THE IRRATIONALITY OF $\log(1 + 1/q) \log(1 - 1/q)$

MASAYOSHI HATA

Abstract. We shall show that the numbers $1$, $\log(1 + 1/q)$, $\log(1 - 1/q)$ and $\log(1 + 1/q) \log(1 - 1/q)$ are linearly independent over $\mathbb{Q}$ for any natural number $q \geq 54$. The key is to construct explicit Padé-type approximations using Legendre-type polynomials.

0. Introduction

Galochkin [3] obtained the lower estimates of polynomials with integral coefficients in the values of certain Siegel’s $G$-functions at algebraic points of a special form. He used Padé-type approximations of the first kind in Mahler’s classification, and all the constants in his estimates are effectively computable. As a simple and instructive application of his general results, which concerns neither a polynomial in one logarithm nor a linear form in several logarithms, the irrationality of the number

$$\log\left(1 + \frac{1}{q}\right) \log\left(1 - \frac{1}{q}\right)$$

(0.1)

is shown for any natural number $q > e^{795}$. Galochkin [4] also pointed out that the bound $e^{795}$ can be improved to $e^{170}$ if one applies Chudnovsky and Chudnovsky’s result [2], which was obtained by Padé-type approximations of the second kind. The authors in both papers used the so-called Siegel’s lemma coming from Dirichlet’s box principle.

Galochkin’s result on the number (0.1) can now be improved remarkably as follows:

Theorem 0.1. The number (0.1) is irrational for any natural number $q \geq 54$.

To see this we will use Padé-type approximations in the same way as in our previous study [7] on the values of the dilogarithm. More precisely, we will construct explicitly a polynomial $P(w) \in \mathbb{Z}[w]$ of degree $n$ satisfying

$$P(z^2)F(z) - Q(z^2) = O\left(z^{2(n+1+\lfloor \lambda n \rfloor)}\right),$$

(0.2)

where $Q(w)$ is some polynomial of degree at most $n$, $\lambda = 1/4$ and

$$F(z) = -\log(1 + z) \log(1 - z) = \sum_{k=1}^{\infty} \left( \sum_{r+s=2k, r,s \geq 1} \frac{(-1)^{r+1}}{rs} \right) z^{2k}.$$
It is still open to find explicit Padé approximations (the case \( \lambda = 1 \)) or Padé-type approximations for some \( \lambda \in (1/4, 1) \) to \( F(z) \).

Put \( P^*(w) = w^n P(w^{-1}) \) for brevity. By noticing that

\[
F(z) = \int_0^1 \frac{1}{z^2 - x} \log \left(1 + \frac{1}{\sqrt{x}}\right) dx
\]

for \( |z| < 1 \), we get

\[
P(z^2) F(z) - z^{2n} \int_0^1 \frac{P^*(z^{-2}) - P^*(x)}{z^{-2} - x} \log \left(1 + \frac{1}{\sqrt{x}}\right) dx
= z^{2n+2} \int_0^1 \frac{P^*(x)}{1 - z^2 x} \log \left(1 + \frac{1}{\sqrt{x}}\right) dx
= \sum_{k=0}^\infty z^{2n+2+2k} \int_0^1 x^k P^*(x) \log \left(1 + \frac{1}{\sqrt{x}}\right) dx.
\]

Since the second term of the left-hand side is a polynomial in \( z^2 \) with rational coefficients of degree at most \( n \), it follows that \( P(z^2) \) satisfies (0.2) if and only if

\[
\int_0^1 x^k P^*(x) \log \left(1 + \frac{1}{\sqrt{x}}\right) dx = 0 \quad \text{for} \quad 0 \leq k < \lfloor \lambda n \rfloor.
\]

(0.3)

However, it seems to be difficult to solve (0.3) directly. Instead of (0.3) we will consider the following orthogonality property:

\[
\int_0^1 x^m P^*(x^2) \log \left(1 + \frac{1}{\sqrt{x}}\right) dx = 0 \quad \text{for} \quad 0 \leq m < 2\lambda n,
\]

(0.4)

which is easier to handle. Indeed, we can construct explicitly the polynomial \( P^*(w) \) which satisfies (0.4) for \( \lambda = 1/4 \) by somewhat modifying the Legendre polynomial \( (x^n(1-x)^n)^{\lfloor n \rfloor}/n! \) in Section 1.

The specific family of linear fractional transformations with real coefficients

\[
T = \left\{ \tau_c(x) = \frac{1-x}{1-cx}; \ c < 1 \right\}
\]

plays an important role in this paper. Note that \( \tau_c : [0, 1] \to [0, 1] \) is an orientation-preserving homeomorphism and satisfies \( \tau_c \equiv \tau_c^{-1} \) for any \( \tau_c \in T \). It is also easily seen that \( d\tau_c/(1 - c\tau_c) = -dx/(1 - cx) \). We call \( \tau_c \) a nice transformation. Some kinds of definite integrals over \([0, 1]^{\ell}, \ \ell \in \mathbb{N}, \) change into simpler and more useful ones by nice transformations. Such an example can be found in Beukers’ paper [1], in which he used the nice transformation \( \tau_{1-xy}(z) \) on some triple integral, so that the asymptotic behavior of the integral was easily obtained. As another example, Rhin and Viola [9] used some birational transformations involving nice transformation \( \tau_y(x) \) in order to choose relevant polynomial factors. In this paper nice transformations will appear in Sections 1 and 5.

In Section 1 we will investigate several properties of our Padé-type approximant \( P(x^2) \). Then a upper estimate of the remainder term in (0.2) will be given in Sections 4 and 5. Our main result on linear independence measures of the numbers \( 1, \log(1+1/q), \log(1-1/q) \) and \( \log(1+1/q)\log(1-1/q) \) will be proved in Section 6.
1. Preliminaries

Lemma 1.1. For any $x \in (0, 1)$ and $0 \leq j < n$, we have

\[
(1.1) \quad \frac{1}{n!} \left( x^j \log \left( 1 + \frac{1}{x} \right) \right)^{(n)} = \frac{(-1)^{n-j}}{(n-j)!} \sum_{k=0}^{n-j-1} \binom{n}{j} \binom{n-j-k}{k} x^{n-j-k(1+x)^n}.
\]

Proof. In the region $C_0$ obtained by omitting the segment $[-1, 0]$ from the complex plane $C$, the principal branch of $\log(1 + 1/z)$ is single-valued and hence analytic. The left-hand side of (1.1) is equal to

\[
\frac{1}{2\pi i} \int_C \frac{\zeta^j}{(\zeta - x)^{n+1}} \log \left( 1 + \frac{1}{\zeta} \right) d\zeta \quad \text{(call it } J),
\]

where $C$ is a small circle in $C_0$ centered at $x$. Since the integrand is estimated by $O(|\zeta|^{-2})$ when $|\zeta|$ is sufficiently large, the contour $C$ can be changed to the (degenerate) curve $C'$ adhering to the branch cut $[-1, 0]$. Hence

\[
J = \frac{(-1)^{n-j+1}}{2\pi i} \int_0^1 \frac{t^j}{(t+x)^{n+1}} \left[ \left\{ \log \left( \frac{1}{t} - 1 \right) - \pi i \right\} - \left\{ \log \left( \frac{1}{t} - 1 \right) + \pi i \right\} \right] dt
\]

\[
= (-1)^{n-j} \int_0^1 \frac{t^j}{(t+x)^{n+1}} dt.
\]

We now use the nice transformation $\tau_{-1/x}(t) \in T$. Substituting $\tau = \tau_{-1/x}(t)$, we get

\[
J = \frac{(-1)^{n-j}}{x^{n-j}(1+x)^n} \int_0^1 (1-\tau)^j (\tau + x)^{n-j-1} d\tau
\]

\[
= \frac{(-1)^{n-j}}{x^{n-j}(1+x)^n} \sum_{k=0}^{n-j-1} \binom{n-j-1}{k} x^k \int_0^1 \tau^{n-j-k-1}(1-\tau)^j d\tau
\]

\[
= \frac{(-1)^{n-j}}{(n-j)!} \sum_{k=0}^{n-j-1} \binom{n}{k} \binom{n-j-k}{k} x^{n-j-k(1+x)^n}.
\]

It follows from Lemma 1.1 that

\[
U_{j,n}(x) = x^n(1+x)^n \left( x^j \log \left( 1 + \frac{1}{x} \right) \right)^{(n)}
\]

is a polynomial satisfying $\deg(U_{j,n}) = n-1$ and $\ord_{x=0}(U_{j,n}) = j$ for $0 \leq j < n$. It seems to be hard to obtain $\deg(U_{j,n}) = n-1$ from the usual differential calculation by Leibniz’ formula.

