

INDUCTION THEOREMS ON THE STABLE RATIONALITY OF THE CENTER OF THE RING OF GENERIC MATRICES

ESTHER BENEISH

ABSTRACT. Following Procesi and Formanek, the center of the division ring of $n \times n$ generic matrices over the complex numbers \mathbf{C} is stably equivalent to the fixed field under the action of S_n , of the function field of the group algebra of a ZS_n -lattice, denoted by G_n . We study the question of the stable rationality of the center C_n over the complex numbers when n is a prime, in this module theoretic setting. Let N be the normalizer of an n -syllow subgroup of S_n . Let M be a ZS_n -lattice. We show that under certain conditions on M , induction-restriction from N to S_n does not affect the stable type of the corresponding field. In particular, $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic and the isomorphism preserves the S_n -action. We further reduce the problem to the study of the localization of G_n at the prime n ; all other primes behave well. We also present new simple proofs for the stable rationality of C_n over \mathbf{C} , in the cases $n = 5$ and $n = 7$.

INTRODUCTION

The question of rationality of the center of the ring of generic matrices has been studied extensively over some decades, in particular for its connections to other problems. Geometrically, the center C_n of $n \times n$ generic matrices over an algebraically closed field is the function field of an algebraic variety under a reductive group, namely $(M_n \times M_n)/\mathrm{PGL}_n$. The classification of quotients under connected reductive groups is one of the main open problems in invariant theory. In the Brauer group setting the stable rationality of this center implies the Merkurjev-Suslin result, which says that the Brauer group of a field k is generated by cyclics, provided k has enough roots of unity. Le Bruyn presents an informative survey of the problem and its applications in [LL].

In 1883 Sylvester showed that for $n = 2$, C_2 was rational over \mathbf{C} , the complex numbers, [S]. In 1979 and 1980 Formanek showed that C_n was rational over \mathbf{C} for the case $n = 3$, [F1], and $n = 4$, [F2]. In 1984, Saltman proved that for all primes n , C_n is retract rational over \mathbf{C} , [SD]. In 1991 Bessenrodt and Le Bruyn showed stable rationality for the cases $n = 5$ and 7 [BL]. Their proof, however, requires the use of a computer. In this paper we present simple proofs for the case $n = 5$ and 7 , Theorems 3.3 and 3.4.

We study the problem in its module theoretic setting. If G is a finite group, and M is a ZG -lattice, then $\mathbf{C}[M]$ denotes the group algebra of the abelian group M , and $\mathbf{C}(M)$ its quotient field. Define the following ZS_n -lattices. $U_n = \sum_{i=1, \dots, n} Zu_i$, where $gu_i = u_{g(i)}$ for all g in S_n . A , usually denoted by A_{n-1} , is the kernel of the

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augmentation map $\varepsilon: U_n \rightarrow Z$. $V_n = \sum_{i,j,i \neq j} Z y_{ij}$ where $g y_{ij} = y_{g(i)g(j)}$ for all g in S_n . G_n is defined by the sequence $0 \rightarrow G_n \rightarrow V_n \rightarrow A \rightarrow 0$, where the map $V_n \rightarrow A$ is given by $y_{ij} \rightarrow u_i - u_j$. Procesi-Formanek show that $C_n \cong \mathbf{C}(G_n \oplus U_n \oplus U_n)^{S_n}$, [F1, Theorem 3]. This implies that C_n is stably isomorphic to $\mathbf{C}(G_n)^{S_n}$ [L, Proposition 1.4]. Let n be an odd prime. Let H be an n -syllow subgroup of G , thus H is cyclic of order n . The normalizer of H in S_n , which we denote by N , is the semidirect product of H by a cyclic group C of order $n - 1$.

The main result of this paper is:

Theorem 2.6. *The fields $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic, and the isomorphism respects the S_n -actions. The center C_n is stably isomorphic to their fields of invariants.*

This theorem is a direct consequence of Proposition 2.4, which says that under certain conditions, induction—restriction from N to G does not affect the stable type of the field. This is a crucial reduction step for the problem. It implies that the stable rationality of C_n depends on the structure of G_n as a ZN -lattice. Set $n = p$, and let Z_p denote the localization of Z at the prime p .

