MORITA EQUIVALENCE FOR CROSSED PRODUCTS
BY HILBERT $C^*$-BIMODULES

BEATRIZ ABADIE, SØREN EILERS, AND RUY EXEL

Abstract. We introduce the notion of the crossed product $A \rtimes_X Z$ of a $C^*$-algebra $A$ by a Hilbert $C^*$-bimodule $X$. It is shown that given a $C^*$-algebra $B$ which carries a semi-saturated action of the circle group (in the sense that $B$ is generated by the spectral subspaces $B_0$ and $B_1$), then $B$ is isomorphic to the crossed product $B_0 \rtimes_{B_1} Z$. We then present our main result, in which we show that the crossed products $A \rtimes_X Z$ and $B \rtimes_Y Z$ are strongly Morita equivalent to each other, provided that $A$ and $B$ are strongly Morita equivalent under an imprimitivity bimodule $M$ satisfying $X \otimes_A M \simeq M \otimes_B Y$ as $A-B$ Hilbert $C^*$-bimodules. We also present a six-term exact sequence for $K$-groups of crossed products by Hilbert $C^*$-bimodules.

1. Introduction

If $A$ is a $C^*$-algebra and $\alpha$ is an automorphism of $A$, let $B$ be the crossed product $C^*$-algebra $B = A \rtimes_\alpha Z$ ([17]). It is well known that $B$ carries a natural action of the circle group, called the dual action, such that the spectral subspaces $B_n$ ([12]) are naturally identified with $Au^n$, where $u$ is the canonical unitary implementing the automorphism. The $B_n$ then form a $\mathbb{Z}$-grading of $B$, which has the property that $B$ is generated by $(B_0$ and) $B_1$.

Now, suppose that one is given a circle action on a $C^*$-algebra $C$, which is semi-saturated in the sense that $C$ is generated by the spectral subspaces $C_0$ and $C_1$. A natural question to ask is whether $C$ is isomorphic to a crossed product of $C_0$ by some automorphism.

The answer to this question is obviously no, as one could be dealing with the trivial action of the circle, a situation that could not possibly arise as a dual action. An affirmative answer can be given, however, when the action is regular ([12]), although this requires an extension of the concept of crossed products to the realm of partial automorphisms. The goal of the present work is, precisely, to study the case in which the regularity property is absent. The important ingredient one must rely upon is the Hilbert $C^*$-bimodule ([6]) over the fixed-point algebra, which is provided by the first spectral subspace.

Received by the editors April 6, 1995.

1991 Mathematics Subject Classification. Primary 46L55, 46L05, 46C50; Secondary 46L45, 46L80.

Key words and phrases. Crossed products, Morita equivalence, $C^*$-algebras, Hilbert $C^*$-bimodules, spectral subspaces, Pimsner-Voiculescu sequence.

The first author was supported by FAPESP, Brazil, on leave from Facultad de Ciencias, Montevideo, Uruguay. The second author was supported by Rejselegat for matematikere, Denmark, on leave from Københavns Universitet. The third author was partially supported by CNPq, Brazil.

©1998 American Mathematical Society
Starting with an arbitrary Hilbert \( C^* \)-bimodule \( X \) over a \( C^* \)-algebra \( A \), we define a crossed product \( B = A \rtimes_X Z \). Here, the bimodule \( X \) replaces, in a sense, the automorphism \( \alpha \), as above. We then prove that, given a semi-saturated circle action on a \( C^* \)-algebra \( B \), one must have \( B = B_0 \rtimes B_1 Z \). Examples of this situation are given for the case of partial automorphisms, and also for quantum Heisenberg manifolds, where application of our techniques is crucial, as there are no partial automorphisms available.

The regularity property, which we do not assume, is granted once one tensors everything by the algebra of compact operators. We may, therefore, use the generalized Pimsner-Voiculescu sequence of [12] to prove a similar result for the situation considered here, provided that we have separability.

The main result of this work, however, is related to the concept of strong Morita equivalence ([19]). That is, suppose we are considering two crossed products by Hilbert \( C^* \)-bimodules as above, say \( A \rtimes_X Z \) and \( B \rtimes_Y Z \), in which the underlying algebras \( A \) and \( B \) are known to be strongly Morita equivalent to each other, under an imprimitivity bimodule \( M \). The question whether \( A \rtimes_X Z \) and \( B \rtimes_Y Z \) are strongly Morita equivalent to each other is our main concern, and we prove this provided that one has that \( X \otimes_A M \) and \( M \otimes_B Y \) are isomorphic as \( A-B \) Hilbert \( C^* \)-bimodules. This should be thought of as a generalization of the result obtained independently by Curto, Muhly and Williams ([9]) on one hand, and Combes ([8]) on the other, in which a necessary condition is given for two strongly Morita equivalent \( C^* \)-algebras to remain strongly Morita equivalent after one takes their crossed product by a locally compact group. Related results were obtained by Bui ([7]), Kaliszewski ([15]), and Echterhoff ([10]).

The construction described by Pimsner in [18] is, in a sense, a generalization of our crossed products by Hilbert \( C^* \)-bimodules. In fact, one can show that the algebra \( \hat{O}_X \), constructed by Pimsner, is isomorphic to the algebra \( A \rtimes_X Z \) which we define here. Conversely, since Pimsner’s algebra \( \hat{O}_X \) carries a semi-saturated circle action, it also follows that his construction is a special case of ours. The drawback is, of course, that no good characterization of the fixed-point algebra for this action is available.

