(H, t)e^{t\xi(H)} \right| \leq Ce^{(N - \varepsilon)t}, \]
for each \( H \in U, t \geq 0 \). Moreover, we say the formal expansion is an asymptotic expansion for \( f \) at \( H_0 \) if for every \( N \in \mathbb{R} \) it is asymptotic to \( f \) of order \( N \) at \( H_0 \).
We write this as
\[ f(tH) \sim \sum_{\xi \in X} p_{\xi}(H, t)e^{t\xi(H)} \quad (t \to \infty). \]

The following result shows that the \( p_{\xi} \)'s are essentially unique.

**Proposition 3.1.** Let \( X \subset a^+_C \), \( \sum_{\xi \in X} p_{\xi}(H, t)e^{t\xi(H)} \) and \( \sum_{\xi \in X} q_{\xi}(H, t)e^{t\xi(H)} \) be formal expansions at \( H_0 \), both assumed to be asymptotic to \( f: a^+ \to V \). Then for each \( \xi \in X \), there is a neighborhood \( U \) of \( H_0 \), such that \( p_{\xi} = q_{\xi} \) on \( U \times \mathbb{R} \).

**Proof.** See Proposition 3.1 in [BS]. \( \square \)

Let \( \lambda - \Lambda \) be a character of \( Z_\tau \) in the sense of Proposition 1.11, where \( \lambda \in a^+_C \), and \( \Lambda \) is given by the infinitesimal character of an irreducible representation of \( M \).
Let \( X(\lambda, \Lambda) \) be the subset of \( a^+_C \) defined by
\[ X(\lambda, \Lambda) = \{ w(\lambda - \Lambda) + \Lambda - \rho - N \cdot \Delta | w \in \hat{W}, \sigma \in \tau, (w(\lambda - \Lambda) + \Lambda - \rho - N \cdot \Delta)|t = 0 \}. \]
Then we have the following results.
Theorem 3.2. (i) For each \( f \in \mathcal{E} \bigl( \Lambda^\infty, \Lambda \bigr) \), \( x \in G \), and \( \xi \in X(\Lambda, \Lambda) \), there exists a unique polynomial \( p_{\lambda, \xi}(f, x, \cdot) \) on \( \mathfrak{a} \) with values in \( V \), such that
\[
f(tH) \sim \sum_{\xi \in X(\Lambda, \Lambda)} p_{\lambda, \xi}(f, x, tH) e^{t\xi(H)} \quad (t \to \infty),
\]
at every \( H_0 \in \mathfrak{a}^+ \), and the polynomials have degree \( \leq d \), where \( d \) is the number of elements in \( \Sigma^+(\mathfrak{g}_C, \mathfrak{h}_C) \).

(ii) Let \( r \in \mathbb{R} \) and \( \xi \in X(\Lambda, \Lambda) \); there exists \( r' \in \mathbb{R} \) such that \( f \to p_{\lambda, \xi}(f, \cdot, \cdot) \) is a continuous map of \( \mathcal{E} \bigl( \Lambda^\infty, \Lambda, r \bigr) \) into \( C^\infty_{\Lambda^\infty}(G, V) \otimes P_d(\mathfrak{a}) \), equivariant for the left action of \( G \) on \( \mathcal{E} \bigl( \Lambda^\infty, \Lambda, r \bigr) \) and in addition holomorphic in \( \lambda \).

Theorem 3.3. Let \( \Omega \) be an open set in \( \mathfrak{a}^* \). Let \( \{ f_\lambda \}_{\lambda \in \Omega} \) be a holomorphic family in \( C^\infty \bigl( \Lambda^\infty, \Lambda \bigr) \) such that \( f_\lambda \in \mathcal{E} \bigl( \Lambda^\infty, \Lambda, r \bigr) \) for each \( \lambda \in \Omega \). Fix \( \lambda_0 \in \Omega \) and \( \xi_0 \in X(\Lambda_0, \Lambda) \). Let
\[
\Xi(\lambda) = \{ w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mu \in X(\Lambda, \Lambda) | w(\lambda_0 - \Lambda) + \Lambda_\sigma - \rho - \mu = \xi_0 \}.
\]
There exist an open neighborhood \( \Omega_0 \subset \Omega \) of \( \lambda_0 \) and a constant \( r' \in \mathbb{R} \) such that the map \( (\lambda, H) \to \sum_{\xi \in \Xi(\lambda)} p_{\lambda, \xi}(f_\lambda, \cdot, H) e^{\xi(H)} \) is continuous from \( \Omega \times \mathfrak{a}^+ \) into \( C^\infty_{\Lambda^\infty}(G, V) \) and in addition holomorphic in \( \lambda \).

We shall prove these results in Sections 5 and 6.

4. SOME ALGEBRAIC RESULTS

This section is a necessary preparation for the proof of the theorems stated in last section. It is strongly influenced by [Ban] and [BS].

Let \( E \) be the set of \( W \)-harmonic polynomials in \( \mathfrak{a}^* \). It is well-known that \( j : E \otimes I(\mathfrak{a}) \to S(\mathfrak{a}) \) is bijective, where \( j(e \otimes h) = eh \).

Now let \( r : I(\mathfrak{h}_C) \to I(\mathfrak{a}) \) be the restriction map. We assume \( r \) is surjective for the rest of the thesis. According to [Helg3] if \( G/K \) is irreducible there are just four exceptions, and they only occur among symmetric spaces of exceptional groups. Pick a set of algebraically independent homogeneous generators of \( I(\mathfrak{a}) \), say, \( p_1, \ldots, p_m \). Choose homogeneous elements \( q_1, \ldots, q_m \) in \( I(\mathfrak{h}_C) \), such that \( r(q_i) = p_i \), for \( i = 1, \ldots, m \). Let \( I_1(\mathfrak{h}_C) \) be the polynomial ring of \( q_1, \ldots, q_m \).

For any \( \mu \in \mathfrak{h}_C^* \), let
\[
I^-_{1, \mu} = \{ (T_\mu p)^- | p \in I_1(\mathfrak{h}_C) \}.
\]
Here \( (T_\mu p)(\nu) = p(\mu + \nu) \), for each \( \nu \in \mathfrak{h}_C^* \), and \( (T_\mu p)^-(\lambda) = p(\mu + \lambda) \), for each \( \lambda \in \mathfrak{a}^* \).

Proposition 4.1. Let \( j_\mu : E \otimes I^-_{1, \mu} \to S(\mathfrak{a}) \) be defined by
\[
j_\mu(e \otimes h) = eh.
\]
Then \( j_\mu \) is bijective.

Proof. Observe \( (T_\mu q_i)^- = p_i + r_i \), with \( \deg r_i < \deg p_i \). Using the fact that \( j \) is bijective and by induction we are done.

Let \( Z_1(\mathfrak{g}) = \gamma^{-1}(I_1(\mathfrak{h})) \). Here \( \gamma \) is the Harish-Chandra homomorphism. For each \( \lambda \in \mathfrak{a}_C^* \), \( \Lambda = \Lambda_\sigma \) for some \( \sigma \in \tau \), let
\[
I(\lambda, \Lambda) = \{ Z \in Z_1(\mathfrak{g}) | \gamma(Z)(\lambda - \Lambda) = 0 \}.
\]
Recall $I(\tau)$ is the left ideal of $U(\mathfrak{g}) \otimes \text{End}(V)$ generated by $X \otimes 1 + 1 \otimes \tau(X)$, for all $X \in \mathfrak{t}$. Let $J(\lambda, \Lambda)$ be the left ideal generated by $I(\lambda, \Lambda)$ and $I(\tau)$. Let

$$\mathfrak{Y}_{\lambda, \Lambda} = U(\mathfrak{g}) \otimes \text{End}(V)/J(\lambda, \Lambda).$$

Our interest in $\mathfrak{Y}_{\lambda, \Lambda}$ comes from the fact that for $f \in \mathcal{E}_{\lambda-\Lambda} \text{Ind}_{K}^{\mathfrak{g}}(\tau)$, the map $u \otimes T \rightarrow TR_{uf}f$ factors through $\mathfrak{Y}_{\lambda, \Lambda}$ since $f$ is killed by $J(\lambda, \Lambda)$. We shall find below an underlying vector space for $\mathfrak{Y}_{\lambda, \Lambda}$ independent of $\lambda$.

Write $\mathfrak{Y} = U(\mathfrak{h}) \otimes E \otimes \text{End}(V)$. We shall construct a linear bijection of $\mathfrak{Y}$ with $\mathfrak{Y}_{\lambda, \Lambda}$. First we identify $\mathfrak{Y}$ with a subspace of $U(\mathfrak{g}) \otimes \text{End}(V)$ as follows: $u \otimes e \otimes T \rightarrow (u \cdot e) \otimes T$, for $u \in U(\mathfrak{h})$, $e \in E$, and $T \in \text{End}(V)$. Here $\cdot$ denotes the multiplication in $U(\mathfrak{a} + \mathfrak{h})$. Let $\Psi : \mathfrak{Y} \otimes Z_{1}(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes \text{End}(V)/I(\tau)$ be the map defined by

$$\Psi(y \otimes Z) = y \cdot (Z \otimes 1) + I(\tau),$$

for $y \in \mathfrak{Y}$, $Z \in Z_{1}(\mathfrak{g})$. Here $\cdot$ means the multiplication in $U(\mathfrak{g}) \otimes \text{End}(V)$.

