WEAK*-CLOSEDNESS OF SUBSPACES
OF FOURIER-STIELTJES ALGEBRAS
AND WEAK*-CONTINUITY OF THE RESTRICTION MAP

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Abstract. Let $G$ be a locally compact group and $B(G)$ the Fourier-Stieltjes algebra of $G$. We study the problem of how weak*-closedness of some translation invariant subspaces of $B(G)$ is related to the structure of $G$. Moreover, we prove that for a closed subgroup $H$ of $G$, the restriction map from $B(G)$ to $B(H)$ is weak*-continuous only when $H$ is open in $G$.

Introduction

Let $G$ be a locally compact group, and let $B(G)$ be the Fourier-Stieltjes algebra of $G$ as defined by Eymard [8]. Recall that $B(G)$ is the linear span of all continuous positive definite functions on $G$ and can be identified with the Banach space dual of $C^*(G)$, the group $C^*$-algebra of $G$. The space $B(G)$, with the norm as dual of $C^*(G)$, is a commutative Banach *-algebra with pointwise multiplication and complex conjugation. The Fourier algebra $A(G)$ of $G$ is the closed *-subalgebra of $B(G)$ generated by the functions in $B(G)$ with compact support. In particular, $A(G)$ is contained in $C_0(G)$, the algebra of complex valued continuous functions on $G$ vanishing at infinity. As is well known $A(G)$ is weak*-dense in $B(G)$ if and only if $G$ is amenable. In [3] translation invariant *-subalgebras $A$ of $B(G)$ were studied, and it was shown that if such $A$ is weak*-closed and point separating, then it must contain $A(G)$. However, apart from this, very little seems to be known about weak*-closed subspaces of $B(G)$.

The first purpose of this paper is to investigate the relation between weak*-closedness of certain interesting norm-closed translation invariant subspaces of $B(G)$ and the structure of $G$. Secondly, we solve the problem of when, for a closed subgroup $H$ of $G$, the restriction map from $B(G)$ to $B(H)$ is weak*-continuous.

A brief outline of the paper is as follows. In Section 2 we establish for almost connected locally compact groups $G$ the relation between weak*-closedness of $B_0(G) = B(G) \cap C_0(G)$ in $B(G)$ and the structure of $G$ (Theorem 2.10). The key result is that for a connected Lie group $G$, $B_0(G)$ is weak*-closed in $B(G)$ if and only if $G$ is a reductive Lie group with compact centre and Kazhdan’s property $(T)$.

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If \( G \) is a compact group, then for any unitary representation \( \pi \) of \( G \), the Fourier space \( A_\pi(G) \) associated to \( \pi \) (see [1] for the definition and properties of \( A_\pi(G) \)) is weak*-closed in \( B(G) \). Note that \( A(G) = A_{\gamma_G}(G) \), where \( \lambda_G \) denotes the left regular representation of \( G \). In Theorem 3.6 we shall show that, conversely, if \( G \) contains an almost connected open normal subgroup and \( A(G) \) is weak*-closed in \( B(G) \), then \( G \) is compact. We also give a characterization of compactness of \( G \) in terms of the weak* and the norm topologies on the unit sphere of \( B(G) \) (Theorem 3.9).

Besides the left regular representation, one of the most interesting representations of a locally compact group \( G \) is the conjugation representation \( \gamma_G \) of \( G \) on \( L^2(G) \). In contrast to \( A(G), A_{\gamma_G}(G) \) need not be a subalgebra and it can at best determine the structure of \( G/Z(G) \), where \( Z(G) \) denotes the centre of \( G \). We prove that if \( G \) is a Lie group with countably many connected components and \( A_{\gamma_G}(G) \) is weak*-closed in \( B(G) \), then \( G/Z(G) \) is compact (Theorem 4.8).

Let \( H \) be a closed subgroup of an arbitrary locally compact group \( G \). Clearly, the restriction map \( B(G) \to B(H) \) is continuous for the weak*-topologies whenever \( H \) is open. In the final section 5 we succeed in showing that conversely weak*-continuity of the restriction map forces \( H \) to be open in \( G \).

1. Preliminaries

Throughout this paper, \( G \) denotes a locally compact group with a fixed left Haar measure \( dx \) and modular function \( \Delta \). \( L^1(G) \) the convolution algebra of integrable functions on \( G \) and \( C^*(G) \) the group C*-algebra of \( G \). The Fourier-Stieltjes algebra \( B(G) \) is the Banach space dual of \( C^*(G) \) and as such carries the weak*-topology (\( w^* \)-topology, for short) \( \sigma(B(G), C^*(G)) \). The basic reference on Fourier and Fourier-Stieltjes algebras is [8].

Next, we have to introduce some notation from representation theory. We use the same letter, for example \( \pi \), for a unitary representation of \( G \) and for the corresponding *-representation of \( C^*(G) \). \( \mathcal{H}_\pi \) will always denote the Hilbert space of \( \pi \) and \( \ker \pi \) the \( C^* \)-kernel of \( \pi \). If \( S \) and \( T \) are sets of unitary representations of \( G \), then \( S \) is weakly contained in \( T \) (\( S \preceq T \)) if \( \bigcap_{\pi \in S} \ker \pi \supseteq \bigcap_{\tau \in T} \ker \tau \) or, equivalently, if every positive definite function associated to \( S \) can be uniformly approximated on compact subsets of \( G \) by sums of positive definite functions associated to \( T \). Also \( S \) and \( T \) are weakly equivalent (\( S \sim T \)) if \( S \preceq T \) and \( T \preceq S \).

The dual space \( \hat{G} \) of \( G \) is the set of equivalence classes of irreducible representations of \( G \), endowed with the Jacobson topology. For a representation \( \pi \) of \( G \), the support of \( \pi \) is the closed subset \( \text{supp} \pi = \{ \rho \in \hat{G} : \rho \preceq \pi \} \) of \( \hat{G} \). In particular, the support of the left regular representation \( \lambda_G \) is the reduced dual \( \hat{G}_r \), and \( \lambda_G(C^*(G)) \) is the so-called reduced group C*-algebra of \( G \) which is denoted by \( C_r^*(G) \). If \( N \) is a closed normal subgroup of \( G \), then every representation of \( G/N \) can be lifted to a representation of \( G \), and in this sense, \( (G/N)^* \) will always be regarded as a subset of \( \hat{G} \). For general references to representation theory and dual spaces we mention [5] and [10].

\( G \) is called amenable if there exists a non-zero positive linear functional \( m \) on the space \( C_b^0(G) \) of bounded continuous complex valued functions on \( G \) such that \( m(\pi f) = m(f) \) for all \( f \in C_b^0(G) \) and \( x \in G \), where \( x f(y) = f(x^{-1}y) \). Recall that amenability is equivalent to a number of different conditions: \( C_r^*(G) = C^*(G) \),...
Let $H$ be a closed subgroup of $G$, and suppose that $\sigma$ and $\pi$ are representations of $H$ and $G$ respectively. The representation of $G$ induced by $\sigma$ is denoted $\text{ind}_G^H \sigma$. Then the tensor product $\pi \otimes \text{ind}_G^H \sigma$ is equivalent to $\text{ind}_G^G (\pi | H \otimes \sigma)$. Furthermore, by the theorem on induction in stages, $\text{ind}_G^G (\pi | H \otimes \sigma)$ for every closed subgroup $K$ of $G$ containing $H$. Finally, we will frequently use that $\pi \prec \text{ind}_G^G (\pi | H)$ if $G$ is amenable [12, Theorem 5.1].

Let $A$ be a $C^*$-algebra and $\hat{A}$ its dual space, i.e. the set of equivalence classes of non-degenerated irreducible *-representations of $A$. We will several times use the fact that if the Banach space dual $A^*$ of $A$ is separable in the norm topology, then $\hat{A}$ is countable (see [15, Theorem 3.1] or [23, Lemma 4.12]).

2. When is $B_0(G)$ $w^*$-closed in $B(G)$?

For a locally compact group $G$, let $B_0(G) = B(G) \cap C_0(G)$ denote the norm-closed and translation invariant subalgebra of $B(G)$ consisting of all functions in $B(G)$ that vanish at infinity. In this section we shall study the problem of when $B_0(G)$ is closed in $B(G)$ with respect to the $w^*$-topology on $B(G)$.

It turns out to be appropriate to reformulate this condition in terms of convergence of positive definite functions. The following lemma, which is a consequence of [8, (2.1) and (2.12)], will be used frequently.

**Lemma 2.1.** Let $P(G)$ denote the set of all normalized continuous positive definite functions on $G$. The following are equivalent:

(i) $B_0(G)$ is $w^*$-closed in $B(G)$.

(ii) If $(\varphi_i)$ is a net in $P(G) \cap C_0(G)$ converging to some $\varphi \in P(G)$ uniformly on compact subsets of $G$, then $\varphi \in C_0(G)$.

**Lemma 2.2.** Suppose that $B_0(G)$ is $w^*$-closed in $B(G)$. Then every amenable closed normal subgroup of $G$ is compact. In particular, for an amenable group $G$, $B_0(G)$ is $w^*$-closed in $B(G)$ only when $G$ is compact.

