WEAK*-CLOSEDNESS OF SUBSPACES
OF FOURIER-STIELTJES ALGEBRAS
AND WEAK*-CONTINUITY OF THE RESTRICTION MAP

M. B. BEKKA, E. KANIUTH, A. T. LAU, AND G. SCHLICHTING

Dedicated to Professor Elmar Thoma on the occasion of his seventieth birthday

ABSTRACT. Let $G$ be a locally compact group and $B(G)$ the Fourier-Stieltjes algebra of $G$. We study the problem of how weak*-closedness of some translation invariant subspaces of $B(G)$ is related to the structure of $G$. Moreover, we prove that for a closed subgroup $H$ of $G$, the restriction map from $B(G)$ to $B(H)$ is weak*-continuous only when $H$ is open in $G$.

INTRODUCTION

Let $G$ be a locally compact group, and let $B(G)$ be the Fourier-Stieltjes algebra of $G$ as defined by Eymard [8]. Recall that $B(G)$ is the linear span of all continuous positive definite functions on $G$ and can be identified with the Banach space dual of $C^*(G)$, the group $C^*$-algebra of $G$. The space $B(G)$, with the norm as dual of $C^*(G)$, is a commutative Banach $*$-algebra with pointwise multiplication and complex conjugation. The Fourier algebra $A(G)$ of $G$ is the closed $*$-subalgebra of $B(G)$ generated by the functions in $B(G)$ with compact support. In particular, $A(G)$ is contained in $C_0(G)$, the algebra of complex valued continuous functions on $G$ vanishing at infinity. As is well known $A(G)$ is weak*-dense in $B(G)$ if and only if $G$ is amenable. In [3] translation invariant $*$-subalgebras $A$ of $B(G)$ were studied, and it was shown that if such $A$ is weak*-closed and point separating, then it must contain $A(G)$. However, apart from this, very little seems to be known about weak*-closed subspaces of $B(G)$.

The first purpose of this paper is to investigate the relation between weak*-closedness of certain interesting norm-closed translation invariant subspaces of $B(G)$ and the structure of $G$. Secondly, we solve the problem of when, for a closed subgroup $H$ of $G$, the restriction map from $B(G)$ to $B(H)$ is weak*-continuous.

A brief outline of the paper is as follows. In Section 2 we establish for almost connected locally compact groups $G$ the relation between weak*-closedness of $B_0(G) = B(G) \cap C_0(G)$ in $B(G)$ and the structure of $G$ (Theorem 2.10). The key result is that for a connected Lie group $G$, $B_0(G)$ is weak*-closed in $B(G)$ if and only if $G$ is a reductive Lie group with compact centre and Kazhdan’s property ($T$).

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If \( G \) is a compact group, then for any unitary representation \( \pi \) of \( G \), the Fourier space \( A_\pi(G) \) associated to \( \pi \) (see [1] for the definition and properties of \( A_\pi(G) \)) is weak*-closed in \( B(G) \). Note that \( A(G) = A_{\lambda_G}(G) \), where \( \lambda_G \) denotes the left regular representation of \( G \). In Theorem 3.6 we shall show that, conversely, if \( G \) contains an almost connected open normal subgroup and \( A(G) \) is weak*-closed in \( B(G) \), then \( G \) is compact. We also give a characterization of compactness of \( G \) in terms of the weak* and the norm topologies on the unit sphere of \( B(G) \) (Theorem 3.9).

Besides the left regular representation, one of the most interesting representations of a locally compact group \( G \) is the conjugation representation \( \gamma_G \) of \( G \) on \( L^2(G) \). In contrast to \( A(G) \), \( A_{\gamma_G}(G) \) need not be a subalgebra and it can at best determine the structure of \( G/Z(G) \), where \( Z(G) \) denotes the centre of \( G \). We prove that if \( G \) is a Lie group with countably many connected components and \( A_{\gamma_G}(G) \) is weak*-closed in \( B(G) \), then \( G/Z(G) \) is compact (Theorem 4.8).

Let \( H \) be a closed subgroup of an arbitrary locally compact group \( G \). Clearly, the restriction map \( B(G) \to B(H) \) is continuous for the weak*-topologies whenever \( H \) is open. In the final section 5 we succeed in showing that conversely weak*-continuity of the restriction map forces \( H \) to be open in \( G \).

1. Preliminaries

Throughout this paper, \( G \) denotes a locally compact group with a fixed left Haar measure \( dx \) and modular function \( \Delta \). \( L^1(G) \) the convolution algebra of integrable functions on \( G \) and \( C^*(G) \) the group \( C^* \)-algebra of \( G \). The Fourier-Stieltjes algebra \( B(G) \) is the Banach space dual of \( C^*(G) \) and as such carries the weak*-topology (\( w^* \)-topology, for short) \( \sigma(B(G), C^*(G)) \). The basic reference on Fourier and Fourier-Stieltjes algebras is [8].

Next, we have to introduce some notation from representation theory. We use the same letter, for example \( \pi \), for a unitary representation of \( G \) and for the corresponding \( \ast \)-representation of \( C^*(G) \). \( \mathcal{H}_\pi \) will always denote the Hilbert space of \( \pi \) and \( \ker \pi \) the \( C^* \)-kernel of \( \pi \). If \( S \) and \( T \) are sets of unitary representations of \( G \), then \( S \) is weakly contained in \( T \) (\( S \prec T \)) if \( \bigcap_{\pi \in S} \ker \pi \supseteq \bigcap_{\tau \in T} \ker \tau \) or, equivalently, if every positive definite function associated to \( S \) can be uniformly approximated on compact subsets of \( G \) by sums of positive definite functions associated to \( T \). Also \( S \) and \( T \) are weakly equivalent (\( S \sim T \)) if \( S \prec T \) and \( T \prec S \).

The dual space \( \hat{G} \) of \( G \) is the set of equivalence classes of irreducible representations of \( G \), endowed with the Jacobson topology. For a representation \( \pi \) of \( G \), the support of \( \pi \) is the closed subset \( \text{supp} \, \pi = \{ \rho \in \hat{G} : \rho \prec \pi \} \) of \( \hat{G} \). In particular, the support of the left regular representation \( \lambda_G \) is the reduced dual \( 
abla \tau \), and \( \lambda_G(C^*(G)) \) is the so-called reduced group \( C^* \)-algebra of \( G \) which is denoted by \( \mathcal{C}_r^*(G) \). If \( N \) is a closed normal subgroup of \( G \), then every representation of \( G/N \) can be lifted to a representation of \( G \), and in this sense, \( (G/N)^\sim \) will always be regarded as a subset of \( \hat{G} \). For general references to representation theory and dual spaces we mention [5] and [10].

\( G \) is called amenable if there exists a non-zero positive linear functional \( m \) on the space \( C_0(G) \) of bounded continuous complex valued functions on \( G \) such that \( m(\varphi f) = m(f) \) for all \( \varphi \in C_0(G) \) and \( f \in G \), where \( \varphi f(y) = f(x^{-1}y) \). Recall that amenability is equivalent to a number of different conditions: \( \mathcal{C}_r^*(G) = C^*(G) \),
Let $G$ be a closed subgroup of $G$, and suppose that $\sigma$ and $\pi$ are representations of $H$ and $G$ respectively. The representation of $G$ induced by $\sigma$ is denoted $\text{ind}_H^G(\sigma)$. Then the tensor product $\pi \otimes \text{ind}_H^G(\sigma)$ is equivalent to $\text{ind}_H^G(\pi[H \otimes \sigma])$. Furthermore, by the theorem on induction in stages, $\text{ind}_H^G(\sigma) = \text{ind}_K^G(\text{ind}_H^K(\sigma))$ for every closed subgroup $K$ of $G$ containing $H$. Finally, we will frequently use that $\pi \prec \text{ind}_H^G(\pi[H])$ if $G$ is amenable [12, Theorem 5.1].

Let $A$ be a $C^*$-algebra and $\hat{A}$ its dual space, i.e. the set of equivalence classes of non-degenerated irreducible *-representations of $A$. We will several times use the fact that if the Banach space dual $A^*$ of $A$ is separable in the norm topology, then $\hat{A}$ is countable (see [15, Theorem 3.1] or [23, Lemma 4.12]).

2. When is $B_0(G)$ $w^*$-closed in $B(G)$?

For a locally compact group $G$, let $B_0(G) = B(G) \cap C_0(G)$ denote the norm-closed and translation invariant subalgebra of $B(G)$ consisting of all functions in $B(G)$ that vanish at infinity. In this section we shall study the problem of when $B_0(G)$ is closed in $B(G)$ with respect to the $w^*$-topology on $B(G)$.

It turns out to be appropriate to reformulate this condition in terms of convergence of positive definite functions. The following lemma, which is a consequence of [8, (2.1) and (2.12)], will be used frequently.

Lemma 2.1. Let $P(G)$ denote the set of all normalized continuous positive definite functions on $G$. The following are equivalent:

(i) $B_0(G)$ is $w^*$-closed in $B(G)$.