**Proposition 4.2.** In the setting above, $\Psi$ is bijective.

**Proof.** By the Iwasawa decomposition we have

$$U(\mathfrak{g}) \otimes \text{End}(V)/I(\tau) \cong U(\mathfrak{h}) \otimes U(\mathfrak{a}) \otimes \text{End}(V).$$

Via this isomorphism the degree on $U(\mathfrak{a})$ induces a degree (denoted by $\text{deg}_{\mathfrak{a}}$) on $U(\mathfrak{g}) \otimes \text{End}(V)/I(\tau)$. Let $\mathfrak{Y} \otimes Z_{1}(\mathfrak{g})$ be filtered by the total degree on $E \otimes Z_{1}(\mathfrak{g})$. Notice

$$\text{deg}_{\mathfrak{a}}(Z \otimes 1 - (T_{p-\Delta, \gamma}(Z))^{-1} \otimes 1 + I(\tau)) < \text{deg}(Z \otimes 1),$$

for $Z \in Z_{1}(\mathfrak{g})$, and each $\sigma \in \tau$. So $\Psi$ preserves the filtrations. It also follows that the graded map

$$gr\Psi : U(\mathfrak{h}) \otimes gr(E \otimes Z_{1}(\mathfrak{g})) \otimes \text{End}(V) \rightarrow U(\mathfrak{h}) \otimes U(\mathfrak{a}) \otimes \text{End}(V)$$

associated to $\Psi$, is given by

$$u \otimes e \otimes Z \otimes T \rightarrow u \cdot e \cdot (T_{p-\Delta, \gamma}(Z))^{-} \otimes T,$$

for $u \in U(\mathfrak{h})$, $e \in E$, $Z \in Z_{1}(\mathfrak{g})$, and $T \in \text{Hom}(V(\sigma), V)$ (here we use Proposition 1.15). This is bijective because of Proposition 4.1.

**Corollary 4.3.** (i) $\Psi$ maps $\mathfrak{Y} \otimes I(\lambda, \Lambda)$ onto $J(\lambda, \Lambda)$ modulo $I(\tau)$; (ii) for each $u \in U(\mathfrak{g}) \otimes \text{End}(V)$ there exists a unique $y \in \mathfrak{Y}$, such that $u - y \in J(\lambda, \Lambda)$.

**Proof.** See Corollary 5.2 in [BS].

From the corollary we obtain a linear bijection $b_{\lambda}$ of $\mathfrak{Y}_{\lambda, \Lambda}$ onto $\mathfrak{Y}$, defined by $u - b_{\lambda}(u + J(\lambda, \Lambda)) \subset J(\lambda, \Lambda)$. Through this bijection $\mathfrak{Y}$ is equipped with a $(\mathfrak{g}, \mathcal{K})$-module structure from $\mathfrak{Y}_{\lambda, \Lambda}$, by making $b_{\lambda}$ a morphism of modules. Recall the $\mathfrak{g}$ action on $\mathfrak{Y}_{\lambda, \Lambda}$ is induced from left multiplication in $U(\mathfrak{g})$, and the $\mathcal{K}$ action is induced from the following $\mathcal{K}$ action on $U(\mathfrak{g}) \otimes \text{End}(V)$,

$$k.(u \otimes T) = \text{Ad}(k)u \otimes T \tau(k^{-1}),$$

for each $k \in \mathcal{K}, u \in U(\mathfrak{g})$, and $T \in \text{End}(V)$. Notice the difference from the action we use to define $U(\mathfrak{g})^{\mathcal{K}}$.

Let $\tau_{\lambda}$ denote the resulting $\mathfrak{g}$ action on $\mathfrak{Y}$. Notice the action of $\mathfrak{h}$ on $\mathfrak{Y}$ is just the left multiplication. The action of $\mathfrak{a}$ can be determined as follows: Let
$y \in \mathfrak{g} \subset U(\mathfrak{g}) \otimes \operatorname{End}(V)$, $H \in \mathfrak{a}$; then $H \cdot y$ can be written (modulo $I(\tau)$) as $\Psi(\sum y_i \otimes Z_i)$ according to Proposition 4.2. Then by the definition of $\tau_\lambda$ we have
\begin{equation}
\tau_\lambda(H)y = \sum \gamma(Z_i)(\lambda - \Lambda)y_i.
\end{equation}

For each $k \in \mathbb{N}$, let $\mathfrak{p}^k$ be the linear span of $k$ times product of $\mathfrak{p}$ in $U(\mathfrak{p})$. Then $\tau_\lambda$ induces a representation $\tau_\lambda^k$ of $\mathfrak{a} + \mathfrak{m}$ on the finite dimensional space $\mathfrak{g}/\mathfrak{p}^k \mathfrak{g}$. In particular $\tau_\lambda^1$ is a representation of $\mathfrak{a} + \mathfrak{m}$ on $\mathfrak{g}/\mathfrak{p} \mathfrak{g} \cong E \otimes \operatorname{End}(V)$. By (\star) we know $\tau_\lambda$ and $\tau_\lambda^k$ are holomorphic in $\lambda$.

Let $\{\lambda_1, \ldots, \lambda_l\}$ be the set of weights of $\tau_\lambda^1$ restricted to $\mathfrak{a}$, and $\Lambda_k \subset -\mathbb{N} \cdot \Delta$ an enumeration of the weights of the $\mathfrak{a}$-module $U(\mathfrak{p})/\mathfrak{p}^k U(\mathfrak{p})$.

**Proposition 4.4.** For each $k \in \mathbb{N}$, $k \geq 1$, the set of weights of $(\tau_\lambda^k, \mathfrak{a})$ is
\[
\{\lambda_i + \mu | i = 1, \ldots, l, \mu \in \Lambda_k\}.
\]

**Proof.** By induction on $k$. It is trivial for $k = 1$. For $k > 1$, the induction step is a consequence of the following two exact sequences of $\mathfrak{a}$-modules:
\begin{align*}
0 \to \mathfrak{p}^{k-1} U(\mathfrak{p})/\mathfrak{p}^k U(\mathfrak{p}) \otimes \mathfrak{g}_{\lambda, \Lambda} \to \mathfrak{g}_{\lambda, \Lambda}/\mathfrak{p} U(\mathfrak{p}) \quad &\to \mathfrak{g}_{\lambda, \Lambda}/\mathfrak{p}^{k-1} U(\mathfrak{p}) \to 0, \\
0 \to \mathfrak{p}^{k-1} U(\mathfrak{p})/\mathfrak{p}^k U(\mathfrak{p}) \to U(\mathfrak{p})/\mathfrak{p}^k U(\mathfrak{p}) &\to U(\mathfrak{p})/\mathfrak{p}^{k-1} U(\mathfrak{p}) \to 0. \quad \Box
\end{align*}

Let $\mathcal{V}_k = \mathfrak{g}/\mathfrak{p}^k \mathfrak{g}$, and $\tilde{V}_k$ be a finite dimensional subspace of $\mathfrak{g}$ mapped bijectively onto $\mathcal{V}_k$ by the canonical projection. Let $\pi: \tilde{V}_k \to \mathcal{V}_k$ be the restriction of the canonical projection. Define $m: \mathfrak{g} \to U(\mathfrak{g}) \otimes \operatorname{End}(V)$ by
\[m(u \otimes e \otimes T) = (u \cdot T),\]
for $u \in U(\mathfrak{p})$, $e \in E$, and $T \in \operatorname{End}(V)$.

Let $V_k$ be the image of $\tilde{V}_k$ under $m$, and $\eta: V_k \to \tilde{V}_k$ be the inverse of $m|\tilde{V}_k$. Let $Z(\mathfrak{a} + \mathfrak{m})$ be the center of $U(\mathfrak{a} + \mathfrak{m})$.

**Proposition 4.5.** For $k \in \mathbb{N}$, $k \geq 1$, there exist
\begin{itemize}
\item[(i)] an algebra homomorphism $b_k(\lambda, \cdot): Z(\mathfrak{a} + \mathfrak{m}) \to \operatorname{End}(V_k)$,
\item[(ii)] a linear map $y_k: Z(\mathfrak{a} + \mathfrak{m}) \otimes V_k \to \mathfrak{p}^k U(\mathfrak{a} + \mathfrak{p}) \otimes \operatorname{End}(V)$, both depending polynomially on $\lambda$, such that for each $\lambda \in \mathfrak{a}_C^\ast$, $D \in Z(\mathfrak{a} + \mathfrak{m})$, and $v \in V_k$,
\[
Dv - b_k(\lambda, D)v - y_k(D, v) \in J(\lambda, \Lambda).
\]
\end{itemize}

**Proof.** Let $p_\lambda: U(\mathfrak{g}) \otimes \operatorname{End}(V) \to \mathfrak{g}$ be the map defined by
\[p_\lambda(u \otimes T) = \tau_\lambda(u)(1 \otimes 1 \otimes T),\]
for $u \in U(\mathfrak{g})$, and $T \in \operatorname{End}(V)$. For $D \in Z(\mathfrak{a} + \mathfrak{m})$, $\tilde{v} \in \tilde{V}_k$ we define the maps
\begin{align*}
\hat{b}_k(\lambda, D) &= \pi^{-1} \circ \tau_\lambda^k \circ \pi \in \operatorname{End} \tilde{V}_k, \\
\hat{y}_k(D, \tilde{v}) &= p_\lambda((D \otimes 1) \cdot m(\tilde{v})) - m(b_k(\lambda, D)\tilde{v}) \in \mathfrak{g}.
\end{align*}
Then $b_k(\lambda, \cdot)$ and $y_k$ are defined by
\begin{align*}
b_k(\lambda, D) &= m \circ \hat{b}_k(\lambda, D) \circ \eta, \\
y_k(D, v) &= m(\hat{y}_k(D, \eta(v))).
\end{align*}
for $D \in Z(\mathfrak{a} + \mathfrak{m})$, $v \in V_k$. \quad \Box

**Corollary 4.6.** As a representation of $\mathfrak{a}$, $b_k(\lambda, \cdot)$ has the same weights as $(\tau_\lambda^k, \mathfrak{a})$, i.e. $\{\lambda_i + \mu | i = 1, \ldots, l, \mu \in \Lambda_k\}$. 


Proof. By definition $b_k(\lambda, D) = m \circ \hat{b}_k(\lambda, D) \circ \eta$, and $\eta = (m|\overline{V}_k)^{-1}$. So $b_k(\lambda, \cdot)$ has the same weights as $\hat{b}_k(\lambda, \cdot)$. Since $\hat{b}_k(\lambda, \cdot) = \pi^{-1} \circ \tau_{k}^* \circ \pi$, the proof is complete. \hfill \Box

Let $V_k^*$ be the dual space of $V_k$, and $b_k^*(\lambda, \cdot)$ be the transpose of $b_k(\lambda, \cdot)$. For each weight $\xi$ of $b_k^*(\lambda, \cdot)$ we denote by $P_{\lambda,\xi}$ the projection map from $V_k^*$ onto the generalized weight space of $\xi$, along the remaining generalized weight spaces. We now consider the holomorphic dependence of $P_{\lambda,\xi}$ on $\lambda$.