**Proof.** Let $N$ be an amenable closed normal subgroup of $G$. Then the trivial one-dimensional representation $1_N$ of $N$ is weakly contained in the left regular representation $\lambda_N$ of $N$. Hence, by continuity of inducing, $\text{ind}_N^G 1_N$ is weakly contained in $\text{ind}_N^G \lambda_N$, which is equivalent to $\lambda_G$.

Let $q : G \to G/N$ denote the quotient homomorphism. Then $\text{ind}_N^G 1_N = \lambda_{G/N} \circ q$. Thus every positive definite function $\varphi$ associated to $\lambda_{G/N} \circ q$ is a uniform on compacta limit of functions in $A(G) \cap P(G) \subseteq B_0(G)$. Hence, by hypothesis, $\varphi \in C_0(G)$. Since such a $\varphi$ is constant on cosets of $N$, $N$ must be compact. \qed

We continue with two inheritance properties.

**Lemma 2.3.** Let $H$ be an open subgroup of the locally compact group $G$. If $B_0(G)$ is $w^*$-closed in $B(G)$, then $B_0(H)$ is $w^*$-closed in $B(H)$.

**Proof.** Let $(\varphi_i)$ be a net in $P(H) \cap C_0(H)$ such that $\varphi_i \to \varphi \in P(H)$ uniformly on compact subsets of $H$. Let $\hat{\varphi}_i$ and $\hat{\varphi}$ denote the trivial extensions of $\varphi_i$ and $\varphi$ to $G$, that is, $\hat{\varphi}_i(x) = \hat{\varphi}_i(x) = 0$ for $x \in G \setminus H$. Clearly, then $\hat{\varphi}_i \in P(G) \cap C_0(G)$, $\hat{\varphi} \in P(G)$ and $\hat{\varphi}_i \to \hat{\varphi}$ uniformly on compact subsets of $G$. By hypothesis, $\hat{\varphi} \in C_0(G)$ and hence $\varphi \in C_0(H)$. \qed
Lemma 2.4. Let $G$ be a locally compact group and $K$ a compact normal subgroup of $G$. Then $B_0(G)$ is $w^*$-closed in $B(G)$ if and only if $B_0(G/K)$ is $w^*$-closed in $B(G/K)$.

Proof. Suppose that $B_0(G)$ is $w^*$-closed in $B(G)$, and let $(\varphi_i)$ be a net in $P(G/K) \cap C_0(G/K)$ converging to some $\varphi \in P(G/K)$ uniformly on compact subsets of $G/K$. Then, with $q : G \to G/K$ the quotient homomorphism, $\varphi_i \circ q \to \varphi \circ q$ uniformly on compact subsets of $G$ and $\varphi_i \circ q \in C_0(G)$ since $K$ is compact. Hence, by hypothesis, $\varphi \circ q \in C_0(G/K)$.

Conversely, suppose that $B_0(G/K)$ is $w^*$-closed in $B(G/K)$, and let $\varphi \in P(G)$ and $(\varphi_i) \subseteq P(G) \cap C_0(G)$ such that $\varphi_i \to \varphi$ uniformly on compact subsets of $G$. Define $\psi_i$ and $\psi$ on $G/K$ by

$$\psi_i(xK) = \int_K |\varphi_i(xk)|^2 dk \text{ and } \psi(xK) = \int_K |\varphi(xk)|^2 dk,$$

$x \in G$ ($dk$ being the normalized Haar measure on $K$). Then

$$\psi \in P(G/K) \text{ and } \psi_i \in P(G/K) \cap C_0(G/K),$$

and $\psi_i \to \psi$ uniformly on compact subsets of $G/K$. Hence $\psi \in C_0(G/K)$.

For $\delta \in \hat{K}$, let $\chi_\delta$ denote the corresponding minimal idempotent in $L^1(K)$. Then by the Cauchy-Schwarz inequality,

$$|\varphi * \chi_\delta(x)| \leq |\psi(xK)|^{1/2}$$

for every $x \in G$. Since $K$ is compact, this implies $\varphi * \chi_\delta \in C_0(G)$ for each $\delta \in \hat{K}$.

Now the linear span of $\{\chi_\delta : \delta \in \hat{K}\}$ is dense in $Z(L^1(K))$, the centre of $L^1(K)$. It follows that for any $f \in Z(L^1(K))$, $\varphi * f$ is a uniform limit on $G$ of finite linear combinations of functions $\varphi * \chi_\delta$, $\delta \in \hat{K}$. Hence $\varphi * f \in C_0(G)$ for every $f \in Z(L^1(K))$. Finally, taking for $f$ functions in $Z(L^1(K))$ with support shrinking to $\{e\}$, we easily conclude that $\varphi \in C_0(G)$. This completes the proof. 

We now turn to connected Lie groups. Theorem 2.7 below is the key result in this section.

Lemma 2.5. Let $G$ be a connected Lie group and $N$ a connected closed normal subgroup. If $B_0(G)$ is $w^*$-closed in $B(G)$, then the centre $Z(N)$ of $N$ is compact and $N/Z(N)$ is semisimple.

Proof. Let $R$ denote the radical of $N$. Then $R$ and $Z(N)$ are amenable normal subgroups of $G$, and therefore both must be compact by Lemma 2.2. Since $R$ is solvable and connected Lie, it is isomorphic to a torus $T^n$. Hence $\text{Aut}(R)$, the automorphism group of $R$, is discrete. Now, $G$ acts by conjugation on $R$, and this defines a continuous homomorphism from $G$ into $\text{Aut}(R)$. $G$ being connected, this homomorphism has to be trivial. This shows that $R$ is contained in the centre of $G$. So $R \subseteq Z(N)$ and hence $N/Z(N)$ is semisimple.

We remind the reader that a locally compact group $G$ is said to have Kazhdan’s property $(T)$ if the trivial representation $1_G$ is an isolated point in the dual space $\hat{G}$. An amenable group satisfies $(T)$ if and only if it is compact. On the other hand, many connected semisimple Lie groups and many discrete groups share property $(T)$. A comprehensive account on groups with property $(T)$ has been given in [14].
Lemma 2.6. Let \( G \) be a connected Lie group such that \( B_0(G) \) is \( w^* \)-closed in \( B(G) \). Then \( G \) has property \((T)\).

Proof. By Lemma 2.5, \( G \) is reductive with compact centre. Let

\[
\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m
\]

be the universal covering group of \( G \), where \( G_1, \ldots, G_m \) are simply connected Lie groups. Denote by \( Z_i \), the (discrete) centre of \( G_i \), \( i = 1, \ldots, m \). Then \( G = \tilde{G}/\Gamma \) for some discrete subgroup \( \Gamma \) of the centre \( Z(\tilde{G}) = \mathbb{R}^n \times Z_1 \times \cdots \times Z_m \) of \( \tilde{G} \). Moreover, \( Z(G) = Z(\tilde{G})/\Gamma \). Hence

\[
G/Z(G) = \tilde{G}/Z(\tilde{G}) = G_1/Z_1 \times \cdots \times G_m/Z_m.
\]

Since \( Z(G) \) is compact, \( B_0(G/Z(G)) \) is \( w^* \)-closed in \( B(G/Z(G)) \) (Lemma 2.4). Assume, towards a contradiction, that \( G \) does not have property \((T)\). Then \( G/Z(G) \) does not have property \((T)\). Hence some factor, say \( G_1/Z_1 \), fails to have property \((T)\) (see [35, Lemma 7.4.1]). Now, recall the following result due to Howe and Moore [35, Theorem 2.2.20]. If \( \pi \) is a unitary representation of a simple Lie group with finite centre and if there are no non-zero \( \pi \)-invariant vectors, then all the matrix coefficients of \( \pi \) vanish at infinity. Therefore, there exists a sequence

\[
(\varphi_n^{(1)}) \subseteq P(G_1/Z_1) \cap C_0(G_1/Z_1)
\]

converging to 1 uniformly on compact subsets of \( G_1/Z_1 \). Observe that \( G_1/Z_1 \) is not compact. Now choose arbitrary

\[
\varphi_k \in P(G_k/Z_k) \cap C_0(G_k/Z_k),
\]

\( k = 2, \ldots, m \), and set

\[
\varphi_n = \varphi_n^{(1)} \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}.
\]

Clearly, \( \varphi_n \) is a sequence of continuous positive definite functions on \( G/Z(G) \) that vanish at infinity, and

\[
\varphi_n \to 1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}
\]

uniformly on compact subsets of \( G/Z(G) \). Since \( 1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)} \) does not vanish at infinity, we have reached a contradiction. Thus \( G \) has property \((T)\). \( \square \)

Theorem 2.7. Let \( G \) be a connected Lie group. Then \( B_0(G) \) is \( w^* \)-closed in \( B(G) \) if and only if \( G \) is a reductive Lie group with compact centre and Kazhdan’s property \((T)\).

Proof. From Lemma 2.5, applied to \( N = G \), and Lemma 2.6 we know that \( w^* \)-closedness of \( B_0(G) \) in \( B(G) \) implies the stated conditions on \( G \).