Let $A(I)$ be the set of all real-analytic and integrable functions defined on an open interval $I$. For any $f(x) \in A(I)$ let $\nu_I(f) \in [0, \infty)$ be the number of zero points of $f(x)$ in $I$ without counting the multiplicities. (Note that $\nu_I(f) = \infty$ if $f(x)$ is identically zero in $I$.) The following basic lemma can be easily shown by a simple application of Rolle’s theorem.

Lemma 1.2. $\nu_I(f') \geq \nu_I(f) - 1$ for any $f(x) \in A(I)$. Put $I = (a, b)$ and suppose further that $\lim_{x \to a^+} f(x) = \lim_{x \to b^-} f(x) = 0$. Then $\nu_I(f') \geq \nu_I(f) + 1$. 

As a corollary, we have

**Corollary 1.3.** Suppose that \( f(x) \in \mathcal{A}(I) \) satisfies \( \int_I x^j f(x) \, dx = 0 \) for \( 0 \leq j < n \). Then \( \nu_1(f) \geq n \).

**Proof.** Put \( I = (a, b) \) and \( f_0(x) \equiv f(x) \). We define \( f_k(x) = \int_a^x f_{k-1}(t) \, dt \) inductively for \( x \in I \) and \( k \in \mathbb{N} \). Obviously \( f_j(x) \in \mathcal{A}(I) \) and \( f_j(a) = 0 \) for \( j \geq 1 \). Moreover we can show that \( f_j(b) = 0 \) for \( 1 \leq j \leq n \) by induction. Therefore it follows from Lemma 1.2 that \( \nu_1(f) = \nu_1(f_0) \geq \nu_1(f_1) + 1 \geq \cdots \geq \nu_1(f_n) + n \geq n \). \( \Box \)

**Lemma 1.4.** Let \( K \) be the open unit interval \((0,1)\). Suppose that \( R(x) \) is a real polynomial in \( x^2 \) and

\[
\int_0^1 x^j R(x) \, dx = 0 = \int_0^1 x^j R(x) \log \left( 1 + \frac{1}{x} \right) \, dx
\]

for \( 0 \leq j < n \). Then \( \nu_K(R) \geq 2n \). Suppose further that \( \int_0^1 R(x) \log x \, dx = 0 \). Then \( \nu_K(R) \geq 2n + 1 \).

**Proof.** Put \( R_0(x) \equiv R(x) \) and define \( R_k(x) = \int_0^x R_{k-1}(t) \, dt \) inductively for \( k \in \mathbb{N} \). Then one can show that \( R_j(0) = R_j(1) = 0 \) for \( 1 \leq j \leq n \) by induction; hence \( R_n(x) = x^n (1-x)^n S(x) \) for some polynomial \( S(x) \). Since \( R_n(x) \equiv (-1)^n R_n(-x) \), we get \( (1-x)^n S(x) \equiv (1+x)^n S(-x) \); therefore \( S(x) = (1+x)^n T(x) \) for some polynomial \( T(x) \).

On the other hand, by \( n \)-fold partial integration, we have

\[
0 = \int_0^1 \left( x^j \log \left( 1 + \frac{1}{x} \right) \right)^{(n)} R_n(x) \, dx = \int_0^1 U_{j,n}(x)(1-x)^nT(x) \, dx
\]

\[
= \int_0^1 U_{j,n}(x) \tilde{T}(x) \, dx
\]

for \( 0 \leq j < n \), where \( \tilde{T}(x) = (1-x)^n T(x) \) is a polynomial. Since \( \deg(U_{j,n}) = n-1 \) and \( \text{ord}_{x=0}(U_{j,n}) = j \), it is easily seen that

\[
\int_0^1 x^k \tilde{T}(x) \, dx = 0 \quad \text{for} \quad 0 \leq k < n.
\]

Hence \( \nu_K(R_n) = \nu_K(S) = \nu_K(T) = \nu_K(\tilde{T}) \geq n \) by Corollary 1.3 and \( \nu_K(R) = \nu_K(R_0) \geq \nu_K(R_1) + 1 \geq \cdots \geq \nu_K(R_n) + n \geq 2n \), as required.

Moreover, if \( \int_0^1 R(x) \log x \, dx = 0 \) in addition, then \( \int_0^1 (\log x)^{(n)} R_n(x) \, dx = 0 \); hence

\[
0 = \int_0^1 x^{-n} R_n(x) \, dx = \int_0^1 (1-x^2)^n T(x) \, dx = \int_0^1 (1+x)^n \tilde{T}(x) \, dx.
\]

Therefore \( \int_0^1 x^n \tilde{T}(x) \, dx = 0 \) and \( \nu_K(R_n) = \nu_K(\tilde{T}) \geq n + 1 \) by Corollary 1.3. We thus have \( \nu_K(R) \geq \nu_K(R_n) + n \geq 2n + 1 \), which completes the proof. \( \Box \)

We now introduce the following Legendre-type polynomial:

\[
L_{2n}(x) = \frac{1}{n!(n/2)!} x^{[n/2]} (1-x^{[n/2]} \left(x^n (1-x^2)^{(n/2)} \right)^{(n/2)})
\]

\[
= \sum_{k=0}^{[n/2]} \sum_{\ell=0}^{(n+1)/2} (-1)^{k+\ell} \binom{[n/2]}{k} \binom{((n+1)/2)}{\ell} \left( \frac{2\ell + n}{n} \right) \left( \frac{2k + 2\ell + [n/2]}{[n/2]} \right) x^{2(k+\ell)}
\]
for \( n \in \mathbb{N} \). Since \( L_{2n}(x) \) is a polynomial in \( x^2 \), one can put \( L_{2n}(x) \equiv A_n(x^2) \), so that \( A_n(z) \) is a polynomial of degree \([n/2] + [(n+1)/2] = n\) with integral coefficients. Concerning the orthogonality of \( A_n(x^2) \), we have

**Lemma 1.5.** For \( 0 \leq j < [n/2] \),

\[
\int_0^1 x^j A_n(x^2) \, dx = 0 = \int_0^1 x^j A_n(x^2) \log \left( 1 + \frac{1}{x} \right) \, dx.
\]

Furthermore

\[
\int_0^1 A_1(x^2) \, dx = 0 \quad \text{and} \quad \int_0^1 A_n(x^2) \log x \, dx = 0
\]

for every odd integer \( n \geq 3 \).

