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GEOMETRIC FAMILIES OF CONSTANT REDUCTIONS AND THE SKOLEM PROPERTY

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Abstract. Let $F|K$ be a function field in one variable and $\mathcal{V}$ be a family of independent valuations of the constant field K. Given $v \in \mathcal{V}$, a valuation prolongation v to F is called a *constant reduction* if the residue fields $F\mathrm{v}|Kv$ again form a function field of one variable. Suppose $t \in F$ is a non-constant function, and for each $v \in \mathcal{V}$ let V_t be the set of all prolongations of the Gauß valuation v_t on $K(t)$ to F. The union of the sets V_t over all $v \in \mathcal{V}$ is denoted by $\boldsymbol{V}_t$.

The aim of this paper is to study families of constant reductions $\boldsymbol{V}$ of F prolonging the valuations of $\mathcal{V}$ and the criterion for them to be principal, that is to be sets of the type $\boldsymbol{V}_t$. The main result we prove is that if either $\mathcal{V}$ is finite and each $v \in \mathcal{V}$ has rational rank one and residue field algebraic over a finite field, or if $\mathcal{V}$ is any set of non-archimedean valuations of a global field K satisfying the strong approximation property, then each geometric family of constant reductions $\boldsymbol{V}$ prolonging $\mathcal{V}$ is principal. We also relate this result to the *Skolem property* for the existence of $\mathcal{V}$-integral points on varieties over K, and Rumely's existence theorem. As an application we give a *birational characterization* of arithmetic surfaces $\mathcal{X}/S$ in terms of the generic points of the closed fibre. The characterization we give implies the existence of finite morphisms to $\mathbb{P}^1_S$.

1. Introduction

Let $F|K$ be a function field in one variable and $\mathcal{V}$ be a family of independent valuations of the exact constant field K. Given $v \in \mathcal{V}$, a valuation prolongation v to F is called a *constant reduction* of F if the residue fields $F\mathrm{v}|Kv$ again form a function field of one variable. Suppose $t \in F$ is a non-constant function and for each $v \in \mathcal{V}$ let V_t be the set of all prolongations of the Gauß valuation v_t on $K(t)$ to F. We denote the union of the sets V_t over all $v \in \mathcal{V}$ by $\boldsymbol{V}_t$.

Suppose $\boldsymbol{V}$ is a family of constant reductions of F prolonging the valuations of $\mathcal{V}$. Then $\boldsymbol{V}$ is said to be *geometric* if there is a non-constant function $t \in F$ such that $\boldsymbol{V}$ and $\boldsymbol{V}_t$ are almost equal, i.e., the symmetric difference of $\boldsymbol{V}$ and $\boldsymbol{V}_t$ is finite. The family $\boldsymbol{V}$ is said to be *principal* if $\boldsymbol{V} = \boldsymbol{V}_t$ for some non-constant function $t \in F$.

Our aim in this paper is to prove the following theorem:

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Theorem 1. *Let $F|K$ be a function field in one variable, $\mathcal{V}$ a family of independent valuations of K and $\boldsymbol{V}$ a geometric family of constant reductions of F prolonging $\mathcal{V}$. Suppose that one of the following conditions is satisfied:*

i) *$\mathcal{V}$ consists of only one valuation with respect to which K is henselian.*
ii) *$\mathcal{V}$ is finite and K is algebraically closed.*
iii) *$\mathcal{V}$ is finite and each valuation $v \in \mathcal{V}$ has rational rank one and residue field algebraic over a finite field.*
iv) *$\mathcal{V}$ is a set of non-archimedean valuations of a global field K satisfying the strong approximation property. Thus for number fields we can take $\mathcal{V}$ to be the set of all non-archimedean valuations and for function fields the set of all but one.*

Then the family $\boldsymbol{V}$ is principal.

The approach we follow in proving this result corresponds to the axiomatic approach from [G–P–R] to Rumely's Local Global Principle. Hence the proof is divided into two parts, the first containing a proof of the result (actually a stronger form) in the semi-local situation, i.e., when $\mathcal{V}$ is finite, and the second the passage from the semi-local to the global situation.

In the local context, when $\mathcal{V}$ consists of only one valuation, the assertion that $\boldsymbol{V}$ is principal is equivalent to the solvability of local Skolem problems. This relationship, together with the proof of the theorem for this case, has been studied in [G–M–P 2] and [G–M–P 4]. In the first part of this paper we prove the corresponding theorem in the semi-local situation by building on the methods developed in these papers. There are two main steps: First we show that if K is algebraically closed then $\boldsymbol{V}$ is principal. Next, using this case we show that the assertion that the sets $\boldsymbol{V}$ are principal is equivalent to the semi-local Skolem property for the existence of $\mathcal{V}$-integral points on varieties defined over K (see section 2), and that this in turn is equivalent to a geometric criterion for the existence of divisors having prescribed poles and zeros in given open sets. When the valuations in $\mathcal{V}$ have rational rank one and residue fields which are algebraic over finite fields we are able to show that this geometric criterion is satisfied by a generalisation of the Rumely existence theorem. Thus by the end of the semi-local part we have proved theorem 1 when any one of the conditions (i), (ii) or (iii) is satisfied.

Globally, this result is more difficult to prove, as now one has to work with infinitely many valuations on the base simultaneously when finding the function t that makes $\boldsymbol{V}$ principal. For geometric families of constant reductions we know that for almost all $v \in \mathcal{V}$ there is a unique prolongation to F in $\boldsymbol{V}$ which is principal and has potential good reduction (see section 3). For the finitely many remaining elements of $\mathcal{V}$ one shows that the subset of prolongations in $\boldsymbol{V}$ is principal by the first part. Using Roquette's unit density lemma, [R2], we show that by suitably approximating the functions making the sets principal, after a finite base extension one obtains a global function which does the job. Finally, by taking the norm one obtains the global result for $\boldsymbol{V}$.

The theorem above can be used to give the following *birational characterization* of arithmetic surfaces, and prove the existence of finite morphisms to $\mathbb{P}^1$. This is the second theorem we want to emphasize here.

Let S be a normal, integral, affine scheme whose local rings at the closed points are valuation rings, and suppose $\mathcal{V}$ is the corresponding set of valuations. Let $\mathcal{X}$ be a proper, normal, integral S-curve (= S-scheme of pure relative dimension 1), with $K := \kappa(S)$ relatively algebraically closed in $F := \kappa(\mathcal{X})$. For each closed point $\mathcal{M}_v \in S$ let $\mathcal{O}_{\eta_i,v} \subseteq F$, $1 \leq i \leq n_v$, be the local rings corresponding to the generic points of the irreducible components of the closed fibre at $\mathcal{M}_v$. We call the set of points of $\mathcal{X}$ corresponding to these local rings the *embedded generic set* at v. For each i, $\mathcal{O}_{\eta_i,v}$ is a valuation ring dominating $\mathcal{O}_v$ whose residue field is a function field of one variable over the residue field of $\mathcal{O}_v$, i.e., a constant reduction of F. Let $\boldsymbol{V}$ denote the family of constant reductions of F corresponding to the embedded generic sets at each $v \in \mathcal{V}$. Using the above terminology we prove:

Theorem 2. *Let S and $\mathcal{X}$ be as defined above and suppose the family $\mathcal{V}$ satisfies one of the conditions in theorem 1. Then there is a finite morphism $\mathcal{X} \longrightarrow \mathbb{P}^1_S$. Moreover, up to isomorphism $\mathcal{X}$ is uniquely determined by $F = \kappa(\mathcal{X})$ and the family $\boldsymbol{V}$, which is geometric. Conversely, up to isomorphism each geometric family of constant reductions $\boldsymbol{V}$ of F prolonging $\mathcal{V}$ determines a unique proper, normal, integral S-curve $\mathcal{X}$.*

I would like to thank Michel Matignon, Florian Pop and Peter Roquette for many discussions related to these problems. The observation that in the global context, by using the norm one obtains a finite morphism to $\mathbb{P}^1_S$ without making a base change, was made by Florian Pop and also by Laurent Moret-Bailly in a private communication. I am grateful to them for suggesting this improvement.