**Proposition 4.7.** There exists for each $\lambda \in \mathfrak{a}^*$, and each weight $\xi$ a unique polynomial $q_{\lambda,\xi}$ on $\mathfrak{a}$ with values in $\text{End}(V_k^*)$, such that

$$P_{\lambda,\xi}q_{\lambda,\xi}(H)P_{\lambda,\xi} = q_{\lambda,\xi}(H),$$

$$\exp b_k^*(\lambda, H) = \sum_{\xi} e^{\xi(H)} q_{\lambda,\xi}(H),$$

for $H \in \mathfrak{a}$.

**Proof.** Let $V_k^*(\xi)$ be the generalized weight space of $\xi$. Then the restriction of $b_k^*(\lambda, \cdot)$ to $V_k^*(\xi)$ gives a representation of $\mathfrak{a}$. $\mathfrak{a}$ is abelian so in particular solvable. Hence by Lie’s theorem one can find a basis such that $b_k^*(\lambda, H)|V_k^*(\xi)$ corresponds to an upper triangular matrix, for each $H \in \mathfrak{a}$. The diagonal entries are $\xi(H)$. So there exists a unique polynomial $q_{\lambda,\xi}(H)$ on $\mathfrak{a}$ with values in $\text{End}(V_k^*)$, such that

$$\exp b_k^*(\lambda, H)|V_k^*(\xi) = e^{\xi(H)} q_{\lambda,\xi}(H).$$

Let $F$ be an $N$-dimensional complex vector space, and $\tau_z$ a family of representations of $\mathfrak{a}$ in $F$, depending on a parameter $z \in \mathbb{C}^n$. For each weight $\xi$ of $\tau_z$, let $P_{z,\xi}$ be the projection map from $F$ onto the generalized weight space $V(\xi)$, along the remaining generalized weight spaces. Fix $z_0 \in \mathbb{C}^n$, and $\xi_0$ a weight of $\tau_{z_0}$.

**Lemma 4.8.** Given any neighborhood $N(\xi_0)$ of $\xi_0$ there exist a neighborhood $V(\xi_0)$ of $\xi_0$ in $N(\xi_0)$, and a neighborhood $\Omega(z_0)$ of $z_0$, such that

$$P(z) = \sum_{\xi \in V(\xi_0)} P_{z,\xi} \in \text{End}(F)$$

is holomorphic in $z$ in $\Omega(z_0)$.

**Proof.** We use the argument in Chapter II in [Kato]. First let us consider the case where $\dim \mathfrak{a} = 1$. Pick a nonzero element $H_0 \in \mathfrak{a}$. Let

$$T(z) = \tau_z(H_0) \in \text{End}(F).$$

Then $\lambda_0 = \xi_0(H_0)$ is an eigenvalue of $T(z_0) = \tau_{z_0}(H_0)$. Define

$$R(z, \lambda) = (T(z) - \lambda)^{-1},$$

for $z \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$. By Theorem 1.5 in Section 3 of Chapter II in [Kato], $R(z, \lambda)$ is holomorphic in the two variables $z$ and $\lambda$ in each domain where $\lambda$ is not an eigenvalue of $T(z)$. Moreover, for each $(z_1, \lambda)$ in such a domain,

$$R(z, \lambda) = R(z_1, \lambda) + \sum_{I \in \mathbb{N}^n} R_I(\lambda)(z - z_1)^I,$$

where $R_I(\lambda)$ are determined by $R(z_1, \lambda)$, and they are holomorphic in $\lambda$. This is called the second Neumann series for the resolvent. It is uniformly convergent for sufficiently small $z - z_1$ and $\lambda \in \Gamma$ if $\Gamma$ is a compact subset of the resolvent set of $T(z_1)$.\hfill \Box
Let $\Gamma$ be a closed positively oriented curve in the resolvent set of $T(z_0)$ enclosing $\lambda_0$ but no other eigenvalues of $T(z_0)$. Then

$$P(z) = -\frac{1}{2\pi i} \int_{\Gamma} R(z, \lambda) d\lambda$$

is holomorphic in $z$, for $z - z_0$ sufficiently small.

It is easy to see $P(z)$ is equal to the sum of the eigenprojections for all eigenvalues of $T(z)$ lying inside $\Gamma$. This basically takes care of the case $\dim \mathfrak{a} = 1$. In general we choose a basis $e_1, \ldots, e_m$ for $\mathfrak{a}$. We can duplicate the above process to $T_i(z) = \tau_z(e_i)$, for $i = 1, \ldots, m$. Thus we get $P_i(z)$, $i = 1, \ldots, m$. Then the composition of $P_i$ is our $P(z)$.

Fix $\lambda_0 \in \mathfrak{a}_c^*$, and $\xi_0$ a weight of $b_k^*(\lambda_0, \cdot)$. For each $\lambda \in \mathfrak{a}_c^*$, let

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda, \rho - \mu \in X(\lambda, \Lambda) | w(\lambda_0 - \Lambda) + \Lambda, \rho - \mu = \xi_0\}.$$

**Proposition 4.9.** There exist a neighborhood $\Omega_0(\lambda_0)$ of $\lambda_0$ and a neighborhood $V(\xi_0)$ of $\xi_0$, such that

$$P(\lambda) = \sum_{\xi \in V(\xi_0)} P_{\lambda, \xi} \in \text{End}(V_k^*)$$

is holomorphic in $\Omega_0(\lambda_0)$, and

$$\{\xi \in V(\xi_0) | \xi \text{ is a weight of } b_k^*(\lambda_0, \cdot) \} \cap X(\lambda, \Lambda) \subset \Xi(\lambda).$$

**Proof.** It follows at once from Lemma 4.8. \qed

5. **Existence of asymptotic expansion**

The methods we use in this section are similar to those used in [Ban], Section 12. Also see [BS], Section 6.

Fix $\lambda \in \mathfrak{a}_c^*$, $H_0 \in \mathfrak{a}^+$ and $r \in \mathbb{R}$. If $A_1, A_2$ are Banach spaces we denote by $B(A_1, A_2)$ the Banach space of bounded linear operators from $A_1$ to $A_2$.

**Proposition 5.1.** There exist, for each $N \in \mathbb{R}$,

(a) open neighborhoods $\Omega$ of $\lambda_0 \in \mathfrak{a}_c^*$ and $U$ of $H_0 \in \mathfrak{a}^+$,

(b) constants $k, q \in \mathbb{N}, r' \geq r$, and $C, \varepsilon > 0$,

(c) a continuous map

$$\Phi: \Omega \times U \rightarrow B(C^r_k(G, V), V_k^* \otimes C_{r'}(G, V)),$$

holomorphic in the first variable, and

(d) a linear form $\eta \in (V_k^*)^*$, such that

(i) $\Phi(\lambda, H)$ intertwines the left actions of $G$ on $C^r_k(G, V)$ and $C_{r'}(G, V)$, for all $\lambda, H \in \Omega \times U$, and (ii)

$$\|R_{e^tH} f - (\eta \circ \exp b_k^*(\lambda, tH) \otimes 1) \Phi(\lambda, H) f\|_{r'} \leq C\|f\|_{q, r} e^{(N+\varepsilon)t},$$

for $f \in \mathcal{E}_{\lambda, \Lambda} \text{Ind}_{K}^{G}(\tau) \cap C^2_k(G, V)$, $\lambda \in \Omega$, $H \in U$, $t \geq 0$.

**Proof.** In the same way as for Proposition 12.6 in [Ban]. \qed

We now begin the proof of Theorem 3.2. Using Proposition 4.7 we can write

$$(\eta \circ \exp b_k^*(\lambda, tH) \otimes 1) \Phi(\lambda, H) = \sum_{\xi} p_{\lambda, \xi}(H, t)e^{\xi(H)},$$
for $\lambda \in \Omega$, $H \in U$, $t \geq 0$, where the summation extends to the weights $\xi$ of $b^*_\lambda (\cdot, \cdot)$ which by Corollary 4.6 is the set
\[
\{ \lambda_i + \mu | i = 1, \ldots, l, \mu \in \Lambda_k \},
\]
and where $p_{\lambda,\xi}(H, t) = (\eta \circ \lambda_{\xi}(tH) \otimes 1)\Phi(\lambda, H) \in B(C^r, C_{pr})$, which is continuous in $H$ and polynomial in $t$. From (d) (ii) of Proposition 5.1 we have
\[
\|R_{\exp tH}f - \sum_\xi e^{t\xi(H)} p_{\lambda,\xi}(H, t)f\|_{r'} \leq C\|f\|_{q,r} e^{t(N-\varepsilon)},
\]
for $f \in \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) \cap C^q_r(G, V)$. Since $N$ is arbitrary we have for each $g \in G$,
\[
f(g \exp tH) \sim \sum_{\xi \in \tilde{X}(\lambda, \Lambda)} (p_{\lambda,\xi}(H, t)f)(g)e^{t\xi(H)} \quad (t \to \infty).
\]
Here $\tilde{X}(\lambda, \Lambda) = \{ \lambda_i + \mu | i = 1, \ldots, l, \mu \in -\mathbb{N} \cdot \Delta \}$.

**Lemma 5.2.** Let $X \subset \mathfrak{a}^*_C$ and $f : \mathfrak{a}^+ \to V$. Assume that for each $H_0 \in \mathfrak{a}^+$ there is a given formal expansion
\[
\sum_{\xi \in X} p_{\xi,H_0}(H, t)e^{t\xi(H)}
\]
which is an asymptotic expansion for $f$ at $H_0$. Then for each $\xi \in X$ there exists a unique continuous function $p_{\xi} : \mathfrak{a}^+ \to V$ such that for each $H_0 \in \mathfrak{a}^+$ there is a neighborhood $U$ with
\[
p_{\xi,H_0}(H, t) = p_{\xi}(tH),
\]
for $H \in U$, and $t > 0$.