**Proof.** Obviously \( \int_0^1 x^j A_n(x^2) \, dx = 0 \) for \( 0 \leq j < [n/2] \), and \( \int_0^1 A_1(x^2) \, dx = 0 \). By \([n/2]\)-fold partial integration,

\[
\int_0^1 x^j A_n(x^2) \log \left( 1 + \frac{1}{x} \right) \, dx = \text{const.} \int_0^1 U_{j,[n/2]}(x) (1-x)^{[n/2]} \left( x^n(1-x^2)^{(n+1)/2} \right)^{(n)} \, dx = 0
\]

for \( 0 \leq j < [n/2] \), since \( \deg(U_{j,[n/2]}) = [n/2] - 1 \). Moreover, if \( n \geq 3 \) is odd,

\[
\int_0^1 A_n(x^2) \log x \, dx = \text{const.} \int_0^1 (1-x^2)^{[n/2]} \left( x^n(1-x^2)^{(n+1)/2} \right)^{(n)} \, dx = 0,
\]

since \( 2[n/2] < n \).

Therefore our polynomial \( A_n(x^2) \) gives a solution to (0.4) for \( \lambda = 1/4 \).

**Lemma 1.6.** All zero points of \( A_n(z) \) are simple and lie on \((0,1)\).

**Proof.** Since \( A_1(z) = 1 - 3z \), we can assume that \( n \geq 2 \). It follows from Lemma 1.4 that \( \nu_K(L_{2n}) \geq 2[n/2] = n \) if \( n \) is even and that \( \nu_K(L_{2n}) \geq 2[n/2] + 1 = n \) if \( n \) is odd. Therefore \( n \geq \nu_K(A_n) = \nu_K(L_{2n}) \geq n \); hence \( \nu_K(A_n) = n \). This completes the proof.

**Lemma 1.7.** For an arbitrarily fixed \( z \in \mathbb{C} \) the sequence \( X_n \equiv A_n(z^2) \) satisfies the linear recurrence

\[
X_{n+1} = (\alpha_n z^2 + \beta_n) X_n + \sum_{k=1}^{5} \gamma_{k,n} X_{n-k}\tag{1.2}
\]

for \( n \geq 6 \), where \( \alpha_n, \beta_n \) and \( \gamma_{k,n} \) (\( 1 \leq k \leq 5 \)) are rational constants depending only on \( n \). Moreover, for an arbitrarily fixed \( z \in \mathbb{C} \setminus [-1,1] \), all the sequences

\[
I_n^\delta(z^2) = \int_0^1 \frac{A_n(x^2)}{x^2 - x^2} x^\delta \, dx \quad \text{and} \quad J_n^\delta(z^2) = \int_0^1 \frac{A_n(x^2)}{x^2 - x^2} x^\delta \log \left( 1 + \frac{1}{x} \right) \, dx
\]

(\( \delta = 0,1 \)) satisfy the same linear recurrence (1.2) for \( n \geq 6 \) as well.
Proof. We define $m_n = 4$ if $n$ is even and $m_n = 5$ if $n$ is odd. Put

$$
\tilde{L}(z) = \tilde{A}(z^2) = A_{n+1}(z^2) - (\alpha_n z^2 + \beta_n) A_n(z^2) - \sum_{k=1}^{m_n} \gamma_{k,n} A_{n-k}(z^2)
$$

for $n \geq 6$, where $\alpha_n$, $\beta_n$ and $\gamma_{k,n}$ $(1 \leq k \leq m_n)$ are rational numbers chosen so that the degree of $\tilde{A}(z^2)$ is less than $2(n-m_n)$. Then $\int_0^1 x^j \tilde{A}(x^2) \, dx = 0$ for $0 \leq j < m$, where

$$
m = \min \left\{ \left[ \frac{n+1}{2} \right], \left[ \frac{n}{2} \right] - 2, \left[ \frac{n-1}{2} \right], \ldots, \left[ \frac{n-m_n}{2} \right] \right\} = \left[ \frac{n-m_n}{2} \right].
$$

Similarly we have $\int_0^1 x^j \tilde{A}(x^2) \log(1+1/x) \, dx = 0$ for $0 \leq j < [(n-m_n)/2]$. Since $n - m_n$ is an even integer, it follows from Lemma 1.4 that

$$
\nu_K(\tilde{A}) = \nu_K(\tilde{L}) \geq 2 \left[ \frac{n-m_n}{2} \right] = n - m_n > \deg(\tilde{A});
$$

therefore $\tilde{A}(z)$ must be identically zero. Hence $X_n = A_n(z^2)$ satisfies (1.2), if we put $\gamma_{0,n} = 0$ for every even $n \geq 6$. Moreover, for any $z \in \mathbb{C} \setminus [-1,1]$, it is easily seen that

$$
I^\delta_n(z^2) - (\alpha_n z^2 + \beta_n) I^\delta_n(z^2) - \sum_{k=1}^{5} \gamma_{k,n} I^\delta_{n-k}(z^2) = -\alpha_n \int_0^1 x^\delta A_n(x^2) \, dx = 0
$$

for $\delta = 0, 1$, since $[n/2] \geq 3$. Similarly $X_n = J^\delta_n(z^2)$ satisfies (1.2). This completes the proof.

Finally, we need the following:

**Lemma 1.8.** Suppose that $g(x)$ is a real-valued integrable function on $(0,1)$ and satisfies

$$
\int_0^1 A_k(x^2)g(x) \, dx = 0
$$

for all $k \geq 2n$. Then $g(x) = V(x) + W(x) \log(1+1/x)$ almost everywhere for some polynomials $V(x)$ and $W(x)$ of degrees less than $n$.

*Proof.* For any vector $\mathbf{v} = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}) \in \mathbb{R}^{2n}$, we put

$$
\phi_{\mathbf{v}}(x) = \sum_{j=0}^{n-1} a_j x^j + \sum_{j=0}^{n-1} b_j x^j \log \left( 1 + \frac{1}{x} \right).
$$

Obviously $\phi_{\mathbf{v}} \in \mathcal{A}(0,1)$. We next define the linear mapping $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$
\Phi(\mathbf{v}) = \left( \int_0^1 \phi_{\mathbf{v}}(x) \, dx, \int_0^1 x \phi_{\mathbf{v}}(x) \, dx, \ldots, \int_0^1 x^{2(2n-1)} \phi_{\mathbf{v}}(x) \, dx \right).
$$

We first show that $\Phi$ is a homeomorphism. To see this, it suffices to show that $\Phi$ is one-to-one. Suppose, on the contrary, that $\Phi(\mathbf{v}) = 0$ for some $\mathbf{v} \neq 0$; that is,

$$
0 = \int_0^1 x^j \phi_{\mathbf{v}}(x) \, dx = \frac{1}{2} \int_0^1 t^j \phi_{\mathbf{v}}(\sqrt{t}) \, dt
$$

for $0 \leq j < 2n$. Since $\varphi(x) \equiv \phi_{\mathbf{v}}(\sqrt{x})/\sqrt{x} \in \mathcal{A}(0,1)$, we get $\nu_K(\phi_{\mathbf{v}}) = \nu_K(\varphi) \geq 2n$ by Corollary 1.3. Thus it follows from Lemma 1.2 that $\nu_K(\phi_{\mathbf{v}}^{(n)}) \geq \nu_K(\phi_{\mathbf{v}}) - n \geq n$.
Therefore, putting \( \tilde{U}(x) \equiv x^n(1+x)^n \phi^{(n)}(x) \), we get \( \nu_K(\tilde{U}) = \nu_K(\phi^{(n)}) \geq n \). Hence we have \( \tilde{U}(x) \equiv 0 \), since

\[
\tilde{U}(x) = \sum_{j=0}^{n-1} b_j x^n(1+x)^n \left( x^j \log \left( 1 + \frac{1}{x} \right) \right) (n) = \sum_{j=0}^{n-1} b_j U_{j,n}(x)
\]

is a polynomial of degree less than \( n \). Thus \( \phi_v(x) \) is some polynomial of degree less than \( n \); hence \( \phi_v(x) \equiv 0 \), since \( \nu_K(\phi_v) \geq 2n \). We thus have

\[ \phi_v(x) = V_0(x) + W_0(x) \log \left( 1 + \frac{1}{x} \right) \equiv 0 \]

for \( x \in (0, 1) \), where \( V_0(x) = \sum_{j=0}^{n-1} a_j x^j \) and \( W_0(x) = \sum_{j=0}^{n-1} b_j x^j \). Then it is easily seen that this occurs if and only if \( V_0(x) = W_0(x) \equiv 0 \); hence \( v = 0 \). This contradiction implies that \( \Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) is a homeomorphism, as required.