2. The semi-local situation

Algebraically closed case. Let $F|K$ be given with K algebraically closed, and suppose $\mathcal{V}$ and $\boldsymbol{V}$ are finite sets of independent valuations of K, respectively constant reductions prolonging $\mathcal{V}$ to F. Our aim is to show that $\boldsymbol{V}$ is principal, i.e., there exists $f \in F$ such that $\boldsymbol{V} = \boldsymbol{V}_f$. The non-constant function f is called an element with the uniqueness property for $\boldsymbol{V}$.

2.1. Suppose $\mathcal{V} = \{v_1, ..., v_n\}$ and $\boldsymbol{V} = \bigcup_i V_i$, where $V_i = \{\mathrm{v} \in \boldsymbol{V} \mid \mathrm{v}|_K = v_i\}$. For each i, we let $\mathbf{v}_i(x) = \inf \mathrm{v}(x)$, for $\mathrm{v} \in V_i$ and $x \in F$. We set $F\mathbf{v}_i := \mathcal{O}_{\mathbf{v}_i}/\mathcal{M}_{\mathbf{v}_i}$, where $\mathcal{O}_{\mathbf{v}_i} = \bigcap_{\mathrm{v}\in V_i} \mathcal{O}_\mathrm{v}$ and $\mathcal{M}_{\mathbf{v}_i} = \bigcap_{\mathrm{v}\in V_i} \mathcal{M}_\mathrm{v}$, the intersections being taken over the respective valuation rings and valuation ideals. Then $F\mathbf{v}_i := \mathcal{O}_{\mathbf{v}_i}/\mathcal{M}_{\mathbf{v}_i} \cong \prod_{\mathrm{v}\in V_i} \mathcal{O}_\mathrm{v}/\mathcal{M}_\mathrm{v} = \prod_{\mathrm{v}\in V_i} F\mathrm{v}$, the product of the residue fields of F with respect to the constant reductions v via the identification $x + \mathcal{M}_{\mathbf{v}_i} \mapsto (x + \mathcal{M}_\mathrm{v})_\mathrm{v}$. For $x \in F$ with $\mathbf{v}_i(x) \geq 0$ we let $x\mathbf{v}_i$ denote its image in $F\mathbf{v}_i$, and for a subset $M \subseteq F$ we denote the image of its elements of inf-norm ≥ 0 by $M\mathbf{v}_i$.

For each i, let $\mathcal{X}_{V_i}$ be the proper, normal, integral $\mathcal{O}_{v_i}$-curve associated with V_i as defined in [G–M–P 3], section 1. Let X be its generic fibre and $\mathcal{X}_{V_i\mathcal{M}_i}$ the special fibre at the closed point $\mathcal{M}_i \in \operatorname{Spec} \mathcal{O}_{v_i}$. By [G–M–P 2], proposition 2.5, and [G–M–P 3], lemma 1.3, the following assertions hold:

i) There exist finite subsets $A_i \subset \mathcal{X}_{V_i\mathcal{M}_i}$ with the property that if D is a positive divisor of $F|K$ such that the specialisations of $\mathrm{supp}(D)$ lie in the complement of A_i and meet all components of the closed fibre $\mathcal{X}_{V_i\mathcal{M}_i}$, then for large $n \in \mathbb{N}$ there exist $f_i \in F$ with $(f_i)_\infty = nD$ and $V_{f_i} = V_i$.

ii) Contraction. For $U_i \subset V_i$ there is a commutative diagram of $\mathcal{O}_{v_i}$-curves:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"] & \mathcal{X}_{V_i} \arrow[d] & \mathcal{X}_{V_i\mathcal{M}_i} \arrow[l] \arrow[d, "pr"] \\ X \arrow[r] & \mathcal{X}_{U_i} & \mathcal{X}_{U_i\mathcal{M}_i} \arrow[l] \end{tikzcd}$$

where the columns are surjective. Moreover there exist finite subsets $B_i \subset \mathcal{X}_{U_i\mathcal{M}_i}$ which have the following properties:

- B_i has the property i).
- The preimage $T_i = pr^{-1}(B_i)$ contains finite subsets $A_i \subset \mathcal{X}_{V_i\mathcal{M}_i}$ having the property i) and satisfying $pr(A_i) = B_i$.
- T_i contains a finite subset which can be prescribed in advance.

Lemma 2.2. *For each i, $1 \le i \le n$, let $U_i \subset V_i$ and $A_i \subset T_i$ be as in ii) above. Let D be a divisor of $F|K$ and suppose that for each i, the specialisation of* $\operatorname{supp}(D)$ *in* $\mathcal{X}_{V_i\mathcal{M}_i}$

- *lies in the complement of the A_i;*
- *meets all the components determined by the U_i;*
- *has no contribution in the components determined by $V_i \setminus U_i$.*

Then there exists $f \in F$ which is simultaneously a U_i element with the uniqueness property for each i, $1 \le i \le n$.

Proof. By the observations above, for each i there exists a U_i element with the uniqueness property f_i with $(f_i)_\infty = mD$, for m large enough. We define f as follows: For each i we apply the strong approximation theorem to obtain $a_i \in K$ with $v_i(a_i - 1) > 0$ and $v_j(a_i) \gg 0$, $j \ne i$. Setting $f = \sum_{i=1}^{n} a_i f_i$, it follows that $\deg f \le m \deg D$, while for each i

$$f\mathbf{v}_i = \sum_j (a_j f_j)\mathbf{v}_i = f_i\mathbf{v}_i, \quad \text{where} \quad \mathbf{v}_i = \inf_{\mathrm{v}\in U_i} \mathrm{v}.$$

Now $\deg f_i\mathbf{v}_i = \deg f_i = m \deg D$, so it follows that $\deg f = \deg f_i\mathbf{v}_i = \deg f\mathbf{v}_i$. Therefore f is a U_i element with the uniqueness property for each i. □

Lemma 2.3. *Suppose that $\#(U_i) = \#(V_i) = 1$ for each i with $1 \le i < n$, and suppose that $f \in F$ is a V_i element with the uniqueness property for such i. Let V_n be the set of all prolongations of the Gauß valuation $v_{n,f}$ from $K(f)$ to F, and fix $U_n \subset V_n$ with $\#(U_n) = 1$. Then there exists a prime divisor P of F such that for i, $1 \le i \le n$, the specialisation is disjoint from A_i, and for $i = n$ it has contribution in the component corresponding to U_n.*

Proof. For each i choose $b_i \in Kv_i$ so that $\big(\operatorname{supp}(f\mathbf{v}_i - b_i)_0\big) \cap A_i = \emptyset$. Next, using the chinese remainder theorem, choose $a \in K$ with $av_i = b_i$ for all i.

We consider $\operatorname{supp}(f-a)_0$: For each i the specialisation is $\operatorname{supp}(f\mathbf{v}_i - b_i)_0$, so disjoint from T_i. For $i = n$ some $P \in \operatorname{supp}(f-a)_0$ has specialisation with contribution in the component of $\mathcal{X}_{V_n\mathcal{M}_n}$ determined by U_n. □

Theorem 2.4. *Let $F|K$, $\mathcal{V}$ and $\boldsymbol{V}$ be as described in 2.1. Then $\boldsymbol{V}$ is principal at $\mathcal{V}$.*

Proof. We have $\mathcal{V} = \{v_1, ..., v_n\}$ and $\boldsymbol{V} = \bigcup_{i=1}^n V_i$. Set $r = \max_i \#(V_i)$. The theorem is proved by induction on r.