Suppose now that \( G \) is a connected reductive Lie group with compact centre \( Z(G) \) and property \((T)\). According to Lemma 2.4 it suffices to show that \( B_0(G/Z(G)) \) is \( w^* \)-closed in \( B(G/Z(G)) \). Recall from the proof of Lemma 2.6 that \( G/Z(G) \) has a decomposition

\[
G/Z(G) = G_1/Z_1 \times \cdots \times G_m/Z_m,
\]

where \( G_1, \ldots, G_m \) are connected simple Lie groups with centres \( Z_1, \ldots, Z_m \), respectively. Set \( H = G/Z(G) \) and \( H_i = G_i/Z_i \) for \( i = 1, \ldots, m \), and observe that every \( H_i \) has property \((T)\). Of course, according to Lemma 2.4 we can assume that none of the \( H_i \) is compact.
Let \( \varphi_n \in P(H) \cap C_0(H) \), \( n \in \mathbb{N} \), such that \( \varphi_n \to \varphi \) uniformly on compact subsets of \( G \) for some \( \varphi \in P(H) \). Let \( \pi_n \) and \( \pi \) denote the representations of \( H \) associated to \( \varphi_n \) and \( \varphi \) through the GNS-construction. Then \( \pi \) is weakly contained in the direct sum \( \bigoplus_{n=1}^{\infty} \pi_n \).

We claim that the restriction \( \pi|_{H_i} \) of \( \pi \) to \( H_i \) does not contain the trivial representation \( 1_{H_i} \). Indeed, otherwise for some \( n \in \mathbb{N} \), \( \pi_n \) contains \( 1_{H_i} \) since \( H_i \) has property (T). However, since \( \varphi_n \in C_0(H) \), all the matrix coefficients of \( \pi_n \) vanish at infinity.

We have thus verified that \( H \) satisfies the hypotheses of the Howe-Moore theorem [35, Theorem 2.2.20]. It follows that all the matrix coefficients of \( \varphi \) vanish at infinity. This proves that \( \varphi \in C_0(H) \).\( \square \)

In order to deal with almost connected groups we need one more lemma.

**Lemma 2.8.** Let \( G \) be a locally compact group and \( H \) a closed subgroup such that \( G/H \) is compact. If \( B_0(H) \) is \( w^* \)-closed in \( B(H) \), then \( B_0(G) \) is \( w^* \)-closed in \( B(G) \).

**Proof.** For any function \( \phi \) on \( G \) and \( x, y \in G \), let

\[
x \phi(y) = \phi(x^{-1}y), \phi_x(y) = \phi(yx) \quad \text{and} \quad \phi^x(y) = \phi(x^{-1}yx).
\]

Notice first that if \( \psi \) is a positive definite function on \( G \), then \( \psi^x + \psi + x\psi + \psi_x \) and \( \psi^x + \psi + i(x\psi - \psi_x) \) are also positive definite for every \( x \in G \). Indeed, for all \( f \in L^1(G) \)

\[
\langle \psi^x + \psi + x\psi + \psi_x, f \rangle = \langle \psi, (\delta_x + \delta_e)^* f \ast (\delta_x + \delta_e) \rangle
\]

and

\[
\langle \psi^x + \psi + i(x\psi - \psi_x), f \rangle = \langle \psi, (\delta_e - i\delta_x)^* f \ast (\delta_e - i\delta_x) \rangle.
\]

Let \( (\varphi_i) \) be a net in \( P(G) \cap C_0(G) \) converging to some \( \varphi \in P(G) \) uniformly on compact subsets of \( G \). Then, uniformly on compact subsets of \( G \), \( \varphi_i^x \to \varphi^x \),

\[
\varphi^x_i + \varphi_i + x(\varphi_i) \to \varphi^x + \varphi + x\varphi + \varphi_x
\]

and

\[
\varphi^x_i + \varphi_i + i(x\varphi_i - (\varphi_i)x) \to \varphi^x + \varphi + i(x\varphi - \varphi_x)
\]

for every \( x \in G \). Thus, since \( B_0(H) \) is \( w^* \)-closed in \( B(H) \),

\[
\varphi|H, \varphi^x|H, (\varphi^x + \varphi + x\varphi + \varphi_x)|H \quad \text{and} \quad (\varphi^x + \varphi + i(x\varphi - \varphi_x))|H
\]

vanish at infinity on \( H \). It follows that \( x\varphi|H \in C_0(H) \) for each \( x \in G \). Since \( G/H \) is compact, employing the uniform continuity of \( \varphi \), it is easily verified that \( \varphi \in C_0(G) \).\( \square \)

The converse to Lemma 2.8 does not hold in general. That is, if \( B_0(G) \) is \( w^* \)-closed in \( B(G) \) and \( H \) is a closed cocompact subgroup of \( G \), then \( B_0(H) \) need not be \( w^* \)-closed in \( B(H) \). As an example, take for \( G \) a simply connected Lie group with finite centre and property (T) and for \( H \) a minimal parabolic subgroup. Then by the Howe-Moore result referred to in the proofs of Lemma 2.6 and Theorem 2.7, \( B_0(G) \) is \( w^* \)-closed in \( B(G) \), while \( B_0(H) \) fails to be \( w^* \)-closed in \( B(H) \) since \( H \) is non-compact and amenable.

**Corollary 2.9.** Let \( G \) be a connected Lie group and \( N \) a connected closed normal subgroup of \( G \) such that \( G/N \) is compact. Then \( B_0(G) \) is \( w^* \)-closed in \( B(G) \) if and only if \( B_0(N) \) is \( w^* \)-closed in \( B(N) \).
Proof. Suppose that $B_0(G)$ is $w^*$-closed in $B(G)$. By Lemma 2.5, $Z(N)$ is compact and $N/Z(N)$ is semisimple. Also, since $G$ has property $(T)$ by Lemma 2.6 and $G/N$ is compact, $N$ has property $(T)$ [34, Theorem 3.7]. By Theorem 2.7 this implies that $B_0(N)$ is $w^*$-closed in $B(N)$.

The converse is a special case of Lemma 2.8.

Theorem 2.10. Let $G$ be an almost connected locally compact group. Then $B_0(G)$ is $w^*$-closed in $B(G)$ if and only if the connected component $G_0$ of $G$ is a projective limit of reductive Lie groups with property $(T)$ and compact centres.

Proof. Suppose first that $G_0$ has the indicated structure. Choose a compact normal subgroup $K$ of $G_0$ such that $G_0/K$ is a reductive Lie group with property $(T)$ and compact centre. By Theorem 2.7, $B_0(G_0/K)$ is $w^*$-closed in $B(G_0/K)$. Since $K$ and $G/G_0$ are compact, an application of Lemmas 2.4 and 2.8 yields that $B_0(G)$ is $w^*$-closed in $B(G)$.

Conversely, suppose that $B_0(G)$ is $w^*$-closed in $B(G)$. $G$ being almost connected it is a projective limit of Lie groups $G/K_i$. Thus there are closed normal subgroups $H_i$ of finite index in $G$ such that $K_i \subseteq H_i$ and $H_i/K_i = (G/K_i)_0$. Then $G_0$ is the projective limit of the groups $G_0/G_0 \cap K_i$, and the $G_0/G_0 \cap K_i$ are connected Lie groups since $G_0/G_0 \cap K_i = G_0K_i/K_i$,

a closed connected subgroup of $G/K_i$. By Theorem 2.7 it suffices to show that $B_0(G_0/G_0 \cap K_i)$ is $w^*$-closed in $B(G_0/G_0 \cap K_i)$.

Now, since $B_0(G)$ is $w^*$-closed in $B(G)$, $B_0(G/K_i)$ is $w^*$-closed in $B(G/K_i)$ by Lemma 2.4, and hence $B_0(H_i/K_i)$ is $w^*$-closed in $B(H_i/K_i)$ by Lemma 2.3. Moreover, $G_0K_i/K_i$ is a cocompact connected normal subgroup of the connected Lie group $H_i/K_i$. Thus, by Corollary 2.9, $B_0(G_0K_i/K_i)$ is $w^*$-closed in $B(G_0K_i/K_i)$. This proves that $B_0(G_0/G_0 \cap K_i)$ is $w^*$-closed in $B(G_0/G_0 \cap K_i)$.

We conclude this section with some remarks.

Remarks 2.11. (i) The connected reductive Lie groups with property $(T)$ and compact centres are precisely the groups of the form $G = (\mathbb{R}^n \times G_1 \times \cdots \times G_m)/\Gamma$, where $G_1, \ldots, G_m$ are simple Lie groups not locally isomorphic to $\text{SO}(k, 1)$, $k \geq 2$, or $\text{SU}(k, 1)$, $k \geq 1$, and $\Gamma$ is a discrete cocompact subgroup of $\mathbb{R}^n \times \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_m$, the centre of $\mathbb{R}^n \times G_1 \times \cdots \times G_m$.

Indeed, if $G$ is of this form, then $G/Z(G) = G_1 \times \cdots \times G_m$, where $Z(G)$ denotes the centre of $G$, has property $(T)$ (see [14, Chap. 2, 13. Corollaire, 9. Remarque and Chap. 9]). As $Z(G)$ is compact, $G$ has property $(T)$ [14, Chap. 1, 9. Proposition].