Since \( \Phi \) is a homeomorphism, there exists a unique vector \( w \in \mathbb{R}^{2n} \) such that

\[
\int_0^1 x^{2j} (g(x) - \phi_w(x)) \, dx = 0 \quad \text{for} \quad 0 \leq j < 2n.
\]

We now have

\[
\int_0^1 A_m(x^2) \phi_w(x) \, dx = 0 \quad \text{for all} \quad m \geq 2n
\]

by Lemma 1.5; hence

\[
\int_0^1 A_m(x^2) (g(x) - \phi_w(x)) \, dx = 0 \quad \text{for all} \quad m \geq 2n.
\]

Combining (1.3) and (1.4), we conclude that

\[
\int_0^1 x^{2j} (g(x) - \phi_w(x)) \, dx = 0 \quad \text{for all} \quad j \geq 0.
\]

Then it can be seen that \( g(x) = \phi_w(x) \) almost everywhere. This completes the proof. \( \Box \)

2. Arithmetical properties of the coefficients

For any \( n \in \mathbb{N} \) and \( \delta = 0,1 \), we define

\[
B_n^\delta(z^2) = \int_0^1 \frac{A_n(z^2) - A_n(x^2)}{x^2 - z^2} x^\delta \, dx,
\]

\[
C_n^\delta(z^2) = \int_0^1 \frac{A_n(z^2) - A_n(x^2)}{x^2 - z^2} x^\delta \log \left( 1 + \frac{1}{x} \right) \, dx.
\]

Obviously \( B_n^\delta(w) \in \mathbb{Q}[w] \). Since it can be seen that

\[
\int_0^1 x^{2k+1} \log \left( 1 + \frac{1}{x} \right) \, dx = \frac{1}{2(k+1)} \sum_{j=1}^{2k+1} \frac{(-1)^{j+1}}{j}
\]

and

\[
\int_0^1 x^{2k} \log \left( 1 + \frac{1}{x} \right) \, dx = \frac{1}{2k+1} \left( 2 \log 2 + \sum_{j=1}^{2k} \frac{(-1)^{j}}{j} \right)
\]
Lemma 2.1. For any positive integers \( j, n \) and any prime number \( p \) satisfying \( j \leq n \) and \( 2([n+1/2]) < p < 2j \), we have \( p | c_{j,n} \).

Proof. Suppose, on the contrary, that some prime number \( p \in (2([n+1/2]), 2j) \) is not a divisor of \( c_{j,n} \). Then there exist \( k \in [0, [n/2]] \) and \( \ell \in [0, ([n+1/2]) \) such that \( k + \ell = j \) and \( p \) is not a divisor of the binomial coefficient \( \binom{2\ell + n}{n} \). Hence we have

\[
\left[ \frac{2\ell + n}{p} \right] = \left[ \frac{2\ell}{p} \right] + \left[ \frac{n}{p} \right]
\]

since \( p > 2([n+1/2]) \geq \sqrt{2\ell + n} \). Put \( \omega = \{n/p\} \) and \( \eta = \{\ell/p\} \) for brevity, where \( \{x\} \) denotes the fractional part of \( x \). Then \( \omega + 2\eta = [2\eta] \). Since \( \ell \leq ((n+1)/2) < p/2 \), we have \( \omega = \ell/p < 1/2 \); hence \( \omega + 2\eta = [2\eta] = 0 \).

On the other hand, it follows that

\[
\ell = j - k \geq j - \left[ \frac{n}{2} \right] \geq \frac{p + 1}{2} - \left[ \frac{n}{2} \right] > \frac{p + 1}{2} - \frac{n}{2} - 1 = \frac{p - n - 1}{2} ;
\]

therefore \( \ell \geq (p - n - 1)/2 + 1/2 = (p - n)/2 \). Thus \( \ell/p \geq 1/2 - n/(2p) \). Since \( n \leq 2((n+1)/2) < p < 2j < 2n \), we have \( 1/2 < n/p < 1 \); hence \( \omega = n/p \) and \( \eta \geq 1/2 - \omega/2 \). Therefore \( \omega + 2\eta \geq 1 \). This contradiction completes the proof. 

Lemma 2.2. Let \( D_n \) be the least common multiple of \( 1, 2, \ldots, n \). Then, for any integers \( 1 \leq \ell < m \leq 2n \),

\[
\frac{1}{\ell m} \in \frac{Z}{D_nD_{2n}}.
\]

Proof. We distinguish two cases: (a) \( \ell \leq n \) and (b) \( \ell > n \). In case (a) the statement is clear. In case (b) we also have

\[
\frac{1}{\ell m} = \frac{1}{m - \ell} \left( \frac{1}{\ell} - \frac{1}{m} \right) \in \frac{Z}{D_nD_{2n}},
\]

since \( m - \ell < 2n - n = n \). 

Theorem 2.3. The polynomials $B^\delta_n(w)$ ($\delta = 0, 1$), $C^1_n(w)$, $C^\delta_n(w) - 2\log 2B^0_n(w)$ all belong to the set $\mathbb{Z}[w]/M_n$, where

$$M_n = \frac{D_nD_{2n}}{\prod_{p\text{ prime } n<p<2n} p}.$$

Proof. We first consider the polynomial

$$C^1_n(w) = \sum_{j=1}^{n} \sum_{r=1}^{j} \frac{2r-1}{2rs} \sum_{s=1}^{\frac{j}{r}} (-1)^{s+1} c_{j,n} w^{j-r}.$$ 

Since $1 \leq s < 2r \leq 2n$, it follows from Lemma 2.2 that $D_nD_{2n}C^1_n(w) \in \mathbb{Z}[w]$. Suppose now that some denominator $2rs$ has a prime factor $p \in (n, 2n)$. Note that $p > 2\left(\frac{n+1}{2}\right)$. We then have $s = p$, because $r \leq n < p$ and $s < 2n < 2p$. Hence $p = s \leq 2r - 1 \leq 2j - 1$; that is, $p < 2j$. Therefore we have $p | c_{j,n}$ by Lemma 2.1. This implies that $D_nD_{2n}C^1_n(w) \in \left(\prod_{p\text{ prime } n<p<2n} p\right) \mathbb{Z}[w]$, as required.

