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SCATTERING THEORY FOR TWISTED AUTOMORPHIC FUNCTIONS

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Abstract. The purpose of this paper is to develop a scattering theory for twisted automorphic functions on the hyperbolic plane, defined by a cofinite (but not cocompact) discrete group Γ with an irreducible unitary representation ρ and satisfying $u(\gamma z) = \rho(\gamma)u(z)$. The Lax-Phillips approach is used with the wave equation playing a central role. Incoming and outgoing subspaces are employed to obtain corresponding unitary translation representations, R_- and R_+, for the solution operator. The scattering operator, which maps $R_- f$ into $R_+ f$, is unitary and commutes with translation. The spectral representation of the scattering operator is a multiplicative operator, which can be expressed in terms of the constant term of the Eisenstein series. When the dimension of ρ is one, the elements of the scattering operator cannot vanish. However when $\dim(\rho) > 1$ this is no longer the case.

1. Introduction

Let Γ be a discrete cofinite but not cocompact group of motions acting on the hyperbolic plane H and let ρ be a unitary representation of Γ acting on a Euclidean vector space V of dimension n. A (Γ, ρ)-automorphic V-valued function on H satisfies the condition:

$$u(\gamma z) = \rho(\gamma)u(\gamma z). \tag{1.1}$$

A theory of such functions was developed early on by Selberg [Se] and extended somewhat later by Venkov [V]. They were concerned with a spectral theory for the Laplace-Beltrami operator:

$$\Delta_\rho = y^2(\partial_x^2 + \partial_y^2) + 1/4, \tag{1.2}$$

with the Eisenstein series and above all with the Selberg trace formula. Scattering theory entered into their work only incidently as an important property of the Eisenstein series.

The purpose of this paper is to systematically develop a scattering theory for the (Γ, ρ)-automorphic wave equation:

$$\partial_t^2 u = \Delta_\rho u, \quad u(z, 0) = f_1(z) \text{ and } \partial_t u(z, 0) = f_2(z), \tag{1.3}$$

all of the functions being (Γ, ρ)-automorphic. To achieve this end we have adapted the Lax-Phillips approach as presented in [LP2] and [LP4]. This theory is built around two translation representations (incoming and outgoing) for the solution operator $U(t)$ of the wave equation. The scattering operator is the mapping from

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the incoming translation representation to the outgoing representation. The corresponding spectral representations are simply the Fourier transforms of the respective translation representations. As expected, the spectral representation of the scattering operator can be expressed in terms of the constant term of the Eisenstein series.

Let P_j be the generator for the parabolic subgroup of the jth cusp and denote by V_j the set of all vectors v such that $\rho(P_j)v = v$. If $r_j = \dim V_j$ is greater than 0, then the jth cusp is called singular. Δ_ρ has a continuous spectrum of multiplicity $r = \sum r_j$. We shall assume throughout that ρ is irreducible. This imposes certain constraints on the system. In Section 2 we illustrate this by studying the kinds of singular cusps that can occur for $\Gamma(2)$ when the dimension of ρ is $1, 2$ and 3. In Section 3 we investigate the discrete spectrum of Δ_ρ which splits into cusp forms (mainly non-positive eigenvalues) and exceptional (positive) eigenvalues. We also introduce the energy form for the wave equation and we show that this form is non-negative on the orthogonal complement of the point spectrum; we denote this subspace by H_E.

Scattering theory takes place on H_E. In Section 4 we introduce the incoming and outgoing subspaces: D_- and D_+ and set

$$H_\pm = \overline{\bigcup U(t)D_\pm}. \tag{1.4}$$

We prove that

(i) $$U(t)D_\pm \subset D_\pm \text{ if } \pm t > 0,$$

(ii) $$\bigcap U(t)D_\pm = \{0\},$$

(iii) $$\overline{\bigcup U(t)D_\pm} = H_E.$$

Property (iii) amounts to proving completeness for the wave operators. Properties (i)–(iii) can be used to establish the existence of translation representations $R_\pm$:

$$(R_\pm U(t)f)(s) = (R_\pm f)(s \mp t), \tag{1.5}$$

mapping H_E unitarily onto $L^2(R)^r$. Also in Section 4 we give an explicit formula for these representations and prove their completeness. Finally in Section 5 we define the scattering operator:

$$S : (R_- f)(-s) \to (R_+ f)(s). \tag{1.6}$$

S is clearly unitary on $L^2(R)^r$ and commutes with translation. In the spectral representation it becomes a multiplicative operator which we identify with the constant term of the various Eisenstein series. We use this machinery to extend a theorem due to Kubota [K], which shows, when ρ is trivial, that no element of the scattering operator vanishes. This remains true when the dimension of ρ equals 1; this is a consequence of various criteria for the nonvanishing of elements of S which are proved for $\dim(\rho) \geq 1$. However we show in Example 5.12 that some elements of the scattering operator for $\Gamma(2)$ can vanish when $\dim(\rho) = 2$.

2. Equivalent representations

Two representations are said to be *equivalent* if the Laplacians for the corresponding automorphic functions have the same spectral properties. It is clear that conjugation of ρ by a unitary map of V will result in an equivalent representation.

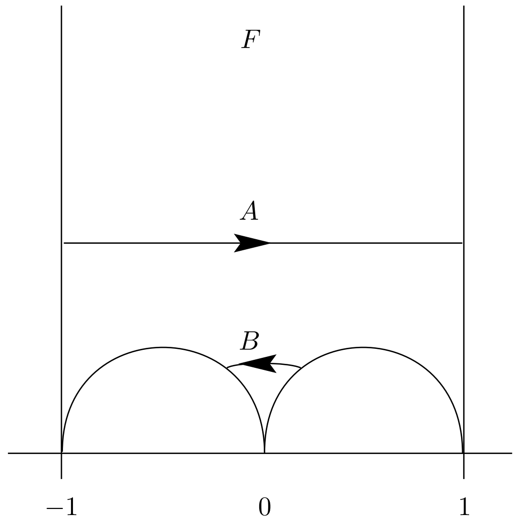

FIGURE 1

Another source of equivalent representations comes from the conjugation of the group elements. In the case of

$$\Gamma(2) = \{\gamma \in \mathrm{PSL}(2, Z); \gamma \equiv \text{ identity } \bmod 2\}$$

we can conjugate by elements of $\Gamma(1) = \mathrm{PSL}(2, Z)$. Now $\Gamma(2)$ is normal in $\Gamma(1)$ and $\Gamma(1)/\Gamma(2)$ is of order 6. We define the mapping

$$g \in \Gamma(1) \to \rho_g(\gamma) = \rho(g^{-1}\gamma g). \tag{2.1}$$

Two elements of the coset $g\Gamma(2)$ map into equivalent representations since for $\beta \in \Gamma(2)$

$$\rho_{g\beta}(\gamma) = \rho(\beta)^{-1}\rho_g(\gamma)\rho(\beta),$$

$\rho(\beta)$ being unitary on V. Conjugation of $\Gamma(2)$ by an element of $g\Gamma(2)$ corresponds to a permutation of the cusps of $\Gamma(2)$ and since the cusps of $\Gamma(2)$ are interchangable the resulting representations will be equivalent.

Now $\Gamma(2)$ is a free group generated by $A = (1\ 2, 0\ 1)$ and $B = (1\ 0, -2\ 1)$. A is the generator of the parabolic subgroup for the cusp at ∞, B for the cusp at 0 and AB for the cusp at 1 (see Figure 1). The matrices $g_1 = (1\ 1, 0\ 1)$ and $g_2 = (0\ 1, -1\ 0)$ generate the cosets of $\Gamma(1)/\Gamma(2)$. We note that $g_1^{-1}Ag_1 = A$, $g_1^{-1}Bg_1 = A^{-1}B^{-1}$ and $g_2^{-1}Ag_2 = B$, $g_2^{-1}Bg_2 = A$. Setting $\rho(A) = a$ and $\rho(B) = b$ we obtain in this way the following pairs of equivalent representations defined by $\{\rho(A), \rho(B)\}$:

$$\{a, b\}, \{b, a\}, \{a, a^{-1}b^{-1}\}, \{a^{-1}b^{-1}, a\}, \{b.a^{-1}b^{-1}\}, \{a^{-1}b^{-1}, b\}. \tag{2.2}$$

Definition 2.1. Suppose P_j is a generator of the parabolic subgroup Γ_j of Γ for the jth cusp and set $V_j = \{v \in V; \rho(P_j)v = v\}$. $r_j = \dim(V_j)$ is the *rank* of the jth cusp. The representation ρ is *singular* at this cusp if $r_j > 0$. The representation is called *singular* if $r = \sum r_j > 0$ and *regular* if $r = 0$.

For the trivial representation each cusp is singular. More generally if ρ is singular in the jth cusp with $\rho(P_j)v = v$ and if f is ρ-automorphic, then $f.v$ will be periodic in x in this cusp. An Eisenstein series can be defined only for such cusps, the first term being $y^s.v$. Likewise the incoming and outgoing subspaces are defined only for such cusps using v-component data.

Definition 2.2. A representation is said to be *irreducible* if there is no proper subspace of V left invariant by $\rho(\Gamma)$.

Remark. One easy way of determining whether ρ is reducible is the following: If W is a maximal invariant subspace of dimension one, then W is an eigenspace for all of the $\rho(P_j)$. If $\dim(V) = 3$ and W is a maximal invariant subspace of dimension two, then the orthogonal complement of W is maximal invariant subspace of dimension one and again the $\rho(P_j)$ will have a common eigenvector. Hence also in this case ρ is irreducible if the $\rho(P_j)$ have no common eigenvector.

In the remainder of this paper we shall consider only irreducible unitary representations of $\Gamma(2)$. This restricts the number of singular cusps occurring in a given representation. For instance when $\dim(V) = 1$ and ρ is nontrivial there can be at most one singular cusp (see [PS]).

Proposition 2.3. *In the case of* $\Gamma(2)$ *if* $\dim(V) = 2$ *then there can exist at most two singular cusps but no singular cusp of rank two.*

Proof. Now A, B and AB are parabolic generators for the cusps at $\infty, 0$ and 1, respectively. Again set $\rho(A) = a$ and $\rho(B) = b$, a and b in $U(2)$. If there is a singular cusp of rank two then either a, b or ab is the identity, in which case ρ is reducible.

Next let $\{v_i\}, \{w_i\}$ and $\{u_i\}$ denote three orthogonal bases for V. We may suppose without loss of generality that

$$w_1 = \mu v_1 + \nu v_2, \quad w_2 = -\overline{\nu} v_1 + \overline{\mu} v_2 \text{ with } |\mu|^2 + |\nu|^2 = 1$$
$$u_1 = \gamma w_1 + \delta w_2 \text{ with } |\gamma|^2 + |\delta|^2 = 1. \tag{2.3}$$

The most general set-up for two singular cusps is

$$av_1 = v_1, \quad av_2 = e^{i\theta} v_2; \quad bw_1 = w_1, \quad bw_2 = e^{i\phi} w_2. \tag{2.4}$$

This representation will be irreducible iff $\mu\nu \neq 0$. If there is an additional singular cusp at 1 we will require $abu_1 = u_1$. We now show that this leads to a contradiction. In the first place if $\gamma\delta = 0$ then either $u_1 = w_1$ or w_2. In either case ρ will be reducible, so we may assume that $\gamma\delta \neq 0$. Writing

$$bu_1 = \gamma w_1 + \delta e^{i\phi} w_2 = (\mu\gamma - \overline{\nu}\delta e^{i\phi}) v_1 + (\nu\gamma + \overline{\mu}\delta e^{i\phi}) v_2,$$

$$\begin{aligned} abu_1 &= (\mu\gamma - \overline{\nu}\delta e^{i\phi}) v_1 + e^{i\theta}(\nu\gamma + \overline{\mu}\delta e^{i\phi}) v_2 = u_1 \\ &= (\mu\gamma - \overline{\nu}\delta) v_1 + (\nu\gamma + \overline{\mu}\delta) v_2. \end{aligned}$$

This requires $\overline{\nu}\delta e^{i\phi} = \overline{\nu}\delta$ and since $\nu\delta \neq 0$ we conclude that ϕ must equal a multiple of 2π, which would make ρ reducible.