Conversely, let $G$ be a connected reductive Lie group with property $(T)$ and compact centre. Let $\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$ be its universal covering group, where $G_1, \ldots, G_m$ are simple Lie groups with centres $Z_1, \ldots, Z_m$. The arguments used in the proof of Lemma 2.6 show that $G_1, \ldots, G_m$ have property $(T)$ and hence are not locally isomorphic to $\text{SO}(k, 1)$ or $\text{SU}(k, 1)$ (see [14, Chap. 6, 23. Corollaire]).

(ii) Let $G$ be a discrete group such that $B_0(G)$ is $w^*$-closed in $B(G)$. Then every element in $G$ has finite order. Indeed, this follows immediately by applying Lemma 2.2 to the cyclic subgroups of $G$.

(iii) Let $G$ be a linear group (that is, a subgroup $\text{GL}(n, K)$ for some field $K$) with the discrete topology. If $G$ is infinite, then $B_0(G)$ is not $w^*$-closed in $B(G)$. In fact, this is clear from (ii) if $G$ has an element of infinite order. On the other
hand, if $G$ is a torsion group, then it is well known to be locally finite [6, Theorem 9.2] and hence amenable, so that the claim follows from Lemma 2.2.

3. CHARACTERIZATIONS OF COMPACT GROUPS

If $G$ is a compact group then $A(G) = B(G)$. It seems likely that the converse is also true; i.e. $w^*$-closedness of $A(G)$ in $B(G)$ already forces $G$ to be compact. We have been able to show this for groups containing an almost connected open normal subgroup (Theorem 3.6). The case that remains open is that of a totally disconnected group.

We start with a lemma which will be generalized in Section 4 (Lemma 4.2).

**Lemma 3.1.** If $A(G)$ is $w^*$-closed in $B(G)$ and $H$ is an open subgroup of $G$, then $A(H)$ is $w^*$-closed in $B(H)$.

**Proof.** It suffices to show that the unit ball of $A(H)$ is $w^*$-closed in the unit ball of $B(H)$. Thus, let $(\varphi_i)$ be a net in $A(H)$ and $\varphi \in B(H)$ such that

$$\|\varphi_i\| \leq 1, \|\varphi\| \leq 1 \text{ and } \varphi_i \to \varphi$$

in the $w^*$-topology. Let $\tilde{\varphi}_i$ and $\tilde{\varphi}$ denote the trivial extensions of $\varphi_i$ and $\varphi$ to $G$. Then $\tilde{\varphi}_i \in A(G)$, $\|\tilde{\varphi}_i\| \leq 1$ and, for each $f \in L^1(G)$,

$$\int_G \tilde{\varphi}_i(x)f(x)dx = \int_H \varphi_i(h)f(h)dh \to \int_H \varphi(h)f(h)dh = \int_G \tilde{\varphi}(x)f(x)dx.$$ 

Hence $\tilde{\varphi}_i \to \tilde{\varphi}$ in the $\sigma(B(G), L^1(G))$-topology. Since $(\tilde{\varphi}_i)$ is a bounded net, it follows that $(\tilde{\varphi}_i)$ is $w^*$-convergent to $\tilde{\varphi}$. By hypothesis, $\tilde{\varphi} \in A(G)$ and so $\varphi \in A(H)$. \qed

**Lemma 3.2.** Let $K$ be a compact normal subgroup of $G$. If $A(G)$ is $w^*$-closed in $B(G)$, then $A(G/K)$ is $w^*$-closed in $B(G/K)$.

**Proof.** Consider the map $T_K : f \to T_K f$ from $L^1(G)$ onto $L^1(G/K)$ given by

$$T_K f(xK) = \int_K f(xk)dk.$$ 

This map extends to a $^*$-homomorphism from $C^*(G)$ onto $C^*(G/K)$ with dual map $T_K^* : B(G/K) \to B(G)$. Furthermore, $T_K^*(B(G/K))$ consists precisely of those functions in $B(G)$ that are constant on cosets of $K$ [8, (2.26)]. Also, since $K$ is compact,

$$T_K^*(A(G/K)) = A(G) \cap T_K^*(B(G/K)).$$

Now, let $\varphi_i \in A(G/K)$ such that $\varphi_i \to \varphi$ in the $w^*$-topology for some $\varphi \in B(G/K)$. Then

$$\langle T_K^*(\varphi_i), f \rangle = \langle \varphi_i, T_K(f) \rangle \to \langle \varphi, T_K(f) \rangle = \langle T_K^*(\varphi), f \rangle$$

for each $f \in C^*(G)$. Thus, by hypothesis, $T_K^*(\varphi) \in A(G)$ and so

$$T_K^*(\varphi) \in A(G) \cap T_K^*(B(G/K)),$$

whence $\varphi \in A(G/K)$. \qed

**Lemma 3.3.** Let $G$ be an almost connected locally compact group. If $A(G)$ is $w^*$-closed in $B(G)$, then $G$ is compact.
Lemma 3.1. Assume that $A$ is a point of continuity of the identity map $\phi$.

Proof. Since $A$ is $w^*$-closed in $B(G/K)$ for every compact normal subgroup $K$ of $G$ (Lemma 3.2), we can assume that $G$ is a Lie group. Being a compactly generated Lie group, $G$ is second countable and hence $A(G)$ is a separable Banach space. By hypothesis,

$$A(G) = B_\lambda(G) = C_r^w(G)^*.$$ 

Now, a $G^*$-algebra $A$ with separable dual Banach space has a countable dual $\hat{A}$ (see Section 1). It follows that $\hat{G}_r$, the reduced dual of $G$, is countable. Finally, by [2, Theorem 2.5] a separable Lie group with countable reduced dual is compact. This shows that $G$ is compact. \hfill \Box

Corollary 3.4. Let $G$ be any locally compact group and suppose that $A(G)$ is $w^*$-closed in $B(G)$. Then $G$ contains a compact open subgroup.

Proof. Since $G/G_0$ is totally disconnected, there exists an open subgroup $H$ of $G$ so that $H/G_0$ is compact. By Lemma 3.1, $A(H)$ is $w^*$-closed and hence $H$ is compact by Lemma 3.3.

Lemma 3.5. If $G$ is a discrete group and $A(G)$ is $w^*$-closed in $B(G)$, then $G$ is finite.

Proof. Assume that $G$ is infinite. Then $G$ has a countable infinite subgroup $H$. By Lemma 3.1, $A(H)$ is $w^*$-closed in $B(H)$. As in the proof of Lemma 3.3 we now conclude that $\tilde{H}_r$ is countable. Applying Baggett’s result again, it follows that $H$ is finite, a contradiction. \hfill \Box

Theorem 3.6. Suppose that $G$ contains an almost connected open normal subgroup. Then $A(G)$ contains a compact open normal subgroup.

Proof. Let $N$ be an almost connected open normal subgroup of $G$. Then $A(N)$ is $w^*$-closed in $B(N)$ by Lemma 3.1, and Lemma 3.3 implies that $N$ is compact. By Lemma 3.2, $A(G/N)$ is $w^*$-closed in $B(G/N)$. Since $N$ is open, Lemma 3.5 gives that $G/N$ is finite. Thus $G$ is compact. \hfill \Box

We now turn to a second characterization of compact groups in terms of certain properties of the $w^*$-topology on $B(G)$. If $G$ is a compact group, then the $w^*$-topology and the norm topology agree on the unit sphere of $B(G) = A(G)$ [11, Corollary 2]. We are going to establish the converse to this (see [22, Theorem 5] for the amenable case). Actually, we prove a stronger result in that we replace the unit sphere of $B(G)$ by the smaller set $P_\lambda(G) = B_\lambda(G) \cap P(G)$ of all normalized positive definite functions on $G$ associated to representations that are weakly contained in the left regular representation. Note that this property implies the Radon-Nikodym property for $B(G)$ but not conversely (see [11] and [33]).

For any locally compact group $G$, let $P_\lambda(G) = B_\lambda(G) \cap P(G)$, the set of all normalized continuous positive definite functions on $G$ associated to representations that are weakly contained in the left regular representation. $P_\lambda(G)$ is a $w^*$-compact convex subset of $B(G)$. We denote by $\text{ex}(P_\lambda(G))$ the set of extreme points of the $w^*$-compact convex subset $P_\lambda(G)$ of $B(G)$.