The similar argument can be applied to the polynomials $B^\delta_n(w) = \sum_{j=1}^{n} \frac{c_{j,n} w^{j-r}}{2r - 1 + \delta} \quad (\delta = 0, 1)$ and

$$C^0_n(w) - 2\log 2B^0_n(w) = \sum_{j=1}^{n} \sum_{r=1}^{j} \frac{2r-2}{(2r-1)s} \sum_{s=1}^{\frac{j-r}{s}} (-1)^{s} c_{j,n} w^{j-r}.$$ 

\[\square\]

3. Simultaneous rational approximations

We recall that

$$I^\delta_n(z^2) = \int_0^1 \frac{A_n(x^2)}{z^2 - x^2} x^\delta \, dx \quad \text{and} \quad J^\delta_n(z^2) = \int_0^1 \frac{A_n(x^2) \log \left(1 + \frac{1}{x}\right)}{z^2 - x^2} dx$$

for any $n \in \mathbb{N}$, $z \in \mathbb{C}\setminus[-1,1]$ and $\delta = 0, 1$. Taking $z = \sqrt{k}$ for any integer $k \geq 2$, it is easily seen that

$$A_n(k) \sqrt{k} \log \frac{\sqrt{k} + 1}{\sqrt{k} - 1} - 2kB^0_n(k) = 2kf^0_n(k),$$

$$A_n(k) \log \left(1 - \frac{1}{k}\right) + 2B^1_n(k) = -2f^1_n(k),$$

$$A_n(k) \log \left(1 + \frac{1}{k}\right) \log \left(1 - \frac{1}{\sqrt{k}}\right) + 2C^1_n(k) = -2J^1_n(k).$$

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and
\[ A_n(k) = \sqrt{k} \Lambda \left( \frac{1}{\sqrt{k}} \right) - 4k \left( C_n^0(k) - 2 \log 2B_n^0(k) \right) = 4k \left( J_n^0(k) - 2 \log 2I_n^0(k) \right), \]
where
\[
\Lambda(x) = \sum_{r=1}^{\infty} \frac{2}{2r+1} \left( \sum_{s=1}^{2r} \frac{(-1)^s}{s} \right) x^{2r+1} = - \int_{-x}^{x} \frac{\log(1-t)}{1+t} \, dt.
\]
Let \( \text{Li}_2(x) = \sum_{r=1}^{\infty} x^r/r^2 \) be the dilogarithm. (For the dilogarithm see Lewin’s book [8, Chapter 1].) Then it can be seen that
\[ \Lambda(x) = \text{Li}_2 \left( \frac{1+x}{2} \right) - \text{Li}_2 \left( \frac{1-x}{2} \right) - \log 2 \log \frac{1+x}{1-x}. \]
Thus (3.1)-(3.4) give a system of simultaneous rational approximations to the numbers including \( \log(1 + 1/\sqrt{k}) \log(1 - 1/\sqrt{k}) \).

We now give an upper bound of \(|A_n(k)|\). It follows that
\[ A_n(k) = L_{2n} \left( \frac{1}{\sqrt{k}} \right) \left( \frac{1}{2\pi i} \right)^{n} \int_{C_0} \int_{C_1} \frac{\zeta \left( 1 - \zeta^2 \right)^{[n/2]} \left( w - \zeta \right)^{[(n+1)/2]} \log(1-w) \, dw \, d\zeta, \]
where \( C_0 \) and \( C_1 \) are the circles centered at \( \zeta = \sqrt{k} \) and \( w = \zeta \) with radii \( \sqrt{k}/4 \) and \( 5\sqrt{k}/4 \) respectively. Then it can be seen that
\[ |A_n(k)| \leq \left( \frac{125k}{16} - 5 \right)^{[n/2]} \frac{2^n \left( \frac{25k}{4} - 1 \right)^{[n+1/2]} \log(1)}{4^n}. \]

We next consider the remainder term \( J_n^0(k) \). It follows from \([n/2]-fold \) partial integration that
\[ J_n^0(k) = (-1)^{[n/2]} \int_{0}^{1} \left( x(1-x^2) \right)^{[n/2]} \frac{1}{n!} \left( x^n (1-x^2)^{(n+1)/2} \right)^{\binom{n}{k}} E_{k,n}^\delta(x) \, dx, \]
where
\[ E_{k,n}^\delta(x) = \frac{1}{[n/2]!} \left( \frac{x^\delta}{k-x^2} \log \left( \frac{1+x}{x} \right) \right)^{[n/2]} \].

Then one has
\[ E_{k,n}^\delta(x) = \frac{1}{2\pi i} \int_{C_2} \frac{1}{(\zeta-x)^{[n/2]+1}} \cdot \frac{\zeta^\delta}{k-\zeta^2} \log \left( \frac{1+\zeta}{\zeta} \right) \, d\zeta \]
\[ = - \frac{1}{2\pi i} \int_{C_2} G_{k,n}^\delta(x, \zeta) \, d\zeta, \quad \text{say,} \]
where \( C_2 \) is a small circle in \( \mathbb{C} \setminus [-1, 0] \) centered at \( x \). Since \( G_{k,n}^\delta(x, \zeta) = O(|\zeta|^{-2}) \) when \( |\zeta| \) is sufficiently large, the contour \( C_2 \) can be changed to the same curve \( C' \) as in the proof of Lemma 1.1 by taking account of the residues of \( G_{k,n}^\delta(x, \zeta) \) at the poles \( \zeta = \pm \sqrt{k} \). Therefore we get
\[ E_{k,n}^\delta(x) = \text{Res}_{\zeta = \sqrt{k}} G_{k,n}^\delta(x, \zeta) + \text{Res}_{\zeta = -\sqrt{k}} G_{k,n}^\delta(x, \zeta) - \frac{1}{2\pi i} \int_{C'} G_{k,n}^\delta(x, \zeta) \, d\zeta. \]
Then it can be seen that
\[ \text{Res}_{\zeta = \pm \sqrt{k}} G_{k,n}^\delta(x, \zeta) = \frac{(-1)^{\kappa} \gamma}{2} k^{(\delta-1)/2} \log \left( \frac{1 \pm 1}{\sqrt{k}} \right) \frac{1}{(\sqrt{k} + x)^{[n/2]+1}} \]
respectively, where \( \kappa^+ = 0 \) and \( \kappa^- = \delta + \lfloor n/2 \rfloor \), and that
\[
- \frac{1}{2\pi i} \int_{C_\epsilon} G_{k,n}(x, \zeta) \, d\zeta = (-1)^\kappa^- \int_0^1 \frac{1}{(t+|x|^{n/2})^{1+\epsilon}} \cdot \frac{t^\delta}{\sqrt{k-t^2}} \, dt.
\]
So it would be convenient to introduce the following:
\[
|\varepsilon_n^\pm(k) = \int_0^1 \frac{1}{n!} \left(x^n(1-x^2)^{(n+1)/2}\right)^{(n)} \frac{(x(1-x^2))^{[n/2]}((\sqrt{k}+|x|^{n/2})+1)}{t^\delta dt} \quad \text{for } \delta = 0, 1.
\]

Similarly it is easily seen that
\[
\mu_n^\delta(k) = \int_0^1 \frac{1}{n!} \left(x^n(1-x^2)^{(n+1)/2}\right)^{(n)} \frac{(x(1-x^2))^{[n/2]}((\sqrt{k}+|x|^{n/2})+1)}{t^\delta dt} \quad \text{for } \delta = 0, 1.
\]

Upper estimates of the terms \( \varepsilon_n^\pm(k) \) and \( \mu_n^\delta(k) \) will be discussed in Sections 4 and 5, respectively.