Pimsner also provides a six-term exact sequence for \( K \)-groups of the algebras arising from his construction; but, due to the fact that the main results in [18] are obtained under the hypothesis that the algebra \( A \) is generated by the scalar products, there is no immediate relationship between Pimsner’s results and ours. It seems reasonable, nevertheless, to expect that a common generalization could be found.

Last, but not least, we would like to mention that crossed products by Hilbert \( C^* \)-bimodules have also been studied by Larry Brown, although no written version of his work is available at the moment.

The second author wishes to express his gratitude for the hospitality that was extended to him at Universidade de São Paulo, where this research was carried out. Also, we would all like to thank Alex Kumjian for valuable comments on a first draft of this paper.

2. Crossed Products by Hilbert \( C^* \)-bimodules

Definition 2.1. Let a \( C^* \)-algebra \( A \) and an \( A - A \) Hilbert \( C^* \)-bimodule \( X \) (in the sense of [6, 1.8]) be given. A covariant representation of \( (A, X) \) is a pair \((\pi_A, \pi_X)\)
of representations into some $\mathcal{B}(\mathcal{F})$ such that both module actions and both inner products become the ones inherited from $\mathcal{B}(\mathcal{F})$, i.e.

(i) $\pi_X(ax) = \pi_A(a)\pi_X(x)$,
(ii) $\pi_X(xa) = \pi_X(x)\pi_A(a)$,
(iii) $\langle x, y \rangle_L = \pi_X(x)\pi_X(y)^*$,
(iv) $\langle x, y \rangle_R = \pi_X(x)^*\pi_X(y)$,

where $a \in A$, $x, y \in X$; cf. [11, 4.5].

We start out by noting that faithful covariant representations always exist. By a restriction $\pi|B$ of a representation $\pi$ into $\mathcal{B}(\mathcal{F})$ of a $C^*$-algebra $A$ to a subalgebra $B$ we will always mean the representation on the essential Hilbert space $B$. We denote by $(\pi_A^a, \delta_B^a)$ the reduced atomic representation of $A$ ([17, 4.3.7]). If $X$ and $Y$ are subsets of a $C^*$-algebra, we denote by $XY$ the closed span of the set $\{xy | x \in X, y \in Y\}$. If $X$ and $Y$ are, respectively, $A-B$ and $B-C$ Hilbert $C^*$-bimodules, we denote by $X^\sim$ the dual $B-A$ Hilbert $C^*$-bimodule, and by $X \otimes_B Y$ the tensor product, with structure as in [19, 6.17,5.9]. We indicate also by the "$\sim$" that an element of the set $X$ is being considered as an element of the Hilbert $C^*$-bimodule $X^\sim$.

**Lemma 2.2.** Let $B$ be a hereditary subalgebra of the $C^*$-algebra $A$. There exists a cardinal $\mathfrak{c}_0$ such that whenever $\mathfrak{c} \geq \mathfrak{c}_0$,

$$\mathfrak{c}\pi_A^a |_{B} \simeq \mathfrak{c}\pi_B^a,$$

where "$\simeq$" denotes unitary equivalence.

**Proof.** An irreducible representation of $B$ extends uniquely to one of $A$ by [17, 4.1.8]. On the other hand, the restriction of any irreducible representation of $A$ to $B$ is also an irreducible representation (or the trivial map) by [17, 4.1.5]. We put $\mathfrak{c}_0 = \max\{\aleph_0, \text{card } \hat{A}\}$, and assume that $\mathfrak{c} \geq \mathfrak{c}_0$. As every irreducible representation of $B$ occurs at least once in $\pi_A^a$, it occurs exactly $\mathfrak{c}$ times in $\pi_A^a$ up to unitary equivalence. \\

**Proposition 2.3.** Given $(A, X)$ as above, for a large enough cardinal $\mathfrak{c}$ there exists a covariant representation $(\pi_A, \pi_X)$ on $\mathfrak{c}\delta_A^a$ with $\pi_A$ faithful (and hence $\pi_X$ isometric by [11, 4.6]).

**Proof.** Let $L$ denote the linking algebra $[A, X]_A$; cf. [6, 2.2]. We denote by $A_1$ (resp. $A_2$) the copy of $A$ in the upper left (resp. lower right) corner. By the above, there exists a cardinal $\mathfrak{c}$ such that

$$\mathfrak{c}\pi_A^a |_{A_1} \simeq \mathfrak{c}\pi_A^a \simeq \mathfrak{c}\pi_A^a |_{A_2}.$$  

Let $\pi' = \mathfrak{c}\pi_A^a$, and denote the unitary implementing the equivalence between $\pi'|A_1$ and $\pi'|A_2$ by $u$. By [11, 4.8], we get a representation $(\pi'_L, \pi'_R, \pi'_X)$ on $\mathfrak{c}\delta_A^a$ of $X$ as an $A-A$ $C^*$-bimodule, where actually $\pi'_L = \pi'|A_1$ and $\pi'_R = \pi'|A_2$ by construction. Setting $\pi_A = \pi'_L$ and $\pi_X = \pi'_X u$, we get a covariant representation, e.g. since

$$\pi_X(x)^* \pi_X(y) = u^* \pi_X(x)^* \pi_X(y) u = u^* \pi'_R ((x, y)_R) u = \pi_A ((x, y)_R).$$  

**Definition 2.4.** Let a $C^*$-algebra $A$ and an $A-A$ Hilbert $C^*$-bimodule $X$ be given. A crossed product of $A$ by $X$ is a $C^*$-algebra $B$ and (structure-preserving) maps
\[ \iota_A : A \to B, \ i_X : X \to B \] such that for every covariant representation \((\pi_A, \pi_X)\) there is a unique \(*\)-homomorphism \(\varphi\) making the following diagram commute:

\[
\begin{array}{ccc}
(A, X) & \xrightarrow{(\iota_A, \iota_X)} & B \\
\downarrow{(\pi_A, \pi_X)} & & \downarrow{\varphi} \\
\mathcal{B}(\mathfrak{H})
\end{array}
\]

Existence of crossed products can be obtained by making use of the results on enveloping \(C^*\)-algebras defined by generators and relations (see \([3], [16]\)). That \(A\) and \(X\) are embedded in this crossed product is essentially a consequence of Proposition 2.3 above. To get a less abstract characterization of the crossed product, we carry out an explicit construction below.