The second result is:

Theorem 2.7. *Let D and R be invertible ZN -lattices. Suppose that R is either 0, or stably permutation and $Z_q C$ -projective for all primes q dividing $p - 1$. If $(G_n \oplus R)_p \cong D_p$, then the fields $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} D)$ are stably isomorphic as G -fields.*

This theorem says that the stable rationality of C_n in fact depends on the structure of $(G_n)_p$ as a $Z_p N$ -lattice.

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1.

Let G be a finite group, a ZG -lattice is finitely generated Z -free ZG -module. A permutation ZG -module is a ZG -lattice with a Z -basis permuted by G . An invertible ZG -module is a direct summand of a permutation module. A ZG -module M is stably permutation if there exists permutation ZG -modules P and P' , such that $M \oplus P \cong P'$.

Definition. A ZG -lattice M is G -faithful if the map $G \rightarrow \text{Aut}_Z(M)$ is injective.

Notation.

- (1) For a ZG -lattice M we denote by $M^* = \text{Hom}_Z(M, Z)$, its dual.
- (2) For any prime q in Z , Z_q denotes the localization of Z at the ideal (q) , and $M_q = Z_q \otimes M$.
- (3) In the remainder of this paper we will denote S_n by G , and $n = p$ will be an odd prime, unless otherwise specified.

The following observations were made in [BL].

The map: $V_n \rightarrow A \otimes U_n$

$$y_{ij} \rightarrow (u_i - u_j) \otimes u_i$$

is a ZG -isomorphism.

Now tensoring the exact sequence

$$0 \rightarrow A \rightarrow U_n \rightarrow Z \rightarrow 0$$

by A , we get

$$0 \rightarrow A \otimes A \rightarrow U_n \otimes A \rightarrow A \rightarrow 0$$

The map $U_n \otimes A \rightarrow A$ sends y_{ij} to $u_i - u_j$, so $G_n \cong A \otimes A$.

Notation. We let $B = A^* \otimes A$.

This lattice is used throughout the paper, in particular if let Q be the field of rational numbers, then $QB \cong QG_n$.

The following proposition was proved by Bessenrodt and Le Bruyn, but not published. We present here a proof.

Proposition 1.1. *We have the following isomorphisms of ZG -lattices $B \oplus U_n \cong V_n \oplus Z$. Hence B is stably permutation.*

Proof. The sequence $0 \rightarrow A \rightarrow U_n \rightarrow Z \rightarrow 0$ splits when localized at all primes q different from n , dividing the order of G , with splitting map given by $1 \rightarrow (\sum_{i=1}^n u_i)/n$. So we have

$$Z_q \oplus A_q \cong (U_n)_q \text{ and taking duals } Z_q \oplus A_q^* \cong (U_n)_q.$$

Now tensoring by A_q we get

$$B_q \oplus A_q \cong (V_n)_q.$$

So B_q is Z_qG -invertible. In particular B_q is Z_qP -invertible for all q -syllow subgroups P of G . To show that B is invertible it suffices by [CT, Lemma 9, section 1] to show that B_p is Z_pH -invertible for a $p = n$ -syllow subgroup of G . Now consider the diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & B & \longrightarrow & V_n & \longrightarrow & A^* \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & U_n \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & Z & \longrightarrow & Z \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array}$$

$(U_n)_p \cong Z_pS_n/S_{n-1}$ and $(V_n)_p \cong Z_pS_n/S_{n-2}$ are Z_pG -projective, since they are induced from groups of order prime to p , hence $X_p \cong (B \oplus U_n)_p \cong (V_n \oplus Z)_p$. So B_p is Z_pG -invertible, and hence Z_pH -invertible. So B is ZG -invertible. This implies by e.g. [B, Lemma 1.2]

$$B \oplus U_n \cong V_n \oplus Z.$$

So B is stably permutation.

Lemma 1.2. *Let $X = Z/pZ$ be the trivial ZG -module with p elements. Then there exists a ZG -exact sequence*

$$0 \rightarrow A \rightarrow A^* \rightarrow X \rightarrow 0.$$

Proof. U_n^* is generated by u_i^* , where $u_i^*(u_j) = \delta_{ij}$, and A^* is generated by $\text{res}(u_i^*)I = 1, \dots, n-1$, where res is restriction from U_n to A . The map $A^* \rightarrow X$ is given by $\text{res}(u_i^*) \rightarrow 1$.