4. UPPER ESTIMATES OF \( |\varepsilon_n^\pm(k)| \)

It follows from \( n \)-fold partial integration that
\[
\varepsilon_n^\pm(k) = (-1)^n \int_0^1 x^n(1-x^2)^{(n+1)/2} \left( \frac{1}{n!} \left(x^n(1-x^2)^{(n+1)/2}\right)^{(n)} \frac{(x(1-x^2))^{[n/2]}((\sqrt{k}+|x|^{n/2})+1)}{t^\delta dt} \right) \, dx
\]

where \( C_x \) is the unit circle centered at \( x \). Then, for any integer \( k \geq 9 \), we have
\[
|\varepsilon_n^\pm(k)| \leq \int_0^1 \frac{x^n(1-x^2)^{(n+1)/2}}{((\sqrt{k}+1)\pm x)^{n/2+1}} \left( \Omega(x) \right)^{[n/2]} \, dx \leq \left( \max_{0 \leq x \leq 1} \frac{x^2(1-x^2)\Omega(x)}{\sqrt{k} - 1 - x} \right)^{n/2},
\]
where \( \Omega(x) = \max_{x \in C_x} |\zeta(1-\zeta^2)|. \) Note that \( \Omega(x) \geq 2 \) for \( x \in [0,1] \). We need a slightly sharp upper estimate of \( \Omega(x) \) as follows:

**Lemma 4.1.** Let \( \rho = (\sqrt[3]{19} + 3\sqrt[3]{33} + \sqrt[3]{19} - 3\sqrt[3]{33} - 2)/3 = 0.83928... \) (\( \rho \) is the unique root in \( (0,1) \) of \( x^3 + 2x^2 - 2 = 0 \)). Then
\[
\Omega(x) \leq \begin{cases} 
(1+x)(2-x^2) & \text{for } 0 \leq x \leq \rho, \\
x(1+x)(2+x) & \text{for } \rho < x \leq 1.
\end{cases}
\]
Since $x^2 = x + e^{i\theta}$, we have $|1 - \zeta^2|^2 = Y(\cos \theta)$, where $Y(w) = 4(x^2 - 1)w^2 + 4x^3w + x^4 + 4$; hence

$$\Omega(x) \leq (1 + x) \sqrt{\max_{|t| \leq 1} Y(t)}.$$  

The maximum of $Y(t)$ as $t$ varies in $[-1, 1]$ is attained at $t = x^3/(2(1 - x^2))$ or $t = 1$ according as $x \in [0, \rho]$ or $x \in (\rho, 1]$ respectively. This completes the proof. \hfill \Box

It thus follows from (4.1) and Lemma 4.1 that

$$|\xi_n^\delta(k)|^{2/n} \leq \max \left\{ \max_{0 \leq x \leq \rho} \frac{x^2(1 - x^2)\Omega(x)}{\sqrt{k - 1 - x}}, \max_{0 \leq x \leq 1} \frac{x^2(1 - x^2)\Omega(x)}{\sqrt{k - 1 - x}} \right\} \leq \max \left\{ \frac{1}{\sqrt{k - 1 - \rho}} \max_{0 \leq x \leq \rho} x^2 \sqrt{1 - x^2}(1 + x)(2 - x^2), \frac{1}{\sqrt{k - 2}} \max_{0 \leq x \leq 1} x^3(1 - x^2)(1 + x)(2 + x) \right\}. \tag{4.2}$$

Since

$$\max_{0 \leq x \leq \rho} x^2 \sqrt{1 - x^2}(1 + x)(2 - x^2)$$

$$< \max_{0 \leq u \leq 1} u \sqrt{1 - u}(2 - u) + \max_{0 \leq v \leq 1} v^{3/2} \sqrt{1 - v}(2 - v)$$

$$= \frac{4}{5^{5/4}} + \frac{2^{7/2}}{27} < 0.9541$$

and

$$\max_{0 \leq x \leq 1} x^3(1 - x^2)(1 + x)(2 + x)$$

$$< 2 \max_{0 \leq u \leq 1} u^{3/2}(1 - u) + 3 \max_{0 \leq v \leq 1} v^2(1 - v) + \max_{0 \leq w \leq 1} w^{5/2}(1 - w)$$

$$= 2\rho^2(1 - \rho^2) + 3\rho^3(1 - \rho^2) + \frac{2 \cdot 5^{5/2}}{7^{7/2}} < 0.9128,$$

it follows from (3.8) and (4.2) that

$$|I_n^\delta(k)| \leq \max \{ |\xi_n^+ (k)|, |\xi_n^- (k)| \} < \left( \frac{0.9541}{\sqrt{k - 1 - \rho}} \right)^{n/2} \tag{4.3}$$

for $\delta = 0, 1$ and for any integer $k \geq 31$.

5. Upper estimates of $|\mu_n^\delta(k)|$

We use the same nice transformation $\tau = \tau_{1/\varepsilon}(t)$ as in the proof of Lemma 1.1. Substituting $\tau = \tau_{1/\varepsilon}(t)$, we get

$$\int_0^1 \frac{1}{(t + x)^{n/2} + 1} \cdot \frac{t^\delta}{k - t^2} dt = x^{\delta - [n/2]}(1 + x)^{-[n/2]} \int_0^1 \frac{(1 - \tau)^\delta (\tau + x)^{[n/2] + 1 - \delta}}{k(\tau + x)^2 - x^2(1 - \tau)^2} d\tau;$$

therefore, by $n$-fold partial integration,

$$\mu_n^\delta(k) = \int_0^1 \int_0^1 x^n(1 - x^2)^{([n/2] + 1)/2} \frac{1}{n!} \times \left( \frac{(1 - \tau)^\delta x^\delta (1 - x)^{[n/2]}(\tau + x)^{[n/2] + 1 - \delta}}{k(\tau + x)^2 - x^2(1 - \tau)^2} \right)^{(n)} d\tau dx.$$
The rational function in $x$ in the big parentheses in the right-hand side can be written in the form

$$P_0(x) + \frac{1}{2\sqrt{k}\tau} \left( \frac{A^{\delta}(1-A)^{[n/2]}(\tau + A)^{[n/2]+1-\delta}}{x-A} - \frac{B^{\delta}(1-B)^{[n/2]}(\tau + B)^{[n/2]+1-\delta}}{x-B} \right),$$

where $A = -\sqrt{k\tau}/(\sqrt{k} - 1 + \tau)$, $B = -\sqrt{k\tau}/(\sqrt{k} + 1 - \tau)$ and $P_0(x)$ is some polynomial in $x$ of degree less than $n$. Hence we have

$$\mu_n(k) = \frac{(-1)^{n+1}}{2} k^{(\delta-1)/2} \left( (-1)^{\delta} \xi_+^{\delta,n}(k) + (-1)^{[n/2]} \xi_-^{\delta,n}(k) \right) \quad (5.1)$$

where

$$\xi_+^{\delta,n}(k) = \int_0^1 \int_0^1 x^n (1-x^2)^{[(n+1)/2]} \tau^{[n/2]} (1-\tau)^{[n/2]+1-\delta} \left( \sqrt{k} \pm (1-\tau) \right)^{n-2[n/2]} \left( \sqrt{k\tau} + \sqrt{k} \pm (1-\tau) \right)^{[n/2]} \left( (\sqrt{k} \pm (1-\tau)x + \sqrt{k\tau}) \right)^{n+1} d\tau \, dx$$

and

$$\xi_-^{\delta,n}(k) = \int_0^1 \int_0^1 x^n (1-x^2)^{[(n+1)/2]} \tau^{[n/2]} (1-\tau)^{[n/2]+1-\delta} \left( \sqrt{k} \pm (1-\tau) \right)^{n-2[n/2]} \left( \sqrt{k\tau} + \sqrt{k} \pm (1-\tau) \right)^{[n/2]} \left( (\sqrt{k} \pm (1-\tau)x + \sqrt{k\tau}) \right)^{n+1} d\tau \, dx$$

respectively. We now use the transformations $T^\pm : [0,1] \to [0,1]$ defined by

$$T^\pm(\tau) = \frac{\sqrt{k\tau}}{\sqrt{k} \pm (1-\tau)} \quad (5.2)$$

respectively. Substituting $T = T^\pm(\tau)$, we get

$$\xi_+^{\delta,n}(k) = k^{(\delta-1)/2} \int_0^1 \int_0^1 x^n (1-x^2)^{[(n+1)/2]} \left( \frac{x+T}{(x+T)^{n+1}} \right)^n T^{[n/2]} (1-T)^{[n/2]+1-\delta} (1+T)^{[n/2]} \left( \sqrt{k+T} \right)^{n+2-\delta} dT \, dx;$$

hence

$$|\xi_+^{\delta,n}(k)| \leq \left( \max_{0 \leq x, T \leq 1} \frac{x^2 (1-x^2)}{(x+T)^x} \cdot \frac{T(1-T^2)}{\sqrt{k} + T} \right)^{[n/2]}$$

for any $k \geq 4$. Using the inequality $x + T \geq 2\sqrt{xy}$, the maximum in the big parentheses in the right-hand side is estimated above by

$$\frac{1}{4} \max_{0 \leq x \leq 1} x(1-x^2) \cdot \max_{0 \leq T \leq 1} \frac{1 - T^2}{\sqrt{k} + T} < \frac{1}{6\sqrt{3}(\sqrt{k} - 1)} < 0.1.$$