We can see immediately that the crossed product must be unique. For if both \(B_1\) and \(B_2\) satisfy the conditions of Definition 2.4, assume that \(B_i \subseteq \mathcal{B}(\mathfrak{H}_i)\) and apply the existence part of the definition to get maps \(\varphi_1 : B_1 \to B_2\) and \(\varphi_2 : B_2 \to B_1\) making the diagram commute. Then \(\varphi_1 \circ \varphi_2\) must intertwine the embeddings of \(A\) and \(X\); and since \(\text{id}_{B_1}\) has the same property, \(\varphi_1 \circ \varphi_2 = \text{id}_{B_1}\) by uniqueness. By symmetry, the \(\varphi_i\) are isomorphisms. We will denote this unique object by \(A \rtimes_X Z\).

We will show later that this generalizes the crossed product by partial isomorphisms introduced in \([12]\).

**Lemma 2.5.** Let \(X\) be an \(A - B\) Hilbert \(C^*\)-bimodule, \(Y\) a \(B - C\) Hilbert \(C^*\)-bimodule. Assume that \(A, B, C, X,\) and \(Y\) are all faithfully represented in \(\mathcal{B}(\mathfrak{H})\) so that the module actions and inner products become the ones inherited from \(\mathcal{B}(\mathfrak{H})\). Then, as an \(A - C\) Hilbert \(C^*\)-bimodule, \(X \otimes_B Y\) is isomorphic to \(XY\).

**Proof.** One checks that the map given by

\[ x \otimes y \mapsto \pi_X(x)\pi_Y(y) \]

is well-defined and a Hilbert \(C^*\)-bimodule isomorphism. \(\square\)

**Corollary 2.6.** When \((\pi_A, \pi_X)\) is a faithful covariant representation of \((A, X)\),

\[ \bigotimes_{1}^{N} X \cong \prod_{1}^{N} \pi_X(X). \]

We construct a \(Z\)-bundle with fibers given by

\[
X_n = \begin{cases} 
X \otimes_A X \otimes_A \cdots \otimes_A X & \text{if } n > 0, \\
A & \text{if } n = 0, \\
X^\sim \otimes_A X^\sim \otimes_A \cdots \otimes_A X^\sim & \text{if } n < 0.
\end{cases}
\]

Given \(m, n \in \mathbb{Z}\), we will build maps \(\cdot : X_m \times X_n \to X_{m+n}\), and must do this in different manners according to the signs of \(m\) and \(n\).

1°: \(m = n = 0\): We employ the multiplication in \(A\).

2°: \(m, n > 0\): Concatenation.

3°: \(m, n < 0\): Concatenation.

4°: \(m = 0, n > 0\): We define maps

\[ \cdot : X_0 \times X_n \to X_n, \quad a \cdot (x_1 \otimes \cdots \otimes x_n) = ax_1 \otimes \cdots \otimes x_n \]
and
\[ \cdot : X_n \times X_0 \to X_n, \quad (x_1 \otimes \cdots \otimes x_n) \cdot a = x_1 \otimes \cdots \otimes x_n a. \]

5°: \( m = 0, n < 0 \): We set
\[ \cdot : X_0 \times X_n \to X_n, \quad a \cdot (\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n}) = (x_1 a^*)^\sim \otimes \cdots \otimes \tilde{x}_{-n} \]

and
\[ \cdot : X_n \times X_0 \to X_n, \quad (\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n}) \cdot a = \tilde{x}_1 \otimes \cdots \otimes (a^* x_{-n})^\sim. \]

6°: \( m > 0, n < 0 \): We define the product recursively by
\[ (x_1 \otimes \cdots \otimes x_m) \cdot (\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n}) = (x_1 \otimes \cdots \otimes x_{m-1} \langle x_m, y_1 \rangle_L) \cdot (\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n}) \]

and
\[ (\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n}) \cdot (x_1 \otimes \cdots \otimes x_m) = (\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n-1}) \cdot (\langle y_{-n}, x_1 \rangle_R x_2 \otimes \cdots \otimes x_m). \]

7°: \( m < 0, n > 0 \): as in 6° above.

To make sense of the last two definitions, one must first define maps from, e.g., \( X_m \otimes X_{-n} \) to \( X_{m-1} \otimes X_{-n-1} \). The existence of such maps follows by facts such as \( \langle xa, y \rangle_L = \langle x, ya^* \rangle_L \) (cf. [6, 1.9]).

We also define maps \( ^* : X_n \to X_{-n} \) by

1°: \( n = 0 \): The involution on \( A \).

2°: \( n > 0 \): We set \( (x_1 \otimes \cdots \otimes x_n)^* = \tilde{x}_n \otimes \cdots \otimes \tilde{x}_1 \).

3°: \( n < 0 \): We set \( (\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n})^* = x_{-n} \otimes \cdots \otimes x_1 \).