Proposition 2.4. *In the case of* $\Gamma(2)$ *if* $\dim(V) = 3$ *there exist representations for which* ∞ *and* 0 *are singular cusps of rank two coexisting with* 1 *being a singular cusp of rank one* (*but not of rank two*).

Proof. In the notation of the previous proof the conditions making ∞ and 0 singular cusps of rank two are

$$
\begin{gathered}
av_1 = v_1, \quad av_2 = v_2, \quad av_3 = e^{i\theta}v_3; \\
bw_1 = w_1, \quad bw_2 = w_2, \quad bw_3 = e^{i\phi}w_3; \\
w_i = \sum_j \nu_{ij} v_j,
\end{gathered} \tag{2.5}
$$

where (ν_{ij}) is a 3-by-3 unitary matrix and $0 < \theta, \phi < 2\pi$. Next we set

$$
\begin{aligned}
u &= \alpha w_1 + \beta w_2 + \gamma w_3 \text{ with } |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1 \\
&= \sum_j (\alpha\nu_{1j} + \beta\nu_{2j} + \gamma\nu_{3j}) v_j
\end{aligned} \tag{2.6}
$$

and compute

$$
\begin{aligned}
abu = \sum_{j\le 2} &(\alpha\nu_{1j} + \beta\nu_{2j} + \gamma e^{i\phi}\nu_{3j}) v_j \\
&+ e^{i\theta}(\alpha\nu_{13} + \beta\nu_{23} + \gamma e^{i\phi}\nu_{33}) v_3.
\end{aligned} \tag{2.7}
$$

In order that 1 be a singular cusp we must have $abu = u$. Combining (2.6) and (2.7) we obtain three homogeneous linear equations in α, β, γ:

$$
\begin{gathered}
0 = \gamma\nu_{31}(1 - e^{i\phi}), \\
0 = \gamma\nu_{32}(1 - e^{i\phi}), \\
0 = \alpha\nu_{13}(1 - e^{i\theta}) + \beta\nu_{23}(1 - e^{i\theta}) + \gamma\nu_{33}(1 - e^{i(\theta+\phi\varepsilon)}).
\end{gathered} \tag{2.8}
$$

There exists a nontrivial solution of (2.8) for α, β, γ for all values of $(\nu_{ij}), \theta$ and ϕ, in particular for values which make ρ irreducible. However for there to exist a two dimensional subspace of solutions (i.e. for 1 to be a singular cusp of rank 2) the determinant of coefficients must be of rank 1 and this requires either $\nu_{31} = \nu_{32} = 0$ or $\nu_{13} = \nu_{23} = 0$. Since (ν_{ij}) is unitary (with orthogonal rows and columns) each of these requirements implies the other. In this case $w_3 = v_3$ and ρ is reducible as we explained in the remark above. This concludes the proof of the proposition.

3. The ρ-automorphic Laplacian

For $n = \dim(V)$ and f, g ρ-automorphic vector valued functions, we define the corresponding Laplacian as $\Delta_\rho = \Delta I_n$; here I_n is the identity matrix on βV and

$$
\Delta = y^2(\partial_x^2 + \partial_y^2) + 1/4. \tag{3.1}
$$

It is routine to show that Δ_ρ is selfadjoint on $L^2(F)^n$ using the quadratic form

$$
C(f,g) = \int_F (y^2 \nabla f . \nabla g + f.g)\, d\mu \text{ with } d\mu = dx\,dy/y^2, \tag{3.2}
$$

where '.' denotes the hermitian inner product on V. The spectrum of Δ_ρ splits into three parts: The exceptional spectrum or positive spectrum, the non-positive cusp form spectrum and the continuous spectrum of multiplicity r. The exceptional spectrum may be empty and when it is not empty there may be cusp forms in it. We also introduce two other forms:

$$
K(f,g) = \int_{F_0} f.g\, d\mu, \tag{3.3}
$$

where $F_0 = F \cap \{y \leq a$ in each cusp$\}$ is a compact subset of F, and C_0 which is the same as C but integrated over F_0 instead of F. We infer from the Rellich compactness criterion that

Lemma 3.1. *The form K is compact with respect to C_0.*

Compactness implies for any $\varepsilon > 0$ that

$$K(f,f) < \varepsilon C_0(f,f) \tag{3.4}$$

on some subspace in $L^2(F)^n$ of finite codimension. Moreover the estimates derived in ([LP2], pp. 94–97), when applied componentwise, show that

$$\begin{aligned} G_1(f,f) &= 2K(f,f) - (\Delta_\rho f, f) \\ &\geq \int_{F_1} (|\partial_x f|^2 + y|\partial_y(f/\sqrt{y})|^2)\, dx\, dy + C_0(f,f)/4; \end{aligned} \tag{3.5}$$

here $F_1 = F \setminus F_0$ and

$$(f,g) = \int_F f.g\, d\mu. \tag{3.6}$$

Now take $\varepsilon < 1/8$. Then it follows from (3.4) and (3.5) that

$$(\Delta_\rho f, f) < 0 \tag{3.7}$$

on a subspace of finite codimension. This proves

Theorem 3.2. *The non-negative spectrum of Δ_ρ on ρ-automorphic data is finite dimensional.*

Let $\{q_i; i = 1, \ldots, m\}$ denote a set of orthonormal eigendata spanning the exceptional eigenspace:

$$\Delta_\rho q_j = \lambda_j^2 q_j, \quad \lambda_j > 0. \tag{3.8}$$

The cusp form eigendata $\{\phi_k\}$ satisfy the following condition in the jth singular cusp (transformed to ∞):

$$\int_{-1}^{1} \phi_k.v\, dx = 0 \tag{3.9}$$

for all y and all $v \in V_j$ in the jth singular cusp. If we exclude the exceptional cusp form eigendata then $\Delta_\rho \phi_k = -\mu_k^2 \phi_k$. It may of course happen that an exceptional eigendata is also a cusp form in the sense of (3.9).

We study the cusp forms using a slightly modified cut-off Laplacian Δ'_ρ defined as follows: Let $L^2(F)'$ denote the space of all square integrable data u which in the jth singular cusp (transformed to ∞) has the zeroth x-Fourier coefficient of $u.v$ vanishing for all y and all v in V_j. Then Δ'_ρ is defined as the Friedrichs' extension of the quadratic form C in (3.2) acting on smooth ρ-automorphic data in $L^2(F)'$. Consequently Δ'_ρ is selfadjoint and, since it is a restriction of the usual cut-off Laplacian, it has a compact resolvent (see section 8 of [LP2] or [C]). This proves

Theorem 3.3. *The cusp form spectrum of Δ_ρ is discrete with finite multiplicity and complete in $L^2(F)'$.*

Next we consider the wave equation. Here we deal with data $u = (u_1, u_2)$ each component of which is a ρ-automorphic n-component vector-valued function on the fundamental domain F. The wave equation in this setting can be written formally as

$$u_t = Au \text{ with initial conditions } u(0) = (f_1, f_2), \tag{3.10}$$

where

$$A = \begin{pmatrix} 0 & I_n \\ \Delta_\rho & 0 \end{pmatrix}. \tag{3.11}$$

As a consequence

$$\partial_t u_1 = u_2 \text{ and } \partial_t^2 u_i = \Delta_\rho u_i, \quad i = 1, 2.$$

The core domain for A consists of $D(\Delta_\rho) \times D(|\Delta_\rho|^{1/2})$. The energy form for $C_0^\infty(F)$ data is

$$\begin{aligned} E(f, g) &= -(\Delta_\rho f_1, g_1) + (f_2, g_2) \\ &= \int_F (y^2 \nabla f_1 . \nabla g_1 - f_1 . g_1/4 + f_2 . g_2)\, d\mu. \end{aligned} \tag{3.12}$$

It is easy to check that A is skew-symmetric with respect to E on the core domain of A; in fact

$$E(Af, g) = -(f_2, \Delta_\rho g_1) + (\Delta_\rho f_1, g_2) = -E(f, Ag). \tag{3.13}$$

It is evident from (3.12) that E need not be a positive form. To correct for this we make use of (3.5) to introduce a new form which is strictly positive:

$$G(f, g) = E(f, g) + 2K(f_1, g_1) = G_1(f_1, g_1) + (f_2, g_2). \tag{3.14}$$

H_G is the closure in the G metric of C_0^∞ ρ-automorphic data. E is clearly continuous in the G-metric. A is defined as the G-closure of A restricted to its core domain.

We now proceed as in section 5 of [LP2] and show that A satisfies the Hille-Yosida criteria to generate a group $\{U(t)\}$ of operators on H_G, which, in view of (3.13), is skew-symmetric with respect to E. The null space of A, namely $N(A)$, consists of those data for which $f_2 = 0$ and $\Delta_\rho f_1 = 0$; and it follows from Theorem 3.2 that $N(A)$ is finite dimensional.

The action of U on ρ-automorphic data can be treated in several ways: (1) A acting on F with suitable boundary conditions on ∂F; (2) A acting on a strip S_j with boundary conditions on ∂S_j; and (3) A acting on the entire hyperbolic plane with no boundary conditions. We will make use of both (1) and (2).

E can be made positive if we project out the positive spectrum of Δ_ρ. To this end we define the following (exceptional) eigendata for A:

$$\begin{aligned} f_j^\sigma &= \begin{pmatrix} q_j \\ -\lambda_j \sigma q_j \end{pmatrix}, \quad \sigma = \pm 1. \\ Af_j^\sigma &= -\lambda_j \sigma f_j^\sigma. \end{aligned} \tag{3.15}$$

We note that

$$E(f_j^\sigma, f_k^\omega) = \lambda_j^2 (\sigma\omega - 1)\delta_{jk}. \tag{3.16}$$

Further we set

$$P_\sigma = \text{ span of the } \{f_j^\sigma; j = 1, \ldots, m\} \text{ and } P = \sum_\sigma P_\sigma. \tag{3.17}$$

Next we define the E-orthogonal projection operator

$$Qf = f + \sum_{j,\sigma} E(f, f_j^\omega) f_j^\sigma / (2\lambda_j^2), \tag{3.18}$$

where $\omega = -\sigma$. Clearly Qf is E-orthogonal to P and each component of Qf is E-orthogonal to the q_j's. It follows that $E \geq 0$ on $H' = QH_G$. We denote the null subspace for E on H' by Z. Moreover $QU(t) = U(t)Q$ so that H' is invariant under the action of U.

Proposition 3.4. $Z = N(A)$.

Proof. For f in $Z \subset H'$ and g in $(C_0^\infty)^{2n}$ we can write

$$0 = E(f, Qg) = E(Qf, g) = E(f, g).$$

Setting $g_1 = 0$ we conclude that $(f_2, g_2) = 0$ for all g_2 in $(C_0^\infty)^n$ and hence that $f_2 = 0$. It then follows that $\Delta_\rho f_1 = 0$, initially in the sense of distributions and then, using elliptic theory, in the usual sense. Hence f lies in $N(A)$. That $N(A) \subset Z$ is obvious from (3.12).