(ii) If $(\varphi_\lambda)$ is a net in $P(G) \cap C_0(G)$ converging to some $\varphi \in P(G)$ uniformly on compact subsets of $G$, then $\varphi \in C_0(G)$.

Lemma 2.2. Suppose that $B_0(G)$ is $w^*$-closed in $B(G)$. Then every amenable closed normal subgroup of $G$ is compact. In particular, for an amenable group $G$, $B_0(G)$ is $w^*$-closed in $B(G)$ only when $G$ is compact.

Proof. Let $N$ be an amenable closed normal subgroup of $G$. Then the trivial one-dimensional representation $1_N$ of $N$ is weakly contained in the left regular representation $\lambda_N$ of $N$. Hence, by continuity of inducing, $\text{ind}_N^G(1_N)$ is weakly contained in $\text{ind}_N^G(\lambda_N)$, which is equivalent to $\lambda_G$.

Let $q : G \to G/N$ denote the quotient homomorphism. Then $\text{ind}_N^G(1_N) = \lambda_{G/N} \circ q$. Thus every positive definite function $\varphi$ associated to $\lambda_{G/N} \circ q$ is a uniform on compacta limit of functions in $A(G) \cap P(G) \subseteq B_0(G)$. Hence, by hypothesis, $\varphi \in C_0(G)$. Since such a $\varphi$ is constant on cosets of $N$, $N$ must be compact.

We continue with two inheritance properties.

Lemma 2.3. Let $H$ be an open subgroup of the locally compact group $G$. If $B_0(G)$ is $w^*$-closed in $B(G)$, then $B_0(H)$ is $w^*$-closed in $B(H)$.

Proof. Let $(\varphi_\lambda)$ be a net in $P(H) \cap C_0(H)$ such that $\varphi_\lambda \to \varphi \in P(H)$ uniformly on compact subsets of $H$. Let $\tilde{\varphi}_\lambda$ and $\hat{\varphi}$ denote the trivial extensions of $\varphi_\lambda$ and $\varphi$ to $G$, that is, $\tilde{\varphi}_\lambda(x) = \varphi_\lambda(x) = 0$ for $x \in G \setminus H$. Clearly, then $\tilde{\varphi}_\lambda \in P(G) \cap C_0(G)$, $\hat{\varphi} \in P(G)$ and $\tilde{\varphi}_\lambda \to \hat{\varphi}$ uniformly on compact subsets of $G$. By hypothesis, $\hat{\varphi} \in C_0(G)$ and hence $\varphi \in C_0(H)$. 

\[ \hat{G}_r = \hat{G} \text{ or } 1_G \prec \lambda_G \] where $1_G$ is the trivial one-dimensional representation of $G$. Concerning the theory of amenable groups, we refer the reader to [29] and [30].
Lemma 2.4. Let $G$ be a locally compact group and $K$ a compact normal subgroup of $G$. Then $B_0(G)$ is $w^*$-closed in $B(G)$ if and only if $B_0(G/K)$ is $w^*$-closed in $B(G/K)$.

Proof. Suppose that $B_0(G)$ is $w^*$-closed in $B(G)$, and let $(\varphi_n)$ be a net in $P(G/K) \cap C_0(G/K)$ converging to some $\varphi \in P(G/K)$ uniformly on compact subsets of $G/K$. Then, with $q : G \to G/K$ the quotient homomorphism, $\varphi_n \circ q \to \varphi \circ q$ uniformly on compact subsets of $G$ and $\varphi_n \circ q \in C_0(G)$ since $K$ is compact. Hence, by hypothesis, $\varphi \circ q \in C_0(G/K)$.

Conversely, suppose that $B_0(G/K)$ is $w^*$-closed in $B(G/K)$, and let $\varphi \in P(G)$ and $(\varphi_n) \subseteq P(G) \cap C_0(G)$ such that $\varphi_n \to \varphi$ uniformly on compact subsets of $G$. Define $\psi_n$ and $\psi$ on $G/K$ by

$$\psi_n(xK) = \int_K |\varphi_n(xk)|^2 dk \quad \text{and} \quad \psi(xK) = \int_K |\varphi(xk)|^2 dk,$$

$x \in G$ ($dk$ being the normalized Haar measure on $K$). Then

$$\psi \in P(G/K) \quad \text{and} \quad \psi_n \in P(G/K) \cap C_0(G/K),$$

and $\psi_n \to \psi$ uniformly on compact subsets of $G/K$. Hence $\psi \in C_0(G/K)$.

For $\delta \in \hat{K}$, let $\chi_\delta$ denote the corresponding minimal idempotent in $L^1(K)$. Then by the Cauchy-Schwarz inequality,

$$|\varphi * \chi_\delta(x)| \leq |\psi(xK)|^{1/2}$$

for every $x \in G$. Since $K$ is compact, this implies $\varphi * \chi_\delta \in C_0(G)$ for each $\delta \in \hat{K}$. Now the linear span of $\{\chi_\delta : \delta \in \hat{K}\}$ is dense in $Z(L^1(K))$, the centre of $L^1(K)$. It follows that for any $f \in Z(L^1(K))$, $\varphi * f$ is a uniform limit on $G$ of finite linear combinations of functions $\varphi * \chi_\delta$, $\delta \in \hat{K}$. Hence $\varphi * f \in C_0(G)$ for every $f \in Z(L^1(K))$. Finally, taking for $f$ functions in $Z(L^1(K))$ with support shrinking to $\{e\}$, we easily conclude that $\varphi \in C_0(G)$. This completes the proof.

We now turn to connected Lie groups. Theorem 2.7 below is the key result in this section.

Lemma 2.5. Let $G$ be a connected Lie group and $N$ a connected closed normal subgroup. If $B_0(G)$ is $w^*$-closed in $B(G)$, then the centre $Z(N)$ of $N$ is compact and $N/Z(N)$ is semisimple.

Proof. Let $R$ denote the radical of $N$. Then $R$ and $Z(N)$ are amenable normal subgroups of $G$, and therefore both must be compact by Lemma 2.2. Since $R$ is solvable and connected Lie, it is isomorphic to a torus $T^n$. Hence $\text{Aut}(R)$, the automorphism group of $R$, is discrete. Now, $G$ acts by conjugation on $R$, and this defines a continuous homomorphism from $G$ into $\text{Aut}(R)$. $G$ being connected, this homomorphism has to be trivial. This shows that $R$ is contained in the centre of $G$. So $R \subseteq Z(N)$ and hence $N/Z(N)$ is semisimple.

We remind the reader that a locally compact group $G$ is said to have Kazhdan’s property $(T)$ if the trivial representation $1_G$ is an isolated point in the dual space $\hat{G}$. An amenable group satisfies $(T)$ if and only if it is compact. On the other hand, many connected semisimple Lie groups and many discrete groups share property $(T)$. A comprehensive account on groups with property $(T)$ has been given in [14].
Lemma 2.6. Let $G$ be a connected Lie group such that $B_0(G)$ is $w^*$-closed in $B(G)$. Then $G$ has property $(T)$.

Proof. By Lemma 2.5, $G$ is reductive with compact centre. Let

$$\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$$

be the universal covering group of $G$, where $G_1, \ldots, G_m$ are simply connected Lie groups. Denote by $Z_i$ the (discrete) centre of $G_i$, $i = 1, \ldots, m$. Then $G = \tilde{G}/\Gamma$ for some discrete subgroup $\Gamma$ of the centre $Z(\tilde{G}) = \mathbb{R}^n \times Z_1 \times \cdots \times Z_m$ of $\tilde{G}$. Moreover, $Z(G) = Z(\tilde{G})/\Gamma$. Hence

$$G/Z(G) = \tilde{G}/Z(\tilde{G}) = G_1/Z_1 \times \cdots \times G_m/Z_m.$$ 

Since $Z(G)$ is compact, $B_0(G/Z(G))$ is $w^*$-closed in $B(G/Z(G))$ (Lemma 2.4).

Assume, towards a contradiction, that $G$ does not have property $(T)$. Then $G/Z(G)$ does not have property $(T)$. Hence some factor, say $G_1/Z_1$, fails to have property $(T)$ (see [35, Lemma 7.4.1]). Now, recall the following result due to Howe and Moore [35, Theorem 2.2.20]. If $\pi$ is a unitary representation of a simple Lie group with finite centre and if there are no non-zero $\pi$-invariant vectors, then all the matrix coefficients of $\pi$ vanish at infinity. Therefore, there exists a sequence

$$(\varphi_n^{(1)}) \subseteq P(G_1/Z_1) \cap C_0(G_1/Z_1)$$

converging to 1 uniformly on compact subsets of $G_1/Z_1$. Observe that $G_1/Z_1$ is not compact. Now choose arbitrary

$$\varphi^{(k)} \in P(G_k/Z_k) \cap C_0(G_k/Z_k),$$

$k = 2, \ldots, m$, and set

$$\varphi_n = \varphi_n^{(1)} \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}.$$ 

Clearly, $(\varphi_n)$ is a sequence of continuous positive definite functions on $G/Z(G)$ that vanish at infinity, and

$$\varphi_n \to 1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}$$

uniformly on compact subsets of $G/Z(G)$. Since $1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}$ does not vanish at infinity, we have reached a contradiction. Thus $G$ has property $(T)$.