We claim that with these operations, the \( \{ X_n \} \) form a \( C^* \)-algebraic bundle over \( \mathbb{Z} \) in the sense of [14]. To avoid tedious computations with simple tensors, we find a family of faithful representations of the \( X_n \) onto the same Hilbert space which respect the structure imposed on the \( X_n \) by these operations. With this in hand, we can reduce the proofs of the relevant axioms to corresponding axioms of operators on a Hilbert space, all of which are well-known.

Let \( (\pi_A, \pi_X) \) denote a representation of \( (X, A) \). We define maps \( \rho_n : X_n \to \mathcal{B}(\mathcal{H}) \) by \( \rho_0(a) = \pi_A(a) \),

\[ \rho_n(x_1 \otimes \cdots \otimes x_n) = \pi_X(x_1) \cdots \pi_X(x_n) \]

for \( n > 0 \), and

\[ \rho_n(\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n}) = \pi_X(x_1)^* \cdots \pi_X(x_{-n})^* \]

We now prove

**Lemma 2.7.** For all \( k, l \in \mathbb{Z} \) and all \( a \in X_k, b \in X_l \),

(i) \( \rho_{k+l}(a \cdot b) = \rho_k(a) \rho_l(b) \),

(ii) \( \rho_{-k}(a^*) = \rho_k(a)^* \).

**Proof.** By totality of the set of simple tensors, we may restrict our attention to these. We obviously have, for \( k > 0 \),

\[ \rho_{-k}(x_1 \otimes \cdots \otimes x_k)^* = \rho_{-k}(x_k \otimes \cdots \otimes x_1) \]

\[ = \pi_X(x_k)^* \cdots \pi_X(x_1)^* \]

\[ = (\pi_X(x_1) \cdots \pi_X(x_k))^* \]

\[ = \rho_k(x_1 \otimes \cdots \otimes x_k)^*, \]

and similarly for negative \( k \).

Proving (i) requires an induction argument based on the recursive definition of the multiplication operation. One first checks the statement for \( k = 0 \) or \( l = 0 \), and
in the cases where \( k \) and \( l \) have the same sign. When \( k > 0 \) and \( l < 0 \), say, one writes
\[
\rho_{k+l}(\langle x_1 \otimes \cdots \otimes x_k \rangle \cdot (y_1 \otimes \cdots \otimes y_{-l})) = \rho_{k-1}(x_1 \otimes \cdots \otimes x_{k-1} \langle x_k, y_1 \rangle_L \cdot (y_2 \otimes \cdots \otimes y_{-l})) = \rho_{k-1}(x_1 \otimes \cdots \otimes x_{k-1} \langle x_k, y_1 \rangle_L \cdot \rho_{l+1}(y_2 \otimes \cdots \otimes y_{-l}))
\]
where the crucial second step is the induction hypothesis.

**Proposition 2.8.** \( B = (X_n, \cdot, *) \) thus defined forms a \( C^* \)-algebraic bundle over \( \mathbb{Z} \) (in the sense of [14] or [13, 2.2]).

**Proof.** We follow the checklist in [13, 2.2]. Axioms (i) and (v) follow by definition. To check all the other axioms, fix a faithful covariant representation \((\pi_A, \pi_X)\) by Proposition 2.3 and define \( \rho_k \) as above. As these representations are all isometric – they are \( A - A \) Hilbert \( C^* \)-bimodule isomorphisms by Corollary 2.6 – we may check all the properties in \( \mathcal{B}(\mathfrak{H}) \) instead. For instance, associativity follows from associativity in \( \mathcal{B}(\mathfrak{H}) \) as
\[
\rho_{k+i+m}(a \cdot (b \cdot c)) = \rho_k(a) \rho_{i+m}(b \cdot c) = \rho_k(a) \rho_l(b) \rho_m(c) = \rho_{k+i+m}((a \cdot b) \cdot c).
\]

**Theorem 2.9.** The cross-sectional algebra \( C^*(B) \) over \( B = (X_n, \cdot, *) \) (as defined in [14]) is the crossed product of \( A \) by \( X \).

**Proof.** There are natural isometric maps \( \iota_A : A \to X_0 \) and \( \iota_X : X \to X_1 \). These extend into first \( L^1(B) \), then \( C^*(B) \). We denote all of these maps by \( \iota_A, \iota_X \).

Given a covariant representation \((\pi_A, \pi_X)\) of \((X, A)\) onto \( \mathfrak{H} \), define \( \rho_k \) as above. Because the \( \rho_k \) respect group structure and the involution by Lemma 2.7, they give rise to a \( * \)-homomorphism \( \varphi : C_c(B) \to \mathcal{B}(\mathfrak{H}) \) by
\[
\sum a_i \delta_i \mapsto \sum \rho_i(a_i).
\]
We may extend \( \varphi \) first to \( L^1(B) \), then to \( C_c(B) \), and the map thus achieved clearly makes the diagram (2.4) commute. Also, as \( X_0 \) and \( X_1 \) generate \( L^1(B) \), this is the only map with this property.

**Corollary 2.10.** The map \( \iota_A \) is faithful. The map \( \iota_X \) is isometric.

**Proof.** Take \((\pi_A, \pi_X)\) isometric, cf. Proposition 2.3, and construct \( \varphi \) as in the proof above. We have \( \varphi \circ \iota_A = \rho_0 = \pi_A \), whence \( \iota_A \) must be norm-preserving. Then so is \( \pi_X \) by [11, 4.6].