**Proof.** See Corollary 3.4 in [BS].

As can be seen in the proof of Proposition 12.6 in [Ban], for $t > 0$, $H \in U$ with $tH \in U$, $\Phi(\lambda, tH) = \Phi(\lambda, H)$. Thus for $t > 0$, $H \in U$ with $tH \in U$, $(p_{\lambda,\xi}(H, t)f)(g) = (p_{\lambda,\xi}(tH, 1)f)(g)$. By Lemma 5.2, for $\lambda \in \mathfrak{a}^*_C$, $r \in \mathbb{R}$, and $\xi \in \tilde{X}(\lambda, \Lambda)$, there exist constants $r' \in \mathbb{R}$, $q \in \mathbb{N}$, and a unique continuous map $p_{\lambda,\xi}(\cdot, \cdot, \cdot) : \mathfrak{a}^+ \to B(\mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) \cap C^q_r(G, V), C_{pr}(G, V))$, such that
\[
f(g \exp tH) \sim \sum_{\xi \in \tilde{X}(\lambda, \Lambda)} p_{\lambda,\xi}(f, g, tH)e^{t\xi(H)} \quad (t \to \infty),
\]
at every $H_0 \in \mathfrak{a}$, for $f \in \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau) \cap C^q_r(G, V)$.

To complete the proof of Theorem 3.2 it remains to show (1) we can replace $\tilde{X}(\lambda, \Lambda)$ by $X(\lambda, \Lambda)$, (2) $p_{\lambda,\xi}(f, g, H)$ is a polynomial in $H$ with order $\leq d$. We shall finish the proof in the next section. We now consider the holomorphic dependence in $\lambda$ in order to prove Theorem 3.3.

Let $r \in \mathbb{R}$ and $\Omega$ be an open set in $\mathfrak{a}^*_C$. Let $\{f_\lambda\}_{\lambda \in \Omega}$ be a holomorphic family in $C^\infty_r(G, V)$, and $f_\lambda \in \mathcal{E}_{\lambda - \Lambda} \text{Ind}^G_K(\tau)$, for each $\lambda \in \Omega$. We now study the asymptotic expansion of $f_\lambda$. Fix $\lambda_0 \in \Omega$, and $\xi_0 \in \tilde{X}(\lambda_0, \Lambda)$.

**Proposition 5.3.** There exist a neighborhood $\Omega(\lambda_0)$ of $\lambda_0$ in $\Omega$ and a neighborhood $V(\xi_0)$ of $\xi_0$ in $\mathfrak{a}^*_C$, such that
\[
(\lambda, H) \to \sum_{\xi \in V(\xi_0)} p_{\lambda,\xi}(f_\lambda, \cdot, H)e^{\xi(H)}
\]
is continuous from $\Omega(\lambda_0) \times U$ to $C^{q'}_r(G, V)$ for some $q' \in \mathbb{N}$, $r' \in \mathbb{R}$, and in addition holomorphic in $\lambda$. Moreover, we can choose $V(\xi_0)$ such that $V(\xi_0) \cap X(\lambda, \Lambda) \subset \Xi(\lambda)$.

**Proof.** It follows from Proposition 4.9. \hfill \square

### 6. Differential Equations for the Coefficients

In this section we derive certain differential equations for the vector-valued functions $p_{\lambda, \xi}(f, g, \cdot)$ on $\mathfrak{a}^t$, where $f \in \mathcal{E}_{\lambda, \Lambda}^\infty \text{Ind}^G_K(\tau)$, and $g \in G$.

Fix $Z \in \mathcal{Z}(\mathfrak{g})$, and $D = \mu(Z) \in \mathcal{Z}$. We can choose finitely many $x_i$ in $\mathbb{R}U(\mathfrak{h})$, and $v_i \in U(\mathfrak{a}) \otimes \text{End}(V)$, such that

$$Z - \tilde{\Gamma}_1(Z \otimes 1) - \sum x_i v_i \in I(\tau),$$

and $ad(\mathfrak{a})$ acts on $x_i$ by a weight $-\eta_i \neq 0$, where $\eta_i \in \mathbb{N} \cdot \Delta$, and $v_i, \tilde{\Gamma}_1(Z \otimes 1) \in U(\mathfrak{a}) \otimes \text{End}(V)$ which can be interpreted as differential operators with constant coefficients on $C^\infty(\mathfrak{a}, V)$.

**Proposition 6.1.** Let $f \in \mathcal{E}_{\lambda, \Lambda}^\infty \text{Ind}^G_K(\tau)$. Then the functions $p_{\lambda, \xi}(f, g, \cdot)e^\xi$ from $G \times \mathfrak{a}^t$ to $V$ satisfy the following recursive equations:

$$1 \otimes \partial(\tilde{\Gamma}_1(Z \otimes 1) - \gamma(Z)(\lambda - \Lambda))(p_{\lambda, \xi}(f, g, \cdot)e^\xi)$$

$$= - \sum_{i, \xi + \eta_i \in \tilde{X}(\lambda, \Lambda)} R_{e^x} \otimes e^{-\eta_i} \partial(v^i)(p_{\lambda, \xi+\eta_i}(f, g, \cdot)e^{\xi+\eta_i}),$$

for all $\xi \in \tilde{X}(\lambda, \Lambda)$.

The proof is the same as for Proposition 7.1 in [BS].

**Proof of Theorem 3.2.** Let

$$V = \bigoplus_{\Lambda_1 \in \mathbf{t}^*} V(\Lambda_1),$$

where $V(\Lambda_1) = \bigoplus_{\tau \in \mathcal{T}_{\Lambda_0} = \Lambda_1} V(\tau)$. Let $P(\Lambda_1)$ be the projection from $V$ to $V(\Lambda_1)$. By Corollary 1.15 $\Gamma_1(Z \otimes 1)V(\Lambda_1) = (T_{\rho - \lambda}, \gamma(Z))^{-} \otimes I_{V(\Lambda_1)}$. For $\xi_1, \xi_2 \in \mathfrak{a}_c^t$, we say $\xi_1 \prec \xi_2$ if there exists $\eta \in \mathbb{N} \cdot \Delta$ such that $\xi_2 = \xi_1 + \eta$. This defines a partial order on $\mathfrak{a}_c^t$. For each $f \in \mathcal{E}_{\lambda, \Lambda}^\infty \text{Ind}^G_K(\tau)$, define $E(\lambda, \Lambda, f)$ by

$$E(\lambda, \Lambda, f) = \{ \xi \in \tilde{X}(\lambda, \Lambda)[p_{\lambda, \xi}(f, \cdot, \cdot) \neq 0].$$

Let $E_L(\lambda, \Lambda, f)$ denote the set of maximal elements in $E(\lambda, \Lambda, f)$. Suppose $\xi \in E_L(\lambda, \Lambda, f)$. Then $p_{\lambda, \xi}(f, g, \cdot) \neq 0$. So one can find $g \in G$, $\Lambda_1 \in \mathbf{t}^*$, such that

$$P(\Lambda_1)p_{\lambda, \xi}(f, g, \cdot) \neq 0.$$

Since the right-hand side of the equation in Proposition 6.1 is zero because $\xi$ is maximal in $E(\lambda, \Lambda, f)$,

$$\partial(\tilde{\Gamma}_1(Z \otimes 1) - \gamma(Z)(\lambda - \Lambda))(p_{\lambda, \xi}(f, g, \cdot)e^\xi) = 0.$$

So

$$\partial((T_{-\lambda, +\rho}^{-}\gamma(Z))^{-} - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda, \xi}(f, g, \cdot)e^\xi) = 0.$$

We extend $p_{\lambda, \xi}(f, g, \cdot)e^\xi$ to a function on $\mathfrak{a}^t + \sqrt{-1}\mathfrak{t} \subset \mathfrak{h} = \mathfrak{a} + \sqrt{-1}\mathfrak{t}$, by abuse of notation still denoted by $p_{\lambda, \xi}(f, g, \cdot)e^\xi$, by the requirement that it be constant in the $\mathfrak{t}$ direction. Hence

$$\partial((T_{-\lambda, +\rho}^{-}\gamma(Z)) - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda, \xi}(f, g, \cdot)e^\xi) = 0.$$
Hence the same as for Theorem 8.4 in [BS].

By Theorem 3.13, Chapter III in [Helg1], $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot) e^{\xi - \Lambda_1 + \rho} = \sum q_i e^{\mu_i}$, where $q_i$ are polynomials on $\mathfrak{h}_1, \mu_i \in \mathfrak{h}_1^\ast$. Recall that $p_{\lambda,\xi}(f, g, tH)$ is a polynomial in $t$. We conclude $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial on $\mathfrak{h}_1$ and

$$
\xi - \Lambda_1 + \rho = w(\lambda - \Lambda),
$$

for some $w \in \tilde{W}$. Also $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a $\tilde{W}(w(\lambda - \Lambda))$-harmonic, where $\tilde{W}(\mu) = \{w \in \tilde{W} | w\mu = \mu\}$, for each $\mu \in \mathfrak{h}_1^\ast$. So

$$
\deg(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)) \leq d.
$$

Here $d$ is the number of elements in $\Sigma^+(g, \mathfrak{h}_1^\ast)$. It follows that we can replace $X(\lambda, \Lambda)$ by $X(\lambda, \Lambda)$ since $E_L(\lambda, \Lambda, f) \subset X(\lambda, \Lambda)$.

By induction on $\xi$ using Proposition 6.1 one can easily show $p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial with degree $\leq d$. Note we only need to show it for $g = e$. So this completes the proof of Theorem 3.2.

The proof of Theorem 3.3 follows from Proposition 5.3. 

7. LEADING EXPONENTS

We further consider the properties of a leading term in the asymptotic expansion of $f \in \mathcal{E}_{\chi, \lambda}^\infty \text{Ind}_{\chi, \lambda}^G(\tau)$.

Proposition 7.1. For each $\xi \in E_L(\lambda, \Lambda, f)$, $man \in B$, $H \in \mathfrak{a}$, and $g \in G$,

$$
p_{\lambda,\xi}(f, g, \mathfrak{man}, H) = e^{(\log a) \tau(m) - 1}p_{\lambda,\xi}(f, g, H + \log a).
$$

Proof. The same as for Theorem 8.4 in [BS].