Remark. Tensoring the sequence of Lemma 1.2 by A we get

$$0 \rightarrow A \otimes A \rightarrow A^* \otimes A \rightarrow X \otimes A \rightarrow 0.$$

Identifying $A \otimes A$ with G_n , and setting $T = X \otimes A \cong A/pA$ we obtain

$$0 \rightarrow G_n \rightarrow B \rightarrow T \rightarrow 0.$$

Let $U = U_n/pU_n$. It is easy checked that the map $U \rightarrow X$ given by $u_i \rightarrow 1$ has kernel T . Thus we have a ZG -exact sequence

$$0 \rightarrow T \rightarrow U \rightarrow X \rightarrow 0.$$

Let $T' = A^*/pA^*$, then it is immediate that the following sequence is ZG -exact

$$0 \rightarrow X \rightarrow U \rightarrow T' \rightarrow 0.$$

These are important sequences which will be used in the following sections.

2.

Throughout the rest of this paper we adopt the following notation unless otherwise specified. We will also keep the notation used in section 1.

- $G = S_n$, $n = p$ is prime; n and p will be used interchangeably without mention.
- $H = p$ -sylow subgroup of G , so H is cyclic of order p .
- $N = N_G(H)$ will be the normalizer of H in G , so $N = H \rtimes C$, where C is cyclic of order $p-1$, and if we let h and c generate H and C respectively, then $chc^{-1} = h^a$ where a is a primitive $(p-1)$ st root of 1 mod p .
- $F = Z/pZ$ field of p elements.
- For any finite group G , and for any ZG -lattice M , we will denote by $\phi(M)$ the flasque class of M , see [CT, section 1] for definitions.
- Z^- will denote the sign representation of G .
- X will denote the trivial ZG -module of p elements.

Definitions. (1) A ZG -lattice M is quasi-permutation [EM], if there exist an exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow Q \rightarrow 0$$

with R and Q permutation.

(2) We will say that 2 fields are stably isomorphic as G -fields, if the fields are stably isomorphic, and the isomorphism respects their G -actions.

Various versions of the following lemma have been proved in the literature, we need a slightly different version here.

Lemma 2.1. *Let G be a finite group.*

(1) *Suppose there exists an exact sequence of ZG -lattices*

$$0 \rightarrow M \rightarrow N \rightarrow E \rightarrow 0$$

with E invertible and M , G -faithful, then $\mathbf{C}(E \oplus M) \cong \mathbf{C}(N)$ as G -fields.

(2) Let G be a finite group, and let M and M' be G -faithful ZG -lattices, with M quasi-permutation. Then the fields $\mathbf{C}(M \oplus M')$ and $\mathbf{C}(M')$ are stably isomorphic as G -fields.

(3) Let M and M' be G -faithful ZG -lattices such that $\phi(M) = \phi(M')$, then $\mathbf{C}(M)$ and $\mathbf{C}(M')$ are stably isomorphic as G -fields.

(4) If $G = S_n$ and M is a G -faithful quasi-permutation ZG -lattice, then $\mathbf{C}(M)^G$ is stably rational over \mathbf{C} .

Proof. (1) The proof is basically that of [L, Proposition 1.5]. The injection $M \rightarrow N$ induces an injection $\mathbf{C}(M) \subset \mathbf{C}(N)$. Now consider the sequence

$$0 \rightarrow \mathbf{C}(M)^* \rightarrow \mathbf{C}(M)^*N \rightarrow E \rightarrow 0.$$

Since G acts faithfully on $\mathbf{C}(M)^*$, $\mathbf{C}(M)^*$ is H^1 -trivial by Hilbert theorem 90. Since E is invertible the sequence splits by [CT, Lemma 1, section 1]. Thus $\mathbf{C}(M)^*N \cong E \oplus \mathbf{C}(M)^*$. This isomorphism yields the required isomorphism of G -fields.

(2) Since M is quasi-permutation there exist an exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow Q \rightarrow 0$$

with Q and R permutation. By (1) $\mathbf{C}(M \oplus Q) \cong \mathbf{C}(R)$. So $\mathbf{C}(M \oplus M' \oplus Q) \cong \mathbf{C}(M' \oplus R)$. Apply [L, Proposition 1.3] to the field $l = \mathbf{C}(M')$ and the l -vector space $W = \sum lr_i$, where $\{r_i\}$ is a Z -basis for R which is permuted by G . Then there exists elements $\{y_i\}$ in W^G which form an L -basis for W . Now $W \subset l(R)$, and hence

$$l(R) = l(y_1, \dots, y_k) = \mathbf{C}(M')(y_1, \dots, y_k).$$

By the same argument there exist indeterminates z_1, \dots, z_t , such that

$$\mathbf{C}(M \oplus M' \oplus Q) = \mathbf{C}(M \oplus M')(z_1, \dots, z_t).$$

So $\mathbf{C}(M')(y_1, \dots, y_k) \cong \mathbf{C}(M \oplus M')(z_1, \dots, z_t)$.