### 3. Applications of Generalized Crossed Products

The following theorem describes the structure of \( C^* \)-algebras carrying a semi-saturated action of the circle. We recall that an action of the circle on a \( C^* \)-algebra \( B \) is said to be semi-saturated ([12, 4.1]) if \( B \) is generated by the fixed-point subalgebra and the first spectral subspace.
As for classical crossed products and crossed products by partial automorphisms ([12]), the crossed products introduced in the previous section carry a natural dual action. This is gotten by extending the maps

\[ \sum a_i \delta_i \mapsto \sum t^i a_i \delta_i \]

from \( C_c(B) \) to maps \( \beta_t : A \times \mathbb{Z} \to A \times \mathbb{Z} \).

**Theorem 3.1.** Suppose \( B \) is a \( C^* \)-algebra with a circle action \( \alpha_t \). Let \( B_0 \) and \( B_1 \) denote the fixed-point algebra and the first spectral subspace for \( \alpha_t \), respectively. Then \( \alpha_t \) is semi-saturated if and only if \( B \) is isomorphic to \( B_0 \times B_1 \mathbb{Z} \) by an isomorphism sending \( \alpha_t \) to the dual action.

**Proof.** Let \( \beta_t \) be the dual action on \( B_0 \times B_1 \mathbb{Z} \). It is clear from our construction that \( \iota_{B_0}(B_0) \) and \( \iota_{B_1}(B_1) \) generate \( B_0 \times B_1 \mathbb{Z} \). These are contained in the fixed-point algebra and the first spectral subspace for \( \beta_t \), respectively. To prove the other implication, we must prove that \( \iota_{B_0}(B_0) \) is in fact the fixed-point space for \( \beta_t \). Consider the conditional expectation onto \( \beta \)'s fixed-point algebra

\[ Q_0(x) = \int_T \beta_t(x) dt \]

as well as the fixed-point algebra \( (B_0 \times B_1 \mathbb{Z})_0 \). We have

\[ \iota_{B_0}(B_0) \subseteq (B_0 \times B_1 \mathbb{Z})_0 = Q_0(B_0 \times B_1 \mathbb{Z}) = Q_0(C_c(B)) \subseteq \iota_{B_0}(B_0) \]

However, as \( \iota_{B_0} \) is an isometry by Corollary 2.10, its image is closed, and \( \iota_{B_0}(B_0) \) is exactly the fixed-point algebra.

Consider now the diagram

\[ \begin{array}{ccc}
(B_0, B_1) & \xrightarrow{\iota_{B_0}, \iota_{B_1}} & B \\
\downarrow \alpha_t & & \downarrow \varphi \\
C^*(B) & \xrightarrow{\varphi} & B
\end{array} \]

where the diagonal maps are the inclusions. The induced map \( \varphi \) is \( \beta - \alpha \) covariant as on \( C_c(B) \),

\[ \varphi(\beta_t(\sum a_i \delta_i)) = \varphi(\sum t^i a_i \delta_i) = \varphi(\sum \alpha_t(a_i) \delta_i) = \sum \alpha_t(a_i) = \alpha_t(\sum a_i) = \alpha_t(\varphi(\sum a_i \delta_i)) \]

It is clear that \( \varphi \) is injective on the fixed-point algebra of \( \beta \), \( \iota_{B_0}(B_0) \), as it sends \( a_0 \delta_0 \) to \( a_0 \). Then \( \varphi \) must be injective according to [12, 2.9]. Finally, \( \varphi \) is onto as a consequence of semisaturation.

**Example 3.2** (Crossed products by partial automorphisms). Let \( (I, J, \theta) \) be a partial automorphism of a \( C^* \)-algebra \( A \), as in [12]. The dual action of the circle on the crossed product \( A \rtimes_\theta \mathbb{Z} \) (cf. the paragraph preceding Theorem 3.1) is semi-saturated ([12, 4.7]) and its fixed-point subalgebra is isomorphic to \( A \) ([12, 3.9]). It follows from Theorem 3.1 that \( A \rtimes_\theta \mathbb{Z} \) is isomorphic to the crossed product of \( A \) by the first spectral subspace. The first spectral subspace \( J_0 \) is the \( A - A \) Hilbert \( C^* \)-bimodule whose underlying vector space is the ideal \( J \) with the usual left action and the right action

\[ x \cdot a = \theta(\theta^{-1}(x)a) \]
and the inner products
\[ \langle x, y \rangle_L = xy^* \quad \text{and} \quad \langle x, y \rangle_R = \theta^{-1}(x^*y). \]

Note that in the case of an automorphism, i.e. when \( I = J = A \) and \( \theta \in \text{Aut}(A) \), the Hilbert \( C^* \)-bimodule \( J_\theta \) is the element of the Picard group of \( A \) corresponding to \( \theta^{-1} \) by the map considered in [5, 3].

Returning to the context of the partial automorphisms, we see that two homomorphisms from \( K_i(J) \) to \( K_i(I) \) arise naturally in this context. One of them is the map \((\theta^{-1})_*\) induced by \( \theta^{-1} \). On the other hand, since \( J_\theta \) is a left-full \( J - I \) Hilbert \( C^* \)-bimodule, it induces, as in [11, 5], homomorphisms
\[ (J_\theta)_*: K_i(J) \to K_i(I). \]

We next show that these two maps agree. As in [11, 3.14], for \( \alpha \in K_0(J) \), let \( p, q \in M_n(J^\sim) \) be such that \( \alpha = [p] - [q] \) and \( p - q \in M_n(J) \). Consider the Fredholm operator
\[ T: pJ^n \to qJ^n \]
defined by \( T((x_i)) = q(x_i) \). Then \((J_\theta)_*\) is, by definition, the index of the operator \( T \circ \id: pJ^n \otimes_J J_\theta \to qJ^n \otimes_J J_\theta \). In general, if \( p \) is any projection in \( M_n(J^\sim) \), the map
\[ \Phi_p: pJ^n \otimes_J J_\theta \to \theta^{-1}(p)I^n \]
given by \( \Phi_p(x_n \otimes x) = \theta^{-1}(x_n)x \), where \( \theta^{-1} \) is extended componentwise to \( J^n \), is a right \( A \)-Hilbert module isomorphism.