Theorem 2.7. Let $G$ be a connected Lie group. Then $B_0(G)$ is $w^*$-closed in $B(G)$ if and only if $G$ is a reductive Lie group with compact centre and Kazhdan’s property $(T)$.

Proof. From Lemma 2.5, applied to $N = G$, and Lemma 2.6 we know that $w^*$-closedness of $B_0(G)$ in $B(G)$ implies the stated conditions on $G$.

Suppose now that $G$ is a connected reductive Lie group with compact centre $Z(G)$ and property $(T)$. According to Lemma 2.4 it suffices to show that $B_0(G/Z(G))$ is $w^*$-closed in $B(G/Z(G))$. Recall from the proof of Lemma 2.6 that $G/Z(G)$ has a decomposition

$$G/Z(G) = G_1/Z_1 \times \cdots \times G_m/Z_m,$$

where $G_1, \ldots, G_m$ are connected simple Lie groups with centres $Z_1, \ldots, Z_m$, respectively. Set $H = G/Z(G)$ and $H_i = G_i/Z_i$ for $i = 1, \ldots, m$, and observe that every $H_i$ has property $(T)$. Of course, according to Lemma 2.4 we can assume that none of the $H_i$ is compact.
Let \( \varphi_n \in P(H) \cap C_0(H) \), \( n \in \mathbb{N} \), such that \( \varphi_n \to \varphi \) uniformly on compact subsets of \( G \) for some \( \varphi \in P(H) \). Let \( \pi_n \) and \( \pi \) denote the representations of \( H \) associated to \( \varphi_n \) and \( \varphi \) through the GNS-construction. Then \( \pi \) is weakly contained in the direct sum \( \bigoplus_{n=1}^{\infty} \pi_n \).

We claim that the restriction \( \pi|_{H_i} \) of \( \pi \) to \( H_i \) does not contain the trivial representation \( 1_{H_i} \). Indeed, otherwise for some \( n \in \mathbb{N} \), \( \pi_n|_{H_i} \) contains \( 1_{H_i} \), since \( H_i \) has property \( (T) \). However, since \( \varphi_n \in C_0(H) \), all the matrix coefficients of \( \pi_n \) vanish at infinity.

We have thus verified that \( H \) satisfies the hypotheses of the Howe-Moore theorem [35, Theorem 2.2.20]. It follows that all the matrix coefficients of \( \pi \) vanish at infinity. This proves that \( \varphi \in C_0(H) \).

In order to deal with almost connected groups we need one more lemma.

**Lemma 2.8.** Let \( G \) be a locally compact group and \( H \) a closed subgroup such that \( G/H \) is compact. If \( B_0(H) \) is \( w^* \)-closed in \( B(H) \), then \( B_0(G) \) is \( w^* \)-closed in \( B(G) \).

**Proof.** For any function \( \phi \) on \( G \) and \( x,y \in G \), let

\[
\varphi(y) = \phi(x^{-1}y), \phi_x(y) = \phi(yx) \text{ and } \varphi^x(y) = \phi(x^{-1}y).
\]

Notice first that if \( \psi \) is a positive definite function on \( G \), then \( \psi^x + \psi + x\psi + \psi_x \) and \( \psi^x + \psi + i(\psi - \psi_x) \) are also positive definite for every \( x \in G \). Indeed, for all \( f \in L^1(G) \)

\[
\langle \psi^x + \psi + x\psi + \psi_x, f \rangle = \langle \psi, (\delta_x + \delta_e) \ast f \ast (\delta_x + \delta_e) \rangle
\]

and

\[
\langle \psi^x + \psi + i(\psi - \psi_x), f \rangle = \langle \psi, (\delta_e - i\delta_x) \ast f \ast (\delta_e - i\delta_x) \rangle.
\]

Let \( \langle \varphi_n \rangle \) be a net in \( P(G) \cap C_0(G) \) converging to some \( \varphi \in P(G) \) uniformly on compact subsets of \( G \). Then, uniformly on compact subsets of \( G \), \( \varphi_n^x \to \varphi^x \),

\[
\varphi^x + \varphi + x(\varphi) + \langle \varphi_n \rangle - x(\varphi) \to \varphi^x + \varphi + x\varphi + \varphi
\]

and

\[
\varphi^x + \varphi + i(\varphi - \varphi_n) \to \varphi^x + \varphi + i(\varphi - \varphi_n)
\]

for every \( x \in G \). Thus, since \( B_0(H) \) is \( w^* \)-closed in \( B(H) \),

\[
\varphi|H, \varphi^x|H, (\varphi^x + \varphi + x\varphi + \varphi_x)|H \text{ and } (\varphi^x + \varphi + i(\varphi - \varphi_x))|H
\]

vanish at infinity on \( H \). It follows that \( \varphi|H \in C_0(H) \) for each \( x \in G \). Since \( G/H \) is compact, employing the uniform continuity of \( \varphi \), it is easily verified that \( \varphi \in C_0(G) \).

The converse to Lemma 2.8 does not hold in general. That is, if \( B_0(G) \) is \( w^* \)-closed in \( B(G) \) and \( H \) is a closed cocompact subgroup of \( G \), then \( B_0(H) \) need not be \( w^* \)-closed in \( B(H) \). As an example, take for \( G \) a simply connected Lie group with finite centre and property \( (T) \) and for \( H \) a minimal parabolic subgroup. Then by the Howe-Moore result referred to in the proofs of Lemma 2.6 and Theorem 2.7, \( B_0(G) \) is \( w^* \)-closed in \( B(G) \), while \( B_0(H) \) fails to be \( w^* \)-closed in \( B(H) \) since \( H \) is non-compact and amenable.

**Corollary 2.9.** Let \( G \) be a connected Lie group and \( N \) a connected closed normal subgroup of \( G \) such that \( G/N \) is compact. Then \( B_0(G) \) is \( w^* \)-closed in \( B(G) \) if and only if \( B_0(N) \) is \( w^* \)-closed in \( B(N) \).
Proof. Suppose that $B_0(G)$ is $w^*$-closed in $B(G)$. By Lemma 2.5, $Z(N)$ is compact and $N/Z(N)$ is semisimple. Also, since $G$ has property $(T)$ by Lemma 2.6 and $G/N$ is compact, $N$ has property $(T)$ [34, Theorem 3.7]. By Theorem 2.7 this implies that $B_0(N)$ is $w^*$-closed in $B(N)$.

The converse is a special case of Lemma 2.8.

Theorem 2.10. Let $G$ be an almost connected locally compact group. Then $B_0(G)$ is $w^*$-closed in $B(G)$ if and only if the connected component $G_0$ of $G$ is a projective limit of reductive Lie groups with property $(T)$ and compact centres.

Proof. Suppose first that $G_0$ has the indicated structure. Choose a compact normal subgroup $K$ of $G_0$ such that $G_0/K$ is a reductive group with property $(T)$ and compact centre. By Theorem 2.7, $B_0(G_0/K)$ is $w^*$-closed in $B(G_0/K)$. Since $K$ and $G/G_0$ are compact, an application of Lemmas 2.4 and 2.8 yields that $B_0(G)$ is $w^*$-closed in $B(G)$.

Conversely, suppose that $B_0(G)$ is $w^*$-closed in $B(G)$. $G$ being almost connected it is a projective limit of Lie groups $G/K_i$. Thus there are connected normal subgroups $H_i$ of finite index in $G$ such that $K_i \subseteq H_i$ and $H_i/K_i = (G/K_i)_0$. Then $G_0$ is the projective limit of the groups $G_0/G_0 \cap K_i$, and the $G_0/G_0 \cap K_i$ are connected Lie groups since

$$G_0/G_0 \cap K_i = G_0 K_i/K_i,$$

a closed connected subgroup of $G/K_i$. By Theorem 2.7 it suffices to show that $B_0(G_0/G_0 \cap K_i)$ is $w^*$-closed in $B(G_0/G_0 \cap K_i)$.

Now, since $B_0(G)$ is $w^*$-closed in $B(G)$, $B_0(G/K_i)$ is $w^*$-closed in $B(G/K_i)$ by Lemma 2.4, and hence $B_0(H_i/K_i)$ is $w^*$-closed in $B(H_i/K_i)$ by Lemma 2.3. Moreover, $G_0 K_i/K_i$ is a cocompact connected normal subgroup of the connected Lie group $H_i/K_i$. Thus, by Corollary 2.9, $B_0(G_0 K_i/K_i)$ is $w^*$-closed in $B(G_0 K_i/K_i)$. This proves that $B_0(G_0/G_0 \cap K_i)$ is $w^*$-closed in $B(G_0/G_0 \cap K_i)$.

We conclude this section with some remarks.