(3) Since $\phi(M) = \phi(M')$, there exist 2 exact sequences

$$0 \rightarrow M \rightarrow E \rightarrow R \rightarrow 0 \text{ and } 0 \rightarrow M' \rightarrow E \rightarrow P \rightarrow 0$$

with P and R permutation by [CT, Lemma 8, section 1]. By (1) $\mathbf{C}(E) \cong \mathbf{C}(M \oplus R) \cong \mathbf{C}(M' \oplus P)$ as G -fields, and by (2) $\mathbf{C}(M)$ and $\mathbf{C}(M')$ are stably isomorphic as G -fields.

(4) Since M is quasi-permutation there is an exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow Q \rightarrow 0$$

with Q and R permutation. By (1) $\mathbf{C}(M \oplus Q) \cong \mathbf{C}(R)$. By (2) $\mathbf{C}(M \oplus Q)$ and $\mathbf{C}(M)$ are stably isomorphic as G -fields. Since $\phi(U_n) = \phi(R) = 0$, $\mathbf{C}(U_n)$ and $\mathbf{C}(R)$ stably isomorphic as G -fields by (3). Since $\mathbf{C}(U_n)^G$ is rational over \mathbf{C} , the result follows.

Proposition 2.2. (1) Let G be a finite group and suppose there exist exact sequences

$$0 \rightarrow E \rightarrow P \rightarrow L \rightarrow 0 \text{ and } 0 \rightarrow E' \rightarrow R \rightarrow J \oplus L \rightarrow 0$$

with P and R , ZG -permutation, E and E' G -faithful, L and J , finite and $J = Q/mQ$ for some ZG -permutation module Q , and some positive integer m . Then the fields $\mathbf{C}(E')$ and $\mathbf{C}(E)$ are stably isomorphic as G -fields.

(2) Let $G = S_n$ and let M be a G -faithful ZG -lattice of trivial cohomology, then $\mathbf{C}(M)^G$ is stably rational over \mathbf{C} . Furthermore if M' is any G -faithful ZG -lattice, then the fields $\mathbf{C}(M \oplus M')$ and $\mathbf{C}(M')$ are stably isomorphic as G -fields.

Proof. (1) We form the commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & E' & \longrightarrow & R & \longrightarrow & J \oplus L \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & E' & \longrightarrow & M_3 & \longrightarrow & Q \oplus P \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & Q \oplus E & \longrightarrow & Q \oplus E \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array}$$

Since R, Q and P are permutation modules, the fields $\mathbf{C}(E)$ and $\mathbf{C}(E')$ are stably isomorphic as G -fields, by Lemma 2.1.

(2) By [BK, Theorem 8.10] M is ZG -projective and by [EM, Theorem 3.3] M is stably permutation. The lemma follows by Lemma 2.1.

Notation. Let $I_{G/N}$ denote the kernel of the augmentation map $Z(G/N) \rightarrow Z$, and let $I = I_{G/N}/pI_{G/N}$.

Proposition 2.3. *The $Z_p G$ -lattice $(I_{G/N})_p$ and the ZG -lattice I are cohomologically trivial. Let $I_{G/N}^- = I_{G/N} \otimes Z^-$, $I^- = I \otimes Z^-$, then $(I_{G/N}^-)$, I^- are also cohomologically trivial.*

Proof. The sequence $0 \rightarrow (I_{G/N})_p \rightarrow Z_p(G/N) \rightarrow Z_p \rightarrow 0$ splits with splitting map $Z_p \rightarrow Z_p(G/N)$ given by $1 \rightarrow (\sum_{g \in G/N} g)/[G : N]$. So $Z_p(G/N) \cong Z_p \oplus (I_{G/N})_p$. By Mackey’s subgroup theorem [CR1, Theorem 10.13] $\text{Res}_H^G ZG/N \cong Z \oplus ZH^k$ for some positive integer k . So as a $\hat{Z}_p H$ -module $(\hat{I}_{G/N})_p$ is free by the Krull-Schmidt-Azumaya and by [CR1, Theorem 30.17] $(I_{G/N})_p$ is a free $Z_p H$ -module. Now for any subgroup K of G , $H^i(K, (I_{G/N})_p)$ injects into $H^i(H, (I_{G/N})_p) = 0$. This proves that $(I_{G/N})_p$ is cohomologically trivial. To prove that I is cohomologically trivial it suffices to take the cohomology of the sequence

$$0 \rightarrow (I_{G/N})_p \rightarrow (I_{G/N})_p \rightarrow I \rightarrow 0.$$

The second statement follows since $I_{G/N}^-, I^-$ are isomorphic to $I_{G/N}$ and I respectively as H -modules.