$r = 1$*:* To prove this case we now perform a second induction on $n = \#(\mathcal{V})$.

For $n = 1$: This is the content of [G–M–P 2], theorem 3.1.

The step $n \to n+1$: Suppose $f \in F$ is a V_i element with the uniqueness property for $1 \leq i \leq n$, where $V_i = \{\mathrm{v}_i\}$. Let v_{n+1} be a constant reduction of F prolonging v_{n+1}, and choose a polynomial $g = p(f) \in K[f]$ such that g is a V_i element with the uniqueness property for $1 \leq i \leq n$ and is v_{n+1} residually transcendental. Set $U_i = V_i$, $1 \leq i \leq n$, $U_{n+1} = \{\mathrm{v}_{n+1}\}$, and let V_{n+1} be the set of prolongations of the Gauß valuation $v_{n+1,g}$ on $K(g)$ to F. Next we apply lemma 2.3 to find a prime divisor P of F with specialisation disjoint from A_i, $1 \leq i \leq n+1$, and for $i = n+1$ with specialisation in the component corresponding to v_{n+1}. Now applying lemma 2.2 we obtain a function, say h, which is an element with the uniqueness property for v_i, $1 \leq i \leq n+1$.

To complete the proof of the theorem it remains to prove the inductive step $r \to r+1$. We partition the sets V_i, $1 \leq i \leq n$, as follows:

i) if $\#(V_i) = 1$ define $W_i = W_i' = V_i$;
ii) if $\#(V_i) > 1$ let $W_i \cup W_i' = V_i$ be a proper partition.

Then for each $i = 1, ..., n$, $\#(W_i)$, $\#(W_i') \leq r$. Therefore by the inductive hypothesis there exist elements with the uniqueness property, f for $W_1, ..., W_n$ and g for $W_1', ..., W_n'$. We next make adjustments to f and g as follows. Replace f by f_1 and g by g_1, elements with the uniqueness property for the respective sets $W_1, ..., W_n$ and $W_1', ..., W_n'$, such that

a) $\mathbf{w}_i'(f_1) > 0$ and $\mathbf{w}_i(g_1) > 0$ if i is in case ii)
(here $\mathbf{w}_i' = \inf\limits_{\mathrm{v} \in W_i'} \mathrm{v}$, resp. $\mathbf{w}_i = \inf\limits_{\mathrm{v} \in W_i} \mathrm{v}$).
b) $f_1\mathrm{v}_i$ and $g_1\mathrm{v}_i$ have disjoint pole divisors if i is in case i).

Now set $h = f_1 + g_1$; then by degree considerations h is an element with the uniqueness property for $V_1, ..., V_n$.

We explain how f_1 and g_1 are obtained from f and g. Repeatedly we will have to use the chinese remainder theorem for the independent valuations $v_1, ..., v_n$ of K to choose suitable coefficients when building f_1 and g_1 as rational functions of f and g.

First we adjust f and g so that for each i with $\#(V_i) = 1$, $f\mathrm{v}_i$ and $g\mathrm{v}_i$ have no common zeros or poles. The second step is to construct f_1 which is a W_i element with the uniqueness property, with $\mathrm{v}(f_1) > 0$ for $\mathrm{v} \in V_i \setminus W_i$, i as in case ii): Consider all $\mathrm{v} \in V_i \setminus W_i$ such that $\mathrm{v}(f) = 0$. Then as $\mathrm{v} \notin W_i$, f isn't v residually transcendental and we can choose $a_\mathrm{v} \in K$ with $f\mathrm{v} = a_\mathrm{v}\mathrm{v}$ and $v_k(a_\mathrm{v}) = 0$, $k \neq i$. We have $\mathrm{v}(f - a_\mathrm{v}) > 0$. Now taking the $f - a_\mathrm{v}$ for each i as in case ii) and $\mathrm{v} \in V_i \setminus W_i$ as above, we form the product

$$u = f \prod_i \prod_\mathrm{v} (f - a_\mathrm{v}).$$

Observe that u is a W_i element with the uniqueness property for $1 \leq i \leq n$. Let $f_1 = u/(1+u)^2$; then for $\mathrm{v} \in V_i \setminus W_i$:

- if $\mathrm{v}(f) = 0$ then $\mathrm{v}(u) > 0$ and so $\mathrm{v}(f_1) > 0$;
- if $\mathrm{v}(f) > 0$ then $\mathrm{v}(u) > 0$ and so $\mathrm{v}(f_1) > 0$
- if $\mathrm{v}(f) < 0$ then $\mathrm{v}(u) < 0$ and so $\mathrm{v}(f_1) = -\mathrm{v}(u) > 0$

Note that $(f_1)_\infty = 2(1+u)_0$.

Next we choose g_1, a W_i' element with the uniqueness property with $\mathrm{v}(g_1) > 0$, for $\mathrm{v} \in V_i \setminus W_i'$, i as in case ii), in the same way as above but satisfying the additional requirement that after constructing

$$t = g \prod_{i} \prod_{\mathrm{v}} (g - b_{\mathrm{v}}),$$

we set $g_1 = t/(c+t)^2$, with $c \in K$ chosen (using the chinese remainder theorem) so that for i as in case i) the support of $(cv_i + t\mathrm{v}_i)_0$ is disjoint from that of $(1+u\mathrm{v}_i)_0$, and for i as in case ii) $cv_i = 1$. It then follows that f_1 and g_1 satisfy both a) and b), finishing the proof. $\square$

The semi-local Skolem property. Our aim in this subsection is to show that theorem 2.4 remains true when we descend from an algebraically closed field to one satisfying the semi-local Skolem property. When considering a global field K equipped with a non-archimedean valuation, the local Skolem property is deduced from the work of Rumely [Ru1], more precisely from his Jacobian Principle which ensures the existence of functions having zeros in a given open and prescribed K-rational poles. In [G–M–P 4] we have shown that this *strong geometric criterion* is satisfied whenever the valuation on K has rational rank 1 and the residue field is algebraic over a finite field. Thus in this situation the local Skolem property holds. Here we shall show that, using this strong geometric criterion, it follows that if K is a field equipped with finitely many distinct valuations all having rational rank 1 and residue fields algebraic over finite fields, then the *semi-local Skolem property* is satisfied.

The results proved in this subsection generalise those of [G–M–P 4] and are needed to treat the semi-local situation.

First we need to recall the definitions.

Let K be any field equipped with finitely many independent valuations $\mathcal{V} = \{v_1, ..., v_n\}$, and $\tilde{K}$ be an algebraic closure equipped with fixed prolongations $\tilde{\mathcal{V}} = \{\tilde{v}_1, ..., \tilde{v}_n\}$. For each i let $(K_i^{\mathrm{h}}, v_i^{\mathrm{h}})$ be the henselisation of (K, v_i) which prolongs to $(\tilde{K}, \tilde{v}_i)$. We set $\mathcal{O}_{\mathcal{V}} = \bigcap_i \mathcal{O}_{v_i}$.

Let X be a geometrically integral curve defined over K with function field F, and denote by $X(\tilde{K})$ its set of $\tilde{K}$-rational points. For a tuple $\boldsymbol{x} = (x_1, ..., x_m)$, $x_j \in F$, and $S \subset \tilde{K}$ we define the S-rational points in $X(\tilde{K})$ with respect to $\boldsymbol{x}$ by $X_{\boldsymbol{x}}(S) := \{P \in X(\tilde{K}) : x_j(P) \in S, 1 \leq j \leq m\}$. The curve X is said to satisfy the semi-local Skolem property at $\mathcal{V}$, if for each tuple $\boldsymbol{x} = (x_1, ..., x_m)$, $x_j \in F$,

$$X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i}) \neq \emptyset \text{ for each } i \;\Rightarrow\; X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{\mathcal{V}}) \neq \emptyset.$$

Density. If X satisfies the semi-local Skolem property, then for each i, (K, v_i) is dense in $(K_i^{\mathrm{h}}, v_i^{\mathrm{h}})$. See [G–M–P 4], the proof of lemma 1.1.