Next we define H'' to be the G-orthogonal complement of Z in H'. H'' is isometric to H'/Z in the E-norm. We see from Lemma 3.1 that K is compact with respect to G and it follows from Theorem 3.7 of [LP3] that

Lemma 3.5. *E and G are equivalent norms on H'' and, in particular, H'' is complete in the E norm.*

It is instructive to introduce the cusp form eigendata:

$$\phi_j^\sigma = \begin{pmatrix} \phi_j \\ i\mu_j \sigma \phi_j \end{pmatrix}, \quad A\phi_j^\sigma = i\mu_j \sigma \phi_j^\sigma. \tag{3.19}$$

Let C denote the span of the nonexceptional $\{\phi_j^\sigma\}$. It is clear that C is invariant under the action of U; further it is easily seen that C is an E-orthogonal set of eigendata which are E-orthogonal to the f_j^σ's; thus $Q\phi_j^\sigma = \phi_j^\sigma$ for ϕ_j^σ in C; so $C \subset H'$.

The main focus of our attention, however, will the ρ-automorphic data B defined as follows: As before let Γ_j denote the parabolic subgroup for the jth singular cusp (transformed to ∞) and denote its fundamental domain by S_j. Then

$$\begin{gathered} B_j = \left\{ f = (f_1, f_2); f_1 = \sum \rho(\gamma^{-1}) v g(\gamma z), f_2 = \sum \rho(\gamma^{-1}) v h(\gamma z) \right\}, \\ B = \sum B_j, \end{gathered} \tag{3.20}$$

where g and h are C_0^∞ scalar functions of y alone in S_j and v ranges over V_j. For each z the defining sums in (3.20) have only a finite number of non-zero terms in F. One sees, by unfolding the integral, that B and C are E-orthogonal and, since $QC = C$, that QB and QC are E-orthogonal.

Proposition 3.6. *B is invariant under the action of U.*

Proof. Making the substitution $w = u(y)/\sqrt{y}, s = \log y$, the scalar wave equation

$$\begin{gathered} u_{tt} = \Delta u \text{ becomes } w_{tt} = w_{ss} \\ \text{and } \int_0^\infty |u|^2 \, dy/y^2 = \int_{-\infty}^\infty |w|^2 \, ds. \end{gathered} \tag{3.21}$$

Further if the initial data $g = (w(s,0), w_t(s,0))$ is $C_0^\infty(R)$ then so is $U_0(t)g = (w(t,s), w_t(s,t))$. It is clear that we can write the action of U on ρ-automorphic data $f = \sum_{\Gamma_j\backslash\Gamma} \rho(\gamma^{-1})vg(\gamma z)$ in B_j $(v \in V_j)$ as

$$U(t)f = \sum_{\Gamma_j\backslash\Gamma} \rho(\gamma^{-1})v(U_0(t)g)(\gamma z), \tag{3.22}$$

which again belongs to B_j, as asserted.

Theorem 3.7. *$Q(B+C)$ is dense in H'/Z.*

Proof. Suppose the assertion is false. Then there exists an $f \in H''$ which is E-orthogonal to $Q(B+C)$. Since Q is an E-orthogonal projection f will be E-orthogonal to $B+C$. Recall that B and C are invariant under the action of U. It follows from this that

$$f_\zeta = \int U(t)f\zeta(t)\,dt, \qquad \zeta \in C_0^\infty(R), \tag{3.23}$$

is also E-orthogonal to $B+C$. The second component of f_ζ is L^2 orthogonal to all second components in B and therefore, by unfolding the integral we see that it lies in $L^2(F)'$. Since f_ζ is also E-orthogonal to C as well as any exceptional cusps in P, it follows from Theorem 3.3 that the second component of f_ζ vanishes. Likewise the second component of $Af_\zeta = -f_{\zeta'}$ vanishes. But then $Af_\zeta = 0$ and f_ζ lies in $N(A)$ and hence Z. Taking the limit as ζ approaches a delta function we see that $f \in Z$ which is the 0 element of H'/Z. This concludes the proof of the theorem.

From now on we shall only work with the E-norm on H'/Z. It follows by Lemma 3.5 that this space is complete. Moreover $U(t)$ is now unitary. Setting $H_E = \overline{QB}$ we conclude from Theorem 3.7 that

$$H_E = H'/Z \ominus C. \tag{3.24}$$

In the next section we shall need a little more information about the scalar wave equation (3.21). The complete solution has two basic modes, incoming $(\sigma = -1)$ and outgoing $(\sigma = 1)$:

$$u_\sigma(s,t) = \begin{pmatrix} \theta_\sigma(s-\sigma t) \\ -\sigma\theta'_\sigma(s-\sigma t) \end{pmatrix} e^{s/2}, \tag{3.25}$$

where θ_σ is a smooth function with support $< \infty$. Given arbitrary smooth initial data $f = e^{s/2}\phi$ with support $< \infty$, we can decompose it into the sum of these two modes as follows: Set

$$\theta_\sigma = \sum_j \psi_j r_j^\sigma \tag{3.26}$$

where

$$\psi_1 = \phi_1, \quad \psi_2 = -\int_s^\infty \phi_2(\tau)\,d\tau, \quad r^\sigma = 1/\sqrt{2}\begin{pmatrix} 1 \\ -\sigma \end{pmatrix};$$

it is easily checked that $\sum_\sigma r_i^\sigma r_j^\sigma = \delta_{ij}$ and $\sum_j r_j^\sigma r_j^\omega = \delta_{\sigma\omega}$. Using the values of θ_σ obtained in this way, we have

$$(U_0(t)f)(s) = \sum_\sigma u_\sigma(s,t). \tag{3.27}$$

Notice that each of the solutions in (3.25) are essentially translation representations. With this in mind we define the σ-representation for $f = e^{s/2}\phi$ so as to pick out the u_σ part of the solution:

$$(R_\sigma^0 f)(s) = 1/\sqrt{2}(\partial_s \phi_1(s) - \sigma\phi_2(s)). \tag{3.28}$$

It follows from (3.25) that

$$(R_\sigma^0 f)(s) = \partial_s \theta_\sigma(s) = (R_\sigma^0 u_\sigma)(s), \tag{3.29}$$

$$(R_\sigma^0 U_0(t) f)(s) = (R_\sigma^0 f)(s - \sigma t). \tag{3.30}$$

Moreover

$$E_0(u_\sigma, u_\sigma) = \int |\sigma_s \theta_\sigma(s)|^2 \, ds = \|R_\sigma^0 f\|^2. \tag{3.31}$$

4. Translation representations

4.1. **Incoming and outgoing subspaces.** We are now ready to start on a scattering theory for ρ-automorphic data. Our theory is based on the notion of incoming and outgoing subspaces (cf. [LP4]). These subspaces have their support in the various singular cusps and remain in that cusp under the action of $U(t)$ for negative t if the subspace is incoming and for positive t if the subspace is outgoing. If the jth cusp is singular let $\{v(jk); k = 1, \ldots, r_j\}$ be an orthonormal basis for V_j and let α denote the pair (jk) associated with $v(jk)$.

Definition 4.1. If the jth cusp is singular we define the αth outgoing ($\sigma = 1$) and incoming ($\sigma = -1$) subspaces on F as

$$D_{\sigma\alpha} = [f = e^{s/2}(\phi_1, \phi_2); \phi_1 = g_k(s)v(jk), \phi_2 = -\sigma g_k'(s)v(jk)], \tag{4.1}$$

where g_k belongs to $C_0^\infty(R)$ with support in $\{s > c\}$. We also define

$$\begin{gathered} D_{\sigma j} = \bigoplus_k D_{\sigma(jk)}, \quad D_\sigma = \bigoplus_j D_{\sigma j}, D'_{\sigma\alpha} = QD_{\sigma\alpha}, \\ D'_{\sigma j} = QD_{\sigma j}, \quad D'_\sigma = QD_\sigma, \\ H_{\sigma\alpha} = \overline{\bigcup U(t)D'_{\sigma\alpha}}, \quad H_{\sigma j} = \bigoplus_k H_{\sigma(jk)} \text{ and } H_\sigma = \bigoplus_j H_{\sigma j}. \end{gathered} \tag{4.2}$$

It is clear that $D_{\sigma j}$ is E-orthogonal to $D_{\omega k}$ if $j \neq k$ since their supports are disjoint and that $D_{\sigma(jk)}$ is orthogonal to $D_{\omega(jm)}$ if $k \neq m$ since $v(jk)$ is orthogonal to $v(jm)$. It is readily checked that $D_{\sigma j}$ is E-orthogonal to $D_{\omega j}$ if $\omega = -\sigma$.

Recall that the solution to the scalar wave equation (3.21) is of the form $e^{s/2}g(s - \sigma t)$. Because of the initial conditions imposed on data in $D_{\sigma j}$ the solution behaves componentwise in F like u_σ in (3.24) when $\sigma t > 0$ and since $QU = UQ$ we infer that

$$U(t)D'_\sigma \subset D'_\sigma \quad \text{for } \sigma t > 0. \tag{4.3$_{\text{i}}$}$$

Now $U(t)D_\sigma$ vanishes in F for $s < c + \sigma t$, $\sigma t > 0$, in each of the singular cusps and hence for d in D_σ and $d' = Qd$, $U(t)d' = QU(t)d$ is a linear combination of data from P for $s < c + \sigma t$. Thus any f in $\bigcap U(t)D'_\sigma$ lies in P and since it is also in the range of Q we have

$$\bigcap_{\sigma t > 0} U(t)D'_\sigma = \{0\}. \tag{4.3$_{\text{ii}}$}$$

We shall prove later on that

$$\overline{\bigcup U(t)D'_\sigma}/Z = H_E. \tag{4.3$_{\rm iii}$}$$

Proposition 4.2. *D_σ and P_σ are orthogonal.*

Proof. Making use of the scalar Laplacian ∂_s^2 (see (3.21)) we find for the 0th Fourier coefficient $f_k^\sigma.v|_0 = e^{s/2}(w_1.w_2), v \in V_j$, that in the jth cusp the relation (3.15) translates into

$$w_2 = -\lambda_j \sigma w_1 \text{ and } \partial_s^2 w_1 = -\lambda_j \sigma w_2.$$

Since w_2 is L^2, the solution to these equations is

$$w_1 = \exp(-\lambda_j s) \text{ and } w_2 = -\sigma\lambda_j \exp(-\lambda_j s). \tag{4.4}$$

For $d = e^{s/2}\phi$ in $D_{\sigma j}$ (which has its support in the jth cusp) we obtain, after performing the x-integration,

$$\begin{aligned} E(d, f_j^\sigma) &= \int (\partial_s \phi_1 \partial_s w_1 - \sigma \partial_s \phi_1 w_2)\, ds \\ &= \int \partial_s \phi_1 (-\lambda_j + \sigma^2 \lambda_j) \exp(-\lambda_j s)\, ds = 0. \end{aligned}$$

Lemma 4.3. *For d in D_σ with $d' = Qd$, we have $d = d' + p_\sigma$ (where p_σ lies in P_σ) and $E(d', d') = E(d, d)$. Further if $\alpha \neq \beta$ then $H_{\sigma\alpha}$ is orthogonal to $H_{\sigma\beta}$.*

Proof. The first assertion follows directly from the previous proposition, the relation (3.16) and the expression (3.18) for Q. Using (3.16) and the fact that d' is orthogonal to P, the second assertion can be read off of

$$E(d, d) = E(d', d') + E(d', p_\sigma) + E(p_\sigma, d') + E(p_\sigma, p_\sigma).$$

We prove that $D'_{\sigma\alpha}$ is orthogonal to $D'_{\sigma\beta}$ for $\alpha \neq \beta$ in a similar fashion, using the fact that $D_{\sigma\alpha}$ is orthogonal to $D_{\sigma\beta}$ and the mutual orthogonality of the f_j^σ's. To prove the last assertion take data $f_\gamma = U(\sigma t_\gamma)d'_\gamma$ for d'_γ in $D'_{\sigma\gamma}$ with $\gamma = \alpha$ and β. Such f_γ's are dense in $H_{\sigma\gamma}$. Next choose τ so that $\sigma(\tau + t_\gamma) > 0$ for $\gamma = \alpha$ and β. Then $U(\sigma\tau)f_\gamma = U(\sigma(\tau + t_\gamma))d'_\gamma$ lies in $D'_{\sigma\gamma}$ by ((4.3)$_{\rm i}$) and hence $U(\sigma\tau)f_\alpha$ is orthogonal to $U(\sigma\tau)f_\beta$. Since U is unitary we conclude that f_α is orthogonal to f_β.