For \( p, q \) and \( T \) as above, the diagram
\[
\begin{array}{ccc}
pJ^n \otimes_J J_\theta & \xrightarrow{T \circ \id} & qJ^n \otimes_J J_\theta \\
\Phi_p & \downarrow & \Phi_q \\
\theta^{-1}(p)I^n & \xrightarrow{T} & \theta^{-1}(q)I^n \\
\end{array}
\]
commutes for \( \hat{T}((i_n)) = \theta^{-1}(q)((i_n)) \). By [11, 3.14] we now have
\[ (J_\theta)_*(\alpha) = \text{Ind}(\hat{T}) = [\theta^{-1}(p)] - [\theta^{-1}(q)] = (\theta^{-1})_*(\alpha), \]
as we wanted to show.

As for the \( K_1 \)-groups, let us denote by \( SB \) the suspension of a \( C^* \)-algebra \( B \). Consider the partial automorphism \((SI, SJ, 1 \otimes \theta)\) of the \( C^* \)-algebra \( SA \). By the result above, the diagram
\[
\begin{array}{ccc}
K_0(SI) & \xrightarrow{SJ} & K_0(SJ) \\
\| & & \| \\
K_0(SI) & \xrightarrow{(1 \otimes \theta)^*} & K_0(SJ) \\
\end{array}
\]
commutes. On the other hand, by [4, 8.2.2], the diagram
\[
\begin{array}{ccc}
K_0(SI) & \xrightarrow{(1 \otimes \theta)^*} & K_0(SJ) \\
\downarrow & & \downarrow \\
K_1(I) & \xrightarrow{(\theta^{-1})_*} & K_1(J) \\
\end{array}
\]
commutes, where the vertical maps are the usual isomorphisms [4, 8.2.2]. It follows that the homomorphisms from \( K_1(I) \) to \( K_1(J) \) induced by \( \theta \) and \( J_\theta \), respectively, agree.
Example 3.3 (Quantum Heisenberg manifolds). The quantum Heisenberg manifolds provide a family of examples of crossed products by $C^*$-Hilbert bimodules not coming from partial automorphisms. For real numbers $\mu, \nu$ and a positive integer $c$, the quantum Heisenberg manifold $D^c_{\mu \nu}$ ([20], [1]) consists of the closure in the multiplier algebra of $C_0(\mathbb{R} \times \mathbb{T}) \rtimes_\alpha \mathbb{Z}$ of the $*$-subalgebra

$$C^c_{\mu \nu} = \{ \Phi \in C_c(\mathbb{Z}, C_b(\mathbb{R} \times \mathbb{T})) | \Phi(p, x + 1, y) = e(-cp(y - p \nu)) \Phi(p, x, y) \},$$

where $(\alpha f)(x, y) = f(x - 2\mu, y - 2\nu)$, and $e(x) = e^{2\pi i x}$. The algebras $D^c_{\mu \nu}$ carry, as in [20, 5.7], an action of the circle $\mathbb{T}$ defined by

$$(\gamma_s \Phi)(p, x, y) = z^p \Phi(p, x, y),$$

for $\Phi \in C^c_{\mu \nu}$. The $n$th spectral subspace $(D^c_{\mu \nu})_n$ consists of the $\delta_n$-maps, i.e.

$$(D^c_{\mu \nu})_n = \{ f \delta_n | f \in C_0(\mathbb{R} \times \mathbb{T}), f(x + 1, y) = e(-cn(y - n \nu)) f(x, y) \}.$$

Thus the fixed-point subalgebra can be identified with $C(\mathbb{T}^2)$.

There is no $\mathbb{T}$-covariant isomorphism between a quantum Heisenberg manifold and a crossed product by a partial automorphism $\theta$. For in that case, since (proof of [2, 2.1])

$$(D^c_{\mu \nu})_1((D^c_{\mu \nu})_1)^* = ((D^c_{\mu \nu})_1)^*(D^c_{\mu \nu})_1 = C(\mathbb{T}^2),$$

$\theta$ would be an automorphism of $C(\mathbb{T}^2)$, and there would be a unitary element $u \in (D^c_{\mu \nu})_1$ implementing $\theta$. If such a unitary existed, it would be a continuous function $u : \mathbb{R} \times \mathbb{T} \to \mathbb{T}$ satisfying

$$u(x + 1, y) = e(-c(y - \nu))u(x, y).$$

But the maps $y \mapsto u(x, y)$ and $y \mapsto u(x + 1, y)$ are homotopic and therefore have the same winding number, which contradicts (1).

We now show that $D^c_{\mu \nu} \cong C(\mathbb{T}^2) \rtimes (D^c_{\mu \nu})_1, \mathbb{Z}$, by proving that the circle action is semi-saturated. Let $A$ denote the $C^*$-algebra generated by $C(\mathbb{T}^2)$ and $(D^c_{\mu \nu})_1$; we show by induction on $n$ that the $n$th spectral subspace is contained in $A$, which will end the proof. So we assume that $(D^c_{\mu \nu})_n \subset A$. As in [2, 2.1], let $\Delta_0, \delta_1 \in (D^c_{\mu \nu})_1, i = 1, 2$, be such that

$$|\Delta_1(x, y)|^2 + |\Delta_2(x, y)|^2 = 1,$$

for all $(x, y) \in \mathbb{R} \times \mathbb{T}$. Given $f \delta_{n+1} \in (D^c_{\mu \nu})_{n+1}$, set

$$F_1(x, y) = f(x, y) \Delta_1(x - 2n\mu, y - 2n\nu).$$

Then $F_1 \delta_n \in (D^c_{\mu \nu})_n \subset A$, and

$$f \delta_{n+1} = F_1 \delta_n * \Delta_1 \delta_1 + F_2 \delta_n * \Delta_2 \delta_1,$$

where $*$ denotes the multiplication in $D^c_{\mu \nu}$.