Let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. We introduce conditions on $\lambda - \Lambda$ and $\lambda$ as follows:

$$
\mathfrak{A}_1 = \{(\lambda - \Lambda) | \lambda \in \mathfrak{a}_1^\ast, \Lambda \in \mathfrak{c}_1^\ast, (\lambda - \Lambda, \alpha^\vee) \notin \mathbb{Z}, \forall \alpha \in \Sigma(g, \mathfrak{h}_1^\ast), \alpha \mathfrak{a} \neq 0\},
$$

$$
\mathfrak{A}_2 = \{\lambda \in \mathfrak{a}_1^\ast | (\lambda, \beta^\vee) \notin -\mathbb{N}, \forall \beta \in \Sigma^+(g, \mathfrak{a})\}.
$$

Let

$$
\tilde{W}_0 = \{w \in \tilde{W} | w|\mathfrak{a} = \text{id}\}, \quad \tilde{W}_1 = \{w \in \tilde{W} | wa = \mathfrak{a}\}.
$$

Proposition 7.2. Suppose $\lambda - \Lambda \in \mathfrak{A}_1$. We have

(i) if $w(\lambda - \Lambda) = \lambda - \Lambda$ for some $w \in \tilde{W}$, then $w \in \tilde{W}_0$;

(ii) if there exist $w \in \tilde{W}$, $\sigma \in \tau$ such that

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|t = 0,$$

then $w \in \tilde{W}_1$, and $\Lambda_\sigma = w\Lambda$.

Proof. (i) Since $w(\lambda - \Lambda) = \lambda - \Lambda$, $w = w_{\alpha_1} \cdots w_{\alpha_s}$, where $\alpha_j \in \Sigma(g, \mathfrak{h}_1^\ast)$, and $\langle \lambda - \Lambda, \alpha_j \rangle = 0$. Then we conclude $\alpha_j |\mathfrak{a} = 0$ from $\mathfrak{A}_1$. So $w \in \tilde{W}_0$. (ii) For any $\beta \in \Sigma(g, \mathfrak{h}_1^\ast)$ with $\beta |\mathfrak{a} = 0$, we have $\langle w(\lambda - \Lambda) + \Lambda_\sigma, \beta \rangle = 0$ since $(w(\lambda - \Lambda) + \Lambda_\sigma)|t = 0$. Hence

$$
\frac{2\langle \lambda - \Lambda, w^{-1}\beta \rangle}{\langle \beta, \beta \rangle} = - \frac{2\langle \Lambda_\sigma, \beta \rangle}{\langle \beta, \beta \rangle},
$$

$$
\frac{2\langle \lambda - \Lambda, w^{-1}\beta \rangle}{\langle w^{-1}\beta, w^{-1}\beta \rangle} = - \frac{2\langle \Lambda_\sigma, \beta \rangle}{\langle \beta, \beta \rangle}.
$$


The right-hand side being integral forces $w^{-1}\beta|a = 0$. This shows $w$ preserves $t$. Therefore $w$ preserves $a$. Trivially $\Lambda_\sigma = w\Lambda$. □

**Proposition 7.3.** Let $f \in E_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$. Suppose $\lambda-\Lambda \in \mathcal{A}_1$, and $\xi$ in $E_L(\lambda, \Lambda, f)$. Then $\xi \in W\lambda - \rho$, and $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $a$ for each $g \in G$.

**Proof.** In the last section we showed if $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot) \neq 0$, then there exists $w \in \tilde{W}$, such that $\xi - \Lambda_\sigma + \rho = w(\lambda - \Lambda)$. So

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|t = 0.$$ 

By Proposition 7.2 (ii) $w \in \tilde{W}_1$. So $\xi + \rho = w\lambda$. Hence $\xi \in W\lambda - \rho$. □

We also showed that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is $\tilde{W}(w(\lambda - \Lambda))$-harmonic. Since $w \in \tilde{W}_1$, $w(\lambda - \Lambda) \in \mathcal{A}_1$. By Proposition 7.2 (i) $\tilde{W}(w(\lambda - \Lambda)) \subset \tilde{W}_0$. We conclude $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is constant in $a$. This shows $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $a$ since $\sigma \in \tau$ is arbitrary. In this case we denote it by $p_{\lambda,\xi}(f, g)$.

**Corollary 7.4.** If $\lambda - \rho \in E_L(\lambda, \Lambda, f)$, and in addition $\lambda$ is regular, i.e., $W(\lambda) = \{w \in W|w\lambda = \lambda\} = e$, then

$$p_{\lambda,\lambda - \rho}(f, g) = P(\Lambda)p_{\lambda,\lambda - \rho}(f, g).$$ 

**Proof.** If for some $\sigma \in \tau$, such that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot) \neq 0$, then there exists $w \in \tilde{W}_1$, with

$$w\lambda = (\lambda - \rho) + \rho, w\Lambda_\sigma = \Lambda.$$ 

$\lambda$ being regular implies $w \in \tilde{W}_0$. But then $P(\Lambda) = P(\Lambda_\sigma)$ by definition. □

By Appendix II in [KKMÖOT] if $\lambda \in \mathcal{A}_2$, then $\lambda - \rho$ is always maximal in $W\lambda - \rho$. So we have the following definition.

**Definition 7.5.** Let $\lambda - \Lambda \in \mathcal{A}_1$, and $\lambda \in \mathcal{A}_2$. For $f \in E_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, $\beta_\lambda(f)$ is defined by

$$\beta_\lambda(f) = p_{\lambda,\lambda - \rho}(f, \cdot).$$ 

We call $\beta_\lambda$ the boundary value map.

**Theorem 7.6.** Let $\lambda - \Lambda \in \mathcal{A}_1$, $\lambda \in \mathcal{A}_2$. Then

(i) $\beta_\lambda$ maps $E_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$ linearly, continuously, and $G$-equivariantly into $C^\infty \text{Ind}_K^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$ for each $r \in \mathbb{R}$, where $\tau(\Lambda)$ is the restriction of $\tau$ to $M$ with representation space $V(\Lambda)$.

(ii) Let $\Omega \subset \mathbb{A}_\mathbb{C}^{\infty}$ be open, $\{f_\lambda\}_{\lambda \in \Omega}$ a holomorphic family in $E_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$; then $\lambda \to \beta_\lambda(f_{\lambda})$ is holomorphic in $\Omega \cap \mathcal{A}_2$.

**Proof.** (i) comes from Theorem 3.2; (ii) is a result of Theorem 3.3. □

Finally we notice for certain $\lambda$ we can obtain the boundary value map by a simple limit procedure.

**Lemma 7.7.** Let $\lambda - \Lambda \in \mathcal{A}_1$. If $\text{Re}(\lambda, \alpha) > 0$, for each $\alpha \in \Sigma^+(g, a)$, then

$$\beta_\lambda(f) = \lim_{t \to \infty} e^{(-\lambda + \rho)(iH)} f(g \exp tH),$$ 

for $f \in E_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, and $H \in \mathfrak{a}^+$. 

Proof. The condition on $\lambda$ implies that $\text{Re} \xi(H) < \text{Re}(\lambda - \rho)(H)$ for all $\xi \in X(\lambda, \Lambda)$ with $\xi \neq \lambda - \rho$. Then the result follows from Theorem 3.2 and the very definition of asymptotic expansion.

For each $\phi \in C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$, we define $P_\lambda \phi$ by

$$P_\lambda \phi(g) = \int_K \tau(k)\phi(gk)dk.$$  

From the proof of Theorem 1.6 we conclude $P_\lambda \phi \in E_{\lambda - \Lambda, r} \text{Ind}_K^\infty(\tau)$. By Example 2.2

$$P_\lambda \phi \in E_{\lambda - \Lambda, r}^\infty \text{Ind}_K(\tau).$$

**Corollary 7.8.** Under the same conditions as in Lemma 7.7,

$$\beta_\lambda P_\lambda \phi = C(\lambda)\phi,$$

for each $\phi \in C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

**Proof.** By Proposition 2.4 and Lemma 7.7.

8. The inversion of the Poisson transform

Let $C(\lambda)$ be the generalized Harish-Chandra $C$-function given by

$$C(\lambda) = \int_N e^{-(\lambda + \rho)H(\eta)} \tau(k(\eta))d\eta.$$  

Recall $P_\lambda: C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \rightarrow E_{\lambda - \Lambda}^\infty \text{Ind}_K(\tau)$ is defined by

$$P_\lambda \phi(g) = \int_K \tau(k)\phi(gk)dk.$$  

**Theorem 8.1.** Let $\lambda - \Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$, and $C_0(\lambda)$ the restriction of $C(\lambda)$ to $V(\Lambda)$. Then

$$\beta_\lambda P_\lambda \phi = C_0(\lambda)\phi,$$

for each $\phi \in C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

**Proof.** If $\text{Re}(\lambda, \alpha) > 0$, for all $\alpha \in \Sigma(g, a)$, then by Corollary 7.8,

$$\beta_\lambda P_\lambda \phi = C_0(\lambda)\phi.$$  

Since $P_\lambda \phi$ is a holomorphic family in $E_{\lambda - \Lambda}^\infty \text{Ind}_K(\tau)$, by Theorem 7.6 the left-hand side is holomorphic. The right-hand side is meromorphic on $\mathfrak{a}_2^\circ$. Hence two sides must coincide.

**Corollary 8.2.** If in addition we assume $\det C_0(\lambda) \neq 0$, then $\beta_\lambda$ is surjective. Hence $P_\lambda$ is injective.

**Theorem 8.3.** Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$, and $\det C_0(\lambda) \neq 0$. Then $P_\lambda$ is bijective, and the inverse of $P_\lambda$ is given by $C_0(\lambda)^{-1} \beta_\lambda$.

For the proof we recall a definition which can be found in [Wall], Section 11.6. Let $\mathfrak{g}$ be a finitely generated $(g, K)$-module.