Taking our cue from (3.28) we now define the σ-translation representors $R_{\sigma j} = \bigoplus_k R_{\sigma(jk)}, R_\sigma = \bigoplus_j R_{\sigma j}$ as follows: In the jth cusp coordinates we write the 0th x-Fourier coefficient as $f_0(s) = e^{s/2}\phi(s)$ for all s in R and set

$$R_{\sigma\alpha} f = (\partial_s \phi_1 - \sigma\phi_2).v(\alpha), \quad \alpha = (jk). \tag{4.5}$$

It follows from (3.5) that $R_{\sigma\alpha} f$ is square integrable on $\{s > c\}$ for f in H_G.

As we noted in Section 3, the ρ-automorphic solution of the wave equation satisfies this equation in both F and S_j with suitable boundary conditions in each case. In particular the 0th x-Fourier coefficient of $U(t)f.v$ with v in V_j will satisfy the scalar wave equation with x-periodic boundary conditions in S_j. If we take f to be smooth with support $< \infty$, it follows from (3.30) that

$$(R_{\sigma\alpha} U(t) f)(s) = (R_{\sigma\alpha} f)(s - \sigma t). \tag{4.6}$$

Keep in mind that the ρ-automorphic data does not vanish outside of F.

Proposition 4.4. *If d lies in D_σ then $(R_\sigma d)(s)$ vanishes for $s < c$.*

Proof. Choose d_α in $D_{\sigma\alpha}$ ($\alpha = (ik)$) and $\sigma t > 0$. In the notation of (4.1), $U(t)d_\alpha.v(ik) = e^{s/2}(g_k(s-\sigma t), -\sigma g_k'(s-\sigma t))$ where the $\operatorname{supp} g_k \subset \{s > c\}$; $U(t)d_\alpha.v = 0$ if v is orthogonal to $v(ik)$. Thus for arbitrary d in D_σ the support of $U(t)d$ tends toward infinity as $\sigma t \to \infty$ in each cusp and in each γF. This means that the support of $R_{\sigma j}U(t)d$ tends toward $\pm\infty$. On the other hand it follows from (4.6) that the support of $R_{\sigma t}U(t)d$ tends only toward $+\infty$. This requires that $d(\gamma z)$ makes a zero contribution to $R_{\sigma j}d$ if γ is not the identity. Likewise if $\gamma = id.$ and $d \in D_{\sigma i}$ with $i \neq j$ then, since $U(t)d$ tends toward infinity in the ith cusp, the same argument shows that $R_{\sigma j}d$ vanishes. Finally if $d \in D_{\sigma j}$ then the support of $R_{\sigma j}d$ is contained in $\{s > c\}$ as desired.

Using this result together with (3.31) we get, after summing over the components of d,

Corollary 4.5. *For f in $D_{\sigma j}$,*

$$E(d,d) = 2\sum_k \int |g_k'(s)|^2\, ds = \|R_{\sigma j} d\|^2. \tag{4.7}$$

The proof of Proposition 4.4 also shows that

Corollary 4.6. *For d in $D_{\sigma\beta}$, $\beta \neq \alpha$, $R_{\sigma\alpha}d = 0$.*

Proposition 4.7. $R_{\sigma j} f_k^\sigma = 0$.

Proof. According to (4.4), $f_k^\sigma.v|_0 = e^{s/2}(1, -\sigma\lambda_k)\exp(-\lambda_k s)$ in the jth cusp coordinates when $v \in V_j$. Substituting this into (4.5) we see that all of the components of $R_{\sigma j} f_k^\sigma$ vanish.

Lemma 4.8. *For d in D_σ and $d' = Qd$ in D_σ', $R_\sigma d' = R_\sigma d$ both of which vanish for $s < c$. For f in $H_{\sigma\alpha}$*

$$E(f,f) = \|R_{\sigma\alpha} f\|^2. \tag{4.8}$$

For f in $H_{\sigma\beta}$ ($\beta \neq \alpha$), $R_{\sigma\alpha} f = 0$.

Proof. According to Proposition 4.2, $d = d' + p_\sigma$ and hence the first assertion follows from Propositions 4.4 and 4.7. Data $f = U(t)d'$, with d' in $D_{\sigma\alpha}'$, form a dense subset of $H_{\sigma\alpha}$. From Lemma 4.3 and the fact that U is unitary we have $E(f,f) = E(d',d') = E(d,d)$. If d lies in $D_{\sigma\alpha}$ then $E(d,d) = \|R_{\sigma\alpha}d\|^2 = \|R_{\sigma\alpha}d'\|^2$ by Corollary 4.5 and $\|R_{\sigma\alpha}d'\| = \|R_{\sigma\alpha}f\|$ follows from (4.6). Finally if $\beta \neq \alpha$ and d in $D_{\sigma\beta}$, we have by Corollary 4.6 and Proposition 4.7 that $0 = R_{\sigma\alpha}d = R_{\sigma\alpha}d'$ and hence by (4.6) that $R_{\sigma\alpha}f = 0$.

Theorem 4.9. *$R_{\sigma\alpha}[R_\sigma]$ is unitary on $H_{\sigma\alpha}[H_\sigma]$ to $L^2(R)$ $[L^2(R)^r]$. If f is orthogonal to $H_{\sigma\alpha}$ in H_E, then $R_{\sigma\alpha}f = 0$.*

Proof. We already know from Lemmas 4.3 and 4.8 that these maps are isometries. So to prove the first assertion we need only show that the range of $R_{\sigma\alpha}$ is all of $L^2(R)$. If this were not true then there would exist a ψ in $L^2(R)$ orthogonal to the range of $R_{\sigma\alpha}$. For d in $D_{\sigma\alpha}$ and $d' = Qd$ we have $R_{\sigma\alpha}d' = R_{\sigma\alpha}d$ and in the notation of (4.1)

$$0 = \int R_{\sigma\alpha} d\psi\, ds = 2\int \partial_s g_k \psi\, ds$$

for all $\{g_k\}$ with support in $\{s > c\}$. Replacing d' by $U(t)d'$ we see that it holds for all $C_0^\infty g_k$'s. Consequently ψ is constant and since it is L^2 it must vanish. A similar argument proves

Corollary 4.10. *$R_{\sigma\alpha}$ maps the closure of $D'_{\sigma\alpha}$ onto $L^2(c,\infty)$.*

To prove the last assertion of the theorem we suppose that f in H_E is orthogonal to $H_{\sigma\alpha}$. Then for $f_0 = e^{s/2}\phi$ and d in $D_{\sigma\alpha}$ with $d' = Qd$, we have in the jth cusp coordinates and in the notation of (4.1)

$$\begin{aligned} 0 = E(d', f) &= \int \partial_s g_k \overline{\partial_s \phi_1 . v(jk)} - \sigma \partial_s g_k \overline{\phi_2 . v(jk)}\, ds \\ &= \int \partial_s g_k (\overline{R_{\sigma\alpha} f})\, ds. \end{aligned}$$

Again replacing d' by $U(t)d'$ we see that this holds for all g_k in C_0^∞ and it follows as before that $R_{\sigma\alpha} f$ is constant. Since it is L^2 on $\{s > c\}$ it must vanish for all s. This concludes the proof of the theorem.

Corollary 4.11. *For f in H_E*

$$\|R_\sigma f\|^2 \leq E(f, f). \tag{4.9}$$

Proof. We decompose f into the sum of two parts: f_1 in H_σ and f_2 orthogonal to H_σ. By Theorem 4.9 $R_\sigma f_2 = 0$ and $\|R_\sigma f_1\|^2 = E(f_1 . f_1)$. Consequently

$$\|R_\sigma f\|^2 = \|R_\sigma f_1\|^2 = E(f_1, f_1) \leq E(f_1, f_1) + E(f_2, f_2) = E(f, f).$$

Corollary 4.12. *For f in H_E, $R_{\sigma\alpha} f = R_{\sigma\alpha} f_{\sigma\alpha}$ where $f_{\sigma\alpha}$ is the orthogonal projection of f in $H_{\sigma\alpha}$.*

4.2. **Completeness.** In this subsection we prove property $(4.3)_{\text{iii}}$ that is

$$H_\sigma = H_E. \tag{4.10}$$

To this end we establish

Lemma 4.13. *$H_E = \sum H_\sigma$ and U has an absolutely continuous spectrum in H_E.*

Proof. According to Theorem 4.9, U has a translation representation on each of the H_σ and it follows that U has an absolutely continuous spectrum on the H_σ and hence on $\overline{\sum H_\sigma}$. Since $H_E = \overline{QB}$ it suffices to show that any Qf, f in B_j, belongs to $\overline{\sum H_\sigma}$. We may suppose that f is described as in (3.20) and set $\phi_1 = e^{-s/2}g$, $\phi_2 = e^{-s/2}h$, both in $C_0^\infty(R)$. To begin with we shall assume that $\int \phi_2\, ds = 0$. We then construct θ_ω, $\omega = \pm 1$, as in (3.26) so that

$$\phi = \phi_\sigma + \phi_{-\sigma} \text{ where } \phi_\omega = \frac{1}{\sqrt{2}}(\theta_\omega, -\omega\theta'_\omega). \tag{4.11}$$

Automorphizing $g_\sigma = e^{s/2}\phi_\sigma|_1$ and $h_\sigma = e^{s/2}\phi_\sigma|_2$ as in (3.20) we end up with f_σ and $f_{-\sigma}$, $f = f_\sigma + f_{-\sigma}$ and Qf_ω lies in H_ω for $\omega = \pm 1$.

If $\int \phi_2\, ds \neq 0$, then we proceed as follows: Choose ζ in C_0^∞ so that $\zeta = 0$ for $s < 1$ and $s > 2$ and $\int \zeta(s)\, ds = 1$; set $\zeta_R = \zeta(s/R)/R$. We now replace ϕ by

$$\phi_R|_1 = \phi_1, \quad \phi_R|_2 = \phi_2 - \left(\int \phi_2\, ds\right)\zeta_R, \tag{4.12}$$

in which case $\phi_R|_2$ has a zero integral. Because of this the lower bound of $\operatorname{supp}\phi_R$ remains the same as that of the g and h and therefore any change in ϕ for s

sufficiently large will only affect f in F in the identity term of the automorphizing sum. We now proceed as before to construct $\theta_{R\omega}$ and $f_{R\omega}$, $\omega = \pm 1$, with all the desired properties. It is easy to check that the $f_{R\omega}$ form a Cauchy sequence in the E-norm as $R \to \infty$. It follows from Lemma 4.3, as before, that the $Qf_{R\omega}$ also converge in the E-norm and the limit will lie in H_ω. It is also clear that $Qf_{R\sigma} + Qf_{R,-\sigma}$ converges to Qf as desired. This completes the proof of the lemma.