Remark 3.4 (A Pimsner-Voiculescu sequence). In case $A$ and $X$ are separable, the circle action $\gamma$ defined in the proof of Theorem 3.1 can be stabilized so that the generalized PV sequence in [12, 7] applies. That is, the action $\gamma \otimes \text{id}$ of the circle on $(A \rtimes X) \otimes \mathbb{K}$ is semi-saturated and regular, and therefore it is the crossed product of the fixed-point algebra $A \otimes \mathbb{K}$ by a partial automorphism $(\theta, I_R \otimes \mathbb{K}, I_L \otimes \mathbb{K})$, where $I_R$ and $I_L$ denote the closed spans of $\langle X, X \rangle_R$ and $\langle X, X \rangle_L$ respectively.
Thus the generalized PV sequence in [12, 7] applies, and we get
\[
\begin{array}{ccc}
K_0(I_L) & \rightarrow & K_0(A) \\
\uparrow & & \downarrow \\
K_1(A \rtimes_X \mathbb{Z}) & \rightarrow & K_1(A) \\
\end{array}
\]

4. Morita equivalence

Let \( X \) and \( Y \) denote \( A - A \) and \( B - B \) Hilbert \( C^* \)-bimodules, respectively, and let \( M \) be an \( A - B \) Morita equivalence bimodule. Our purpose is to show that if \( X \otimes_A M \) and \( M \otimes_B Y \) are isomorphic as \( A - B \) Hilbert \( C^* \)-bimodules, then the crossed products \( A \rtimes_X \mathbb{Z} \) and \( B \rtimes_Y \mathbb{Z} \) are Morita equivalent. When \( X = A_\alpha \) and \( Y = B_\beta \), for some \( \alpha \in \text{Aut}(A) \) and \( \beta \in \text{Aut}(B) \), the condition \( X \otimes_A M \simeq M \otimes_B Y \) is equivalent to the conditions stated in [9] and [8] to get Morita equivalent crossed products. In order to prove the result mentioned above we construct a Hilbert \( C^* \)-bimodule \( W \) over the linking algebra \( L \) of \( M \) and then prove that \( A \rtimes_X \mathbb{Z} \) and \( B \rtimes_Y \mathbb{Z} \) are complementary full corners of \( L \rtimes_W \mathbb{Z} \).

The next representation lemma will help us avoid computations.

**Lemma 4.1.** Let \( A, B, L, M, X, \) and \( Y \) be as above. Then there are faithful representations \( \pi, \pi_X \) and \( \pi_Y \) of \( L, X, \) and \( Y \), respectively, on the same Hilbert space \( \mathcal{H} \) such that \((\pi|_A, \pi_X)\) and \((\pi|_B, \pi_Y)\) are covariant.

Besides, if the \( A - B \) Hilbert \( C^* \)-bimodules \( X \otimes_A M \) and \( M \otimes_B Y \) are isomorphic, then \( \pi_X \) can be chosen so that \( \pi_X(X)\pi(M) = \pi(M)\pi_Y(Y) \).

**Proof.** By Lemma 2.2, for a large enough cardinal \( \mathfrak{c} \), \( L \) can be faithfully represented on \( \mathcal{H} = \mathcal{H}_A^\mathfrak{c} \oplus \mathcal{H}_B^\mathfrak{c} \) by a representation \( \pi \) such that \( \pi|A \) and \( \pi|B \) are \( \mathcal{H}_A^\mathfrak{c} \) and \( \mathcal{H}_B^\mathfrak{c} \), respectively. Now, by Proposition 2.3, there are faithful representations \( \pi_X \) and \( \pi_Y \) on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, such that \((\pi|_A, \pi_X)\) and \((\pi|_B, \pi_Y)\) are covariant. To represent \( X \) and \( Y \) on \( \mathcal{H} \), set

\[
\pi_X(x) = \begin{bmatrix} \pi_X(x) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi_Y(y) = \begin{bmatrix} 0 & 0 \\ 0 & \pi_Y(y) \end{bmatrix}.
\]

Assume now that \( X \otimes_A M \) and \( M \otimes_B Y \) are isomorphic. Set \( \mathcal{H}_0 = \mathcal{H}_A \) and \( \mathcal{H}_1 = \mathcal{H}_B \). To simplify notation we view \( X, A \subset B(\mathcal{H}_0), Y, B \in B(\mathcal{H}_1) \) and \( M \subset B(\mathcal{H}_1, \mathcal{H}_0) \), by identifying those sets with their images under the representations described above, restricted to their essential subspaces. Then, by Lemma 2.5 this implies there is an isomorphism \( \tau : XM \rightarrow MY \).