**Definition 8.4.** $\mathfrak{g}_{mod}^*$ denotes the set of all $\mu \in \mathfrak{g}^*$, such that there exists $d_\mu \in \mathbb{R}$ and for each $\nu \in \mathfrak{g}$ there exist an analytic function $f_{\mu, \nu}$ and a constant $C_{\mu, \nu} > 0$ with the following properties:

(i) $L_uf_{\mu, \nu}(k) = \mu(k^{-1}.(u, \nu))$, for $u \in U(\mathfrak{g})$, $k \in K$,

(ii) $|f_{\mu, \nu}(g)| \leq C_{\mu, \nu}||g||^{d_\mu}$, for each $g \in G$.  


Recall that \((C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1))'\) is the strong topological dual of \(C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1)\). The following result can also be found in [Wall], Section 11.7.

**Proposition 8.5.** Let \((C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1))_{K,\text{finite}}\) denote the space of \(K\)-finite elements in \(C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1)\). Then

\[
(C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1))_{K,\text{finite}}^\prime = (C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1))'.
\]

Before we go ahead with the proof of Theorem 8.3, we mention the following result about the irreducibility of the principal series representations. Let \(\sigma \in \widehat{M}\).

**Lemma 8.6.** As a \((g, K)\)-module \(C^\infty \text{Ind}_{B}^{G}(\sigma \otimes (\lambda) \otimes 1)_{K,\text{finite}}\) is irreducible if \(\lambda - \Lambda \in \mathfrak{A}_{1}\).

**Proof.** This is a direct consequence of Theorem 1.1 in [SV].

**Proof of Theorem 8.3.** It suffices to show \(\beta_{\lambda}\) is injective. Assume the opposite. Then there exists \(f_{0} \in E_{\xi - \Lambda}^{\infty} \text{Ind}_{K}^{G}(\tau)\), such that \(\beta_{\lambda} f_{0} = 0\), and \(f_{0} \neq 0\). We can assume \(f_{0}(e) \neq 0\) since \(\beta\) is \(G\)-equivariant. Define \(f_{K}\) by

\[
f_{K}(g) = \int_{K} \text{tr} \tau(k)f_{0}(kg)dk.
\]

Then \(f_{K}\) is \(K\)-finite, and \(f_{K}(e) = \frac{1}{\dim(\tau)}f_{0}(e) \neq 0\). Let

\[
\mathfrak{W} = L_{U(g)L_{K}}f_{K}.
\]

Then \(\mathfrak{W}\) is a finitely generated \((g, K)\)-module. Let \(\mathfrak{W}_{1}\) be an irreducible submodule of \(\mathfrak{W}\). By the subrepresentation theorem and Lemma 8.4 there exists \(\sigma \in \widehat{M}\), such that \(\mathfrak{W}_{1} \cong C_{\infty} \text{Ind}_{B}^{G}(\sigma \otimes (-\lambda) \otimes 1)_{K,\text{finite}}\). So there is a \((g, K)\) map

\[
P_{\sigma}: C_{\infty} \text{Ind}_{B}^{G}(\sigma \otimes (-\lambda) \otimes 1)_{K,\text{finite}} \rightarrow \mathfrak{W}.
\]

It is easy to see \(\Lambda = \Lambda_{\sigma}\). Define \(\mu \in \mathfrak{W}^{\ast} \otimes V\) by

\[
\mu(\nu) = \nu(e),
\]

for each \(\nu \in \mathfrak{W}\).

Taking \(f_{\mu, \nu} = \nu \in E_{\xi - \Lambda}^{\infty} \text{Ind}_{K}^{G}(\tau)\) in Definition 8.4, we can verify that (i) and (ii) are satisfied. So \(\mu \in \mathfrak{W}_{1}^{\ast} \otimes V\). Hence

\[
\mu^{\sharp} = \mu \circ P_{\sigma} \in (C_{\infty} \text{Ind}_{B}^{G}(\sigma \otimes (-\lambda) \otimes 1))_{mod}^{\ast} \otimes V.
\]

Then by Proposition 8.5,

\[
\mu^{\sharp} \in (C_{\infty} \text{Ind}_{B}^{G}(\sigma \otimes (-\lambda) \otimes 1))' \otimes V.
\]

Now define \(P_{\sigma}^{\sharp} : C_{\infty} \text{Ind}_{B}^{G}(\sigma \otimes (-\lambda) \otimes 1) \rightarrow C_{\infty} \text{Ind}_{K}^{G}(\tau)\) by

\[
P_{\sigma}^{\sharp}\phi(g) = \mu^{\sharp}(L_{g^{-1}}\phi).
\]

Since \(P_{\sigma}\) is a \(g\) map and eigensections are analytic we can show that for \(\phi\) in \(C_{\infty} \text{Ind}_{B}^{G}(\sigma \otimes (-\lambda) \otimes 1)_{K,\text{finite}}\),

\[
P_{\sigma}\phi = P_{\sigma}^{\sharp}\phi,
\]

by showing they are identical at \(e\) along with their derivatives.
We observe that $P^g_\sigma$ is a linear, continuous, and $G$-equivariant map from $C^\infty \text{Ind}^K_B(\sigma \otimes (-\lambda) \otimes 1)$ to $C^\infty \text{Ind}^K_B(\tau)$. By Proposition 1.8 we conclude $\sigma \in \tau$, and there exists $T \in \text{Hom}_M(V_\sigma, V)$ such that $P^g_\sigma = P_T$. Hence

$$P_\sigma = P_T : C^\infty \text{Ind}^K_B(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}} \rightarrow \mathcal{W}.$$ 

Taking $\sigma \in C^\infty \text{Ind}^G_B(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$ such that $0 \neq f = P_T \phi$, then $f = P_\lambda(T\phi)$. Notice $T\phi \in C^\infty \text{Ind}^G_B(\tau(\bar{\Lambda}) \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$. So

$$B_\lambda f = \beta_\lambda P_\lambda(T\phi) = C(\lambda)T\pi \neq 0.$$ 

This contradicts the assumption $f \in \mathcal{W} \subset \ker(\beta_\lambda)$. \hfill \Box

9. Vector-valued distributions

Suppose $K$ is a Lie group and $V$ a finite dimensional space over $C$. Let $C^{-\infty}(K, V)$ denote all continuous $C$-linear maps from $C^\infty_c(K, C)$ to $V$. Let $M$ be a compact subgroup of $K$, and $(\pi, V)$ a finite dimensional representation of $M$. Let $C^{-\infty} \text{Ind}_M^K(\pi)$ be the space defined by

$$\{ \in C^{-\infty}(K, V) | R_m f(\phi) = \pi(m^{-1}) f(\phi), \forall \phi \in C^\infty_c(K, C), \forall m \in M \}.$$ 

Here $R_m f(\phi) = f(R_m^{-1} \phi)$, with $R_m^{-1} \phi(k) = \phi(km^{-1})$.

Let $(\bar{\pi}, V^*)$ be the dual representation of $(\pi, V)$, and $(\langle , \rangle)$ the nondegenerate bilinear form on $V \times V^*$. Let $(C^\infty_c \text{Ind}_M^K(\pi))^\prime$ be the strong dual of $C^\infty_c \text{Ind}_M^K(\pi)$. For each $T \in (C^\infty_c \text{Ind}_M^K(\pi))^\prime$, $\phi \in C^\infty_c(K, C)$, and $v \in V$, we define $\xi_1(T)(\phi) \in V^*$ by

$$\langle v, \xi_1(T)(\phi) \rangle = T(\xi_1(\phi, v)),$$

where $\xi_1(\phi, v)(k) = \int_M \phi(km) \pi(m) v dm$. It is easy to show that

$$\xi_1(T) \in C^{-\infty} \text{Ind}_M^K(\bar{\pi}).$$

**Proposition 9.1.** The map $\xi_1 : (C^\infty_c \text{Ind}_M^K(\pi))^\prime \rightarrow C^{-\infty} \text{Ind}_M^K(\bar{\pi})$ is bijective.

**Proof.** Define

$$\eta_1 : C^{-\infty} \text{Ind}_M^K(\bar{\pi}) \rightarrow (C^\infty_c \text{Ind}_M^K(\pi))^\prime,$$

as follows: for each $f \in C^{-\infty} \text{Ind}_M^K(\bar{\pi})$, and $\phi \in C^\infty_c \text{Ind}_M^K(\pi)$, the map

$$f_\phi : u \mapsto f(\langle \phi, u \rangle)$$

is a linear map from $V^*$ to $V^*$. Then we define

$$\eta_1(f) = \text{tr}(f_\phi).$$

It is a long but rather straightforward calculation to show $\xi_1$ and $\eta_1$ are inverses to each other. \hfill \Box

Now let $G = KAN$, and $(\delta, V_\delta)$ be a finite dimensional representation of $B = MAN$. Let

$$C^\infty \text{Ind}_B^G(\delta) = \{ f \in C^\infty(G, V_\delta) | R_{\text{man}} f = a^{-\rho} \delta^{-1}(\text{man}) f, \forall \text{man} \in B \},$$

$$C^{-\infty} \text{Ind}_B^G(\delta) = \{ f \in C^{-\infty}(G, V_\delta) | R_{\text{man}} f = a^{-\rho} \delta^{-1}(\text{man}) f, \forall \text{man} \in B \}.$$

For $T \in (C^\infty \text{Ind}_B^G(\delta))^\prime$, $\xi(T)$ is defined by

$$\langle v, \xi(T)(\phi) \rangle = T(\xi(\phi, v)),$$

where $\xi(\phi, v)(k) = \int_B \phi(km) \delta(m) v dm$. It is easy to show that

$$\xi(T) \in C^{-\infty} \text{Ind}_B^G(\bar{\delta}).$$

**Proposition 9.2.** The map $\xi : (C^\infty \text{Ind}_B^G(\delta))^\prime \rightarrow C^{-\infty} \text{Ind}_B^G(\bar{\delta})$ is bijective.