Theorem 4.14. *$H_\sigma = H_E$.*

Proof. If the theorem were false there would exist a non-zero f in $H_E \ominus H_\sigma$. According to Theorem 4.9 $R_\sigma f = 0$. Since H_σ is invariant under the action of U the data f_ζ, defined by (3.23), also lies in the complement of H_σ. Now f is the limit of such data so we will attain a contradiction if we can show that $f_\zeta = 0$ for all ζ. Moreover

$$w = Af_\zeta = -\int U(t) f \zeta'(t)\, dt \tag{4.13}$$

is well defined and since A has an absolutely continuous spectrum in H_E, it follows that $w = 0$ will imply $f_\zeta = 0$. Clearly $R_\sigma w = 0$. The remainder of the argument, which consists of three steps, shows that $w = 0$.

Step 1. Local energy decay. Notice that $A^{2N} = \Delta_\rho^N$ and that w lies in the domain of A^∞. Hence $\Delta_\rho^N U(t) w = U(t)\Delta_\rho^N w$ and the second component $w(t)_2$ of $U(t)w$ remains bounded in t in any Sobolev norm. It follows by Rellich's compactness criterion that the $\{w(t)_2\}$ form a compact subset in $L^2(F)^{\mathrm{loc}}$. If we now represent $E(U(t)w, h)$ in terms of its spectral resolution, we conclude from the absolute continuity of the spectrum and the Riemann-Lebesgue lemma that $U(t)w$ converges to zero weakly in H_E as $\sigma t \to \infty$. In particular for $h = (0, \psi)$ with ψ in $C_0^\infty(F)$, we see that $w(t)_2$ converges to zero weakly in L^2 and this together with the compactness shows that it converges to zero locally in the L^2 norm as $\sigma t \to \infty$.

Step 2. Control of the 0th x-Fourier coefficient in the jth singular cusp. Let $\varepsilon_k = \{\delta_{jk};\, j = 1, \dots, r_j\}$. Then in the jth cusp with $w(t)_2|_0 = e^{s/2}\phi_2(t,s)$ we see from (4.5) that

$$\phi_2(t,.).v(jk) = -\tfrac{1}{2} \sum_\omega \omega R_{\omega j} w(t).\varepsilon_k. \tag{4.14}$$

Recall that $R_\sigma w = 0$. Moreover according to Corollary 4.11 $\|R_\omega w(t)\|^2 \le E(w,w)$. Hence for $\sigma t > 0$, $0 < k \le r_j$ and any $\varepsilon > 0$

$$\begin{aligned} \sum_k \int_b^\infty |\phi_2(t,s).v(jk)|^2\, ds & \\ &\le \int_b^\infty |(R_{-\sigma j} w)(s + \sigma t)|^2\, ds \\ &\le \int_b^\infty |(R_{-\sigma} w)(s)|^2\, ds < \varepsilon \end{aligned}$$

for b sufficiently large. We can rewrite this inequality as

$$\sum_k \int_b^\infty e^{-s} |w(t)_2|_0.v(jk)|^2\, ds < \varepsilon. \tag{4.15}$$

Step 3. Control of the other Fourier components. Again let P_j denote a generator of the parabolic subgroup for the jth cusp of $\Gamma(2)$. Next we introduce an orthonormal basis for V consisting of the eigenvectors of $\rho(P_j)$:

$$\begin{aligned} \rho(P_j)v(jk) &= v(jk) \quad \text{for } 0 < k \leq r_j, \\ \rho(P_j)v(jk) &= \exp(2\pi i\theta_k)v(jk) \quad \text{for } r_j < k \leq n, 0 < \theta_k < 1. \end{aligned} \tag{4.16}$$

In this step we will estimate all of the x-Fourier coefficients not treated in Step 2. The basic inequality comes from (3.12) and the fact that $E \geq 0$ componentwise in H_E:

$$\begin{aligned} \int_F y^2(|\partial_x w_2|^2 + |\partial_y w_2|^2)\, d\mu &= -(w_2, \Delta_\rho w_2) + (w_2, w_2)/4 \\ &\leq (\Delta_\rho w_2, \Delta_\rho w_2) + (w_2, w_2)/2 \leq E(\Delta_\rho w, \Delta_\rho w) + E(w, w). \end{aligned} \tag{4.17}$$

This inequality holds as well for $w(t)_2$ and in this case we can then replace $w(t)$ on the RHS by $w(0)$:

$$\int_F y^2|\partial_x w(t)_2|^2\, d\mu \leq E(\Delta_\rho w(0), \Delta_\rho w(0)) + E(w(0), w(0)). \tag{4.18}$$

Now $\exp(-\pi i\theta_k x)w_2.v(jk)$ is x-periodic, $-1 < x \leq 1$, in the jth cusp and can be expanded in a Fourier series:

$$\begin{aligned} \exp(-\pi i\theta_k x)w_2(s,x).v(jk) &= \sum a_m(s)\exp(\pi i m x), \\ a_m(s) &= \frac{1}{2}\int_{-1}^{1} \exp(-\pi i(\theta_k + m)x)w_2(s,x).v(jk)\, dx, \\ 2\sum |a_m(s)|^2 &= \int_{-1}^{1} |w_2(s,x).v(jk)|^2\, dx. \end{aligned}$$

Since $\exp(-\pi i\theta_k s)\partial_x w_2.v(jk)$ is also x-periodic in the jth cusp, so we can apply the same analysis as above to get

$$2\sum_m \pi^2(\theta_k + m)^2|a_m(s)|^2 = \int_{-1}^{1} |\partial_x w_2(s,x).v(jk)|^2\, dx.$$

If we let $\sum'$ denote the sum over all m when $r_j < k \leq n$ and all but $m = 0$ when $0 < k \leq r_j$, then it follows from this and (4.18) that

$$\sum{}' \int_b^\infty \int_{-1}^{1} e^{-s}|w(t,s)_2.v(jk)|^2\, dx\, ds < Ce^{-2b}. \tag{4.19}$$

Combining the results of Steps 2 and 3 we see that in the jth cusp for a given $\varepsilon > 0$

$$\int_b^\infty \int_{-1}^{1} e^{-s}|w(t,s)_2|^2\, dx\, ds < \varepsilon \tag{4.20}$$

for b sufficiently large. Now (4.20) is valid for all of the cusps and this together with Step 1 implies that

$$\lim_{\sigma t \to \infty} \int_F |w(t)_2|^2\, d\mu = 0. \tag{4.21}$$

This holds for any data which is the 2nd component of data of the form Af_ζ. Now $w(t)_1 = U(t)f_\zeta|_2$ is uniformly bounded in the L^2 norm. (4.20) applies to both

$\Delta_\rho w(t)_1 = U(t)A^2 f_\zeta|_2$ and $w(t)_2 = U(t)Af_\zeta|_2$. It therefore follows from (3.12) that

$$E(w,w) = \lim_{\sigma t\to\infty} E(w(t),w(t)) = 0. \tag{4.22}$$

This completes the proof of Theorem 4.14.

5. The scattering operator

In Part 1 of this section we use the translation representors to define the scattering operator. We then exhibit its spectral representation in terms of the constant term of the Eisenstein series. In Part 2 we find conditions under which the various matrix elements of the scattering operator do not vanish and we conclude with an example with $\dim(\rho) = 2$ in which the diagonal elements of the scattering operator do vanish.

Part 1.

Definition 5.1. The scattering operator

$$S : v'(s) = (T_- f)(s) = (R_- f)(-s) \to v(s) = (R_+ f)(s); \tag{5.1}$$

here f ranges over H_E, '$-$' replaces $\sigma = -1$ and '$+$' replaces $\sigma = 1$.

It follows from Theorem 4.9 that S is unitary on $L^2(R)^r$. Its action on the various $H_{-\alpha}$ subspaces is given by

$$S_{\alpha\beta} : v'_\alpha(s) = (T_{-a} f)(s) = (R_{-\alpha} f)(-s) \to v_\beta(s)(R_{+\beta} f)(s). \tag{5.2}$$

Recall (see Corollary 4.12) that $R_{\omega\alpha} f = R_{\omega\alpha} f_{\omega\alpha}$ where $f_{\omega\alpha}$ is the orthogonal projection of f onto $H_{\omega\alpha}$. We deduce from (4.6) that

$$v'(s-t) \to v(s-t) \tag{5.3}$$

so that S commutes with translation. This shows that S is a convolution operator on $L^2(R)^r$ and hence will become a multiplicative operator in the spectral representation.

We obtain the spectral representation from the translation representation by taking the Fourier transform:

$$\tilde{f}_\alpha(\eta) = \int e^{i\eta s} T_{-\alpha} f(s)\, ds. \tag{5.4}$$

We shall treat only data f in the dense subspace $\bigcup U(t)D'_+$. For such f, $R_+ f$ is smooth with compact support (see Propositions 4.4 and 4.7). Moreover

Lemma 5.2. *For f in $\bigcup U(t)D'_+, T_{-\alpha} f = \sum a_j \exp(\lambda_j s)$ for $s \ll 0$.*

Proof. Since we are working with translation representations we may suppose that f lies in D'_+; set $f = d'_+$. By Lemma 4.3 for d'_σ in D'_σ there exists a d_σ in D_σ and a p_σ in P_σ such that $d'_\sigma = d_\sigma + p_\sigma$. Let $D''_- = D_- \cap D'_-$. Then d_- belongs to D''_- iff $E(d_-, f_k^+) = 0$ for all k. Recall (see (4.4)) that $T_{-\alpha} f_k^+ = \text{const}\,.\exp(\lambda_k s)$. We denote by g_k the data in H_E for which

$$T_- g_k(s) = \begin{cases} T_- f_k^+(s) & \text{for } s < -c, \\ 0 & \text{for } s > -c. \end{cases} \tag{5.5}$$

Then g_k lies in the closure of D'_- by Corollary 4.10. Further one can read from (4.8) that

$$E(d_-, f_k^+) = E(d_-, f_k^+|_0) = \int T_- d_- . T_- f_k^+ \, ds$$
$$= \int T_- d_- . T_- g_k \, ds = E(d_-, g_k) = E(d'_- . g_k).$$

Hence d'_- belongs to D''_- iff $E(d'_-, g_k) = 0$ for all k. Now let P denote the orthogonal projection onto $\{g_k\}$. Then $(I - P)D'_- = D''_-$.

For d''_- in D''_- we have

$$E(d'_+, d''_-) = E(d_+, d''_-) = 0$$

since D_+ is E-orthogonal to D_-. It follows that

$$\int T_- d'_+ . T_- d'_- ds = E(d'_+, d'_-) = E(d'_+, (I - P)d'_-) + E(d'_+, Pd'_-)$$
$$= E(d'_+, Pd'_-) = E(Pd'_+, d'_-)$$
$$= \int T_- Pd'_+ . T_- d'_- \, ds.$$

Since $T_- D'_-$ is dense in $L^2(-\infty, -c)$, this shows that the restriction of $T_- d'_+$ to $\{s < -c\}$ is equal to $T_- Pd'_+$. According to (4.4) and (5.5) $T_{-\alpha} g_k = \text{const} . \exp(\lambda_k s)$ for $s < -c$; the assertion of the lemma follows.

Setting $\zeta = \xi + i\eta$, with f in $\bigcup U(t)D'_+$, we can therefore express $\tilde{f}_\alpha$ as

$$\tilde{f}_\alpha(\eta) = \text{l. i. m.}_{\xi \downarrow 0} \int e^{-\bar{\zeta} s} T_{-\alpha} f \, ds, \tag{5.6}$$

where l.i.m. denotes the limit in the mean.