Remarks 2.11. (i) The connected reductive Lie groups with property $(T)$ and compact centres are precisely the groups of the form $G = (\mathbb{R}^n \times G_1 \times \cdots \times G_m)/\Gamma$, where $G_1, \ldots, G_m$ are simple Lie groups not locally isomorphic to $SO(k, 1)$, $k \geq 2$, or $SU(k, 1)$, $k \geq 1$, and $\Gamma$ is a discrete cocompact subgroup of $\mathbb{R}^n \times Z_1 \times \cdots \times Z_m$, the centre of $\mathbb{R}^n \times G_1 \times \cdots \times G_m$.

Indeed, if $G$ is of this form, then $G/Z(G) = G_1 \times \cdots \times G_m$, where $Z(G)$ denotes the centre of $G$, has property $(T)$ (see [14, Chap. 2, 13. Corollaire, 9. Remarque and Chap. 9]). As $Z(G)$ is compact, $G$ has property $(T)$ [14, Chap. 1, 9. Proposition].

Conversely, let $G$ be a connected reductive Lie group with property $(T)$ and compact centre. Let $\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$ be its universal covering group, where $G_1, \ldots, G_m$ are simple Lie groups with centres $Z_1, \ldots, Z_m$. The arguments used in the proof of Lemma 2.6 show that $G_1, \ldots, G_m$ have property $(T)$ and hence are not locally isomorphic to $SO(k, 1)$ or $SU(k, 1)$ (see [14, Chap. 6, 23. Corollaire]).

(ii) Let $G$ be a discrete group such that $B_0(G)$ is $w^*$-closed in $B(G)$. Then every element in $G$ has finite order. Indeed, this follows immediately by applying Lemma 2.2 to the cyclic subgroups of $G$.

(iii) Let $G$ be a linear group (that is, a subgroup $GL(n, K)$ for some field $K$) with the discrete topology. If $G$ is infinite, then $B_0(G)$ is not $w^*$-closed in $B(G)$.

In fact, this is clear from (ii) if $G$ has an element of infinite order. On the other
hand, if $G$ is a torsion group, then it is well known to be locally finite [6, Theorem 9.2] and hence amenable, so that the claim follows from Lemma 2.2.

3. Characterizations of compact groups

If $G$ is a compact group then $A(G) = B(G)$. It seems likely that the converse is also true; i.e. $w^*$-closedness of $A(G)$ in $B(G)$ already forces $G$ to be compact. We have been able to show this for groups containing an almost connected open normal subgroup (Theorem 3.6). The case that remains open is that of a totally disconnected group.

We start with a lemma which will be generalized in Section 4 (Lemma 4.2).

Lemma 3.1. If $A(G)$ is $w^*$-closed in $B(G)$ and $H$ is an open subgroup of $G$, then $A(H)$ is $w^*$-closed in $B(H)$.

Proof. It suffices to show that the unit ball of $A(H)$ is $w^*$-closed in the unit ball of $B(H)$. Thus, let $(\varphi_i)$ be a net in $A(H)$ and $\varphi \in B(H)$ such that

$$\|\varphi_i\| \leq 1, \|\varphi\| \leq 1 \text{ and } \varphi_i \to \varphi$$

in the $w^*$-topology. Let $\tilde{\varphi}_i$ and $\tilde{\varphi}$ denote the trivial extensions of $\varphi_i$ and $\varphi$ to $G$. Then $\tilde{\varphi}_i \in A(G)$, $\|\tilde{\varphi}_i\| \leq 1$ and, for each $f \in L^1(G)$,

$$\int_G \tilde{\varphi}_i(x)f(x)dx = \int_H \varphi_i(h)f(h)dh \to \int_H \varphi(h)f(h)dh = \int_G \tilde{\varphi}(x)f(x)dx.$$  

Hence $\tilde{\varphi}_i \to \tilde{\varphi}$ in the $\sigma(B(G), L^1(G))$-topology. Since $(\tilde{\varphi}_i)$ is a bounded net, it follows that $(\tilde{\varphi}_i)$ is $w^*$-convergent to $\tilde{\varphi}$. By hypothesis, $\tilde{\varphi} \in A(G)$ and so $\varphi \in A(H)$.  

Lemma 3.2. Let $K$ be a compact normal subgroup of $G$. If $A(G)$ is $w^*$-closed in $B(G)$, then $A(G/K)$ is $w^*$-closed in $B(G/K)$.

Proof. Consider the map $T_K : f \to T_Kf$ from $L^1(G)$ onto $L^1(G/K)$ given by

$$T_Kf(xK) = \int_K f(xk)dk.$$  

This map extends to a $^*$-homomorphism from $C^*(G)$ onto $C^*(G/K)$ with dual map $T^*_K : B(G/K) \to B(G)$. Furthermore, $T^*_K(B(G/K))$ consists precisely of those functions in $B(G)$ that are constant on cosets of $K [8, (2.26)]$. Also, since $K$ is compact,

$$T^*_K(A(G/K)) = A(G) \cap T^*_K(B(G/K)).$$  

Now, let $\varphi_i \in A(G/K)$ such that $\varphi_i \to \varphi$ in the $w^*$-topology for some $\varphi \in B(G/K)$. Then

$$\langle T^*_K(\varphi_i), f \rangle = \langle \varphi_i, T_K(f) \rangle \to \langle \varphi, T_K(f) \rangle = \langle T^*_K(\varphi), f \rangle$$

for each $f \in C^*(G)$. Thus, by hypothesis, $T^*_K(\varphi) \in A(G)$ and so

$$T^*_K(\varphi) \in A(G) \cap T^*_K(B(G/K)),$$

whence $\varphi \in A(G/K)$.  

Lemma 3.3. Let $G$ be an almost connected locally compact group. If $A(G)$ is $w^*$-closed in $B(G)$, then $G$ is compact.
Assume that $G$ is finite.

Proof. Since an almost connected group is a projective limit of Lie groups and $A(G/K)$ is $w^*$-closed in $B(G/K)$ for every compact normal subgroup $K$ of $G$ (Lemma 3.2), we can assume that $G$ is a Lie group. Being a compactly generated Lie group, $G$ is second countable and hence $A(G)$ is a separable Banach space. By hypothesis,

$$A(G) = B_\lambda(G) = C_r^*(G)^*.$$  

Now, a $G^*$-algebra $A$ with separable dual Banach space has a countable dual $\hat{A}$ (see Section 1). It follows that $\hat{G}_r$, the reduced dual of $G$, is countable. Finally, by [2, Theorem 2.5] a separable Lie group with countable reduced dual is compact. This shows that $G$ is compact. \qed

Corollary 3.4. Let $G$ be any locally compact group and suppose that $A(G)$ is $w^*$-closed in $B(G)$. Then $G$ contains a compact open subgroup.

Proof. Since $G/G_0$ is totally disconnected, there exists an open subgroup $H$ of $G$ so that $H/G_0$ is compact. By Lemma 3.1, $A(H)$ is $w^*$-closed and hence $H$ is compact by Lemma 3.3. \qed

Lemma 3.5. If $G$ is a discrete group and $A(G)$ is $w^*$-closed in $B(G)$, then $G$ is finite.

Proof. Assume that $G$ is infinite. Then $G$ has a countable infinite subgroup $H$. By Lemma 3.1, $A(H)$ is $w^*$-closed in $B(H)$. As in the proof of Lemma 3.3 we now conclude that $\hat{H}_r$ is countable. Applying Baggett’s result again, it follows that $H$ is finite, a contradiction. \qed

Theorem 3.6. Suppose that $G$ contains an almost connected open normal subgroup. Then $A(G)$ is $w^*$-closed in $B(G)$ if and only if $G$ is compact.

Proof. Let $N$ be an almost connected open normal subgroup of $G$. Then $A(N)$ is $w^*$-closed in $B(N)$ by Lemma 3.1, and Lemma 3.3 implies that $N$ is compact. By Lemma 3.2, $A(G/N)$ is $w^*$-closed in $B(G/N)$. Since $N$ is open, Lemma 3.5 gives that $G/N$ is finite. Thus $G$ is compact. \qed

We now turn to a second characterization of compact groups in terms of certain properties of the $w^*$-topology on $B(G)$. If $G$ is a compact group, then the $w^*$-topology and the norm topology agree on the unit sphere of $B(G) = A(G)$ [11, Corollary 2]. We are going to establish the converse to this (see [22, Theorem 5] for the amenable case). Actually, we prove a stronger result in that we replace the unit sphere of $B(G)$ by the smaller set $P_\lambda(G) = B_\lambda(G) \cap P(G)$ of all normalized positive definite functions on $G$ associated to representations that are weakly contained in the left regular representation. Note that this property implies the Radon-Nikodym property for $B(G)$ but not conversely (see [11] and [33]).

For any locally compact group $G$, let $P_\lambda(G) = B_\lambda(G) \cap P(G)$, the set of all normalized continuous positive definite functions on $G$ associated to representations that are weakly contained in the left regular representation. $P_\lambda(G)$ is a $w^*$-compact convex subset of $B(G)$. We denote by $\text{ex}(P_\lambda(G))$ the set of extreme points of the $w^*$-compact convex subset $P_\lambda(G)$ of $B(G)$.