Lemma 3.7. Let $G$ be a locally compact group and $\varphi \in \text{ex}(P_\lambda(G))$. Suppose that $\varphi$ is a point of continuity of the identity map

$$(\text{ex}(P_\lambda(G)), w^*) \to (\text{ex}(P_\lambda(G)), \| \cdot \|).$$

Then $\pi_\varphi$ is an isolated point in $\hat{G}_r$. 
Proof. Notice first that \( \text{ex}(P_\lambda(G)) \subseteq \text{ex}(P(G)) \) because if \( \varphi \in P_\lambda(G) \) and \( \psi \in P(G) \) are such that \( c\varphi - \psi \) is positive definite for some \( c \geq 0 \), then \( \psi \in P_\lambda(G) \). By [5, 2.12.1], if \( \varphi_1, \varphi_2 \in \text{ex}(P(G)) \) and \( \pi_{\varphi_1} \) and \( \pi_{\varphi_2} \) are not equivalent, then \( \| \varphi_1 - \varphi_2 \| \geq 2 \). By assumption there exists a \( w^* \)-open subset \( U \) of \( \text{ex}(P_\lambda(G)) \) such that

\[
U \subseteq \{ \psi \in \text{ex}(P_\lambda(G)) : \| \psi - \varphi \| < 2 \}.
\]

It follows that \( \pi_\psi = \pi_\varphi \) for all \( \psi \in U \). Now, by [5, Theorem 3.4.11], the map \( q : \psi \to \pi_\psi \) from \( \text{ex}(P_\lambda(G)) \) onto \( \hat{G}_r \) is open. Thus \( \{ \pi_\varphi \} = q(U) \) is open in \( \hat{G}_r \). \( \square \)

**Lemma 3.8.** Let \( H \) be an open subgroup of \( G \). If the identity map from \( (P_\lambda(G), w^*) \) to \( (P_\lambda(H), \| \cdot \|) \) is continuous, then the identity map from \( (P_\lambda(H), w^*) \) to \( (P_\lambda(H), \| \cdot \|) \) is continuous.

**Proof.** For any \( \varphi \in P_\lambda(H) \), the trivial extension \( \tilde{\varphi} \) belongs to \( P_\lambda(G) \). Indeed, \( \tilde{\varphi} \) is a positive definite function associated to the induced representation \( \text{ind}_H^G \pi_\varphi \), and \( \pi_\varphi \prec \lambda_H \) implies

\[
\text{ind}_H^G \pi_\varphi \prec \text{ind}_H^G \lambda_H = \lambda_G.
\]

Let \( (\tilde{\varphi}_\alpha) \) be a net in \( P_\lambda(H) \) converging to \( \varphi \in P_\lambda(H) \) in the \( w^* \)-topology. Then \( \tilde{\varphi}_\alpha \to \tilde{\varphi} \) in the \( w^* \)-topology on \( P_\lambda(G) \) (compare the proof of Lemma 3.1). By hypothesis, \( \| \tilde{\varphi}_\alpha - \tilde{\varphi} \| \to 0 \) and hence \( \| \varphi_\alpha - \varphi \| \to 0 \). \( \square \)

**Theorem 3.9.** For any locally compact group \( G \) the following conditions are equivalent.

(i) \( G \) is compact.

(ii) The \( w^* \)-topology and the norm topology agree on the unit sphere of \( B(G) \).

(iii) The \( w^* \)-topology and the norm topology agree on \( P_\lambda(G) \).

**Proof.** As mentioned above, (i) \( \Rightarrow \) (ii) is due to Granirer and Leinert [11]. Since (ii) \( \Rightarrow \) (iii) is trivial, it only remains to prove (iii) \( \Rightarrow \) (i).

Assume that \( G \) fails to be compact. Then \( G \) contains a non-compact, \( \sigma \)-compact, open subgroup \( H \). By Lemma 3.8, the \( w^* \)-topology and the norm topology coincide on \( P_\lambda(H) \). It follows from Lemma 3.7 that \( \hat{H}_r \) is discrete. Since \( H \) is \( \sigma \)-compact, Theorem 7.6 of [34] now shows that \( H \) is compact, a contradiction. \( \square \)

4. **When is \( A_\gamma(G) \) \( w^* \)-closed in \( B(G) \)?**

For a locally compact group \( G \) and any unitary representation \( \pi \) of \( G \), the Fourier space \( A_\pi(G) \) associated to \( \pi \) is defined to be the norm-closed linear subspace of \( B(G) \) generated by all the coordinate functions of \( \pi \) [1], that is, the functions of the form \( x \to \langle \pi(x)\xi, \eta \rangle, \xi, \eta \in \mathcal{H}_\pi \).

The conjugation representation \( \gamma_G \) (or simply \( \gamma \), if no confusion can arise) on \( L^2(G) \) is defined by

\[
\gamma_G(x)f(y) = \Delta(x)^{1/2}f(x^{-1}yx),
\]

\( f \in L^2(G), x, y \in G \). The purpose of this section is to investigate the question of when \( A_\gamma(G) \) is \( w^* \)-closed in \( B(G) \). It will turn out that this is closely related to problems on the support of \( \gamma \) as studied in [21]. We start with two simple facts on \( A_\pi(G) \) for general representations \( \pi \).
Lemma 4.1. If \( G \) is a compact group, then \( A_\pi(G) \) is \( w^* \)-closed in \( B(G) \) for every representation \( \pi \) of \( G \).

Proof. The \( w^* \)-closure \( \overline{A_\pi(G)} \) of \( A_\pi(G) \) is the dual space of the \( C^* \)-algebra \( \pi(C^*(G)) \), which is a quotient of \( C^*(G) \). Hence each \( \varphi \in \overline{A_\pi(G)} \) is a linear combination of positive definite functions in \( A_\pi(G) \). Therefore, it suffices to prove that every positive definite \( \varphi \in \overline{A_\pi(G)} \) actually is in \( A_\pi(G) \).

For that, notice that there is a net \( (\varphi_\alpha) \) in \( A_\pi(G) \) such that \( \varphi_\alpha \to \varphi \) in the \( w^* \)-topology and \( \| \varphi_\alpha \| \to \| \varphi \| \) (compare [10, p. 565]). By [11, Theorem A] it follows that \( \| \varphi_\alpha \psi - \varphi \psi \| \to 0 \) for every \( \psi \in A(G) \). In particular, \( \| \varphi_\alpha - \varphi \| \to 0 \) by setting \( \psi = 1 \in B(G) = A(G) \). This shows that \( \varphi \in A_\pi(G) \).

Lemma 4.2. Suppose that \( \pi \) is a representation of \( G \) such that \( A_\pi(G) \) is \( w^* \)-closed in \( B(G) \). Then \( A_{\pi|H}(H) \) is \( w^* \)-closed in \( B(H) \) for every open subgroup \( H \) of \( G \).

Proof. Recall that, by [1, Theorem 2.2], \( \varphi \in B(G) \) belongs to \( A_\pi(G) \) if and only if \( \varphi \) can be written as

\[
\varphi = \sum_{n=1}^{\infty} \langle \pi(\cdot) \xi_n, \eta_n \rangle
\]

where \( \xi_n, \eta_n \in H_\pi \) and \( \sum_{n=1}^{\infty} \| \xi_n \| \cdot \| \eta_n \| < \infty \). In particular, \( A_{\pi|H}(H) = A_{\pi}(G)|H \).

It suffices to show that the unit ball of \( A_{\pi|H}(H) \) is \( w^* \)-closed in the unit ball of \( B(H) \). Thus, let \( \varphi_i \in A_{\pi|H}(H) \), \( i \in I \), and \( \varphi \in B(H) \) such that \( \| \varphi_i \| \leq 1, \| \varphi \| \leq 1 \) and \( \varphi_i \to \varphi \) in the \( w^* \)-topology. Choose representations

\[
\varphi_i = \sum_{n=1}^{\infty} \langle \pi(\cdot) \xi_{in}, \eta_{in} \rangle
\]

such that \( \sum_{n=1}^{\infty} \| \xi_{in} \| \cdot \| \eta_{in} \| \leq 2 \) (see [1, Proposition 2.9]). Define

\[
\psi_i(x) = \sum_{n=1}^{\infty} \langle \pi(x) \xi_{in}, \eta_{in} \rangle
\]

for all \( x \in G \) and \( i \in I \). Then \( \psi_i \in A_\pi(G) \) and \( \| \psi_i \| \leq 2 \). Since the unit ball in \( B(G) \) is \( w^* \)-compact, we can assume that \( \psi_i \to \psi \) in the \( w^* \)-topology for some \( \psi \in B(G) \). Now, \( A_\pi(G) \) is \( w^* \)-closed in \( B(G) \), so that \( \psi \in A_\pi(G) \) and hence \( \psi|H \in A_{\pi|H}(H) \).

On the other hand, the restriction map \( B(G) \to B(H) \) is \( w^* \)-continuous as \( H \) is open (see Section 5). Indeed, this follows from the fact that \( C^*(H) \) is a subalgebra of \( C^*(G) \) whenever \( H \) is open in \( G \). Thus

\[
\varphi_i = \psi_i|H \to \psi|H \text{ and } \varphi_i \to \varphi
\]

in the \( w^* \)-topology. This proves \( \varphi = \psi|H \in A_{\pi|H}(H) \).

We now apply the preceding lemmas to the conjugation representation. The following corollary will be used several times in the sequel.

Corollary 4.3. Suppose that \( G \) is second countable and \( A_{\gamma G}(G) \) is \( w^* \)-closed in \( B(G) \). Then, for every open subgroup \( H \) of \( G \), \( \text{supp } \gamma_H \) is countable.