Thus it follows from (5.1) that

$$|\mu_n^{\delta}(k)| \leq \max \left\{ |\xi_+^{\delta,n}(k)|, |\xi_-^{\delta,n}(k)| \right\} < \left( \frac{0.1}{\sqrt{k} - 1} \right)^{[n/2]} ;$$

hence, from (3.7) and (4.3), we obtain

$$|J_n^{\delta}(k)| \leq \max \left\{ |\xi_+^{\delta,n}(k)|, |\xi_-^{\delta,n}(k)| \right\} + |\mu_n^{\delta}(k)| \leq 2 \left( \frac{0.9541}{\sqrt{k} - 1 - \rho} \right)^{[n/2]} \quad (5.3).$$
Remark 5.1. The transformations $T^{\pm}(\tau)$ defined in (5.2) are not nice transformations, since they are orientation-preserving homeomorphisms. However it is easily seen that both $T^{\pm} \circ \tau$ and $\tau \circ T^{\pm}$ belong to $T$ for any $c < 1$. In particular, $T^{\pm} \in T^2 = \{\tau_c \circ \tau_{c'}; c, c' < 1\}$. Note that $T^3 = \{\tau_c \circ \tau_{c'} \circ \tau_{c''}; c, c', c'' < 1\}$ coincides with $T$.

6. Main results

Although we gave the upper estimates for the remainder terms in the system of simultaneous rational approximations (3.1)-(3.4) in the previous sections, it seems difficult to give their exact asymptotic behaviors as $n$ tends to infinity. This is the reason why we need the various lemmas concerning our Legendre-type polynomials in Section 1. To derive linear independence measures from the system (3.1)-(3.4), we need the following lemma, which is a generalization of [5, Lemma 3.2].

Lemma 6.1. Let $M \in N$, and let $\gamma_1, \gamma_2, \ldots, \gamma_M$ be given real numbers. Let $d$ be a fixed positive number. For given sequences $\{q_n\}_{n \geq 1}$ and $\{p_{m,n}\}_{n \geq 1}$ in $Z + idZ$ satisfying $q_n \gamma_m - p_{m,n} = \varepsilon_{m,n}$ ($1 \leq m \leq M$), suppose that $q_n \neq 0$ for all $n \in N$, and

$$\limsup_{n \to \infty} \frac{1}{n} \log |q_n| \leq \sigma, \quad \max_{1 \leq m \leq M} \limsup_{n \to \infty} \frac{1}{n} \log |\varepsilon_{m,n}| \leq -\tau$$

for some positive numbers $\sigma, \tau$. Suppose further that there exists a positive integer $N$ satisfying

$$\sum_{j=0}^{N} |n_0 q_{n+j} + \sum_{m=1}^{M} n_m p_{m,n+j}| > 0$$

for all $n \in N$ and any $(n_0, n_1, \ldots, n_M) \in Z^{M+1}\{0,0,\ldots,0\}$. Then, for any $\varepsilon > 0$, there exists an effectively computable constant $H_0 \equiv H_0(\varepsilon)$ such that

$$\left|n_0 + \sum_{m=1}^{M} n_m \gamma_m\right| \geq H^{-\sigma/\tau - \varepsilon}$$

for any $(n_1, n_2, \ldots, n_M) \in Z^M$ with $H = \max_{1 \leq m \leq M} |n_m| \geq H_0$.

Proof. We put $\Theta(n_0, n_1, \ldots, n_M) = n_0 + \sum_{m=1}^{M} n_m \gamma_m$ for brevity. Then

$$q_0 \Theta(n_0, n_1, \ldots, n_M) = \left(n_0 q_0 + \sum_{m=1}^{M} n_m p_{m,n}\right) + \sum_{m=1}^{M} n_m \varepsilon_{m,n} \equiv S_n + \omega_n, \quad \text{say},$$

for all $n \in N$. The condition (6.1) implies that there exists an integer $r(n) \in [n, n + N]$ satisfying $S_{r(n)} \neq 0$ for all $n \in N$; hence $|S_{r(n)}| \geq \kappa_d$ for some constant $\kappa_d \in (0, 1]$ depending only on $d$, because $S_{r(n)} \in Z + idZ$.

For any $\varepsilon > 0$, we can define a sufficiently small $\varepsilon' \in (0, \tau)$ satisfying $\sigma/\tau + \varepsilon/2 > (\sigma + \varepsilon')/\tau + \varepsilon')$. Then there exists an integer $n^* \equiv n^*(\varepsilon)$ such that $|q_n| \leq e^{(\sigma + \varepsilon')n}$ and $|\varepsilon_{m,n}| \leq e^{-(\tau - \varepsilon')n}$ for $1 \leq m \leq M$ and any $n \geq n^*$. Let $H_0 \equiv H_0(\varepsilon)$ be the least positive integer satisfying

$$2H_0 M e^{-(\tau - \varepsilon)n^*} \geq \kappa_d \quad \text{and} \quad H_0^{\varepsilon/2} \geq \left(\frac{2M}{\kappa_d}\right)^{(\sigma + \tau)/(\tau - \varepsilon')} e^{(\sigma + \varepsilon')(N+1)}.$$
Then, for any \((n_1, n_2, \ldots, n_M) \in \mathbb{Z}^M\) with \(H = \max_{1 \leq m \leq M} |n_m| \geq H_0\), let \(\tilde{n}\) be the least positive integer satisfying \(2HM e^{-(\tau - \epsilon')\tilde{n}} < \kappa_d\). Obviously \(n^* < \tilde{n} \leq r(\tilde{n})\).