If there were data $e_\alpha(\zeta)$ in $H_{-\alpha}$ such that

$$(T_- e_\alpha(\zeta))(s) = \{\delta_{\alpha\beta} \exp(-\zeta s); \beta\}, \tag{5.7}$$

then we could write $\tilde{f}_\alpha$ in terms of the energy form as

$$\tilde{f}_\alpha(\eta) = \text{l. i. m.}_{\xi \downarrow 0} \int T_- f . T_- e_\alpha(\zeta) \, ds = \text{l. i. m.}_{\xi \downarrow 0} E(f, e_\alpha(\zeta)). \tag{5.8}$$

Similarly we could write

$$(\widetilde{Sf})_\beta(\eta) = \lim_{\xi \downarrow 0} \int R_+ f . T_- e_\beta(\zeta)) \, ds \tag{5.9}$$

as a pointwise limit. Moreover, since S is linear and commutes with translation we would have

$$e^{-\zeta t}(ST_- e_\alpha(\zeta))(s) = (S(T_- e_\alpha(\zeta)(. + t)))(s) = (ST_- e_\alpha(\zeta))(s + t).$$

Thus $ST_- e_\alpha(\zeta)$ is an eigendata of $U(t)$ in the translation representation with eigenvalue $e^{-\zeta t}$. Interchanging the roles of s and t and setting $s = 0$ in the above relation, we see that $ST_- e_\alpha(\zeta)$ is a linear combination of the $e_\beta(\zeta)$'s; that is

$$ST_- e_\alpha(\zeta) = \sum_\beta s_{\alpha\beta}(\zeta) T_- e_\beta(\zeta) = \{s_{\alpha\beta}(\zeta); \beta\}. \tag{5.10}$$

The $r \times r$ matrix $S(\zeta) = (s_{\alpha\beta}(\zeta))$ is called the *scattering matrix*.

In order to make this rigorous we have to construct $e_\alpha(\zeta)$ to satisfy (5.7). After doing this we shall express $s_{\alpha\beta}$ in terms of the constant coefficient of the Eisenstein series. Since $e^{-\zeta s}$ is not square integrable we shall have to extend the space H_E. Here we follow Chapter 7 of [LP2] and introduce a λ-norm on functions $g(s)$ in $L^2(R)^r$:

$$\|g\|_\lambda^2 = \int_{-\infty}^{0} e^{2\lambda s}|g|^2\,ds + \int_0^\infty |g|^2\,ds. \tag{5.11}$$

It is proved in the above reference that for f in H_E and $0 < \lambda < \min\{\lambda_k\}$ (or, if there is no residual spectrum, $0 < \lambda < 1/2$) that

$$\|R_+ f\|_\lambda \le c\|T_- f\|_\lambda. \tag{5.12}$$

Incidently, it follows from this that $S(\zeta)$ is analytic in ζ for $0 < \xi < \lambda$. Next we define a λ-norm for data f in H_E, namely

$$\|f\|_\lambda = \|T_- f\|_\lambda \tag{5.13}$$

and denote by $H(\lambda)$ the completion of H_E in this norm. The relation (5.12) continues to hold for all f in $H(\lambda)$. Notice that if $0 < \xi < \lambda$ then $T_- e_\alpha(\zeta)$ has a finite λ-norm; we shall prove that $e_\alpha(\zeta)$ can be realized in $H(\lambda)$.

The fact that the exponential is a generalized eigendata of A in the translation representation suggests that we use the Eisenstein series to devise $e_\alpha(\zeta)$. The Eisenstein series $E(.,\zeta,\alpha)$ with $\alpha = (j,k)$ is generated by automorphizing $y^{1/2+\zeta}v(jk)$. The 0th Fourier coefficient of the mth component of $E(.,\zeta,\alpha)$ in the ith cusp ($\beta = (im)$) is (see Venkov [V])

$$E(.,\zeta.\alpha).v(im)|_0 = \delta_{\alpha\beta}e^{(1/2+\zeta)s} - \theta_{\alpha\beta}(\zeta)e^{(1/2-\zeta)s}. \tag{5.14}$$

We now set

$$e_\alpha(\zeta) = \frac{1}{2}\left(\frac{1}{\zeta}E, E\right). \tag{5.15}$$

$e_\alpha(\zeta)$ is clearly ρ-automorphic.

We see from (5.14) that the 0th x-Fourier coefficient of the mth component of $e_\alpha(\zeta)$ in the ith cusp is

$$e_\alpha(\zeta).v(im)|_0 = e^{s/2}\left(\frac{1}{\zeta}\phi_{\alpha\beta}, \phi_{\alpha\beta}\right), \tag{5.16}$$

where

$$\phi_{\alpha\beta} = \frac{1}{2}(\delta_{\alpha\beta}e^{\zeta s} - \theta_{\alpha\beta}(\zeta)e^{-\zeta s}).$$

Using (4.5) a straightforward calculation gives

$$R_{-\beta}e_\alpha(\zeta) = \{\delta_{\alpha\beta}e^{\zeta s}\} \text{ and } ST_-e_\alpha(\zeta) = \{R_{+\beta}e_\alpha(\zeta)\} = \{\theta_{\alpha\beta}(\zeta)e^{-\zeta s}\}. \tag{5.17}$$

Thus $e_\alpha(\zeta)$ satisfies (5.7) and, comparing (5.10) with (5.17), we see that

$$s_{\alpha\beta}(\zeta) = \theta_{\alpha\beta}(\zeta). \tag{5.18}$$

In order to show that $e_\alpha(\zeta)$ belongs to $H(\lambda)$ we must approximate it in the λ-norm by data in H_E. To this end we choose ψ_L in $C_0^\infty(R)$ so that

$$\psi_L(s) = \begin{cases} 1 & \text{for } c+1 < s < L, \\ 0 & \text{for } s < c \text{ and } s > L+1; \end{cases}$$

and set

$$d_{\alpha L}(\zeta) = \frac{1}{2} e^{s/2}/\zeta(\psi_L e^{\zeta s}, (\psi_L e^{\zeta s})')v(jk) \tag{5.19}$$

in the jth cusp and 0 elsewhere. It is clear that $d_{\alpha L}(\zeta)$ is an element in $D_{-\alpha}$ and hence Proposition 4.4 applies. We therefore get

$$R_- d_{\alpha L}(\zeta) = \{\delta_{\alpha\beta}(\psi_L(s)e^{\zeta s} + \psi'_L(s)e^{\zeta s}/\zeta). \tag{5.20}$$

Since $d_{\alpha L}$ belongs to $D_{-\alpha}$ Lemma 4.8 applies and we get

$$R_{-\beta} d_{\alpha L}(\zeta)' = R_{-\beta} d_{\alpha L}(\zeta). \tag{5.21}$$

Proposition 5.3. $Qe_\alpha(\zeta) = e_\alpha(\zeta)$.

Proof. Both $e_\alpha(\zeta)$ and f_m^ω are eigendata of A with eigenvalues ζ and $-\omega\lambda_m$ respectively. Further the exponential decay of f_m^ω in the cusps makes the integrand in $E(e_\alpha(\zeta), f_m^\omega)$ integrable. The assertion now follows from

$$(\zeta - \omega\lambda_m)E(e_\alpha(\zeta), f_m^\omega) = E(Ae_\alpha(\zeta), f_m^\omega) + E(e_\alpha(\zeta), Af_m^\omega) = 0;$$

here we have used the skew symmetry of A.

We now set

$$e_{\alpha L}(\zeta) = U(-L/2)d_{\alpha L}(\zeta) \text{ and } e_{\alpha L}(\zeta)' = Qe_{\alpha L}(\zeta). \tag{5.22}$$

Then $e_{\alpha L}(\zeta)$ is ρ-automorphic and $e_{\alpha L}(\zeta)'$ lies in H_E.

Theorem 5.4. *$e_{\alpha L}(\zeta)$ lies in $H(\lambda)$ and $e_{\alpha L}(\zeta)'$ converges to $e_\alpha(\zeta)$ in the λ-norm as $L \to \infty$ if $0 < \xi < \lambda$.*

Proof. We see from (5.17), (5.20) and (5.21) that

$$T_- e_\alpha(\zeta) - T_- e_{\alpha L}(\zeta)' = \{\delta_{\alpha\beta}(1 - \psi_L(-s + L/2) - \psi'_L(-s + L/2)/\zeta)e^{-\zeta s}\},$$

which clearly converges 0 in the λ-norm as $L \to \infty$ when $0 < \xi < \lambda$. This validates the assertion of the theorem. It follows from (5.12) that

Corollary 5.5. *$R_+ e_{\alpha L}(\zeta)'$ converges in the λ-norm to $R_+ e_\alpha(\zeta)$.*

Since S is unitary on H_E we can write

$$\begin{aligned} \int T_- f.T_- e_{\alpha L}(\zeta)'\, ds &= \int ST_- f.ST_- e_{\alpha L}(\zeta)'\, ds \\ &= \int R_+ f.R_+ e_{\alpha L}(\zeta)'\, ds. \end{aligned} \tag{5.23}$$

For f in $\bigcup U(t)D'_{+\alpha}$ we now have

$$\begin{aligned} \tilde{f}_\alpha(\eta) &= \mathrm{l.\,i.\,m.}_{\xi\downarrow 0} \lim_{L\to\infty} \int T_- f.T_- e_{\alpha L}(\zeta)'\, ds \\ &= \lim_{\xi\downarrow 0} \lim_{L\to\infty} \int R_+ f.R_+ e_{\alpha L}(\zeta)'\, ds \\ &= \lim_{\xi\downarrow 0} \int R_+ f.R_+ e_\alpha(\zeta)\, ds. \end{aligned}$$

Making use of (5.10) this becomes

$$\tag{5.24}\begin{aligned}\tilde{f}_\alpha(\eta) &= \lim_{\xi\downarrow 0}\int R_+f.\left(\sum_\beta s_{\alpha\beta}(\zeta)T_-e_\beta(\zeta)\right\}\,ds\\ &= \lim_{\xi\downarrow 0}\sum_\beta\int\overline{s_{\alpha\beta}(\zeta)}\{R_+f\}.T_-e_\beta(\zeta)\,ds\\ &= \sum_\beta\overline{s_{\alpha\beta}(i\eta)}(\widetilde{Sf})_\beta.\end{aligned}$$

Since this relates $\tilde{f}$ and $\widetilde{Sf}$ it follows that $(s_{\alpha\beta}(i\eta))$ is indeed spectral representive of the inverse scattering matrix. It is shown in Corollary 4.2 of Chapter 4 in [LP1] that $(s_{\alpha\beta}(i\eta))$ is unitary on C^r. Since $s_{\alpha\beta}(i\eta) = \theta_{\alpha\beta}(i\eta)$ this also follows from known properties of the Eisenstein series. In any case we can now write

$$\tag{5.25}(\widetilde{Sf})_\beta(\eta) = \sum_\alpha s_{\alpha\beta}(i\eta)\tilde{f}_\alpha.$$

Another treatment of the scattering matrix, which does not make explicit use of the Eisenstein series, can be found in Chapter 7 of [LP2].

Part 2. The previous development gives us a new insight into a theorem due to Kubota which played a prominant role in [PS]. In the case of the trivial representation Kubota proved ([K], p. 16) that $\theta_{jk}(i\eta)$ cannot be identically zero. We now treat the corresponding problem for the general representations considered in this paper. In view of (5.18) this amounts to showing that $s_{\alpha\beta}$ is not zero. Now $s_{\alpha\beta} \neq 0$ iff there is an f in $H_{-\alpha}$ which has a nontrivial projection into $H_{+\beta}$ or what amounts to the same thing.