Proof. Since \( G \) is second countable, \( A_{\gamma G}(G) \) is norm separable. Since the restriction map from \( B(G) \) to \( B(H) \) is norm continuous, \( A_{\gamma G|H}(H) = A_{\gamma G}(G)|H \) is norm...
separable. Now, since $\gamma_H$ is a subrepresentation of $\gamma_G|H$ and $A_{\gamma_G|H}(H)$ is $w^*$-closed in $B(H)$ by Lemma 4.2,

$$A_{\gamma_H}(H)^{w^*} \subseteq A_{\gamma_G|H}(H),$$

so that $A_{\gamma_H}(H)^{w^*}$ is norm separable. Thus $\gamma_H(C^*(H))$ has a norm separable dual Banach space, $A_{\gamma_H}(H)^{w^*}$, and hence

$$\text{supp } \gamma_H = \gamma_H(C^*(H))^\sim$$

is countable. \hfill $\square$

**Corollary 4.4.** Let $Z(G)$ denote the centre of $G$. If $G/Z(G)$ is compact, then $A_{\gamma_G}(G)$ is $w^*$-closed in $B(G)$.

**Proof.** For $z \in Z(G)$, $\gamma_G(z)$ is the identity on $L^2(G)$. Thus $\pi(xZ(G)) = \gamma_G(x)$, $x \in G$, defines a representation of $G/Z(G)$, and therefore $A_{\pi}(G/Z(G))$ is $w^*$-closed in $B(G/Z(G))$ by Lemma 4.1. Denoting by $q : G \to G/Z(G)$ the quotient homomorphism, we have

$$A_{\gamma_G}(G) = A_{\pi}(G/Z(G)) \circ q.$$

By [1, (2.10)], $A_{\gamma_G}(G)$ is $w^*$-closed in $B(G)$. \hfill $\square$

Our goal is to establish the converse to Corollary 4.4 for Lie groups with countably many connected components (Theorem 4.8). Apart from using various results from [21], a major step in proving the theorem will be the next lemma.

We remind the reader that a group $G$ is called an FC-group if all its conjugacy classes are finite. Such a group, more generally every locally compact group all of whose conjugacy classes are relatively compact, is amenable.

**Lemma 4.5.** Let $G$ be a countable discrete FC-group. If $\text{supp } \gamma_G$ is countable, then $G$ has a finite commutator subgroup.

**Proof.** Let $S = \text{supp } \gamma_G$ and notice first that points in $S$ are closed in $\hat{G}$. Indeed, the primitive ideal space of any FC-group is a $T_1$ space (see [28, Theorem 5.2]) and $C^*_\sigma(G)$, being a separable $C^*$-algebra with countable dual, is of type I. Thus the points of $S$ are closed in $\hat{G}$. Since $C^*(G)$ is unital, it follows that every $\sigma \in S$ is finite dimensional.

Next, employing the facts that points in $S$ are closed, that $S$ is countable and that duals of $C^*$-algebras are Baire spaces [5, (3.4.13)], a straightforward argument yields the existence of some dense subset $D$ of $S$ consisting of points that are also open in $S$.

Since $G$ is a countable amenable group,

$$\bigcup_{\pi \in \hat{G}} \text{supp}(\pi \otimes \pi)$$

is a dense subset of $S$ by [19, Theorem]. Let $C(S)$ denote the set of all closed subsets of $S$, endowed with Fell’s topology [10, p. 427]. By [18, Proposition 2], the mapping

$$\pi \mapsto \text{supp}(\pi \otimes \pi), \hat{G} \to C(S)$$

is continuous. It follows that

$$V = \{\pi \in \hat{G} : \text{supp}(\pi \otimes \pi) \cap D \neq \emptyset\}$$

is non-empty and open in $\hat{G}$. \hfill $\square$
Now, as points in $D$ are open in $S$, supp($\pi \otimes \pi$) contains a finite dimensional subrepresentation for each $\pi \in V$. However, $\pi \otimes \pi$ then also contains the trivial representation $1_G$. This can be seen as follows. Suppose that $\tau$ is finite dimensional and that $\pi \leq \pi \otimes \pi$. Then

$$1_G \leq \tau \otimes \tau \leq \pi \otimes \pi \otimes \tau,$$

and $\pi \otimes \tau$ is a (finite) direct sum of irreducible representations $\rho_1, \ldots, \rho_n$. Thus $1_G \leq \pi \otimes \pi_i$ for some $i$, which is impossible unless $\rho_i \sim \pi$ (see [17, Proposition 2.4]).

As is well-known, $1_G$ is finite dimensional. If $\pi$ is finite dimensional and $\pi \otimes \pi \otimes \tau$, then the commutator subgroup has to be finite. In fact, this has been shown in [7, Theorem 3.2.3] as an application of representation theory of crossed product $C^*$-algebras. We prefer to outline a direct argument for this within the framework of Mackey’s unitary representation theory of group extensions [26].

Finally, for a discrete FC-group $G$, the existence of a non-empty open subset in $\hat{G}$ consisting of finite dimensional representations implies that $G$ has a finite commutator subgroup. In fact, in this case the left regular representation of $G$ has a subrepresentation of type I, and then the commutator subgroup has to be finite by [16, Satz 1] (see also [32, Theorem 3]).

**Lemma 4.6.** Let $G$ be a locally compact group with open centre. Then, for each $a \in G$,

$$\text{ind}_{C(a)}^G 1_{C(a)} \sim \gamma_G,$$

where $C(a)$ denotes the centralizer of $a$ in $G$.

**Proof.** Since ind$^G_{C(a)} 1_{C(a)}$ is the cyclic representation of $G$ defined by the positive definite function $\chi_{C(a)}$, the characteristic function of $C(a)$, and since $C(a)$ is open, it suffices to show that given any compact subset $K$ of $G \setminus C(a)$, there is a positive definite function $\varphi$ associated to $\gamma_G$ such that $\varphi(x) = 1$ for all $x \in C(a)$ and $\varphi(x) = 0$ for all $x \in K$. Now, since

$$C = \{a^{-1}x^{-1}ax : x \in K\}$$

is compact and $e \notin C$, we find an open neighborhood $V$ of $e$, contained in the centre of $G$, such that $CV \cap V = \emptyset$. Let

$$\varphi(x) = |V|^{-1} \langle \gamma_G(x) \chi_{aV}, \chi_{aV} \rangle$$

for $x \in G$. Then it is easily verified that $\varphi(x) = 1$ for all $x \in C(a)$ and $\varphi(x) = 0$ for all $x \in K$. \hfill $\Box$

**Lemma 4.7.** Suppose that $G$ is second countable and contains an open normal subgroup $N$ such that $G/N$ is abelian and every irreducible representation of $N$ is finite dimensional. If supp $\gamma_G$ is countable, then every irreducible representation of $G$ is finite dimensional.

**Proof.** Given an irreducible representation $\pi$ of $G$, there exist a subgroup $H$ of $G$ and a finite dimensional irreducible representation $\tau$ of $H$ such that $N \leq H$ and $\pi \sim \text{ind}_H^G \tau$.

In fact, this has been shown in [7, Theorem 3.2.3] as an application of representation theory of crossed product $C^*$-algebras. We prefer to outline a direct argument for this within the framework of Mackey’s unitary representation theory of group extensions [26].

Choose $\gamma \in \hat{N}$ such that $\pi|N \sim \gamma_N$, the $G$-orbit of $\gamma$ in $\hat{N}$ under the action of $G$. Let $S$ denote the stability subgroup of $\gamma$. By [26, Theorems 8.2 and 8.3], there
exist a multiplier $\omega$ on $S/N$, an irreducible $\omega$-representation $\rho$ of $S$ in $\mathcal{H}_\omega$ and an irreducible $\omega$-representation $\sigma$ of $S/N$ such that
\[ \pi = \text{ind}^G_S(\rho \otimes \sigma). \]
Now, an irreducible $\omega$-representation of an abelian group is weakly equivalent to the $\omega$-representation induced from some one-dimensional $\omega$-representation of a certain subgroup. Hence there exist a subgroup $H$ of $S$ containing $N$ and a one-dimensional $\omega$-representation $\lambda$ of $H/N$ such that
\[ \rho \otimes \sigma \sim \rho \otimes \text{ind}^H_S(\lambda). \]
Let $\pi = \rho|H \otimes \lambda$, a finite dimensional ordinary representation. Then
\[ \pi \sim \text{ind}^G_S(\text{ind}^H_S(\rho|H \otimes \lambda)) = \text{ind}^G_H \tau, \]
as required.
It follows that
\[ \pi \otimes \pi = \text{ind}^G_H(\tau \otimes \pi|H) \succ \text{ind}^G_H(\tau \otimes \pi) \succ \text{ind}^G_H(G/H). \]
On the other hand, $\pi \otimes \pi$ is weakly contained in $\gamma_G$ since $G$ is amenable. Now, as $\text{supp} \gamma_G$ is countable and $H$ is open and $G/H$ is abelian, $H$ must have finite index in $G$. Thus, $\text{ind}^G_H \tau$ is finite dimensional and hence so is $\pi$. \hfill \Box

**Theorem 4.8.** Suppose that $G$ is a Lie group with countably many connected components. If $A_{\gamma_G}(G)$ is $w^*$-closed in $B(G)$, then $G/Z(G)$ is compact.