Lemma 3.7. Let $G$ be a locally compact group and $\varphi \in \text{ex}(P_\lambda(G))$. Suppose that $\varphi$ is a point of continuity of the identity map

$$(\text{ex}(P_\lambda(G)), w^*) \rightarrow (\text{ex}(P_\lambda(G)), \| \cdot \|).$$

Then $\pi_\varphi$ is an isolated point in $\hat{G}_r$.
Proof. Notice first that \( \text{ex}(P_\lambda(G)) \subseteq \text{ex}(P(G)) \) because if \( \varphi \in P_\lambda(G) \) and \( \psi \in P(G) \) are such that \( c\varphi \sim \psi \) is positive definite for some \( c \geq 0 \), then \( \psi \in P_\lambda(G) \). By \([5, 2.12.1]\), if \( \varphi_1, \varphi_2 \in \text{ex}(P(G)) \) and \( \pi_{\varphi_1} \) and \( \pi_{\varphi_2} \) are not equivalent, then \( \|\varphi_1 - \varphi_2\| \geq 2 \).

By assumption there exists a \( w^\ast \)-open subset \( U \) of \( \text{ex}(P_\lambda(G)) \) such that
\[
U \subseteq \{ \psi \in \text{ex}(P_\lambda(G)) : \|\psi - \varphi\| < 2 \}.
\]
It follows that \( \pi_\psi = \pi_\varphi \) for all \( \psi \in U \). Now, by \([5, \text{Theorem 3.4.11}]\), the map \( q : \psi \rightarrow \pi_\psi \) from \( \text{ex}(P_\lambda(G)) \) onto \( \hat{G}_r \) is open. Thus \( \{\pi_\varphi\} = q(U) \) is open in \( \hat{G}_r \). \( \square \)

Lemma 3.8. Let \( H \) be an open subgroup of \( G \). If the identity map from \( (P_\lambda(G), w^\ast) \) to \( (P_\lambda(H), \|\cdot\|) \) is continuous, then the identity map from \( (P_\lambda(H), w^\ast) \) to \( (P_\lambda(H), \|\cdot\|) \) is continuous.

Proof. For any \( \varphi \in P_\lambda(H) \), the trivial extension \( \tilde{\varphi} \) belongs to \( P_\lambda(G) \). Indeed, \( \tilde{\varphi} \) is a positive definite function associated to the induced representation \( \text{ind}_H^G \pi_{\varphi} \), and \( \pi_\varphi \prec \lambda_H \) implies
\[
\text{ind}_H^G \pi_{\varphi} \prec \text{ind}_H^G \lambda_H = \lambda_G.
\]
Let \( (\tilde{\varphi}_\alpha) \) be a net in \( P_\lambda(H) \) converging to \( \varphi \in P_\lambda(H) \) in the \( w^\ast \)-topology. Then \( \tilde{\varphi}_\alpha \rightarrow \tilde{\varphi} \) in the \( w^\ast \)-topology on \( P_\lambda(G) \) (compare the proof of Lemma 3.1). By hypothesis, \( \|\tilde{\varphi}_\alpha - \tilde{\varphi}\| \rightarrow 0 \) and hence \( \|\varphi_\alpha - \varphi\| \rightarrow 0 \). \( \square \)

Theorem 3.9. For any locally compact group \( G \) the following conditions are equivalent.

(i) \( G \) is compact.
(ii) The \( w^\ast \)-topology and the norm topology agree on the unit sphere of \( B(G) \).
(iii) The \( w^\ast \)-topology and the norm topology agree on \( P_\lambda(G) \).

Proof. As mentioned above, (i) \( \Rightarrow \) (ii) is due to Granirer and Leinert \([11]\). Since (ii) \( \Rightarrow \) (iii) is trivial, it only remains to prove (iii) \( \Rightarrow \) (i).

Assume that \( G \) fails to be compact. Then \( G \) contains a non-compact, \( \sigma \)-compact, open subgroup \( H \). By Lemma 3.8, the \( w^\ast \)-topology and the norm topology coincide on \( P_\lambda(H) \). It follows from Lemma 3.7 that \( \hat{H}_r \) is discrete. Since \( H \) is \( \sigma \)-compact, Theorem 7.6 of \([34]\) now shows that \( H \) is compact, a contradiction. \( \square \)

4. When is \( A_\gamma(G) \) \( w^\ast \)-closed in \( B(G) \)?

For a locally compact group \( G \) and any unitary representation \( \pi \) of \( G \), the Fourier space \( A_\pi(G) \) associated to \( \pi \) is defined to be the norm-closed linear subspace of \( B(G) \) generated by all the coordinate functions of \( \pi \) \([1]\), that is, the functions of the form \( x \rightarrow \langle \pi(x)\xi, \eta \rangle, \xi, \eta \in \mathcal{H}_x \).

The conjugation representation \( \gamma_G \) (or simply \( \gamma \), if no confusion can arise) on \( L^2(G) \) is defined by
\[
\gamma_G(x)f(y) = \Delta(x)^{1/2}f(x^{-1}yx),
\]
where \( f \in L^2(G) \), \( x, y \in G \). The purpose of this section is to investigate the question of when \( A_\pi(G) \) is \( w^\ast \)-closed in \( B(G) \). It will turn out that this is closely related to problems on the support of \( \gamma \) as studied in \([21]\). We start with two simple facts on \( A_\pi(G) \) for general representations \( \pi \).
Lemma 4.1. If $G$ is a compact group, then $A_{\pi}(G)$ is $w^*$-closed in $B(G)$ for every representation $\pi$ of $G$.

Proof. The $w^*$-closure $\overline{A_{\pi}(G)^{w^*}}$ of $A_{\pi}(G)$ is the dual space of the $C^*$-algebra $\pi(C^*(G))$, which is a quotient of $C^*(G)$. Hence each $\varphi \in \overline{A_{\pi}(G)^{w^*}}$ is a linear combination of positive definite functions in $\overline{A_{\pi}(G)^{w^*}}$. Therefore, it suffices to prove that every positive definite $\varphi \in \overline{A_{\pi}(G)^{w^*}}$ actually is in $A_{\pi}(G)$.

For that, notice that there is a net $(\varphi_\alpha)$ in $A_{\pi}(G)$ such that $\varphi_\alpha \to \varphi$ in the $w^*$-topology and $\|\varphi_\alpha\| \to \|\varphi\|$ (compare [10, p. 565]). By [11, Theorem A] it follows that $\|\varphi_\alpha \psi - \varphi \psi\| \to 0$ for every $\psi \in A(G)$. In particular, $\|\varphi_\alpha - \varphi\| \to 0$ by setting $\psi = 1 \in B(G) = A(G)$. This shows that $\varphi \in A_{\pi}(G)$.

Lemma 4.2. Suppose that $\pi$ is a representation of $G$ such that $A_{\pi}(G)$ is $w^*$-closed in $B(G)$. Then $A_{\pi|H}(H)$ is $w^*$-closed in $B(H)$ for every open subgroup $H$ of $G$.

Proof. Recall that, by [1, Theorem 2.2], $\varphi \in B(G)$ belongs to $A_{\pi}(G)$ if and only if $\varphi$ can be written as

$$\varphi = \sum_{n=1}^{\infty} \langle \pi(\cdot)\xi_n, \eta_n \rangle$$

where $\xi_n, \eta_n \in \mathcal{H}_\pi$ and $\sum_{n=1}^{\infty} \|\xi_n\| \cdot \|\eta_n\| < \infty$. In particular, $A_{\pi|H}(H) = A_{\pi}(G)|H$.

It suffices to show that the unit ball of $A_{\pi|H}(H)$ is $w^*$-closed in the unit ball of $B(H)$. Thus, let $\varphi_i \in A_{\pi|H}(H)$, $i \in I$, and $\varphi \in B(H)$ such that $\|\varphi_i\| \leq 1$, $\|\varphi\| \leq 1$ and $\varphi_i \to \varphi$ in the $w^*$-topology. Choose representations

$$\varphi_i = \sum_{n=1}^{\infty} \langle \pi(\cdot)\xi_{in}, \eta_{in} \rangle$$

such that $\sum_{n=1}^{\infty} \|\xi_{in}\| \cdot \|\eta_{in}\| \leq 2$ (see [1, Proposition 2.9]). Define

$$\psi_i(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_{in}, \eta_{in} \rangle$$

for all $x \in G$ and $i \in I$. Then $\psi_i \in A_{\pi}(G)$ and $\|\psi_i\| \leq 2$. Since the unit ball in $B(G)$ is $w^*$-compact, we can assume that $\psi_i \to \psi$ in the $w^*$-topology for some $\psi \in B(G)$.