We now take \(n = r(\tilde{n})\) in (6.2). Then

\[
|\Theta(n_0, n_1, \ldots, n_M)| \geq \frac{|S_{r(\tilde{n})}| - |\varnothing_{r(\tilde{n})}|}{|q_{r(\tilde{n})}|} \geq \frac{\kappa_d - HMe^{-(\tau - \epsilon')\tilde{n}}}{e^{(\sigma + \epsilon')(\tilde{n} + N)}} \geq \frac{\kappa_d}{2e^{(\sigma + \epsilon')(\tilde{n} + N)}},
\]

Since \(2HM e^{-(\tau - \epsilon')\tilde{n}} \geq \kappa_d\), we get

\[
|\Theta(n_0, n_1, \ldots, n_M)| \geq \frac{\kappa_d}{2e^{(\sigma + \epsilon')(N+1)}} \cdot \left( \frac{\kappa_d}{2MH} \right)^{(\sigma + \epsilon')(\tau - \epsilon')}
\]

\[
\geq H_0^{\epsilon/2}H^{-(\sigma + \epsilon')(\tau - \epsilon')/2} > H^{-\sigma/\tau - \epsilon}.
\]

This completes the proof.\(\square\)

For an arbitrarily fixed integer \(k \geq 2\), we now put \(q_n = M_nA_n(k) \in \mathbb{Z}\). Then \(q_n \neq 0\) for all \(n \in \mathbb{N}\) by Lemma 1.6. Since \(\lim_{n \to \infty}(\log M_n)/n = 2\) by the prime number theorem, it follows from (3.6) that

\[
(6.3) \limsup_{n \to \infty} \frac{1}{n} \log |q_n| \leq \log \left( \frac{5^{5/2}k}{4} \right) + 2 = \sigma(k), \quad \text{say.}
\]

Putting \(p_{1+\delta,n} = (-1)^{\delta}2k^{1-\delta}M_nB_n^\delta(k)\) (\(\delta = 0, 1\)), \(p_{3,n} = -2M_nC_n^1(k)\) and \(p_{4,n} = 4kM_n(C_n^0(k) - 2 \log 2B_n^0(k))\), we have \(p_{m,n} \in \mathbb{Z}\) for \(1 \leq m \leq 4\) and \(n \in \mathbb{N}\) by Theorem 2.3. We also put \(\varepsilon_{1+\delta,n} = (-1)^{\delta}2k^{1-\delta}M_nI_n^\delta(k)\) (\(\delta = 0, 1\)), \(\varepsilon_{3,n} = -2M_nJ_n^1(k)\) and \(\varepsilon_{4,n} = 4kM_n(J_n^0(k) - 2 \log 2I_n^0(k))\). Then it follows from (4.3) and (5.3) that

\[
(6.4) \max_{1 \leq m \leq 4} \limsup_{n \to \infty} \frac{1}{n} \log |\varepsilon_{m,n}| \leq \frac{1}{2} \log \left( \frac{0.9541}{\sqrt{k-1-\rho}} \right) + 2 = -\tau(k), \quad \text{say,}
\]

for any integer \(k \geq 31\). We have \(\tau(k) > 0\) for any integer \(k \geq 2909\).

We next show that our sequences \(\{q_n\}\) and \(\{p_{m,n}\}\) (\(1 \leq m \leq 4\)) defined above satisfy the condition (6.1) for \(N = 5\). Suppose, on the contrary, that

\[
S_{r+j} = 0, q_{r+j} + \sum_{m=1}^{4} r_m p_{m,r+j} = 0 \quad (0 \leq j \leq 5)
\]

for some \(r \in \mathbb{N}\) and some \((r_0, r_1, \ldots, r_4) \in \mathbb{Z}_5^5 \setminus \{0, 0, \ldots, 0\}\). Since the sequence \(X_n = S_n/M_n\) can be expressed as a linear combination of \(A_n(k), I_n^0(k)\) and \(J_n^0(k)\) (\(\delta = 0, 1\)) with coefficients independent of \(n\), it follows from Lemma 1.7 that \(\{X_n\}\) also satisfies the recurrence (1.2). Therefore \(X_n = 0\); hence \(q_n \Theta(r_0, r_1, \ldots, r_4) = \omega_n\) for all \(n \geq r\). If \(k \geq 2909\), then \(\omega_n \to 0\) as \(n \to \infty\); thus we get \(\Theta(r_0, r_1, \ldots, r_4) = 0\) and \(\omega_n = 0\) for all \(n \geq r\). Since

\[
0 = \sum_{m=1}^{4} r_m \varepsilon_{m,n} = k(r_1 - 4r_4 \log 2)I_n^0(k) - r_2 I_n^1(k) + 2kr_4 J_n^0(k) - r_3 J_n^1(k)
\]

\[
= \int_0^1 \frac{k(r_1 - 4r_4 \log 2) - r_2 x + (2kr_4 - r_3 x) \log(1 + 1/x)}{k - x^2} A_n(x^2) dx
\]

for all \(n \geq r\), it follows from Lemma 1.8 that

\[
\frac{k(r_1 - 4r_4 \log 2) - r_2 x}{k - x^2} - U(x) + \left( \frac{2kr_4 - r_3 x}{k - x^2} - W(x) \right) \log \left( 1 + \frac{1}{x} \right) = 0
\]
for $0 < x < 1$ and for some polynomials $U(x)$ and $W(x)$. Therefore we have $k(r_1 - 4r_4 \log 2) - r_2 x \equiv (k - x^2)U(x)$ and $2kr_4 - r_3 x \equiv (k - x^2)W(x)$; hence $U(x) = W(x) \equiv 0$ and $r_1 = r_2 = r_3 = r_4 = 0$. We thus have $r_0 = 0$, because $0 = S_* = r_0q_r$. This contradiction implies that our sequences satisfy the condition (6.1), as required.

Thus Lemma 6.1 can be applied to the system (3.1)-(3.4), so that we have

**Theorem 6.2.** Let $k \geq 2909$ be an integer. Then, for any $\varepsilon > 0$, there exists an effectively computable constant $H_0 = H_0(\varepsilon, k)$ such that

$$
|n_0 + n_1 \sqrt{k} \log \sqrt{k} + 1 + n_2 \log \left(1 - \frac{1}{k}\right) + n_3 \log \left(1 + \frac{1}{\sqrt{k}}\right) \log \left(1 - \frac{1}{\sqrt{k}}\right) + n_4 \sqrt{k} \Lambda \left(\frac{1}{\sqrt{k}}\right)| \geq H^{-\sigma(k)/\tau(k) - \varepsilon}
$$

for any $(n_0, n_1, ..., n_4) \in \mathbb{Z}^5$ satisfying $H = \max_{1 \leq m \leq 4} |n_m| \geq H_0$. (For the definitions of $\sigma(k), \tau(k)$ and the function $\Lambda(x)$, see (6.3), (6.4) and (3.5) respectively.)

In particular, taking $k = q^2$, we have

**Corollary 6.3.** Let $q \geq 54$ be an integer. Then, for any $\varepsilon > 0$, there exists an effectively computable constant $H_1 = H_1(\varepsilon, q)$ such that

$$
|n_0 + n_1 \log \left(1 + \frac{1}{q}\right) + n_2 \log \left(1 - \frac{1}{q}\right) + n_3 \log \left(1 + \frac{1}{q}\right) \log \left(1 - \frac{1}{q}\right) + n_4 \Lambda \left(\frac{1}{q}\right)| \geq H^{-\sigma(q^2)/\tau(q^2) - \varepsilon}
$$

for any $(n_0, n_1, ..., n_4) \in \mathbb{Z}^5$ satisfying $H = \max_{1 \leq m \leq 4} |n_m| \geq H_1$.

Theorem 0.1 in the Introduction follows immediately from this corollary.

The linear independence measure $\sigma(k)/\tau(k)$ in Theorem 6.2 is fairly large when $k$ is not too large. For example, one has $\sigma(2909)/\tau(2909) = 349075.6...$. However it is easily seen that $\sigma(k)/\tau(k)$ tends to 4 as $k \to \infty$.

Linear independence results for other sets of four numbers will be obtained if we put $z = \sqrt{k}$ or $z = e^{\pi i/3} \sqrt{k}$ instead of $z = \sqrt{k}$. To such cases we will be able to apply Lemma 6.1 for $d = 1$ or $d = \sqrt{3}$ respectively.

**References**


IRRATIONALITY OF $\log(1 + 1/q) \log(1 - 1/q)$


Division of Mathematics, Faculty of Integrated Human Studies, Kyoto University, Kyoto 606-01, Japan

E-mail address: hata@i.h.kyoto-u.ac.jp