**Proof.** For any normal subgroup $N$ of $G$, let $q_N : G \to G/N$ denote the quotient homomorphism. Since $A_{\gamma_G}(G)$ is $w^*$-closed in $B(G)$ and $G$ is second countable, and since the connected component $G_0$ of $e$ is open, $\text{supp} \gamma_{G_0}$ is countable by Corollary 4.3. Theorem 3.4 of [21] yields that $G_0 = V \times C$, the direct product of a vector group $V$ and a compact group $C$. Clearly, $C$ is normal in $G$.

We claim that $V = G_0/C$ is contained in the centre of $G/C$. To that end, fix $a \in G$ and consider the open subgroup $H$ of $G$ generated by $a$ and $G_0$. Then $\text{supp} \gamma_H$ is countable (Corollary 4.3), and since
\[ \gamma_{G/C} \circ q_C \prec \gamma_H \]
due to the compactness of $C$ [19, Remark 1], it follows that $\gamma_{H/C}$ has a countable support. Since $G_0/C$ is a vector group and $(H/C)/(G_0/C) = H/G_0$ is abelian, an application of Lemma 3.2 in [21] shows that $G_0/C$ is in the centre of $H/C$. Since $a$ is arbitrary, $G_0/C$ is central in $G/C$.

Next we are going to prove that $G/G_0$ has a finite commutator subgroup. Passing to $F = G/C$, we know that $\gamma_F$ has countable support, and since $V$ is central in $F$, by [21, Lemma 1.1]
\[ \gamma_{F/V} \circ q_V \prec \gamma_F, \]
so that $\text{supp} \gamma_{F/V}$ is countable. Let $D = F/V$ and denote by $D_f$ the finite conjugacy class subgroup of $D$. Then, by [20, Theorem 1.8]
\[ \lambda_{D/D_f} \circ q_{D_f} \prec \gamma_D, \]
where $\lambda_{D/D_f}$ is the regular representation of $D/D_f$. Thus the reduced dual of $D/D_f$ is countable, and hence $D/D_f$ is finite (see [2, 34]). At this stage we know in particular that $G$ is amenable, so that $\gamma_{G/N} \circ q_N \prec \gamma_G$ for each closed normal
subgroup \( N \) of \( G \) [21, Lemma 1.1]. Also, \( \gamma_{D_f} \) has countable support and therefore the commutator subgroup \( D_f' \) of \( D_f \) is finite by Lemma 4.5. Let \( N \) be the inverse image of \( D_f' \) in \( G \) and let \( E = G/N \). Then \( \gamma_E \) has countable support, and \( E \) possesses an abelian normal subgroup \( A \) of finite index, namely \( A = D_f/D_f' \). We apply Lemma 4.6 to \( E \) and obtain that
\[
\text{ind}_{C(x)}^E 1_{C(x)} \prec \gamma_E
\]
for every \( x \in E \). It follows that
\[
\text{ind}_{C(x)}^A 1_{C(x)} \prec (\text{ind}_{C(x)}^E 1_{C(x)})|A \prec \gamma_E|A,
\]
and \( \gamma_E|A \) has countable support since \( \gamma_E \) does and \( E/A \) is finite. However, \( A \) being abelian, countability of
\[
(A/C(x) \cap A)^\gamma = \text{supp}(\text{ind}_{C(x)}^A 1_{C(x)}|A)
\]
implies that \( C(x) \cap A \) is of finite index in \( A \). Hence \( C(x) \) has finite index in \( E \) for every \( x \in E \). Therefore \( G/N \) is an FC-group, and Lemma 4.5 implies that \( E \) has a finite commutator subgroup. Recalling that \( N/G_0 = D_f' \) is finite, we conclude that \( G/G_0 \) has a finite commutator subgroup.

Since \( G/G_0 \) has a finite commutator subgroup, there exists a normal subgroup \( N \) of \( G \) such that \( G_0 \subseteq N \), \( N/G_0 \) is finite and \( G/N \) is abelian. Now, \( G_0 = V \times C \), and this implies that all the irreducible representations of \( N \) are finite dimensional. Since \( \text{supp} \gamma_G \) is countable, Lemma 4.7 shows that \( G \) has only finite dimensional irreducible representations.

On the other hand, \( G \) has a relatively compact commutator subgroup. To see this, notice first that since \( V \) is contained in the centre of \( N/C \), \( N/C \) has a finite commutator subgroup. Denoting its inverse in \( G \) by \( K \), \( N/K \) is isomorphic to \( V \). By arguments that have previously been used, \( V \) is central in \( G/K \), and applying Lemma 4.6 again, this time to \( G/K \), we conclude that \( G/K \) is a group with finite conjugacy classes.

Thus \( G \) is a group with relatively compact conjugacy classes all of whose irreducible representations are finite dimensional. Since, in addition, \( G \) is a Lie group, combining Theorem 2 of [27] and Lemma 5.4 of [24] shows that \( G/Z(G) \) is compact.

\( \square \)

5. \( w^* \)-continuity of the restriction map

Let \( G \) be a locally compact group, and let \( H \) be a closed subgroup of \( G \). In this section we study the question of when the restriction map
\[
\Phi : B(G) \to B(H), \varphi \to \varphi|H
\]
is continuous for the \( w^* \)-topologies. Clearly, if \( H \) is open then \( C^*(H) \) is a subalgebra of \( C^*(G) \) and hence \( \Phi \) is \( w^* \)-continuous. We are going to establish the following converse.

**Theorem 5.1.** Let \( H \) be a closed subgroup of the locally compact group \( G \). If the restriction map \( B(G) \to B(H) \) is continuous for the \( w^* \)-topologies, then \( H \) is open in \( G \).

Let \( M(G) \) denote the algebra of all bounded measures on \( G \). The canonical embedding of \( L^1(G) \) into \( M(G) \) extends to an isometric \( * \)-homomorphism of \( C^*(G) \)
into $C^*(M(G))$, the enveloping $C^*$-algebra of $M(G)$. Therefore we may (and shall) identify $C^*(G)$ with a closed two-sided ideal of $C^*(M(G))$.

Let $H$ be a closed subgroup of $G$. For $f \in L^1(H)$, let $\mu_f \in M(G)$ denote the measure on $G$ defined by $f$. The mapping $f \mapsto \mu_f$ from $L^1(H)$ into $M(G)$ extends to a *-homomorphism $\Psi: C^*(H) \to C^*(M(G))$. By general principles, the restriction map $\Phi : B(G) \to B(H)$ is $w^*$-continuous if and only if $\Phi$ is the transpose of some continuous linear mapping $\Theta : C^*(H) \to C^*(G)$. It is easy to verify that, in this case, $\Theta$ has to agree with $\Psi$ on $L^1(H)$. Hence we have the following lemma.

**Lemma 5.2.** The restriction map $\Phi : B(G) \to B(H)$ is $w^*$-continuous if and only if the range of $\Psi : C^*(H) \to C^*(M(G))$ is contained in $C^*(G)$.

**Remark 5.3.** It is interesting to notice that the homomorphism $\Psi : C^*(H) \to C^*(M(G))$ need not always be injective (see [4]).

Now, let $\rho$ denote the right regular representation of $G$ on $L^2(G)$ and $C_r^*(M(G))$ the $C^*$-subalgebra of $L(L^2(G))$ generated by the set of all operators $\rho(\mu)$, $\mu \in M(G)$. The mapping $f \mapsto \mu_f$ from $L^1(H)$ into $M(G)$ extends to a *-homomorphism

$$\Psi_r : C^*(H) \to C_r^*(M(G)).$$

It is clear that $\Psi_r$ is the composition of $\Psi$ and the canonical homomorphism from $C^*(M(G))$ onto $C_r^*(M(G))$.

In view of Lemma 5.2, we thus observe that Theorem 5.1 will be a consequence of the following stronger result.

**Theorem 5.4.** Let $H$ be a closed subgroup of the locally compact group $G$. If the range of the homomorphism $\Psi_r : C^*(H) \to C_r^*(M(G))$ is contained in $C_r^*(G)$, then $H$ is open in $G$.

The proof of Theorem 5.4 depends on two elementary lemmas, the first of which appears also in [25]. For the sake of completeness, however, we give a very short and different proof.

**Lemma 5.5.** Let $G$ be a locally compact group and let $T \in C_r^*(G)$. Suppose that $K$ is a compact subset of $G$ and regard $L^2(K)$ as a closed subspace of $L^2(G)$ in the usual manner. Then the restriction

$$T|L^2(K) : L^2(K) \to L^2(G)$$

of $T$ to $L^2(K)$ is a compact operator.

**Proof.** Of course, it suffices to prove the statement for operators $T$ of the form $T = \rho(f)$ where $f \in C_c(G)$. Let $K' = \text{supp} \ f \cdot K$. Then $Tg \in L^2(K')$ for all $g \in L^2(K)$. Choose $f_j \in C_c(K')$ and $g_j \in C_c(K)$ such that, as $n \to \infty$,

$$\int_G \int_G \left| \sum_{j=1}^n f_j(x)g_j(y) - f(xy) \right|^2 \ dx \ dy \to 0.$$ 

It is straightforward to verify that this implies that $T : L^2(K) \to L^2(K')$ is a norm limit of finite rank operators.