Theorem 5.6. *$s_{\alpha\beta} \neq 0$ iff $H_{-\alpha}$ and $H_{+\beta}$ are not orthogonal.*

Let ε denote either $\alpha = (jk)$ or $\beta = (im)$. According to (4.4)

$$\tag{5.26} f_n^\omega.v(\varepsilon)|_0 = c_n^\omega(\varepsilon)(1, -\omega\lambda_n)\exp((1/2-\lambda_n)s)$$

in S_j [or S_i] if $\varepsilon = \alpha$ [or β]. It is clear from (3.5) that $c_n^\omega(\varepsilon)$ does not depend on ω and we will omit the superscript from now on.

Proposition 5.7. *$s_{\alpha\beta} \neq 0$ if for some exceptional eigenvalue λ_p we have*

$$\tag{5.27}\sum_{\lambda_n=\lambda_p} c_n(\alpha)\overline{c_n(\beta)} \neq 0.$$

Proof. Now d_ε in $D_{\omega\varepsilon}$ can be expressed as $d_\varepsilon = e^{s/2}(\theta_\varepsilon, -\omega\theta'_\varepsilon)v(\varepsilon)$ in the associated cusp. According to (3.16) and Lemma 4.3,

$$\tag{5.28} d'_\varepsilon = Qd_\varepsilon = d_\varepsilon + \sum_n E(d_\varepsilon, f_n^{\omega'})f_n^\omega/(2\lambda_n^2),$$

where $\omega' = -\omega$ and

$$\tag{5.29} E(d_\varepsilon, f_n^{\omega'}) = E(d_\varepsilon, f_n^{\omega'}|_0) = -2\lambda_n\overline{c_n(\varepsilon)}\int\theta'_\varepsilon\exp(-\lambda_n s)\,ds.$$

Keeping in mind that $D_{-\alpha}$ and $D_{+\beta}$ are orthogonal, we get for d'_α in $D'_{-\alpha}$ and d'_β in $D'_{+\beta}$

$$E(d'_\alpha, d'_\beta) = 2\sum_n \overline{c_n(\alpha)} c_n(\beta) \int \theta'_\alpha e^{-\lambda_n s}\, ds \int \overline{\theta'_\beta e^{-\lambda_n s}\, ds}. \tag{5.30}$$

Finally we can choose θ'_α and θ'_β in $L^2(c,\infty)$ (see Corollary 4.10) so that

$$\int \theta'_\alpha e^{-\lambda_n s}\, ds = \delta_{np} = \int \theta'_\beta e^{-\lambda_n s}\, ds.$$

For such a choice it follows by our hypothesis that the resulting d'_α and d'_β are not orthogonal and hence that $s_{\alpha\beta}$ is not zero.

It could happen that the hypothesis of Proposition 5.7 is not satisfied by a particular representation. Indeed for some representations there are no exceptional eigendata (see [PS]).

Corollary 5.8. *Let $d_\alpha \in D_{-\alpha}$ and $d_\beta \in D_{+\beta}$ and suppose that the hypothesis of Proposition 5.7 does not hold. Then*

$$E(U(t)d'_\alpha, d'_\beta) = E(U(t)d_\alpha, d_\beta). \tag{5.31}$$

Proof. We proceed as above with d_α replaced by $U(t)d_\alpha$. Notice that

$$E(U(t)d_\alpha, f_n^+) = E(d_\alpha, U(-t)f_n^+) = \exp(\lambda_n t)E(d_\alpha, f_n^+).$$

Replacing $E(d_\alpha, f_n^{\omega'})$ in (5.28) by this, (5.30) becomes

$$E(U(t)d_\alpha, d_\beta) = E(U(t)d'_\alpha, d'_\beta) - 2\sum_n e^{\lambda_n t}\overline{c_n(\alpha)} c_n(B) \int \theta_\alpha e^{-\lambda s}\, ds \int \overline{\theta_\beta e^{-\lambda_n s}\, ds}.$$

Since all of the partial sums (5.27) vanish, we conclude that the entire sum in this expression vanishes and we obtain (5.31).

Proposition 5.9. *For $\alpha = (jk)$ and $\beta = (im)$, if $v(\alpha).v(\beta) \neq 0$ and $i \neq j$, then $s_{\alpha\beta} \neq 0$.*

Proof. If Proposition 5.7 applies then the assertion is true. Otherwise we can assume that (5.31) holds. Again take d_α in $D_{-\alpha}$ and d_β in $D_{+\beta}$. Set the 0th x-Fourier coefficient of d_β on S_j equal to $e^{s/2}\phi$. If we now unfold the integral for $E(U(t)d_\alpha, d_\beta)$ in the jth cusp and perform the x-integration we get (in the notation of Proposition 5.7)

$$E(U(t)d_\alpha, d_\beta) = \int_{S_j} \theta'_\alpha(s+t)v(\alpha).(\phi'_1 + \phi_2)\, ds. \tag{5.32}$$

In the ith cusp, $\phi = (\theta_\beta, -\theta'_\beta)v(\beta)$. To make (5.32) different from zero we choose θ_α and θ_β to have positive slope in $c < s < c+\delta$ in the jth and ith cusp coordinates, respectively. The second component of $d_\beta.v(\alpha)$ will be negative in this region and so will the derivative of ϕ_1 in the jth cusp. The stability group of the ith cusp forces d_β to be constant on all horocycles about the point at infinity of the ith cusp. Next we choose t so that the supports of $\theta_\alpha(s+t)$ and $d_\beta|_F$ overlap only on the region of positive slope for $\theta_\alpha(s+t)$ as in Figure 2. In these horocycles ϕ will be in the $v(\beta)$ direction and in the overlap region $\theta_\alpha(s+t)' > 0$ and $(\phi'_1.v(\alpha))(\phi_2.v(\alpha)) > 0$. As a consequence (5.32) will be different from zero, as desired.

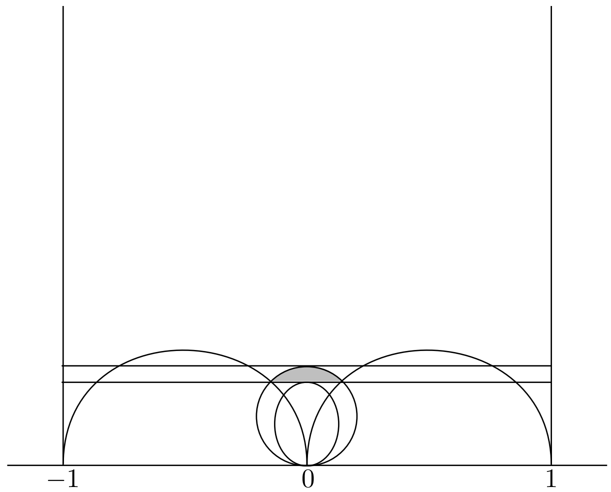

FIGURE 2

Remark 5.10. In case $\dim \rho = 1$, all of the $v(\varepsilon)$'s are the same. For $\Gamma = \Gamma(2)$ and $\dim \rho = 2$, then $v(\alpha).v(\beta) \neq 0$ if ρ is irreducible and $\alpha \neq \beta$. Hence in these cases none of the off diagonal $s_{\alpha\beta}$ vanish.

Remark 5.11. If there is only one singular cusp ($r = 1$), say at α, then $H_{-\alpha} = H_E = H_{+\alpha}$ and hence $s_{\alpha\alpha} \neq 0$.

Since ρ is irreducible there will exist γ in Γ for which $v(\alpha).\rho(\gamma)v(\beta) \neq 0$. Hence even if $v(\alpha).v(\beta) = 0$ we can still pursue the method used in Proposition 5.9. The main complication in using a γ not the identity arises from the fact that d_β will be supported on several horocycles of the same size all of which contribute to (5.32). The γ's in Γ associated with any one such horocycle differ only by a factor δ on the right belonging to the stability group of the ith cusp. Since $\rho(\delta)v(\beta) = v(\beta)$ it follows that $v(\alpha).\rho(\gamma)v(\beta)$ (which enters in (5.32)) remains the same throughout any such horocycle.

To illustrate this procedure we now treat the diagonal element $s_{\alpha\alpha}$ of the scattering operator for $\Gamma(2)$ when ρ is of dimension 2 and there are two singular cusps (we shall use the notation introduced in Proposition 2.3). Again we assume that Proposition 5.7 does not apply. We may take $\alpha = (j, k)$ with the jth cusp at ∞. Now any element $\gamma = (a\ b, c\ d)$ of $\Gamma(2)$ takes the region $\{y > \beta\}$ into the horocycle of height $1/(\beta c^2)$ which is tangent at a/c. The highest such horocycle will be of height $1/(4\beta)$ and will come from the group elements $B^{\pm 1}A^n$ (see Figure 3), where, as before, the matrices $A = (1\ 2, 0\ 1)$ and $B = (1\ 0, -2\ 1)$ freely generate $\Gamma(2)$. As in (2.4) we set $av_1 = v_1, av_2 = e^{i\theta}v_2, bw_1 = w_1$ and $bw_2 = e^{i\phi}w_2$, where $0 < \theta, \phi < 2\pi$;

$$w_1 = \overline{\mu}v_1 + \nu v_2, \quad w_2 = \overline{\nu}v_1 - \mu v_2;$$
$$v_2 = \overline{\mu}w_1 + \nu w_2, \quad v_2 = \overline{\nu}w_1 - \mu w_2,$$

where $|\mu|^2 + |\nu|^2 = 1$ and, since ρ is irreducible, $\mu\nu \neq 0$. Here $v(\alpha) = v_1$ corresponds to a singular cusp.

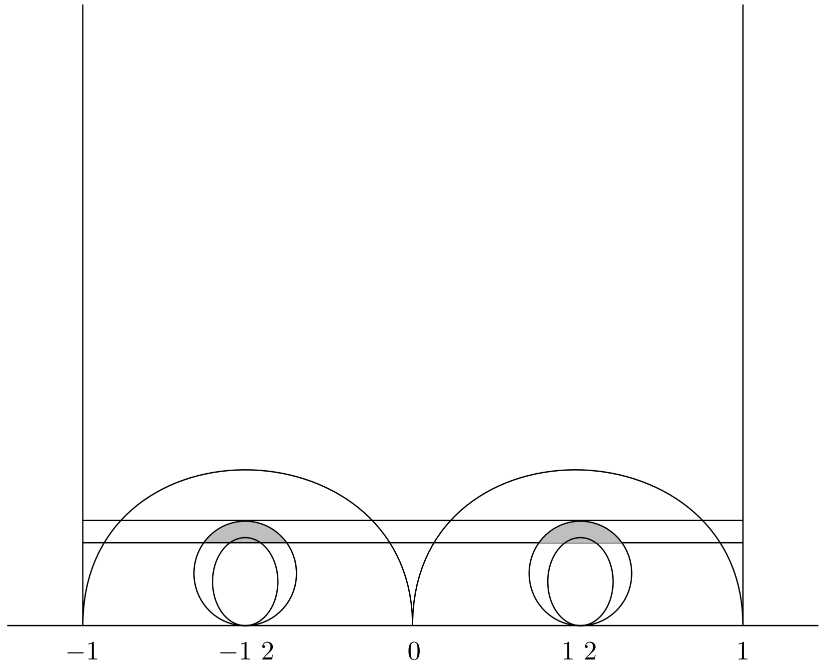

FIGURE 3

The relevant factor in (5.32) is $I = I_+ + I_-$, where

$$I_\pm = v_1.b^{\pm 1}v_1 = v_1.(\mu w_1 + \nu e^{\pm i\phi} w_2) = |\mu|^2 + |\nu|^2 e^{i\phi}, \qquad (5.33)$$
$$I = 2(|\mu|^2 + |\nu|^2 \cos\phi).$$

Defining θ_α as before we see that $s_{\alpha\alpha} \neq 0$ if $I \neq 0$.