Reciprocity and observations concerning the sets $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$. Let i be fixed, $\tilde{W}_i$ a set of constant reductions of $F\tilde{K}$ prolonging $\tilde{v}_i$ and suppose $x \in F\tilde{K}$ is a $\tilde{W}_i$ element with the uniqueness property. To simplify notation we supress the index i. Let $\mathcal{X}_{\tilde{W}}$ be the associated $\mathcal{O}_{\tilde{v}}$-curve as in 2.1, with special fibre $\mathcal{X}_{\tilde{W}\tilde{\mathcal{M}}} := \overline{X}$, and let $\overline{X}'$ be its normalisation. The ring $\mathcal{O}_{\tilde{v}}[x]'$ determines an affine open of $\mathcal{X}_{\tilde{W}}$, and $\mathcal{O}_{\tilde{v}}[x]'\mathbf{v} \cong \mathcal{O}_{\tilde{v}}[x]' \otimes_{\tilde{K}} \tilde{K}\tilde{v}$ the corresponding open dense subset of $\overline{X}$. Here $\tilde{\mathbf{v}} = \inf \tilde{\mathrm{v}}$ ($\tilde{\mathrm{v}} \in \tilde{W}$), the inf-norm on $F\tilde{K}$ determined by $\tilde{W}$, and the ring of rational functions of $\overline{X}$ is $F\tilde{K}\tilde{\mathbf{v}} = \prod F\tilde{K}\tilde{\mathrm{v}}$. The integral closure of $\mathcal{O}_{\tilde{v}}[x]'\tilde{\mathbf{v}}$ in $F\tilde{K}\tilde{\mathbf{v}}$ is $\prod \tilde{K}\tilde{v}[x\tilde{\mathrm{v}}]'$ and determines an affine open of $\overline{X}'$.

Given $P \in X(\tilde{K})$, *let* $\overline{P}$ *be its specialisation in* $\overline{X}$ *and* $\overline{Q}$ *any point of* $\overline{X}'$ *lying over* $\operatorname{supp}(\overline{P})$. *Then if* $t \in \mathcal{O}_{\tilde{v}}[x]'$ *and* $P \in X_x(\mathcal{O}_{\tilde{v}})$, *it follows that*

$$t(P)\tilde{v} = t\tilde{\mathrm{v}}(\overline{Q}),$$

where $\overline{Q}$ *lies in the component of* $\overline{X}'$ *determined by* $\tilde{\mathrm{v}}$.

Indeed, as $t \in \mathcal{O}_{\tilde{v}}[x]'$ and $P \in X_x(\mathcal{O}_{\tilde{v}})$, it follows that $\tilde{v}(t(P)) \geq 0$. Using square brackets to denote the ideals of the points in the respective affine rings, we have

$$(t - t(P)) \subset [P] \subset \mathcal{O}_{\tilde{v}}[x]', \qquad (t - t(P))\tilde{\mathbf{v}} \subset [\overline{P}] \subset \mathcal{O}_{\tilde{v}}[x]'\tilde{\mathbf{v}},$$

and hence $(t - t(P))\tilde{\mathrm{v}} \subset [\overline{Q}] \subset \tilde{K}\tilde{v}[x\tilde{\mathrm{v}}]'$. It follows that

$$t\tilde{\mathrm{v}}(\overline{Q}) - t(P)\tilde{v} = (t - t(P))\tilde{\mathrm{v}}(\overline{Q}) = 0.$$

Using the observations above we see that for $P \in X_x(\mathcal{O}_{\tilde{v}})$ and $\overline{Q}$ and $\tilde{\mathrm{v}}$ as above, if $t(P)$ is a $\tilde{v}$-unit then

- $\overline{Q} \notin \operatorname{supp}(t\tilde{\mathrm{v}})$.
- t is a $\tilde{\mathrm{v}}$-unit.

Set $\boldsymbol{x} = (x, t, t^{-1})$; then the scheme theoretical specialisation of $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ in $\overline{X}$ is disjoint from $\operatorname{supp}(t\tilde{\mathbf{v}})$ and the components determined by those $\tilde{\mathrm{v}} \in \tilde{W}$ for which t isn't a $\tilde{\mathrm{v}}$-unit.

Prolongations of constant reductions. For a finite set V_i of constant reductions of $F|K$ prolonging v_i let V_i^{h} be the set of all constant reductions $\mathrm{v}_i^{\mathrm{h}}$ of $FK_i^{\mathrm{h}}|K_i^{\mathrm{h}}$ which prolong both V_i and v_i^{h} to FK_i^{h}. We define $\tilde{V}_i$ correspondingly on $F\tilde{K}$, and remark that $\tilde{V}_i|_{FK_i^{\mathrm{h}}} = V_i^{\mathrm{h}}$ and $V_i^{\mathrm{h}}|_F = V_i$. Further by [G–M–P 4], for each $\mathrm{v}_i \in V_i$ there is a unique $\mathrm{v}_i^{\mathrm{h}} \in V_i^{\mathrm{h}}$ prolonging it.

Theorem 2.5. *Let* X *be a geometrically integral curve defined over* K *with function field* F, *and* $\mathcal{V}$ *be a finite set of independent valuations of* K. *Then the following are equivalent.*

i) *X has the semi-local Skolem property at $\mathcal{V}$.*
ii) *Each finite set of constant reductions $\boldsymbol{V}$ of F prolonging $\mathcal{V}$ is principal.*
iii) *Geometric criterion: For each i, (K, v_i) is dense in its henselisation $(K_i^{\mathrm{h}}, v_i^{\mathrm{h}})$, and for every $\boldsymbol{x} = (x_1, \ldots, x_m)$, $x_j \in F$, if $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i}) \neq \emptyset$ for all i, then there exists a positive K-rational divisor D and, for each i, $P_{i,1}, \ldots, P_{i,d} \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ such that $D \sim P_{i,1} + P_{i,2} + \cdots + P_{i,d}$ over $\tilde{K}$.*

Proof. i)⇒ii) First let $x \in F$ be chosen residually transcendental for each $\mathrm{v} \in \boldsymbol{V}$. For each i, $1 \leq i \leq n$, we let W_i be the set of prolongations of $v_{i,x}$ to F and set $\boldsymbol{W} = \bigcup_i W_i$. Observe that as $V_i \subset W_i$ for each i, $\boldsymbol{V} \subset \boldsymbol{W}$. Let $\tilde{\boldsymbol{V}} = \bigcup_i \tilde{V}_i$, where $\tilde{V}_i$ is the set of constant reductions of $F\tilde{K}$ prolonging both V_i and $\tilde{v}_i$; we define $\tilde{\boldsymbol{W}} = \bigcup_i \tilde{W}_i$ similarly.

For each i, let A_i be a finite subset of $\mathcal{X}_{\tilde{W}_i\tilde{\mathcal{M}}_i}$. Then as x is W_i residually transcendental, there exists a polynomial $z \in \mathcal{O}_{\mathcal{V}}[x]$ such that A_i is contained in the specialisation to $\mathcal{X}_{\tilde{W}_i\tilde{\mathcal{M}}_i}$ of the divisor support of z, for each i. We choose the sets A_i to satisfy 1.1 i) for the closed fibre $\mathcal{X}_{\tilde{W}_i\tilde{\mathcal{M}}_i}$. Next we choose $y \in \mathcal{O}_{\mathcal{V}}[x]' \subset F$ such that for each i, $y\mathrm{v} \neq 0$ for $\mathrm{v} \in V_i$ and $y\mathrm{v} = 0$ for $\mathrm{v} \in W_i \setminus V_i$.