As in the preceding proof we have $FG/N \cong F \oplus I$. The following proposition is a crucial reduction step. In particular it implies Theorem 2.6 which says that $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic as G -fields, so that we can basically study G_n at the N -level and induce up.

Proposition 2.4. *Let L be a finite ZG -mdoule of exponent p . Suppose we have*

(1) An FG -extension of X or X^- by L , or of L by X or X^- , whose middle term is FG -projective.

(2) A ZG -exact sequence

$$0 \rightarrow M \rightarrow P \rightarrow L \oplus Q/mQ \rightarrow 0$$

with P and Q stably permutation, where m is a positive integer.

Then the fields $\mathbf{C}(M)$ and $\mathbf{C}(ZG \otimes_{ZN} M)$ are stably isomorphic as G -fields.

Proof. Tensoring the sequence

$$(s1) \quad 0 \rightarrow M \rightarrow P \rightarrow L \oplus Q/mQ \rightarrow 0$$

by ZG over ZN we get

$$(s2) \quad 0 \rightarrow ZG \otimes_{ZN} M \rightarrow ZG \otimes_{ZN} P \rightarrow FG \otimes_{FN} L \oplus ZG \otimes_{ZN} Q/mQ \rightarrow 0.$$

We form the sequence

$$(s3) \quad 0 \rightarrow Q \rightarrow Q \rightarrow Q/mQ \rightarrow 0$$

and add it to (s2) to obtain

$$(s4) \quad \begin{aligned} 0 \rightarrow ZG \otimes_{ZN} M \oplus Q &\rightarrow ZG \otimes_{ZN} P \oplus Q \\ &\rightarrow FG \otimes_{FN} L \oplus ZG \otimes_{ZN} Q/mQ \oplus Q/mQ \rightarrow 0. \end{aligned}$$

We have $FG \otimes_{FN} L \cong L \oplus I \otimes L$, and we assume that we have an FG -exact sequence of the form

$$(1) \quad 0 \rightarrow L \rightarrow U \rightarrow X \rightarrow 0.$$

The proof extends directly to the other cases. Tensoring (1) by I of F preserves FG -exactness since I is F -free, and we get

$$0 \rightarrow L \otimes I \rightarrow U \otimes I \rightarrow I \rightarrow 0.$$

Since U is FG -projective, so is $U \otimes I$. By looking at the cohomology of the ZG -exact sequence

$$0 \rightarrow ZG \rightarrow ZG \rightarrow FG \rightarrow 0,$$

we see that FG is cohomologically trivial as a ZG -module, therefore so is $U \otimes I$. By Proposition 2.3 so is I . Thus $L \otimes I$ is also ZG -cohomologically trivial. Now consider the exact sequence

$$(s5) \quad 0 \rightarrow M' \rightarrow M'' \rightarrow L \otimes I \rightarrow 0 \text{ with } M'' \text{ free.}$$

Since M'' and $L \otimes I$ are cohomologically trivial so is M' , thus M' is ZG -projective by [BK, Theorem 8.10], and by [EM, Theorem 3.3] it is stably permutation. Now add the sequences (s1), (s5) and

$$0 \rightarrow ZG \otimes_{ZN} Q \rightarrow ZG \otimes_{ZN} Q \rightarrow ZG \otimes_{ZN} Q/mQ \rightarrow 0$$

and form a commutative diagram with (s4),

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
 0 \rightarrow & ZG \otimes_{ZN} M \oplus Q & \longrightarrow & ZG \otimes_{ZN} P \oplus Q & \longrightarrow & ZG \otimes_{ZN} Q/mQ \oplus Q/mQ & \rightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 \rightarrow & ZG \otimes_{ZN} M \oplus Q & \longrightarrow & M_3 & \longrightarrow & M'' \oplus P \oplus ZG \otimes_{ZN} Q & \rightarrow 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & M' \oplus M \oplus ZG \otimes_{ZN} Q & \longrightarrow & M' \oplus M \oplus ZG \otimes_{ZN} Q & \rightarrow 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
 \end{array}$$

The result now follows by Lemma 2.1, since M'', P, M', Q and $ZG \otimes_{ZN} Q$ are stably permutation for this case. A similar argument applies to the other cases.