For \( m \in M, x \in X \) and \( \xi \in \mathcal{H}_1 \), we have
\[
\langle \tau(xm)\xi, \tau(xm)\xi \rangle_{\mathcal{H}_0} = \langle \tau(xm), \tau(xm) \rangle_{R \xi \xi} \mathcal{H}_0 = \langle \langle x\xi \rangle_{R \xi \xi} \rangle_{\mathcal{H}_0} = \langle \langle x\xi \rangle_{R \xi \xi} \rangle_{\mathcal{H}_0} = \langle x\xi, x\xi \rangle_{\mathcal{H}_0}.
\]

Therefore there is a partial isometry \( u \in B(\mathcal{H}_0) \) with initial space \( XM\mathcal{H}_0 \) and final space \( MY\mathcal{H}_1 \), such that \( \tau(xm) = uxm \). Notice that \( u \) commutes with \( A \), because \( \tau \) is an \( A \)-module isomorphism and \((XM\mathcal{H}_0) \) is \( A \)-invariant. Besides, for \( x_0, x_1 \in X \)

\[
(ux_0)^*(ux_1) = x_0^*u^*ux_1 = x_0^*x_1 = (x_0, x_1)_R,
\]

and

\[
(ux_0)(ux_1)^* = u^*x_0^*u^*x_1 = x_0^*u^*u^*x_1 = x_0x_1 = (x_0, x_1)_L.
\]

Finally, we just showed that \((ux)M = MY \). It suffices now to replace the representation \( \pi_X \) by \( u\pi_X \). \(\square\)
Theorem 4.2. Let $X$ and $Y$ be $A - A$ and $B - B$ Hilbert $C^*$-bimodules, respectively. Let $M$ be an $A - B$ Morita equivalence bimodule such that $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic as $A - B$ Hilbert $C^*$-bimodules. Then $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are Morita equivalent.

Proof. We view $A, B, L, M, X,$ and $Y$ as acting on a Hilbert space $\mathfrak{H}$ as in Lemma 4.1, so that $XM = MY$, and all module actions and inner products are the ones inherited from $\mathbb{B}(\mathfrak{H})$. We also assume that, as in Lemma 4.1, the span of $A\mathfrak{H} \cup B\mathfrak{H}$ is dense in $\mathfrak{H}$.

We then have
\[ M^*X = M^*XMM^* = M^*MYM^* = YM^* \]
by [6, 1.7]. Now, the subspace $W = \left\{ \frac{X}{YM} \right\}$ is a right Hilbert $C^*$-module over $L$ for the structure inherited from $\mathbb{B}(\mathfrak{H})$. On the other hand $W = \left\{ \frac{X}{YM} \right\}$, so it is a left Hilbert $C^*$-module over $L$ for the structure inherited from $\mathbb{B}(\mathfrak{H})$. It follows that $W$ is an $L - L$ Hilbert $C^*$-bimodule.

We next show that $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are complementary full corners of $L \rtimes_W \mathbb{Z}$, which, in view of [5, 1.1], will end the proof. By Theorem 2.9 $L \rtimes_W \mathbb{Z} \simeq C^*(B_W)$ and $A \rtimes_X \mathbb{Z} \simeq C^*(B_X)$, where $B_W$ and $B_X$ are the $C^*$-algebraic bundles whose fibers are given by

\[ W_n = \begin{cases} W^n & \text{if } n > 0, \\ L & \text{if } n = 0, \\ (W^*)^{-n} & \text{if } n < 0, \end{cases} \]

and

\[ X_n = \begin{cases} X^n & \text{if } n > 0, \\ A & \text{if } n = 0, \\ (X^*)^{-n} & \text{if } n < 0, \end{cases} \]

with the structure inherited from $\mathbb{B}(\mathfrak{H})$.

The projection $p = \left[ \begin{smallmatrix} 1_{M^*(A)} & 0 \\ 0 & 0 \end{smallmatrix} \right]$ in $\mathbb{B}(\mathfrak{H})$ is a multiplier of degree zero of $B_W$, in the sense of [14, VIII, 2.14], and therefore can be viewed as a projection in $M(C^*(B_W))$. We now show that $C^*(B_X) \simeq p(C^*(B_W))$. It follows from the identities $XM = MY$ and $M^*X = YM^*$ that $W^n = \left\{ \frac{X^n}{YM} \right\}$ for $n > 0$, and therefore $pW_n = X_n$.

Thus the Banach $*$-algebras $L^1(B_X)$ and $pL^1(B_W)$ can be identified. By taking their respective enveloping $C^*$-algebras we get $C^*(B_X) = pC^*(B_W)$. (To see that the enveloping $C^*$-algebra of $pL^1(B_W)$ is $pC^*(B_W)$, notice that $pL^1(B_W)$ is invariant under the dual action of the circle on $L^1(B_W)$ defined in the proof of Theorem 3.1. Besides, the obvious map from $C^*(pL^1(B_W))$ to $p(C^*(B_W))$ is covariant under the extensions of that action, and it is one-to-one when restricted to the fixed-point algebras. So [12, 2.9] applies.) Analogous reasoning proves that, for $q = \left[ \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right]$, we have $B \rtimes_Y \mathbb{Z} \simeq q(L \rtimes_W \mathbb{Z})$. Finally, a straightforward computation shows that $LpW^n = W^n = W^nqL$ for all $n \geq 0$, which proves that $p$ and $q$ are full projections. 

References


DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE SÃO PAULO, RUA DO MATÃO 1010, 05508-900 SÃO PAULO, BRAZIL
*Current address*: Centro de Matemáticas, Facultad de Ciencias, Universidad de la República, Eduardo Acevedo 1139, CP 11200 Montevideo, Uruguay
*E-mail address*: abadie@cmat.edu.uy

MATHEMATISCH INSTITUT, KÖBENHAVNS UNIVERSITET, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK
*E-mail address*: eilers@math.ku.dk

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE SÃO PAULO, RUA DO MATÃO 1010, 05508-900 SÃO PAULO, BRAZIL
*E-mail address*: exel@ime.usp.br