Now, $A_{\pi}(G)$ is $w^*$-closed in $B(G)$, so that $\psi \in A_{\pi}(G)$ and hence $\psi|H \in A_{\pi|H}(H)$. On the other hand, the restriction map $B(G) \to B(H)$ is $w^*$-continuous as $H$ is open (see Section 5). Indeed, this follows from the fact that $C^*(H)$ is a subalgebra of $C^*(G)$ whenever $H$ is open in $G$. Thus

$$\varphi_i = \psi_i|H \to \psi|H$$

and $\varphi_i \to \varphi$ in the $w^*$-topology. This proves $\varphi = \psi|H \in A_{\pi|H}(H)$.

We now apply the preceding lemmas to the conjugation representation. The following corollary will be used several times in the sequel.

Corollary 4.3. Suppose that $G$ is second countable and $A_{\gamma G}(G)$ is $w^*$-closed in $B(G)$. Then, for every open subgroup $H$ of $G$, $\text{supp}\gamma_H$ is countable.

Proof. Since $G$ is second countable, $A_{\gamma}(G)$ is norm separable. Since the restriction map from $B(G)$ to $B(H)$ is norm continuous, $A_{\gamma|H}(H) = A_{\gamma G}(G)|H$ is norm
separable. Now, since $\gamma_H$ is a subrepresentation of $\gamma_G|H$ and $A_{\gamma_G|H}(H)$ is $w^*$-closed in $B(H)$ by Lemma 4.2,
\[ A_{\gamma_H}(H)^{w^*} \subseteq A_{\gamma_G|H}(H), \]
so that $A_{\gamma_H}(H)^{w^*}$ is norm separable. Thus $\gamma_H(C^*(H))$ has a norm separable dual Banach space, $A_{\gamma_H}(H)^{w^*}$, and hence
\[ \text{supp} \gamma_H = \gamma_H(C^*(H))^\wedge \]
is countable.

Corollary 4.4. Let $Z(G)$ denote the centre of $G$. If $G/Z(G)$ is compact, then $A_{\gamma_G}(G)$ is $w^*$-closed in $B(G)$.

Proof. For $z \in Z(G)$, $\gamma_G(z)$ is the identity on $L^2(G)$. Thus $\pi(xZ(G)) = \gamma_G(x)$, $x \in G$, defines a representation of $G/Z(G)$, and therefore $A_\pi(G/Z(G))$ is $w^*$-closed in $B(G/Z(G))$ by Lemma 4.1. Denoting by $q : G \to G/Z(G)$ the quotient homomorphism, we have
\[ A_{\gamma_G}(G) = A_\pi(G/Z(G)) \circ q. \]
By [1, (2.10)], $A_{\gamma_G}(G)$ is $w^*$-closed in $B(G)$.

Our goal is to establish the converse to Corollary 4.4 for Lie groups with countably many connected components (Theorem 4.8). Apart from using various results from [21], a major step in proving the theorem will be the next lemma.

We remind the reader that a group $G$ is called an FC-group if all its conjugacy classes are finite. Such a group, more generally every locally compact group all of whose conjugacy classes are relatively compact, is amenable.

Lemma 4.5. Let $G$ be a countable discrete FC-group. If $\text{supp} \gamma_G$ is countable, then $G$ has a finite commutator subgroup.

Proof. Let $S = \text{supp} \gamma_G$ and notice first that points in $S$ are closed in $\hat{G}$. Indeed, the primitive ideal space of any FC-group is a $T_1$ space (see [28, Theorem 5.2]) and $C^*_v(G)$, being a separable $C^*$-algebra with countable dual, is of type I. Thus the points of $S$ are closed in $\hat{G}$. Since $C^*(G)$ is unital, it follows that every $\sigma \in S$ is finite dimensional.

Next, employing the facts that points in $S$ are closed, that $S$ is countable and that duals of $C^*$-algebras are Baire spaces [5, (3.4.13)], a straightforward argument yields the existence of some dense subset $D$ of $S$ consisting of points that are also open in $S$.

Since $G$ is a countable amenable group,
\[ \bigcup_{\pi \in \hat{G}} \text{supp}(\pi \otimes \overline{\pi}) \]
is a dense subset of $S$ by [19, Theorem]. Let $C(S)$ denote the set of all closed subsets of $S$, endowed with Fell’s topology [10, p. 427]. By [18, Proposition 2], the mapping
\[ \pi \to \text{supp}(\pi \otimes \overline{\pi}), \hat{G} \to C(S) \]
is continuous. It follows that
\[ V = \{ \pi \in \hat{G} : \text{supp}(\pi \otimes \overline{\pi}) \cap D \neq \emptyset \} \]
is non-empty and open in $\hat{G}$.
Now, as points in $D$ are open in $S$, $\text{supp}(\pi \otimes \tau)$ contains a finite dimensional subrepresentation for each $\pi \in V$. However, $\pi \otimes \tau$ then also contains the trivial representation $1_G$. This can be seen as follows. Suppose that $\tau$ is finite dimensional and that $\pi \leq \pi \otimes \tau$. Then
\begin{align*}
1_G \leq \tau \otimes \tau \leq \pi \otimes \tau \otimes \tau,
\end{align*}
and $\pi \otimes \tau$ is a (finite) direct sum of irreducible representations $\rho_1, \ldots, \rho_n$. Thus $1_G \leq \pi \otimes \rho_i$ for some $i$, which is impossible unless $\rho_i \sim \pi$ (see [17, Proposition 2.4]). As is well-known, $1_G \leq \pi \otimes \pi$ forces $\pi$ to be finite dimensional. Hence every $\pi \in V$ is finite dimensional.

Finally, for a discrete FC-group $G$, the existence of a non-empty open subset in $\widehat{G}$ consisting of finite dimensional representations implies that $G$ has a finite commutator subgroup. In fact, in this case the left regular representation of $G$ has a subrepresentation of type I, and then the commutator subgroup has to be finite by [16, Satz 1] (see also [32, Theorem 3]).

**Lemma 4.6.** Let $G$ be a locally compact group with open centre. Then, for each $a \in G$,
\begin{align*}
\text{ind}^{G}_{C(a)} 1_{C(a)} \prec \gamma_G,
\end{align*}
where $C(a)$ denotes the centralizer of $a$ in $G$.

**Proof.** Since $\text{ind}^{G}_{C(a)} 1_{C(a)}$ is the cyclic representation of $G$ defined by the positive definite function $\chi_{C(a)}$, the characteristic function of $C(a)$, and since $C(a)$ is open, it suffices to show that given any compact subset $K$ of $G \setminus C(a)$, there is a positive definite function $\varphi$ associated to $\gamma_G$ such that $\varphi(x) = 1$ for all $x \in C(a)$ and $\varphi(x) = 0$ for all $x \in K$. Now, since
\begin{align*}
C = \{a^{-1}x^{-1}ax : x \in K\}
\end{align*}
is compact and $e \notin C$, we find an open neighborhood $V$ of $e$, contained in the centre of $G$, such that $CV \cap V = \emptyset$. Let
\begin{align*}
\varphi(x) = |V|^{-1} \langle \gamma_G(x) \chi_{aV}, \chi_{aV} \rangle
\end{align*}
for $x \in G$. Then it is easily verified that $\varphi(x) = 1$ for all $x \in C(a)$ and $\varphi(x) = 0$ for all $x \in K$. 

**Lemma 4.7.** Suppose that $G$ is second countable and contains an open normal subgroup $N$ such that $G/N$ is abelian and every irreducible representation of $N$ is finite dimensional. If $\text{supp} \gamma_G$ is countable, then every irreducible representation of $G$ is finite dimensional.

**Proof.** Given an irreducible representation $\pi$ of $G$, there exist a subgroup $H$ of $G$ and a finite dimensional irreducible representation $\tau$ of $H$ such that $N \subseteq H$ and $\pi \sim \text{ind}^{G}_{H} \tau$.

In fact, this has been shown in [7, Theorem 3.2.3] as an application of representation theory of crossed product $C^*$-algebras. We prefer to outline a direct argument for this within the framework of Mackey’s unitary representation theory of group extensions [26].

Choose $\gamma \in \widehat{N}$ such that $\pi|N \sim G(\gamma)$, the $G$-orbit of $\gamma$ in $\widehat{N}$ under the action of $G$. Let $S$ denote the stability subgroup of $\gamma$. By [26, Theorems 8.2 and 8.3], there
exist a multiplier \( \omega \) on \( S/N \), an irreducible \( \omega \)-representation \( \rho \) of \( S \) in \( \mathcal{H} \) and an irreducible \( \omega \)-representation \( \sigma \) of \( S/N \) such that

\[
\pi = \text{ind}_S^G(\rho \otimes \sigma).
\]

Now, an irreducible \( \omega \)-representation of an abelian group is weakly equivalent to the \( \omega \)-representation induced from some one-dimensional \( \omega \)-representation of a certain subgroup. Hence there exist a subgroup \( H \) of \( S \) containing \( N \) and a one-dimensional \( \omega \)-representation \( \lambda \) of \( H/N \) such that

\[
\rho \otimes \sigma \sim \rho \otimes \text{ind}_H^S(\lambda).
\]

Let \( \pi = \rho|H \otimes \lambda \), a finite dimensional ordinary representation. Then

\[
\pi \sim \text{ind}_S^G(\text{ind}_H^S(\rho|H \otimes \lambda)) = \text{ind}_H^G \tau,
\]

as required.