**Lemma 5.6.** Let $(X, \mu)$ be a probability measure space without atoms. Then the canonical embedding $L^2(X) \to L^1(X)$ fails to be compact.
Proof. It suffices to show that there is an orthonormal sequence \((f_n)_n\) in \(L^2(X)\) such that \(\|f_n\|_\infty \leq 1\). Indeed, we then have

\[ 2 = \|f_n - f_m\|_2^2 = \int_X |f_n(x) - f_m(x)|^2 d\mu(x) \leq \|f_n - f_m\|_1 \]

for all \(n, m \in \mathbb{N}\), \(n \neq m\), and hence no subsequence of \((f_n)_n\) can converge in \(L^1(X)\).

Since \(\mu\) has no atoms, there exists, for every \(0 \leq r \leq 1\), a measurable subset \(A\) of \(X\) with \(\mu(A) = r\) (see [13, Section 4.1, Exercise 1]). Therefore, we can choose inductively, for every \(n \in \mathbb{N}\), disjoint measurable subsets

\[ A_1^{(n)}, A_2^{(n)}, \ldots, A_{2^n-1}^{(n)} \]

of \(X\) with the following properties:

(i) \(\mu(A_i^{(n)}) = \frac{1}{2^n}\) for \(i = 1, \ldots, 2^n-1\).

(ii) \(A_{2i-1}^{(n+1)}, A_{2i}^{(n+1)} \subseteq A_i^{(n)}\) for \(i = 1, \ldots, 2^{n-1}\).

For each \(n \in \mathbb{N}\), define a function \(f_n \in L^2(X)\) (a kind of Rademacher function) by setting

\[ f_n(x) = (-1)^i \text{ for } x \in A_i^{(n)} \text{ and } i = 1, \ldots, 2^{n-1}. \]

By construction and (1) and (2), we obviously have

\[ \|f_n\|_\infty = \|f_n\|_2 = 1 \text{ and } \int_X f_n(x)f_m(x)d\mu(x) = 0, \quad n \neq m. \]

This completes the proof. \(\square\)

Proof of Theorem 5.4. Let \(H\) be a closed subgroup of \(G\) and suppose \(\Psi_r(C^*(H))\) is contained in \(C^*_c(G)\). Fix a Bruhat function \(\beta\) on \(G\) for \(H\) (see [31, Chapter 8]), that is, a non-negative continuous function \(\beta\) on \(G\) with the following properties:

(i) For every compact subset \(K\) of \(G\), \(\beta\) agrees on \(KH\) with the restriction of some function from \(C_c(G)\).

(ii) \(\int_H \beta(xh)dh = 1\) for all \(x \in G\).

Let \(\Delta_G\) and \(\Delta_H\) denote the modular functions of \(G\) and \(H\), respectively, and define a function \(q\) on \(G\) by

\[ q(x) = \int_H \beta(xh)\Delta_G(h)\Delta_H(h)^{-1} dh. \]

Since \(q(xh) = q(x)\Delta_G(h)^{-1}\Delta_H(h)\) for all \(x \in G\) and \(h \in H\), there exists a quasi-invariant measure \(d_q\hat{\beta}\) on \(\hat{G} = G/H\) such that

\[ \int_{G} \left( \frac{f(xh)}{q(xh)} \right) d_q\hat{\beta} = \int_{\hat{G}} f(x)dx \]

for all \(f \in L^1(G)\) [31, Chapter 8].

Now let \(\hat{K} \subseteq \hat{G}\) be a compact neighbourhood of \(\hat{e}\) in \(\hat{G}\). We are going to prove that \(\hat{K}\) is finite. Clearly, this will imply that \(H\) is open. Let \(\pi : G \to \hat{G}\) denote the quotient map and choose a compact neighbourhood \(K\) of \(e\) in \(G\) so that \(\pi(K) = \hat{K}\).

Fix a continuous function \(f\) with compact support on \(H\) such that

\[ \int_H f(h)\Delta_G(h)^{-1} dh \neq 0. \]
Let \( K' = KH \cap \text{supp} \beta \), which is a compact subset of \( G \). By Lemma 5.5, the restriction of \( \rho(\mu_f) \), the right convolution operator defined by \( \mu_f \), to \( L^2(K') \) is a compact operator. Define a linear mapping

\[
T : L^2(\dot{K}, d_{\dot{q}} \dot{x}) \to L^2(K')
\]

by \( T \dot{g}(x) = \dot{g}(\dot{x}) \beta(x)^{1/2} \) for \( \dot{g} \in L^2(\dot{K}, d_{\dot{q}} \dot{x}) \) and \( x \in K \). Since

\[
\int_H \frac{\beta(xh)}{q(xh)} dh = \frac{1}{q(x)} \int_H \beta(xh) \Delta_G(h) \Delta_H(h)^{-1} dh = 1
\]

for all \( x \in G \), we have

\[
\|T \dot{g}\|_2^2 = \int_G \left( \int_H \frac{|\dot{g}(\dot{x})|^2 \beta(xh)}{q(xh)} dh \right) d_{\dot{q}} \dot{x} = \int_G |\dot{g}(\dot{x})|^2 d_{\dot{q}} \dot{x} = \|\dot{g}\|_2^2.
\]

Thus \( T \) is a bounded linear operator. Set

\[
q_1(x) = \int_H \beta(xh)^{1/2} \Delta_G(h) \Delta_H(h)^{-1} dh
\]

for \( x \in G \). Then, for all \( x \in G \) and \( h \in H \),

\[
q_1(xh) = q_1(x) \Delta_G(h)^{-1} \Delta_H(h).
\]

Hence there exists a quasi-invariant measure \( d_{q_1} \dot{x} \) on \( \dot{G} \) such that

\[
\int_{\dot{G}} \left( \int_H \frac{q(xh)}{q_1(xh)} dh \right) d_{q_1} \dot{x} = \int_G g(x) dx
\]

for all \( g \in L^1(G) \). Let

\[
S : L^1(G) \to L^1(\dot{G}, d_{q_1} \dot{x})
\]

be the linear operator defined by

\[
Sg(\dot{x}) = \int_H \frac{g(xh)}{q_1(xh)} dh
\]

for \( g \in C_c(G) \). \( S \) is bounded since

\[
\|Sg\|_{L^1(\dot{G}, d_{q_1} \dot{x})} \leq \int_{\dot{G}} \left( \int_H \frac{|g(xh)|}{q_1(xh)} dh \right) d_{q_1} \dot{x} = \|g\|_1.
\]

Observe next that \( \rho(\mu_f) \) maps \( L^2(K') \) into \( L^2(K''') \) for some compact subset \( K''' \) of \( G \). Hence \( \rho(\mu_f)T \) map \( L^2(\dot{K}, d_{\dot{q}} \dot{x}) \) into \( L^1(G) \), and we may consider the bounded linear operator

\[
S\rho(\mu_f)T : L^2(\dot{K}, d_{\dot{q}} \dot{x}) \to L^1(\dot{G}, d_{q_1} \dot{x}).
\]

For \( \dot{g} \in C(\dot{K}) \), we have

\[
S\rho(\mu_f)T(\dot{g})(\dot{x}) = \dot{g}(\dot{x}) \int_H \frac{1}{q_1(xh)} \left( \int_H \beta(xhk)^{1/2} f(k) dk \right) dh
\]

\[
= \dot{g}(\dot{x}) \int_H f(k) \left( \int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)} dh \right) dk.
\]
However, for $x \in G$ and $k \in K$,
\[
\int_{\mathcal{H}} \frac{\beta(xhk)}{q_1(xh)} \, dh = \Delta_H(k)^{-1} \int_{\mathcal{H}} \frac{\beta(xh)}{q_1(xh)} \, dh,
\]
\[
= \Delta_G(k)^{-1} \int_{\mathcal{H}} \frac{\beta(xh)}{q(xh)} \, dh = \Delta_G(k)^{-1}.
\]
Thus, setting $\alpha = \int_{\mathcal{H}} f(k) \Delta_G(k)^{-1} \, dk \neq 0$, we obtain that
\[
S\rho(\mu_{f\mathcal{H}})T(\hat{g}) = \alpha \hat{g}
\]
for all $\hat{g} \in L^2(\hat{K}, d_q \hat{x})$. Now, by hypothesis and Lemma 5.5, the restriction of $\rho(\mu_{f\mathcal{H}})$ to $L^2(K')$ is compact. Hence
\[
\alpha I : L^2(\hat{K}, d_q \hat{x}) \to L^1(\hat{K}, d_q \hat{x})
\]
is a compact operator.

Finally, notice that since
\[
d_{q_1, \hat{x}} = \frac{q_1(x)}{q(x)} d_q \hat{x}
\]
and $\frac{q_1}{q}$ is a strictly positive continuous function on the compact set $\hat{K}$, the corresponding $L^1$-spaces are equal and the $L^1$-norms are equivalent. Hence the embedding
\[
L^2(\hat{K}, d_q \hat{x}) \to L^1(\hat{K}, d_q \hat{x})
\]
is compact. By Lemma 5.6, $\hat{K}$ has to have atoms. However, this implies that $\hat{K}$ is finite.

\section*{References}

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