**Proof.** Define

$$\eta_2 : C^{-\infty} \text{Ind}_B^G(\bar{\delta}) \rightarrow (C^\infty \text{Ind}_B^G(\delta))^\prime,$$

as follows: for each $f \in C^{-\infty} \text{Ind}_B^G(\bar{\delta})$, and $\phi \in C^\infty \text{Ind}_B^G(\delta)$, the map

$$f_\phi : u \mapsto f(\langle \phi, u \rangle)$$

is a linear map from $V^*$ to $V^*$. Then we define

$$\eta_2(f) = \text{tr}(f_\phi).$$

It is a long but rather straightforward calculation to show $\xi$ and $\eta_2$ are inverses to each other. \hfill \Box
for each \( v \in V_\delta \), and \( \phi \in C_c^\infty (G, \mathbb{C}) \). Here \( \xi (\phi, v) \) is defined as follows: for each \( g \in G \),

\[
\xi (\phi, v)(g) = \int_{\text{MAN}} \phi(g \delta(\text{man})a^\rho \delta(\text{man})v) \, d\mu \, d\nu.
\]

Now we show \( \xi (T) \in C^\infty \text{Ind}_G^G (\delta) \). By definition,

\[
\langle v, \xi (T)(R_{(\text{man})^{-1}} \phi) \rangle = T(\xi (R_{(\text{man})^{-1}} \phi, v)).
\]

However, it is a simple calculation to see

\[
\xi (R_{(\text{man})^{-1}} \phi, v) = \xi (\phi, a^{-\rho} \delta(\text{man})v).
\]

Hence

\[
\langle v, R_{\text{man}} \xi (T)(\phi) \rangle = \langle v, \xi (T)(R_{(\text{man})^{-1}} \phi) \rangle
= T(\xi (\phi, a^{-\rho} \delta(\text{man})v))
= \langle a^{-\rho} \delta(\text{man})v, \xi (T)(\phi) \rangle
= \langle v, a^{-\rho} \delta((\text{man})^{-1})T(\phi) \rangle.
\]

This proves \( \xi (T) \in C^\infty \text{Ind}_G^G (\delta) \).

**Theorem 9.2.** Let \( \xi \) be defined as above. Then \( \xi \) is a \( G \)-equivariant bijection from \( (C^\infty \text{Ind}_G^G (\delta))' \) to \( C^\infty \text{Ind}_G^G (\delta) \).

**Lemma 9.3.** Let \( L \) be a Lie group and \((\pi, V)\) a finite dimensional representation of \( L \) on \( V \). Suppose \( f \in C^\infty (L, V) \), satisfying

\[
R_l f = \pi (l^{-1}) f.
\]

for each \( l \in L \). Let \( dl \) be the right invariant Haar measure on \( L \). Then there exists a unique vector \( v \in V \), such that

\[
f(\phi) = \int_L \phi(l) \pi(l^{-1}) v \, dl,
\]

for each \( \phi \in C_c^\infty (L, \mathbb{C}) \).

**Proof.** We use an argument due to Helgason. For \( \phi \) and \( \psi \) in \( C_c^\infty (L, \mathbb{C}) \), we define \( \phi * \psi \) in \( C_c^\infty (L, \mathbb{C}) \) by

\[
\phi * \psi (x) = \int_L \phi(l) \psi(xl^{-1}) \, dl.
\]

Then

\[
f(\phi * \psi) = \int_L \phi(l) f(R_{l^{-1}} \psi) \, dl = \int_L \phi(l) \pi(l^{-1}) f(\psi) \, dl.
\]

Choose a sequence \( \psi_n \) such that \( \psi_n \to \delta \), the delta function, as \( n \to +\infty \). Here \( \psi_n (l) = \psi_n (l^{-1}) \). Let \( v_n = f(\psi_n) \). Then

\[
\phi * \psi = \int_L \phi(l) \pi(l^{-1}) v_n \, dl.
\]

We can choose an appropriate \( \phi \) (e.g. close to \( \delta \)), such that \( \int_L \phi(l) \pi(l^{-1}) v \, dl \) is invertible. Since \( \phi * \psi \to \phi \), by letting \( n \to +\infty \) in \((*)\), we conclude there exists \( v \in V \), such that \( v_n \to v \), and

\[
f(\phi) = \int_L \phi(l) \pi(l^{-1}) v \, dl.
\]
The uniqueness follows from the fact that there is \( \phi \) such that \( \int_L \phi(l)\pi(l^{-1})dl \) is invertible.

**Proof of Theorem 9.2.** First we construct the inverse \( \eta \) of \( \xi \) as follows: Take \( f \in C^{-\infty} \text{Ind}_{B}^{K}(\delta), \) and \( \psi \in C^{\infty}(K, \mathbb{C}). \) Then \( \phi \to f(\psi \otimes \phi) \) defines a continuous linear map from \( C_{c}^{\infty}(A \times N, \mathbb{C}) \) to \( V_{s}^{*} \), where

\[
(\psi \otimes \phi)(kan) = \psi(k)\phi(an).
\]

It is easy to check this map satisfies all the conditions as in Lemma 9.3 if we take \( L = AN, \) \( \pi(an) = a^{\rho}\delta(an) \). So there exists a unique element in \( V_{s}^{*} \), which we denote by \( f^{-}(\psi) \), such that

\[
f(\psi \otimes \phi) = \int_{A \times N} \phi(an)a^{\rho}\delta^{-1}(an)f^{-}(\psi)dadn.
\]

Notice \( a^{2\rho}dadn \) gives a right invariant Haar measure on \( AN \). It is fairly easy to see \( f^{-} \in C^{-\infty} \text{Ind}_{M}^{K}(\delta|M) \). Then by Proposition 9.1 \( \eta_{1}(f^{-}) \) gives an element in \( (C^{\infty} \text{Ind}_{M}^{K}(\delta|M))' \). Since \( C^{\infty} \text{Ind}_{M}^{K}(\delta|M) \cong C^{\infty} \text{Ind}_{B}^{K}(\delta) \), one can view \( \eta_{1}(f^{-}) \) as an element in \( (C^{\infty} \text{Ind}_{B}^{K}(\delta))' \). Finally we define \( \eta(f) \) by

\[
\eta(f) = \eta_{1}(f^{-}).
\]

The final step of the proof is to show \( \eta \circ \xi = \text{id} \), and \( \eta \circ \xi = \text{id} \). For each \( T \in (C^{\infty} \text{Ind}_{B}^{K}(\delta))' \), \( \psi \in C^{\infty}(K, \mathbb{C}) \), and \( \phi \in C_{c}^{\infty}(A \times N, \mathbb{C}) \),

\[
\xi(T)(\psi \otimes \phi) = \int_{A \times N} \phi(an)a^{\rho}\delta^{-1}(an)(\xi(T))^{-}(\psi)dadn.
\]

So for each \( v \in V \),

\[
(\ast \ast) \quad \langle v, \xi(T)(\psi \otimes \phi) \rangle = \langle v, \int_{A \times N} \phi(an)a^{\rho}\delta^{-1}(an)(\xi(T))^{-}(\psi)dadn \rangle.
\]

By definition

\[
\xi(\psi \otimes \phi, v)(k) = \int_{MAN} (\psi \otimes \phi)(kan)a^{\rho}\delta(man)vdmadadn
\]

\[
= \int_{MAN} \psi(km)\delta(m)\phi(an)a^{\rho}\delta(an)vdmadadn
\]

\[
= \xi_{1}(\psi, v_{1}),
\]

where \( v_{1} = \int_{A \times N} a^{\rho}\phi(an)\delta(an)vdadn \). So by \( (\ast \ast) \)

\[
\langle v, \xi(T)(\psi \otimes \phi) \rangle = T(\xi_{1}(\psi, v_{1}))
\]

\[
= \langle v_{1}, \xi_{1}(T)(\psi) \rangle
\]

\[
= \langle v, \int_{A \times N} \phi(an)a^{\rho}\delta^{-1}(an)\xi_{1}(T)(\psi)dadn \rangle.
\]

Hence

\[
\int_{A \times N} \phi(an)a^{\rho}\delta^{-1}(an)(\xi(T))^{-}(\psi)dadn
\]

\[
= \int_{A \times N} \phi(an)a^{\rho}\delta^{-1}(an)\xi_{1}(T)(\psi)dadn.
\]

By comparing both sides we have \( \xi_{1}(T) = (\xi(T))^{-} \). Hence

\[
T = \xi_{1}^{-1}((\xi(T))^{-}) = \eta_{1}((\xi(T))^{-}) = \eta(\xi(T)).
\]
Similarly we can verify $\xi \circ \eta = \text{id}$. Note it is enough to check on functions of the form $\psi \otimes \phi$.

Now suppose $V_\lambda$ is a Hilbert space. Let $\delta^*$ be the representation defined as follows: for each $g \in G$, $w, v \in V_\delta$, we have $(\delta(g)v, w) = (v, \delta(g)^*w)$; then $\delta^*(g) = (g^{-1})^t$. Let $C^{-\infty} \text{Ind}^G_B(\delta^*)$ be the space of conjugate linear maps $f$ from $C_c^\infty(G, \mathbb{C})$ to $V_\delta$, such that

$$R_{man}f = a^{-\rho}(\delta^*\text{man})^f.$$ 

For each $T \in (C^\infty \text{Ind}^G_B(\delta))'$, and $\phi \in C_c^\infty(G, \mathbb{R})$, $\xi(T)(\phi)$ is defined by

$$(v, \xi(T)(\phi)) = T(\xi(\phi, v)),$$

for each $v \in V_\delta$. Here

$$\xi(\phi, v)(g) = \int_{MAN} \phi(g)\text{man}d\mu.$$ 

**Corollary 9.4.** $\xi$ is a bijection from $(C^\infty \text{Ind}^G_B(\delta))'$ onto $C^{-\infty} \text{Ind}^G_B(\delta^*)$.

Let $\sigma$ be a unitary representation of $M$ and $\lambda \in a^*_c \otimes 1$ is the representation of $B$ defined by $\text{man} \rightarrow a^*_\lambda \sigma(m)$. Then

$$(\sigma \otimes 1)^* = \sigma \otimes (\rho) \otimes 1.$$ 

**Corollary 9.5.** The map

$$\xi: (C^\infty \text{Ind}^G_B(\sigma \otimes 1))' \rightarrow C^{-\infty} \text{Ind}^G_B(\sigma \otimes (\rho) \otimes 1)$$

is a bijection.