It follows that

\[
\pi \otimes \pi = \text{ind}_S^G(\tau \otimes \pi|H) \succ \text{ind}_S^G(\tau \otimes \pi)
\]

\[
\succ \text{ind}_H^G \tau \sim \frac{G}{H}.
\]

On the other hand, \( \pi \otimes \pi \) is weakly contained in \( \gamma_G \) since \( G \) is amenable. Now, as \( \text{supp} \gamma_G \) is countable and \( H \) is open and \( G/H \) is abelian, \( H \) must have finite index in \( G \). Thus, \( \text{ind}_H^G \tau \) is finite dimensional and hence so is \( \pi \). \( \square \)

**Theorem 4.8.** Suppose that \( G \) is a Lie group with countably many connected components. If \( A_{\gamma_G}(G) \) is \( w^* \)-closed in \( B(G) \), then \( G/Z(G) \) is compact.

**Proof.** For any normal subgroup \( N \) of \( G \), let \( q_N : G \rightarrow G/N \) denote the quotient homomorphism. Since \( A_{\gamma_G}(G) \) is \( w^* \)-closed in \( B(G) \) and \( G \) is second countable, and since the connected component \( G_0 \) of \( e \) is open, \( \text{supp} \gamma_{G_0} \) is countable by Corollary 4.3. Theorem 3.4 of [21] yields that \( G_0 = V \times C \), the direct product of a vector group \( V \) and a compact group \( C \). Clearly, \( C \) is normal in \( G \).

We claim that \( V = G_0/C \) is contained in the centre of \( G/C \). To that end, fix \( a \in G \) and consider the open subgroup \( H \) of \( G \) generated by \( a \) and \( G_0 \). Then \( \text{supp} \gamma_H \) is countable (Corollary 4.3), and since

\[
\gamma_{G/C} \circ q_C \prec \gamma_H
\]

due to the compactness of \( C \) [19, Remark 1], it follows that \( \gamma_{H/C} \) has a countable support. Since \( G_0/C \) is a vector group and \( (H/C)/(G_0/C) = H/G_0 \) is abelian, an application of Lemma 3.2 in [21] shows that \( G_0/C \) is in the centre of \( H/C \). Since \( a \) is arbitrary, \( G_0/C \) is central in \( G/C \).

Next we are going to prove that \( G/G_0 \) has a finite commutator subgroup. Passing to \( F = G/C \), we know that \( \gamma_F \) has countable support, and since \( V \) is central in \( F \), by [21, Lemma 1.1]

\[
\gamma_{F/V} \circ q_V \prec \gamma_F,
\]

so that \( \text{supp} \gamma_{F/V} \) is countable. Let \( D = F/V \) and denote by \( D_f \) the finite conjugacy class subgroup of \( D \). Then, by [20, Theorem 1.8]

\[
\lambda_{D/D_f} \circ q_{D_f} \prec \gamma_D,
\]

where \( \lambda_{D/D_f} \) is the regular representation of \( D/D_f \). Thus the reduced dual of \( D/D_f \) is countable, and hence \( D/D_f \) is finite (see [2, 34]). At this stage we know in particular that \( G \) is amenable, so that \( \gamma_{G/N} \circ q_N \prec \gamma_G \) for each closed normal
subgroup \( N \) of \( G \) [21, Lemma 1.1]. Also, \( \gamma_{D_f} \) has countable support and therefore the commutator subgroup \( D_f' \) of \( D_f \) is finite by Lemma 4.5. Let \( N \) be the inverse image of \( D_f' \) in \( G \) and let \( E = G/N \). Then \( \gamma_E \) has countable support, and \( E \) possesses an abelian normal subgroup \( A \) of finite index, namely \( A = D_f/D_f' \). We apply Lemma 4.6 to \( E \) and obtain that
\[
\text{ind}^{E}_{C(\gamma_{\bar{E}})} 1_{C(x)} \prec \gamma_E
\]
for every \( x \in E \). It follows that
\[
\text{ind}^{A}_{C(x)} 1_{C(x)} \prec (\text{ind}^{E}_{C(x)} 1_{C(x)})|A \prec \gamma_E|A,
\]
and \( \gamma_E|A \) has countable support since \( \gamma_E \) does and \( E/A \) is finite. However, \( A \) being abelian, countability of
\[
(A/C(x) \cap A)^{\sim} = \text{supp}(\text{ind}^{A}_{C(x)} 1_{C(x)}|A)
\]
implies that \( C(x) \cap A \) is of finite index in \( A \). Hence \( C(x) \) has finite index in \( E \) for every \( x \in E \). Therefore \( G/N \) is an FC-group, and Lemma 4.5 implies that \( E \) has a finite commutator subgroup. Recalling that \( N/G_0 = D_f' \) is finite, we conclude that \( G/G_0 \) has a finite commutator subgroup.

Since \( G/G_0 \) has a finite commutator subgroup, there exists a normal subgroup \( N \) of \( G \) such that \( G_0 \subseteq N \), \( N/G_0 \) is finite and \( G/N \) is abelian. Now, \( G_0 = V \times C \), and this implies that all the irreducible representations of \( N \) are finite dimensional. Since \( \text{supp} \gamma_G \) is countable, Lemma 4.7 shows that \( G \) has only finite dimensional irreducible representations.

On the other hand, \( G \) has a relatively compact commutator subgroup. To see this, notice first that since \( V \) is contained in the centre of \( N/C \), \( N/C \) has a finite commutator subgroup. Denoting its inverse in \( G \) by \( K \), \( N/K \) is isomorphic to \( V \). By arguments that have previously been used, \( V \) is central in \( G/K \), and applying Lemma 4.6 again, this time to \( G/K \), we conclude that \( G/K \) is a group with finite conjugacy classes.

Thus \( G \) is a group with relatively compact conjugacy classes all of whose irreducible representations are finite dimensional. Since, in addition, \( G \) is a Lie group, combining Theorem 2 of [27] and Lemma 5.4 of [24] shows that \( G/Z(G) \) is compact.

\[ \square \]

5. \( w^* \)-Continuity of the Restriction Map

Let \( G \) be a locally compact group, and let \( H \) be a closed subgroup of \( G \). In this section we study the question of when the restriction map
\[
\Phi : B(G) \to B(H), \varphi \to \varphi|H
\]
is continuous for the \( w^* \)-topologies. Clearly, if \( H \) is open then \( C^*(H) \) is a subalgebra of \( C^*(G) \) and hence \( \Phi \) is \( w^* \)-continuous. We are going to establish the following converse.

Theorem 5.1. Let \( H \) be a closed subgroup of the locally compact group \( G \). If the restriction map \( B(G) \to B(H) \) is continuous for the \( w^* \)-topologies, then \( H \) is open in \( G \).

Let \( M(G) \) denote the algebra of all bounded measures on \( G \). The canonical embedding of \( L^1(G) \) into \( M(G) \) extends to an isometric \(^*\)-homomorphism of \( C^*(G) \)
Lemma 5.5. Let $G$ be a locally compact group and let $T \in C_{r}^{*}(G)$. Suppose that $K$ is a compact subset of $G$ and regard $L^{2}(K)$ as a closed subspace of $L^{2}(G)$ in the usual manner. Then the restriction

$$T|L^{2}(K) : L^{2}(K) \to L^{2}(G)$$

of $T$ to $L^{2}(K)$ is a compact operator.

Proof. Of course, it suffices to prove the statement for operators $T$ of the form $T = \rho(f)$ where $f \in C_{c}(G)$. Let $K' = \text{supp } f \cdot K$. Then $Tg \in L^{2}(K')$ for all $g \in L^{2}(K)$. Choose $f_{j} \in C_{c}(K')$ and $g_{j} \in C_{c}(K)$ such that, as $n \to \infty$,

$$\int_{G} \int_{G} \left| \sum_{j=1}^{n} f_{j}(x)g_{j}(y) - f(xy) \right|^{2} dx dy \to 0.$$  

It is straightforward to verify that this implies that $T : L^{2}(K) \to L^{2}(K')$ is a norm limit of finite rank operators. \hfill \Box

Lemma 5.6. Let $(X, \mu)$ be a probability measure space without atoms. Then the canonical embedding $L^{2}(X) \to L^{1}(X)$ fails to be compact.
Proof. It suffices to show that there is an orthonormal sequence \((f_n)_n\) in \(L^2(X)\) such that \(\|f_n\|_\infty \leq 1\). Indeed, we then have
\[
2 = \|f_n - f_m\|^2 = \int_X |f_n(x) - f_m(x)|^2 d\mu(x) \leq \|f_n - f_m\|_1
\]
for all \(n, m \in \mathbb{N}, n \neq m\), and hence no subsequence of \((f_n)_n\) can converge in \(L^1(X)\).