Let $\boldsymbol{x} = (x, y, y^{-1}, z, z^{-1})$ and for each i let $\mathcal{D}_i$ be the scheme theoretical specialisation of $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ in $\mathcal{X}_{\tilde{W}_i\tilde{\mathcal{M}}_i}$. Then, by the discussion preceeding the theorem, for each i, $\mathcal{D}_i$ is disjoint from the set A_i and the components of $\mathcal{X}_{\tilde{W}_i\tilde{\mathcal{M}}_i}$ corresponding to $\tilde{\mathrm{v}} \in \tilde{W}_i \setminus \tilde{V}_i$. It follows that $\mathcal{X}_{\tilde{W}_i\tilde{\mathcal{M}}_i} \setminus \mathcal{D}_i$ has the properties required of T_i in 1.1 ii) (here W_i replaces V_i and V_i replaces U_i). In particular, for every K-rational divisor D with $\mathrm{supp}(D) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ whose specialisation meets all the components of $\mathcal{X}_{\tilde{V}_i\tilde{\mathcal{M}}_i}$, there exist $\tilde{V}_i$ elements with the uniqueness property $h_i \in F\tilde{K}$ such that $(h_i)_\infty = lD$ for $l \in \mathbb{N}$ big enough.

Since $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i}) \neq \emptyset$ for each i, by hypothesis there exists $P \in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{\mathcal{V}})$. We let $D = r\sum P'$, where P' is a system of G_K conjugates of P and r is a large enough power of the characteristic exponent of K so that D is K-rational. By the discussion above we conclude that for each i there exists a $\tilde{V}_i$ element with the uniqueness property $h_i \in F\tilde{K}$ with $(h_i)_\infty = lD$, provided l is large enough.

Directly from the definition it follows that $\tilde{V}_i$ is $G_{K_i^{\mathrm{h}}}$ invariant. Hence for each $\sigma \in G_{K_i^{\mathrm{h}}}$ one has $\tilde{V}_i^{\sigma} = \tilde{V}_i$, and consequently σh_i is a $\tilde{V}_i$ element with uniqueness property with $(h_i)_\infty = (\sigma h_i)_\infty$. Set $t_i = \left(\prod \overline{\sigma} h_i\right)^r$, where the $\overline{\sigma} h_i$ are a system of representatives for the $G_{K_i^{\mathrm{h}}}$ conjugates of h_i and r is a large enough power of the characteristic exponent of K so that $t_i \in FK_i^{\mathrm{h}}$. As the $\overline{\sigma} h_i$ all have the same pole divisor, it follows that t_i is a $\tilde{V}_i$ element with the uniqueness property. Since it lies in FK_i^{h}, it is an element with the uniqueness property for V_i^{h}. We denote the pole divisor of t_i by mD.

Now as (K, v_i) is dense in $(K_i^{\mathrm{h}}, v_i^{\mathrm{h}})$ we can find g_i in the linear space $\mathcal{L}_K(mD)$ such that $(g_i)_\infty = mD$ and $\mathrm{v}_i^{\mathrm{h}}(t_i - g_i) > 0$ for $\mathrm{v}_i^{\mathrm{h}} \in V_i^{\mathrm{h}}$. Then, using the fundamental inequality,

$$\begin{aligned}
[FK_i^{\mathrm{h}} : K_i^{\mathrm{h}}(t_i)] &= [FK_i^{\mathrm{h}} : K_i^{\mathrm{h}}(g_i)] \\
&\geq \sum e_{\mathrm{v}_i^{\mathrm{h}}} \delta_{\mathrm{v}_i^{\mathrm{h}}} [FK_i^{\mathrm{h}}\mathrm{v}_i^{\mathrm{h}} : K_i^{\mathrm{h}}(g_i)\mathrm{v}_i^{\mathrm{h}}] \quad \text{as } \mathrm{v}_i^{\mathrm{h}} \in V_{ig_i}^{\mathrm{h}} \\
&\geq \sum e_{\mathrm{v}_i^{\mathrm{h}}} \delta_{\mathrm{v}_i^{\mathrm{h}}} [FK_i^{\mathrm{h}}\mathrm{v}_i^{\mathrm{h}} : K_i^{\mathrm{h}}(t_i)\mathrm{v}_i^{\mathrm{h}}] = [FK_i^{\mathrm{h}} : K_i^{\mathrm{h}}(t_i)].
\end{aligned}$$

Thus $[FK_i^{\mathrm{h}} : K_i^{\mathrm{h}}(g_i)] = \sum e_{\mathrm{v}_i^{\mathrm{h}}} \delta_{\mathrm{v}_i^{\mathrm{h}}} [FK_i^{\mathrm{h}}\mathrm{v}_i^{\mathrm{h}} : K_i^{\mathrm{h}}(g_i)\mathrm{v}_i^{\mathrm{h}}]$, and hence g_i is an element with the uniqueness property for V_i^{h}. Since $g_i \in F$, it is an element with uniqueness property for V_i.

Setting $g = \sum_i a_i g_i$, $a_i \in K$ chosen as in 1.2 so that $g\mathbf{v}_i = g_i\mathbf{v}_i$, it follows that for each i, g is a V_i element with the uniqueness property, completing the proof of i)$\Rightarrow$ii).

ii)$\Rightarrow$i) Suppose $\boldsymbol{x} = (x_1, ..., x_m)$, $x_j \in F$, and for each i, $1 \leq i \leq n$, $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i}) \neq \emptyset$. We show that $X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{\mathcal{V}}) \neq \emptyset$. We first show that for each i there exists $f_i \in FK_i^{\mathrm{h}}$ such that $X_{f_i}(\mathcal{O}_{\tilde{v}_i}) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$.

Note that for $P \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ all its $\mathrm{G}_{K_i^{\mathrm{h}}}$-conjugates P' lie in $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$. Set $A = p^e \sum P'$, where p is the characteristic exponent of K and e is sufficiently large so that A is K_i^{h}-rational and by Riemann-Roch there exists $g_i \in FK_i^{\mathrm{h}}$ with $(g_i)_0 = A$. Next let $u_j(T) := \mathrm{Irr}(x_j(P)|K_i^{\mathrm{h}})$ and set $y_j = u_j(x_j)$. Clearly $\mathrm{supp}\, A \subset \mathrm{supp}\,(y_j)_0$; hence g_i^{-1} is integral over $\tilde{K}[y_j^{-1}]$. From the irreducible equations of g_i^{-1} over the $\tilde{K}[y_j^{-1}]$, one deduces that for all $\pi_i \in K_i^{\mathrm{h}}$ with $v_i^{\mathrm{h}}(\pi_i)$ sufficiently large, $f_i = g_i/\pi_i$ satisfies the claim.

Let $V_{f_i}^{\mathrm{h}}$ be the set of all prolongations of the Gauß valuation $v_{if_i}^{\mathrm{h}}$ to FK_i^{h}, and V_i its restriction to F. By the earlier remarks we have $V_{f_i}^{\mathrm{h}} = V_i^{\mathrm{h}}$.