10. **Distribution boundary values**

We now introduce a weak growth condition in the eigenspace $E_{\lambda-\Lambda} \text{Ind}^G_K(\tau)$. Recall in Section 2 we have

$$C^q_r(G, V) = \{ f \in C^q(G, V) \mid ||f||_{q,r} < \infty \},$$

$q \in \mathbb{N}$ and $r \in \mathbb{R}$. $C^\infty_r(G, V) = \bigcap_q C^q_r(G, V)$. We define $\mathfrak{g}$ to be the space

$$\mathfrak{g} = \bigcap_r C^\infty_r(G, V) = \bigcup_q C^q_r(G, V),$$

defined with the projective limit topology for the intersection over $q$ and $r$ (i.e., the topology given by the family of forms $\| \cdot \|_{q,r}$). Using the same argument as on p. 142 in [BS] we conclude $\mathfrak{g}$ is a Fréchet space. It follows from Section 2 that $L$ and $R$ act smoothly on $\mathfrak{g}$.

Let $\mathfrak{g}'$ be the space dual to $\mathfrak{g}$, equipped with the strong dual topology. For each $T \in \mathfrak{g}'$, $q \in \mathbb{N}$, and $r \in \mathbb{R}$, we define

$$||T||_{q,r} = \text{sup} \{ T(\varphi) \mid \varphi \in \mathfrak{g}, \|\varphi\|_{q,r} \leq 1 \}.$$ 

The space $C^q_0(G, V)' = \{ T \in \mathfrak{g}' \mid ||T||_{q,r} < \infty \}$ with this norm is the dual space of $C_0^q(G, V)$. Moreover, we have $\mathfrak{g}' = \bigcup_{q,r} C^q_0(G, V)'$. By duality $\mathfrak{g}'$ is the inductive limit of these spaces. Using Lemma 2.1 we can prove that for some $b \in \mathbb{R}$,

$$\int_G g \|b\|dg < \infty.$$ 

It follows that there is a continuous injection of $C^q_0(G, V)$ into $C^q_b(G, V)'$ defined by integration over $G$. Hence there is a continuous injection of $C^q_r(G, V)$ into $\mathfrak{g}'$. 
Let $q' \geq q$, and $r \in \mathbb{R}$. For each $T \in C^q_r(G, V)'$, and $\varphi \in C^{q'}_r(G, \mathbb{R})$, we define an element $L^r(\varphi)T$ in $C^{q'-q}_r(G, V)$ by
\[
\langle v, L^r(\varphi)T(x) \rangle = T(R_{x^{-1}}\varphi \cdot v).
\]
Note if $f \in C^q_r(G, V)$, and $\varphi \in C^{q'}_r(G, \mathbb{C})$, then
\[
L^r(\varphi)f(x) = \int_G \varphi(g)f(gx)dg.
\]

**Lemma 10.1.** Let $q, q' \in \mathbb{N}$ with $q \leq q'$. There exist $s \geq 0$ and $C \geq 0$ such that
\[
\|L^r(\varphi)T\|_{q'-q,r} \leq C\|T\|_{q',r}^s\|\varphi\|_{q',r-s},
\]
for all $r \in \mathbb{R}$, $T \in C^q_r(G, V)'$, and $\varphi \in C^{q'}_r(G, \mathbb{R})$.

**Proof.** See Lemma 11.1 in [BS].

Let $E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$ denote the closed subspace $E^*_{\lambda-L} \text{Ind}_{K}^G(\tau) \cap G'$. We call the elements of $E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$ eigensections of weak moderate growth. Notice if $f \in E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$, and $\varphi \in C^\infty(\mathbb{C}, \mathbb{R})$, then $L^r(\varphi)f \in E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$ by Lemma 10.1. For $\lambda - L \in \mathbb{R}_1$, $\lambda \in \mathbb{R}_2$, and $f \in E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$, we define a vector-valued distribution $\overline{\beta}_\lambda f$ on $G$ by
\[
\overline{\beta}_\lambda f(\varphi) = \beta_\lambda(L^r(\varphi)f)(e),
\]
for each $\varphi \in C^\infty(G, \mathbb{R})$.

**Proposition 10.2.** $\overline{\beta}_\lambda$ is a linear, continuous, and $G$-equivariant map from $E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$ to $C^{-\infty} \text{Ind}_{B}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

**Proof.** It suffices to show $\overline{\beta}_\lambda f \in C^{-\infty} \text{Ind}_{B}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$. By definition,
\[
L^r(R_{(\text{man})^{-1}}\varphi)f(x) = f(R_{x^{-1}}R_{(\text{man})^{-1}}\varphi) = f(R_{(\text{man}x)^{-1}}\varphi) = L^r(\varphi)f(manx).
\]
However, $\beta_\lambda$ is $G$-equivariant. Hence
\[
B_\lambda(L^r(R_{(\text{man})^{-1}}\varphi)f)(e) = \beta_\lambda(L^r(\varphi)f)(\text{man}) = \tau(\Lambda)(m^{-1})a^{\lambda-r}\beta_\lambda(L^r(\varphi)f)(e).
\]
This proves $\overline{\beta}_\lambda f \in C^{-\infty} \text{Ind}_{B}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

For each $T \in (C^\infty \text{Ind}_{B}^G(\tau(\Lambda) \otimes \overline{\Lambda} \otimes 1))'$, we define $\overline{P}_\lambda T$ as follows:
\[
\langle v, \overline{P}_\lambda T(g) \rangle = T(P(\Lambda)L_g\Phi_\lambda \cdot v)),
\]
for each $v \in V$. Here $\Phi_\lambda(x)$ is the transpose of $\Psi_\lambda(x^{-1})$, and $P(\Lambda)$ the projection from $V$ to $V(\Lambda)$. The motivation of this definition is from Corollary 1.10.

**Proposition 10.3.** $\overline{P}_\lambda T \in E^*_{\lambda-L} \text{Ind}_{K}^G(\tau)$, for $T \in (C^\infty \text{Ind}_{B}^G(\tau(\Lambda) \otimes \overline{\Lambda} \otimes 1))'$. And $\overline{P}_\lambda$ is linear, continuous, and $G$-equivariant.

**Proof.** Similar to the proof for Corollary 11.3 in [BS].

**Lemma 10.4.** Let $T \in (C^\infty \text{Ind}_{B}^G(\tau(\Lambda) \otimes \overline{\Lambda} \otimes 1))'$, and $\varphi \in C^\infty(G, \mathbb{R})$. Then $L^r(\varphi)\overline{P}_\lambda T = P_\lambda(L^r(\varphi)\xi(T))$. Here $\xi$ is the isomorphism in Corollary 9.5, and $L^r(\varphi)\xi(T)(x) = \xi(T)(R_{x^{-1}}\varphi)$. 
Proof. \( L^\vee(\varphi), \overline{P}_\lambda, \) and \( P_\lambda \) are continuous. So it is enough to check for \( T \in C^\infty \text{Ind}_K^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \). The proof follows from the \( G \)-equivariance of \( P_\lambda \). \( \square \)

By a similar argument we get

Lemma 10.5. Let \( f \in \mathcal{E}_{\Lambda}^* \text{Ind}_K^G(\tau), \) and \( \varphi \in C_c^\infty(G, \mathbb{R}) \). Then
\[
L^\vee(\varphi) \overline{\beta}_\lambda f = \beta_\lambda(L^\vee(\varphi)f).
\]

Theorem 10.6. Under the same conditions as in Theorem 8.3, \( \overline{P}_\lambda \) defines a \( G \)-equivariant topological isomorphism from \( (C^\infty \text{Ind}_K^G(\tau(\Lambda) \otimes X \otimes 1))' \) onto \( \mathcal{E}_{\Lambda}^* \text{Ind}_K^G(\tau) \). And \( \eta \circ C_0(\lambda)^{-1} \circ \overline{\beta}_\lambda \) gives the inverse of \( \overline{P}_\lambda \).

Proof. By Theorem 8.1 and Lemma 10.4, 10.5,
\[
L^\vee(\varphi)\overline{\beta}_\lambda P_\lambda T = \beta_\lambda P_\lambda L^\vee(\varphi)\xi(T) = C_0(\lambda)L^\vee(\varphi)\xi(T),
\]
for \( T \in (C^\infty \text{Ind}_K^G(\tau(\Lambda) \otimes X \otimes 1))' \). Similarly, for each \( f \in \mathcal{E}_{\Lambda}^* \text{Ind}_K^G(\tau) \),
\[
L^\vee(\varphi)\overline{P}_\lambda \eta(C_0(\lambda)^{-1}\beta_\lambda f) = P_\lambda C_0(\lambda)^{-1}\beta_\lambda L^\vee(\varphi)f = L^\vee(\varphi)f.
\]
So we have
\[
\beta_\lambda \circ \overline{P}_\lambda = C_0(\lambda) \circ \xi, \quad \overline{P}_\lambda \circ \eta \circ C_0(\lambda)^{-1}\beta_\lambda = \text{id}.
\]

Remark 10.7. Let \( \mathcal{E}_{\Lambda}^* \text{Ind}_K^G(\tau) = \mathcal{E}_{\Lambda} \text{Ind}_K^G(\tau) \cap C_r(G, V) \) be equipped with the Banach space topology inherited from \( C_r(G, V) \). Then \( \mathcal{E}_{\Lambda}^* \text{Ind}_K^G(\tau) \) is identical with the inductive limit topology for the union \( \mathcal{E}_{\Lambda}^* \text{Ind}_K^G(\tau) = \bigcup \mathcal{E}_{\Lambda,r} \text{Ind}_K^G(\tau) \). See p. 146 in [BS].

A classical result asserts that the left \( K \)-finite elements in \( \mathcal{E}_{\Lambda} \text{Ind}_K^G(\tau) \) increase at most exponentially. So by the remark above we easily get

Corollary 10.8. Under the same conditions as in Theorem 8.3, \( P_\lambda \) is a bijection from \( C^\infty \text{Ind}_K^G(\tau(\Lambda) \otimes (\lambda) \otimes 1)_{K\text{-finite}} \) to \( \mathcal{E}_{\Lambda} \text{Ind}_K^G(\tau)_{K\text{-finite}} \).

Remark 10.9. I think by Schmid’s method indicated in [Sch] one should be able to get a bijection on the level of hyperfunctions from Corollary 10.8.

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