Since \(\mu\) has no atoms, there exists, for every \(0 \leq r \leq 1\), a measurable subset \(A\) of \(X\) with \(\mu(A) = r\) (see [13, Section 4.1, Exercise 1]). Therefore, we can choose inductively, for every \(n \in \mathbb{N}\), disjoint measurable subsets
\[
\mathcal{A}_1^{(n)}, \mathcal{A}_2^{(n)}, \ldots, \mathcal{A}_{2^{n-1}}^{(n)}
\]
of \(X\) with the following properties:

(i) \(\mu(\mathcal{A}_1^{(n)}) = \frac{1}{2^{n-r}}\) for \(i = 1, \ldots, 2^{n-1}\).
(ii) \(\mathcal{A}_{2i-1}^{(n+1)}, \mathcal{A}_{2i}^{(n+1)} \subseteq \mathcal{A}_i^{(n)}\) for \(i = 1, \ldots, 2^{n-1}\).

For each \(n \in \mathbb{N}\), define a function \(f_n \in L^2(X)\) (a kind of Rademacher function) by setting
\[
f_n(x) = (-1)^i \text{ for } x \in \mathcal{A}_i^{(n)} \text{ and } i = 1, \ldots, 2^{n-1}.
\]

By construction and (1) and (2), we obviously have
\[
\|f_n\|_\infty = \|f_n\|_2 = 1 \text{ and } \int_X f_n(x)f_m(x)d\mu(x) = 0, \quad n \neq m.
\]

This completes the proof. \(\square\)

Proof of Theorem 5.4. Let \(H\) be a closed subgroup of \(G\) and suppose \(\Psi, (C^*(H))_r\) is contained in \(C^*_c(G)\). Fix a Bruhat function \(\beta\) on \(G\) for \(H\) (see [31, Chapter 8]), that is, a non-negative continuous function \(\beta\) on \(G\) with the following properties:

(i) For every compact subset \(K\) of \(G\), \(\beta\) agrees on \(K\) with the restriction of some function from \(C_c(G)\).
(ii) \(\int_G \beta(xh)dh = 1\) for all \(x \in G\).

Let \(\Delta_G\) and \(\Delta_H\) denote the modular functions of \(G\) and \(H\), respectively, and define a function \(q\) on \(G\) by
\[
q(x) = \int_H \beta(xh)\Delta_G(h)\Delta_H(h)^{-1}dh.
\]

Since \(q(xh) = q(x)\Delta_G(h)^{-1}\Delta_H(h)\) for all \(x \in G\) and \(h \in H\), there exists a quasi-invariant measure \(d_q\hat{\cdot}\) on \(\hat{G} = G/H\) such that
\[
\int_{\hat{G}} \left( \frac{f(xh)}{q(xh)}dh \right) d_q\hat{\cdot} = \int_G f(x)dx
\]
for all \(f \in L^1(G)\) [31, Chapter 8].

Now let \(\hat{K} \subseteq \hat{G}\) be a compact neighbourhood of \(\hat{e}\) in \(\hat{G}\). We are going to prove that \(\hat{K}\) is finite. Clearly, this will imply that \(H\) is open. Let \(\pi : G \to \hat{G}\) denote the quotient map and choose a compact neighbourhood \(K\) of \(e\) in \(G\) so that \(\pi(K) = \hat{K}\). Fix a continuous function \(f\) with compact support on \(H\) such that
\[
\int_H f(h)\Delta_G(h)^{-1}dh \neq 0.
\]
Let $K' = KH \cap \text{supp } \beta$, which is a compact subset of $G$. By Lemma 5.5, the restriction of $\rho(\mu_f)$, the right convolution operator defined by $\mu_f$, to $L^2(K')$ is a compact operator. Define a linear mapping

$$T : L^2(\hat{K}, d_q \hat{x}) \to L^2(K')$$

by $T\hat{g}(x) = \hat{g}(\hat{x})\beta(x)^{1/2}$ for $\hat{g} \in L^2(\hat{K}, d_q \hat{x})$ and $x \in K$. Since

$$\int_H \frac{\beta(xh)}{q(xh)} \frac{dh}{\|g\|_2} = \frac{1}{q(x)} \int_H \beta(xh) \Delta_G(h) \Delta_H(h)^{-1} dh \delta x = 1$$

for all $x \in G$, we have

$$\|T\hat{g}\|_2 = \int_G \left( \int_H \frac{|\hat{g}(\hat{x})|^2 \beta(xh)}{q(xh)} dh \right) d_q \hat{x}$$

$$= \int_G |\hat{g}(\hat{x})|^2 d_q \hat{x} = \|\hat{g}\|_2^2.$$ 

Thus $T$ is a bounded linear operator. Set

$$q_1(x) = \int_H \beta(xh)^{1/2} \Delta_G(h) \Delta_H(h)^{-1} dh$$

for $x \in G$. Then, for all $x \in G$ and $h \in H$,

$$q_1(xh) = q_1(x) \Delta_G(h)^{-1} \Delta_H(h).$$

Hence there exists a quasi-invariant measure $d_{q_1} \hat{x}$ on $\hat{G}$ such that

$$\int_G \left( \int_H \frac{q(xh)}{q_1(xh)} dh \right) d_{q_1} \hat{x} = \int_G g(x) dx$$

for all $g \in L^1(G)$. Let

$$S : L^1(G) \to L^1(\hat{G}, d_{q_1} \hat{x})$$

be the linear operator defined by

$$Sg(\hat{x}) = \int_H \frac{g(xh)}{q_1(xh)} dh$$

for $g \in C_c(G)$. $S$ is bounded since

$$\|Sg\|_{L^1(\hat{G}, d_{q_1} \hat{x})} \leq \int_G \left( \int_H \frac{|g(xh)|}{q_1(xh)} dh \right) d_{q_1} \hat{x} = \|g\|_1.$$ 

Observe next that $\rho(\mu_f)$ maps $L^2(K')$ into $L^2(K''')$ for some compact subset $K'''$ of $G$. Hence $\rho(\mu_f)T$ map $L^2(\hat{K}, d_q \hat{x})$ into $L^1(G)$, and we may consider the bounded linear operator

$$S\rho(\mu_f)T : L^2(\hat{K}, d_q \hat{x}) \to L^1(\hat{G}, d_{q_1} \hat{x}).$$

For $\hat{g} \in C(\hat{K})$, we have

$$S\rho(\mu_{f|H})T(\hat{g})(\hat{x}) = \hat{g}(\hat{x}) \int_H \frac{1}{q_1(xh)} \left( \int_H \beta(xhk)^{1/2} f(k) dk \right) dh$$

$$= \hat{g}(\hat{x}) \int_H f(k) \left( \int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)} dh \right) dk.$$
However, for \( x \in G \) and \( k \in K \),
\[
\int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)} \, dh = \Delta_H(k)^{-1} \int_H \frac{\beta(xh)^{1/2}}{q_1(xh^{-1})} \, dh = \Delta_G(k)^{-1} \int_H \frac{\beta(xh)^{1/2}}{q(xh)} \, dh = \Delta_G(k)^{-1}.
\]
Thus, setting \( \alpha = \int_H f(k) \Delta_G(k)^{-1} \, dk \neq 0 \), we obtain that
\[
S\rho(\mu_{fH})T(\hat{g}) = \alpha \hat{g}
\]
for all \( \hat{g} \in L^2(\hat{K}, d_\mu \hat{x}) \). Now, by hypothesis and Lemma 5.5, the restriction of \( \rho(\mu_{fH}) \) to \( L^2(K') \) is compact. Hence
\[
\alpha I : L^2(\hat{K}, d_\mu \hat{x}) \to L^1(\hat{K}, d_\mu \hat{x})
\]
is a compact operator.

Finally, notice that since
\[
d_{q_1} \hat{x} = \frac{q_1(x)}{q(x)} d_q \hat{x}
\]
and \( \frac{q_1}{q} \) is a strictly positive continuous function on the compact set \( \hat{K} \), the corresponding \( L^1 \)-spaces are equal and the \( L^1 \)-norms are equivalent. Hence the embedding
\[
L^2(\hat{K}, d_\mu \hat{x}) \to L^1(\hat{K}, d_\mu \hat{x})
\]
is compact. By Lemma 5.6, \( \hat{K} \) has to have atoms. However, this implies that \( \hat{K} \) is finite. \( \square \)

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DÉPARTEMENT DE MATHEMATICIQUES, UNIVERSITÉ DE METZ, F - 57045 METZ, FRANCE

E-mail address: bekka@poncelet.univ-metz.fr

FACHBEREICH MATHEMATIK/INFORMATIK, UNIVERSITÄT PADENBOURN, D - 33095 PADENBOURN, GERMANY

E-mail address: kaniuth@uni-paderborn.de

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, CANADA T6G 2G1

E-mail address: tlau@vega.math.ualberta.ca

MATHEMATISCHES INSTITUT, TECHNISCHE UNIVERSITÄT MÜNCHEN, D - 80290 MÜNCHEN, GERMANY

E-mail address: gschlich@mathematik.tu-muenchen.de

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