Let $g \in F$ be an element with the uniqueness property for V_i for each i, which exists by hypothesis. Then by the earlier remarks it follows that g is an element with the uniqueness property for V_i^{h}, for each i. It follows that f_i is integral over

$\mathcal{O}_{v^{\mathrm{h}}_{ig}}$ and so also over $\mathcal{O}_{\tilde{v}_{ig}}$ when viewed in $F\tilde{K}|\tilde{K}$. Hence one has

$$f_i^{n_i} + r_{i,n_i-1}(g)f_i^{n_i-1} + \cdots + r_{i,0}(g) = 0,$$

where $r_{i,k}(g) = p_{i,k}(g)/q_{i,k}(g)$ and $\tilde{v}_{ig}(q_{ik}(g)) = 0 \le \tilde{v}_{ig}(p_{i,k}(g))$. By the maximum principle there exists $\alpha_i \in \mathcal{O}_{\tilde{v}_i}$ such that $\tilde{v}_i(q_{i,k}(\alpha_i)) = 0 \le \tilde{v}_i(p_{i,k}(\alpha_i))$. Using the strong approximation theorem we choose $\alpha \in \tilde{K}$ approximating the α_i closely enough with respect to $\tilde{v}_i$ so that

$$\tilde{v}_i(q_{i,k}(\alpha)) = 0 \le \tilde{v}_i(p_{i,k}(\alpha)).$$

We show that the zeros P of $g-\alpha$ lie in $X_{f_i}(\mathcal{O}_{\tilde{v}_i})$: Let P be a zero of $g-\alpha$, i.e. $g(P) = \alpha$. Then

$$(f_i(P))^{n_i} + r_{i,n_i-1}(\alpha)(f_i(P))^{n_i-1} + \cdots + r_{i,0}(\alpha) = 0.$$

As the $r_{i,k}(\alpha) \in \mathcal{O}_{\tilde{v}_i}$ it follows that $f_i(P) \in \mathcal{O}_{\tilde{v}_i}$. Hence all the zeros P of $h = g-\alpha$ lie in $X_{f_i}(\mathcal{O}_{\tilde{v}_i}) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$.

Next we set $h_i = \big(\prod_{\overline{\sigma}} \overline{\sigma}h\big)^r$, where the $\overline{\sigma}h$ are a system of representatives for the $G_{K_i^{\mathrm{h}}}$-conjugates of h and r is a suitable power of the characteristic exponent such that $h_i \in FK_i^{\mathrm{h}}$. Note that for each i we have $\operatorname{supp}(h_i)_0 \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$. Let $L|K$ be a finite galois extension over which each of the h_i are defined, and set $v_{iL} = \tilde{v}_i|_L$. For each i the set of all prolongations of v_i to L can be described by $(v_{iL}^{\overline{\tau}_i})_{\overline{\tau}_i}$, where the $\overline{\tau}_i$ form a system of representatives for the cosets of $\mathrm{Gal}(L|K)/Z(v_{iL}|v_i)$. For each $\overline{\tau}_i$ let $\overline{\tau}_i h_i$ be the corresponding conjugate of h_i. As K is dense in K_i^{h} it follows that the valuations $(v_{iL}^{\overline{\tau}_i})_{\overline{\tau}_i,i}$ are independent. Therefore the diagonal embedding $L \hookrightarrow \prod_i \prod_{\overline{\tau}} \big(L, v_{iL}^{\overline{\tau}_i}\big)$ is dense, and so is the embedding $\mathcal{L}_L(D) \hookrightarrow \prod_i \prod_{\overline{\tau}_i} \big(\mathcal{L}_L(D), v_{iL}^{\overline{\tau}_i}\big)$. Hence, there exist $t \in \mathcal{L}_L(D)$ such that $t \approx \overline{\tau}_i h_i$ in the $v_{iL}^{\overline{\tau}_i}$-adic topology for each $\overline{\tau}_i$. Given $\sigma \in G_K$, for each i let $\overline{\tau}_i$ be its representative in $\mathrm{Gal}\big(L|K\big)/Z(v_{iL}|v_i)$. Then $\sigma h_i = \overline{\tau}_i h_i$ and hence $\sigma^{-1}t \approx h_i$ in the $\tilde{v}_i$-adic topology, as $t \approx \overline{\tau}_i h_i$ in the $v_{iL}^{\overline{\tau}_i}$-adic topology. Consequently for each i the zeros of $\sigma^{-1}t$ are $\tilde{v}_i$-close to those of h_i. By the choice of h_i, $\mathrm{supp}(h_i)_0 \subseteq X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$; hence the same is true for the zeros of $\sigma^{-1}t$. Therefore, if P is a zero of t then $P^\sigma \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ for all $\sigma \in G_K$ and i. Equivalently, $P \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i^\sigma})$ for all $\sigma \in G_K$ and i. Hence $P \in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{\mathcal{V}})$, completing the proof of ii)⇒i).

Equivalence with iii). The interested reader can easily modify the proof of ii)⇒i), which essentially goes through iii). The only additional information needed for the direction iii)⇒i), is that if the $P_{i,j} \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ are not disjoint from $\operatorname{supp} D$ then one can replace them by points for which this is the case; see [G–M–P 4] for details. □

For our discussion we shall not use the geometric criterion above, but a stronger form which is valid when the valuations v_i have rational rank 1 and residue fields which are algebraic over finite fields. In this situation we have:

2.6. The strong geometric criterion. Let X be a geometrically integral curve defined over K with function field F, and $\mathcal{V} = \{v_1, ..., v_n\}$ a finite set of distinct valuations of K having rational rank 1 and residue fields algebraic over finite fields. Then for every $\boldsymbol{x} = (x_1, \ldots, x_m)$, $x_j \in F$, if $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i}) \neq \phi$ for all i, then *for each* positive K-rational divisor D there exist $P_{i,1}, \ldots, P_{i,d} \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$ such that, for some multiple, $mD \sim P_{i,1} + P_{i,2} + \cdots + P_{i,d}$ over $\tilde{K}$.

This result is part of the proof of [G–M–P 4], theorem 3.1, and is established using the non-archimedean uniformisation of abelian varieties when $\mathcal{V}$ consists of only 1 valuation, and so holds generally. Using this strong geometric criterion, we obtain:

Theorem 2.7. *Let X be a geometrically integral curve defined over K with function field F and $\mathcal{V}$ a finite set of distinct valuations of K having rational rank 1 and residue fields algebraic over finite fields. Then X has the semi-local Skolem property at $\mathcal{V}$.*

Proof. By the discussion above, X satisfies the strong geometric criterion at $\mathcal{V}$. Therefore, given $\boldsymbol{x} = (x_1, \dots, x_m)$, $x_j \in F$, if $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i}) \neq \emptyset$ for all i, then for each positive K-rational divisor D there exist $f_i \in F\tilde{K}$ with $(f_i)_\infty = mD$ for some multiple and $\operatorname{supp}(f_i)_0 \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}_i})$. Using the f_i we construct functions h_i defined over FK_i^{h} as in the last part of 2.5 ii)$\Rightarrow$i), and so deduce there exist $P \in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{\mathcal{V}})$. □

Remarks. 1) As a corollary to theorem 2.7 we obtain theorem 1, stated at the beginning of the paper, subject to the conditions i), ii) or iii) (the semi-local part).

2) For an arbitrary field K equipped with finitely many independent valuations $\mathcal{V}$ the semi-local Skolem property implies the local Skolem property for each $v \in \mathcal{V}$. When the v have rational rank 1 and residue fields algebraic over finite fields, the converse also holds. *However, it is an open question whether this equivalence holds in general.*

3. The passage from the semi-local to the global situation

Our aim in this section is to extend the results obtained in the semi-local situation to the global situation. The main ingredients used here are Roquette's Unit Density Property and principal basis construction with regularity criterion for constant reductions.

We begin by fixing the notations to be used throughout this section.