If $I = 0$ then we will have to check out the next lower level, that is for horocycle height $1/(16\beta)$ tangent at $\pm 1/4$ and $\pm 3/4$. The corresponding γ's are

$$B^2 = (1\ 0, -4\ 1), BAB = (-3\ \ -2, -4\ \ -3)$$

and their inverses. The relevant factor in this case is

$$I' = v_1.(b^2 v_1 + b^{-2} v_1 + babv_1 + b^{-1}a^{-1}b^{-1}v_1).$$

It turns out that both I and I' can vanish simultaneously for a restricted set of parameters.

In essence the scheme which we have outlined above is a way of determining whether or not the successive terms in the series expansion for $s_{\alpha\alpha}$ vanish (see [K]). The main difficulty in pursuing this program is factoring an arbitrary γ in $\Gamma(2)$ into A and B factors.

The relation (5.33) leaves open the possibility that $s_{\alpha\alpha}$ may be zero for the following $\Gamma(2)$-representation of dimension 2 in which $\phi = \pi$ and $\mu = \nu$:

$$\underline{a}v_1 = v_1, \quad \underline{a}v_2 = -v_2, \quad \underline{b}w_1 = w_1, \quad \underline{b}w_2 = -w_2; \qquad (5.34)$$
$$v_1 = 1/\sqrt{2}(w_1 + w_2), \quad v_2 = 1/\sqrt{2}(w_1 - w_2),$$
$$w_1 = 1/\sqrt{2}(v_1 + v_2), \quad w_2 = 1/\sqrt{2}(v_1 - v_2).$$

It is easy to see that

$$\underline{a}w_1 = w_2, \quad \underline{a}w_2 = w_1, \quad \underline{b}v_1 = v_2, \quad \underline{b}v_2 = v_1. \tag{5.35}$$

Example 5.12. $s_{\alpha\alpha} = 0$ for the 2-dimensional representation of $\Gamma(2)$ described in (5.34); here $\alpha = (\infty 1)$ or (01).

Remark 5.13. If $s_{\alpha\alpha} = 0$ then it follows from Proposition 5.7 that $c_n(\alpha) = 0$ for all of the exceptional eigenvalues λ_n; in other words any exceptional eigendata must be a cusp form.

The proof of this assertion will be presented in several steps. It suffices to treat only the case $\alpha = (\infty 1)$. At this point it is convenient to introduce a modified incoming subspace:

$$D''_{-\alpha} = D_{-\alpha} \cap D'_{-\alpha}, \tag{5.36}$$

consisting of those data in $D_{-\alpha}$ which are orthogonal to P. It is easy to show that

$$H_{-\alpha} = \overline{\bigcup U(t) D''_{-\alpha}}. \tag{5.37}$$

In fact since for $d''_- = e^{s/2}(\theta, \partial_s\theta)v(\alpha)$ in $D''_{-\alpha}$ we have (by unfolding the integral)

$$E(U(t)d''_-, f_n^\sigma) = -\lambda_n \overline{c_n(\alpha)}(1+\sigma) \int e^{-\lambda_n s}\partial_s\theta(s+t)\, ds = 0, \tag{5.38}$$

which holds for a dense set of $\partial_s\theta$'s because no linear combination of these exponentials is square integrable. To be sure that the dense set consists of derivatives of test functions, we can include the constant function into the set of exponentials.

Set $v(\alpha) = v_1$ (of (5.34)). Because of Theorem 5.6 and (5.37) it is enough to show that

$$E(U(t)d''_-, d_+) = E(U(t)d''_-, Qd_+) = 0$$

for every d_- in $D''_{-\alpha}$ and d_+ in $D_{+\alpha}$. In F we have

$$d_\sigma = (y^{1/2}\psi_\sigma(y), -\sigma y^{3/2}\psi'_\sigma(y))v_1,$$

where $\operatorname{supp}\psi_\sigma \subset \{y > 1\}$ and d_- satisfies (5.38) (with θ replaced by ψ_-). Recall that $U(t)d_- = \rho$-automorphised

$$g = (y^{1/2}\psi_-(ye^t), y^{3/2}\psi'_-(ye^t))v_1.$$

We now denote the energy form restricted to a subset S by E_S. Then by unfolding the integral expression for $E_F(U(t)d_-, d_+)$ we get

$$E_F(U(t)d_-, d_+) = E_{S_\infty}(g, d_+) = \sum_{\Gamma_\infty\backslash\Gamma} E_{\gamma F}(g, d_+).$$

Since $d_+(\gamma z) = \rho(\gamma)d_+(z)$ this expression can be rewritten as

$$E_F(U(t)d_-, d_+) = \sum_{\Gamma_\infty\backslash\Gamma} E_F(g(\gamma z), d_+)I(\gamma), \tag{5.39}$$

where $I(\gamma) = \rho(\gamma)v_1.v_1$. Since g has compact support this is a finite sum.

In the above sum we choose the elements $\gamma = (a\ b, c\ d)$ with $|a| < c$ to represent the Γ_∞ cosets. As noted above γ sends $\{y = \beta\}$ onto the horocycle of height $1/(\beta c^2)$ and point of tangency $x = a/c$. Multiplying γ on the right by A moves γz along such a horocycle. If $\beta > 1$ the horocycle remains within S_∞. Notice also that $I(\gamma A) = \rho(\gamma)\underline{a}v_1.v_1 = I(\gamma)$ so this factor in (5.39) remains the same for all

γA^p's. Now for a fixed c there are only a finite set of a's with $|a| < c$ and $(a, c) = 1$ and when summed over the p's the value of $g(\gamma z)$ are the same on these horocycles. Hence if we group the terms in (5.39) into partial sums over $\{\gamma A^p\}_p$ and then sum over all subsums with fixed c, it will suffice to prove for each such c that

$$J(c) = \sum I(\gamma) = 0; \tag{5.40}$$

here the sum is over the (odd) a's with $(a, c) = 1$ and $|a| < c$.

We now represent γ in factored form:

$$\gamma = A^{p_1} B^{q_1} \cdots A^{p_k} B^{q_k}. \tag{5.41}$$

Let sc denote a change in the sign of the off diagonal terms of γ. Then $(\gamma_1\gamma_2)^{sc} = \gamma_1^{sc}\gamma_2^{sc}$ and hence γ^{sc} is the same as (5.41) with each A replaced by A^{-1} and each B by B^{-1}. Since $\underline{a}^{-1} = \underline{a}$ and $\underline{b}^{-1} = \underline{b}$, we see that $I(\gamma^{sc}) = I(\gamma)$. In particular this means that we need only consider the case of $0 < a < c$.

Step 1. If $c = 2n$ and n is odd then $I(\gamma) = 0$. Notice that

$$\begin{aligned} A^p(a\ \times, c\ \times) &= (a + 2pc\ \times,\ c\ \times), \\ B^q(a\ \times, c\ \times) &= (a\ \times, c - 2qa\ \times). \end{aligned} \tag{5.42}$$

Now any γ in $\Gamma(2)$ can be constructed from the identity by adding factors of A and B on the left. We see from (5.42) that c changes under such a factor only under the action of B^q. According to (5.35) $\rho(B) = \underline{b}$ takes $v_1[v_2]$ into $v_2[v_1]$. Hence if $q = \sum |q_i|$ is odd then $\rho(\gamma)v_1 = \pm v_2$ and $I(\gamma) = 0$.

Suppose next that $c = 4n$ and $a = 1, 3, \ldots, c - 1$. We are concerned only with a's relatively prime to c; these are the only a's which occur in $\Gamma(2)$.

Step 2. $I(\gamma) = 1$ if $a \equiv 1$ or $7 \bmod 8$ and $I(\gamma) = -1$ if $a \equiv 3$ or $5 \bmod 8$. We see from (5.34) that $I(\gamma)$ is not affected if we replace some of the A's or B's by their inverses; since this takes a/c into $-a/c$ we can as well assume that the q_i are ≥ 0. We shall use an induction argument on q and since q is even when $c = 4n$, q will increase by steps of 2. Notice that by (5.42) for $\gamma' = A^p\gamma$ we have $a' = a + 2pc \equiv a \bmod 8$ (since $2c \equiv 0 \bmod 8$); hence adding the factor A^p on the left does not change $a \bmod 8$. Moreover $I(A^p\gamma) = \underline{a}^p\rho(\gamma)v_1.v_1 = \rho(\gamma)v_1.\underline{a}^p v_1 = I(\gamma)$. In particular for $q = 0$ where $\gamma' = A^p id$, $a \equiv 1 \bmod 8$ and $I(\gamma') = 1$.

Next we suppose the assertion holds for some $q \geq 0$ and we add on the left all possible factors which contain two B's to an arbitrary γ with q B's. In view of the above we need only consider two factors, namely B^2 and BA^pB. For $\gamma' = B^2\gamma$ we see from (5.42) that $a' = a$ and since $\rho(B^2) = id.$ it is clear that $I(\gamma') = I(\gamma)$. Finally when $\gamma' = BA^pB\gamma$ a straightforward calculation using (5.42) shows that $a' = a + 2pc - 4pa \equiv a - 4pa \bmod 8$; that is $a' \equiv a \bmod 8$ if p is even and $a' \equiv a + 4 \bmod 8$ if p is odd. On the other hand $\rho(\gamma') = \underline{b}\underline{a}^p\underline{b}\rho(\gamma)$ so that $I(\gamma') = \rho(\gamma)v_1.\underline{b}\underline{a}^p\underline{b}v_1$ which equals $I(\gamma)$ if p is even and $-I(\gamma)$ if p is odd. This completes the induction for Step 2.

Step 3. For a fixed c the set of a's with $(a, c) = 1$ and $0 < a < c$ has the same number of a's congruent to 1 or 7 as a's congruent to 3 or $5 \bmod 8$. It will follow from this and Step 2 that $J(c) = 0$, as desired. This is clearly true for $s(0)$ which consists of all the odd a's (not necessarily prime to c) going from 1 through $c - 1$. Since this is an uninterrupted string of $2n = c/2$ odd integers, it consists modulo 8 of groups of 4, namely $(1, 3, 5, 7)$, ending with either a last such group or $(1, 3)$.

Suppose next that $c = mp^\alpha$ with $4|m$ and $(m,p)=1$. Then the a's divisible by p are

$$s(p) = p, 3p, \ldots, (mp^{\alpha-1}-1)p. \tag{5.43}$$

This is a string of $mp^{\alpha-1}/2$ successive odd multiples of p which repeats itself moduli 8 in groups of 4, ending with either a group of 4 or 2. Clearly $(p, 3p, 5p, 7p)$ goes into $(1,3,5,7)$ if $p \equiv 1 \bmod 8$; $(3,1,7,5)$ if $p \equiv 3 \bmod 8$; $(5,7,1,3)$ if $p \equiv 5 \bmod 8$; and $(7,5,3,1)$ if $p \equiv 7 \bmod 8$. In each case there will be as many a's congruent to 1 or $7 \bmod 8$ as 3 or $5 \bmod 8$ in the sequence.

Finally let $c = mp_1^\alpha p_2^\alpha \cdots p_k^\alpha$ with $m = 2^\alpha$ $(\alpha \geq 2)$ and the p_i' distinct prime numbers > 2. Then the set of a's prime to c is given by

$$\begin{aligned} s = s(0) &- \sum s(p_i) + \sum s(\text{pairs of } p_i'\text{'s}) \\ &- \sum s(\text{triples of } p_i\text{'s}) + \cdots \\ &+ (-1)^k \sum s(k\text{-tuples of } p_i\text{'s}). \end{aligned} \tag{5.44}$$

As we have seen above each of the sequences has the desired property and hence so does s. This completes the proof of Example 5.12.

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