3.1. 1) Let K be an arbitrary field endowed with a set $\mathcal{V}$ of non-archimedean valuations of finite character. This means that for each $a \in K$, $v(a) = 0$ for almost all $v \in \mathcal{V}$. Further we assume that the set of valuations $\mathcal{V}$ satisfies the strong approximation property. Note that if K' is any finite extension of K, then the prolongation $\mathcal{V}'$ of $\mathcal{V}$ has the same properties. See for example [Grif].

2) Next we recall those results from constant reduction theory which are fundamental to our discussion. For details the reader is referred to [G–M–P 1], [P1], [P2], [R1], and [R3].

Let $F|K$ be a function field in one variable and $t \in F$ a non-constant function. Then by [G–M–P 1], 1.3, and [R1], there exists a finite set $A \subset K$ such that for each non-archimedean valuation v of K, if $v(A) \geq 0$ then the Gauß valuation v_t has a unique prolongation v to F and t is regular at v, i.e. $\deg t = \deg t\mathrm{v}$. Further there exists a finite purely inseparable constant extension $FK'|K'$ of $F|K$ such that the unique prolongation v' of v to FK' is a good reduction. We say that v is a potential good reduction of $F|K$.

The families of constant reductions we are interested in satisfy the following geometric property.

Definition. *Geometric families of constant reductions.* Let $\boldsymbol{V}$ be a set of constant reductions of F prolonging $\mathcal{V}$. Then $\boldsymbol{V}$ is said to be geometric if there exists a non-constant function $t \in F$ such that $\boldsymbol{V}$ and $\boldsymbol{V}_t := \bigcup_{v\in\mathcal{V}} V_t$ are almost equal. Here by almost equal we mean that the symmetric difference of $\boldsymbol{V}$ and $\boldsymbol{V}_t$ is finite.

Observations. 1) The set $\boldsymbol{V}|\mathcal{V}$ determined by the S-curve $\mathcal{X}$ from the introduction (theorem 2) is a geometric set of constant reductions. (See the proof of theorem 2 at the end of the paper).

2) By the remarks in 3.1 2), if t and u are non-constant functions of F, then $\boldsymbol{V}_t$ and $\boldsymbol{V}_u$ are almost equal. Hence it follows that any two geometric sets of constant reductions prolonging $\mathcal{V}$ are almost equal.

3) Using 2) above together with 3.1 2), we deduce that $\boldsymbol{V}$ is a geometric set of constant reductions if and only if its prolongation to any finite extension $FK'|K'$ is.

Therefore without loss of generality we can assume that for almost all $v \in \mathcal{V}$ there is a unique prolongation v to F in $\boldsymbol{V}$ and that v is a potential good reduction. Let $\mathcal{G} \subset \mathcal{V}$ be the set of all such valuations.

3.2. The Unit Density Property. The pair $(K, \mathcal{V})$ is said to satisfy the Unit Density Property if for each finite subset $\mathcal{S} \subset \mathcal{V}$ and $a \in \tilde{K}^{\times}$ there exists $c \in \tilde{K}$, such that:

- $c - a$ is arbitrarily close to 0 v-adically for all $v \in \mathcal{S}$. (In general $c, a \notin K$, and to say that $c - a$ is arbitrarily close to 0 v-adically means that this is the case for every prolongation of v to $\tilde{K}$.)
- c is a unit at v for all $v \in \mathcal{V} \setminus \mathcal{S}$. (Again, this means that c is a unit at each prolongation of v to $\tilde{K}$.)

The unit density property is satisfied whenever $\mathcal{O}_{\mathcal{V}}$ is a Dedekind domain whose ideal class group is torsion and the valuations have residue fields which are algebraic over a finite field. Furthermore, the algebraic number c can be chosen so that each prime from $\mathcal{S}$ splits completely in $K(c)$, i.e., $K(c)$ is contained in $K^{\mathcal{S}}$, the field of totally $\mathcal{S}$-adic numbers. In particular, if K is a global field and $\mathcal{V}$ is any set of non-archimedean valuations satisfying the strong approximation property, then the unit density property is satisfied. This is a non-trivial theorem, which together with the Rumely existence theorem are the main ingredients needed to prove the Local-Global Principle for the solvability of algebraic diophantine equations in the ring of all algebraic integers over a global field. A proof of the unit density theorem can be found in [C–R], and the application to prove the Local-Global Principle in [G–P–R].

Throughout the rest of this section we shall assume that K is equipped with a set of non-archimedean valuations having the unit density property.

3.3. The principal basis and regularity criterion. Suppose D is a positive divisor of $F|K$, and for each prime divisor $Q \in \text{supp}(D)$ let r_Q denote its multiplicity and $l_Q := \deg Q$. From the Riemann-Roch theorem it follows that for each $Q \in \text{supp}(D)$ and $m > 2g_F$ there exist functions u_{mQi}, $1 \leq i \leq l_Q$,

- with pole divisor mQ and zeros disjoint from the primes occuring in D, and
- which are K-linearly independent and hence form a K-basis for $\mathcal{L}(mQ)$ modulo $\mathcal{L}((m-1)Q)$.

Now let $n > 2g_F$, and for each Q set $n_Q = nr_Q$. Then, again applying Riemann-Roch, the functions $(u_{n_QQi})_{Q,i}$ form a K-basis of $\mathcal{L}(nD)$ modulo $\mathcal{L}(B)$, where

$B = \sum_Q (n_Q - 1)Q$. We will call this basis a *principal basis for D of level n*, and to simplify the notation we set $u_{Qi} = u_{n_Q Qi}$. We next choose a K-basis of $\mathcal{L}(B)$, say $(w_j)_j$. Then each function $f \in \mathcal{L}(nD)$ has a unique representation in the form

$$f = \sum_j c_j w_j \; + \sum_{Q,i} c_{Qi} u_{Qi}$$

with coefficients c_j, $c_{Qi} \in K$. The pole divisor of f is precisely nD if and only if $c_{Qi} \neq 0$ for all Q and some i.

Regularity criterion. Let $v \in \mathcal{V}$, and v a constant reduction of $F|K$ prolonging v. For the basis $(w_j)_j \cup (u_{Qi})_{Q,i}$ of $\mathcal{L}(nD)$ we assume that:

i) The w_j and the u_{Qi} are regular functions for v.
ii) For each Q, the functions u_{Qi}, $1 \leq i \leq l_Q$, form a v valuation basis for $\mathcal{L}(n_Q Q)$ modulo $\mathcal{L}((n_Q - 1)Q)$, and for $Q \neq Q'$ one has

$$\operatorname{supp}(u_{Qi}\mathrm{v})_\infty \cap \operatorname{supp}(u_{Q'k}\mathrm{v})_\infty = \emptyset.$$

From these assumptions it follows that

- for each Q and $i \neq k$, $(u_{Qi}\mathrm{v})_\infty = (u_{Qk}\mathrm{v})_\infty = n_Q \overline{Q}$, for some divisor $\overline{Q} \in \operatorname{Div}(F\mathrm{v}|Kv)$;
- for each function w_j, $\operatorname{supp}(w_j\mathrm{v})_\infty \subset \bigcup_Q \operatorname{supp} \overline{Q}$;
- the functions $(u_{Qi})_{Q,i}$ form a v valuation basis for $\mathcal{L}(nD)$ modulo $\mathcal{L}(B)$.

Let $f \in F$ with $(f)_\infty = nD$, so that it can be written in the form

$$f = \sum_j c_j w_j \; + \sum_{Q,i} c_{Qi} u_{Qi}, \qquad c_j,\, c_{Qi} \in K.$$

Suppose that $v(c_j) \geq 0$ for each j, and $v(c_{Qi}) = 0$ for each Q and i. Then f is a regular function for v. Indeed $(w_j\mathrm{v})_\infty < \sum_{\overline{Q}} n_Q \overline{Q}$, so for f we obtain

$$\begin{aligned} \deg f\mathrm{v} &= \sum_Q \deg w_Q\mathrm{v}, \qquad \text{where } w_Q\mathrm{v} := \sum_i c_{Qi} u_{Qi}\mathrm{v}, \\ &= \sum_Q n_Q \deg \overline{Q} = \deg f. \end{aligned}$$

To simplify notation we shall simply write $(u_i)_i$ when we mean the basis $(w_j)_j \cup (u_{Qi})_{Q,i}$ of $\mathcal{L}(nD)$.

We now prove the main theorem of this section.

Theorem 3.4. *Let $F|K$ be a function field in one variable and $\mathcal{V}$ a set of valuations of the constant field K satisfying:*

i) *the semi-local Skolem property for $F|K$;*
ii) *the unit density property.*

Let $\boldsymbol{V}|\mathcal{V}$ be a geometric set of constant reductions of F. Then $\boldsymbol{V}$ is principal at $\mathcal{V}$.

Proof. It suffices to prove the theorem when $F|K$ is a regular function field. For otherwise, let $E|K \subset F|K$ be a function field in one variable with E separably closed in F. Then $E|K$ is regular and each constant reduction of E has a unique prolongation to F. Hence there is a bijective correspondence between the constant reductions of F and E, via restriction and prolongation. Therefore if a geometric set of constant reductions of E is principal, so is its prolongation to F.

Suppose $\mathcal{S}_1 = \mathcal{V} \setminus \mathcal{G}$, and let $\boldsymbol{V}_{\mathcal{S}_1}$ be the set of all prolongations of the valuations of $\mathcal{S}_1$ in $\boldsymbol{V}$. Then by definition both these sets are finite. Hence by the semi-local Skolem property there exists $f \in F$ which is a $\boldsymbol{V}_{\mathcal{S}_1}$ element with the uniqueness property. Suppose $(f)_\infty = nD$, with $n > 2g_F$. (We replace f by some power if necessary.)

Next let $(u_i)_i$ be a principal basis for nD. Then, by assumption, for almost all $v \in \mathcal{G}$ the u_i satisfy the conditions 3.3 i) and ii) for the potential good reduction v prolonging v. Suppose $\mathcal{S}_2 \subset \mathcal{G}$ is the finite subset for which these conditions are not satisfied. Then after a finite purely inseparable extension of F, the potential good reduction v prolongs to a good reduction which we again denote by v. Using the divisor reduction map for each such v, see [R3], plus the chinese remainder theorem, we construct a function h, simultaneously v-regular for each v$|v$, $v \in \mathcal{S}_2$, with $(h)_\infty = nD$.

Using the principal basis construction, we have

$$f = \sum a_i u_i \quad \text{and} \quad h = \sum b_i u_i, \quad \text{with} \quad a_i, \in K,\ b_i \in \tilde{K}.$$

We shall now use the unit density property to construct a function having the required properties.

Using the unit density property, we find $c_i \in \tilde{K}$ such that

- $c_i - a_i$ is arbitrarily close to 0 v-adically for $v \in \mathcal{S}_1$;
- $c_i - b_i$ is arbitrarily close to 0 v-adically for $v \in \mathcal{S}_2$;
- c_i is a unit at v for all $v \in \mathcal{V} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$.

Setting $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $K' = K\big[(c_i)_i\big]$, it follows that $K' \subset K^{\mathcal{S}}$.

We set $t = \sum c_i u_i \in FK'$ and claim it is a $\boldsymbol{V}'|\mathcal{V}'$ element with the uniqueness property. Here $\boldsymbol{V}'|\mathcal{V}'$ is the prolongation of $\boldsymbol{V}|\mathcal{V}$ to $FK'|K'$.

For $v' \in \mathcal{S}_1'$, by writing $t = f + (t - f)$, observing that for each v$' \in V'$, $t\mathrm{v}' = f\mathrm{v}' + (t-f)\mathrm{v}' = f\mathrm{v}'$ while $\deg t = \deg f = n \deg D$, and applying the fundamental valuation equality (possibly with defect contribution), it follows that t is a V' element with the uniqueness property.

For $v' \in \mathcal{S}_2'$ we write $t = h + (t - h)$ and note that it is v$'$-regular, as here $\deg t = \deg h = \deg h\mathrm{v}' = \deg t\mathrm{v}'$.

For $v' \in \mathcal{V}' \setminus (\mathcal{S}')$, we use the regularity criterion to ensure that t is v$'$-regular.

In the last part of the proof we show that, considering the norm of t in F, we obtain an element with the uniqueness property for $\boldsymbol{V}|\mathcal{V}$. For each $\sigma \in \mathrm{Aut}(FK'|F) = \mathrm{Aut}(K'|K)$ (as $F|K$ is assumed to be regular) we have:

- $(\sigma t)_\infty = (t)_\infty$, as the pole divisor is K-rational.
- $\boldsymbol{V}'^{\sigma} = \boldsymbol{V}'$ and $\mathcal{V}'^{\sigma} = \mathcal{V}'$.

We deduce that σt is an element with the uniqueness property for $\boldsymbol{V}'|\mathcal{V}'$. Finally, consider $h = \prod_\sigma \sigma t$. By the observations above, h is also an element with the uniqueness property for $\boldsymbol{V}'|\mathcal{V}'$. Since $h \in F$, on restricting to F we see that the same holds for $\boldsymbol{V}|\mathcal{V}$, completing the proof of the theorem. $\square$

Remarks. The remaining part of theorem 1, i.e., subject to the condition iv) (the global part), follows as a corollary to theorem 3.4.

Combining the above result with the characterisation for curves over valuation rings given in [G], we can now prove theorem 2:

Proof of theorem 2. Recall, S is a normal, integral, affine scheme whose local rings at the closed points are valuation rings, and $\mathcal{V}$ is the corresponding set of valuations, which is assumed to satisfy one of the conditions in theorem 1. $\mathcal{X}$ is a proper, normal, integral S-curve and $\boldsymbol{V}$ is the family of constant reductions of $F := \kappa(\mathcal{X})$ prolonging $\mathcal{V}$ which are determined by its embedded generic set.

We first observe that by [G], theorem 3.6, it follows that up to a birational isomorphism $\mathcal{X}$ is uniquely determined by the family $\boldsymbol{V}$.

Next we note that because of the conditions imposed on $\mathcal{V}$, it follows that $\boldsymbol{V}$ is a geometric family of constant reductions. In the semi-local situation this is clear from the definitions. In the global situation $S = \operatorname{Spec}\mathcal{O}$, where $\mathcal{O}$ is some affine normal model for a global field K. For almost all $v \in \mathcal{V}$, the fibre over $\mathcal{M}_v \in S$ has potential good reduction. Therefore for such v there is a unique prolongation v in $\boldsymbol{V}$, and this is a potential good reduction of F. By the observations made at the beginning of this section it follows that if t is any non-constant function in F, the symmetric difference of $\boldsymbol{V}$ and $\boldsymbol{V}_t$ is finite. Hence $\boldsymbol{V}$ is a geometric family of constant reductions.

Applying theorem 1, we deduce that $\boldsymbol{V}$ is principal, with an element with the uniqueness property, f, say. We conclude that $\mathcal{X}$ is S-isomorphic to the normalisation of $\mathbb{P}^1_S$ relative to the extension of function fields, $F|K(f)$.

Finally note that on the other hand, if $\boldsymbol{V}$ is any geometric family of constant reductions of F prolonging $\mathcal{V}$, one obtains the unique proper, normal, integral S-model $\mathcal{X}$ having $\boldsymbol{V}$ as embedded generic set by taking the normalisation of $\mathbb{P}^1_S$ relative to the extension $F|K(f)$. This completes the proof of theorem 2. $\square$

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