Corollary 2.5. *The fields $\mathbf{C}(A^*)$ and $\mathbf{C}(ZG \otimes_{ZN} A^*)$ are stably isomorphic as G -fields. So are the fields $\mathbf{C}(A^{*-})$ and $\mathbf{C}(ZG \otimes_{ZN} A^{*-})$.*

Proof. To prove the first statement of the corollary, we note that there exists an exact sequence

$$0 \rightarrow A \oplus Z \rightarrow U_n \rightarrow X \rightarrow 0$$

where the map from U_n to X is given by $u_i \rightarrow 1$. Tensoring the sequence by A^* over Z we get

$$0 \rightarrow B \oplus A^* \rightarrow V \rightarrow T' \rightarrow 0,$$

where $T' = A^*/pA^*$ as in section 1.

By the remark at the end of section 1 we have an exact sequence

$$0 \rightarrow X \rightarrow U \rightarrow T' \rightarrow 0.$$

Hence by Proposition 2.4, $\mathbf{C}(A^* \oplus B)$ and $\mathbf{C}(ZG \otimes_{ZN}(A^* \oplus B))$ are stably isomorphic as G -fields. The result follows by Lemma 2.1 since B is stably permutation. For the second statement we have the exact sequence

(1) $0 \rightarrow A^{*-} \oplus Z^- \rightarrow U_n^- \rightarrow T'^- \rightarrow 0$, where the map $U_n^- \rightarrow T'^- \rightarrow 0$ is given by $U_n \rightarrow A^* \rightarrow T' \rightarrow 0$. We also have the sequence $0 \rightarrow Z \oplus Z^- \rightarrow ZG/A_n \rightarrow J \rightarrow 0$ where $J = Z/2Z$.

Tensoring by U_n we get

(2) $0 \rightarrow U_n^- \oplus U_n \rightarrow ZG \otimes_{ZA_n} U_n \rightarrow J' \rightarrow 0$ where $J' = U_n/2U_n$, and A_n is the alternating group.

Adding U_n to the first 2 terms of (1) and combining with (2) we get

$$0 \rightarrow A^{*-} \oplus Z^- \oplus U_n \rightarrow ZG \otimes_{ZA_n} U_n \rightarrow T'^- \oplus J' \rightarrow 0$$

since T'^- and J' are of relatively prime orders.

So $\mathbf{C}(A^{*-} \oplus Z^- \oplus U_n)$ and $\mathbf{C}(ZG \otimes_{ZN}(A^{*-} \oplus Z^- \oplus U_n))$ are stably isomorphic as G -fields, by Proposition 2.4. Now the result follows by Lemma 2.1 and by observing that Z^- is quasi-permutation.

The main theorem is a direct consequence of Proposition 2.4.

Theorem 2.6. *The fields $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic as G -fields. The center C_n is isomorphic to their fixed fields.*

Proof. By the remark at the end of section 1, there is an exact sequence

$$0 \rightarrow T \rightarrow U \rightarrow X \rightarrow 0.$$

Applying Proposition 2.4 to the ZG -exact sequence: $0 \rightarrow G_n \rightarrow B \rightarrow T \rightarrow 0$ from section 1, we get that $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} G_n)$ are stably isomorphic as G -fields. By [F1, Theorem 3] and [L, Proposition 1.4] we have C_n and $\mathbf{C}(G_n)^G$ stably isomorphic.

Theorem 2.7. *Let D and R be invertible ZN -lattices. Suppose that R is either 0, or stably permutation and Z_qC -projective for all primes q dividing $p - 1$. If $(G_n \oplus R)_p \cong D_p$, then the fields $\mathbf{C}(G_n)$ and $\mathbf{C}(ZG \otimes_{ZN} D)$ are stably isomorphic as G -fields.*

Proof. We have $(G_n \oplus R)_p \cong D_p$. By [CR1, Lemma 31.4], this implies the existence of a ZN -exact sequence

$$0 \rightarrow G_n \oplus R \rightarrow D \rightarrow Y \rightarrow 0$$

where Y is a finite ZN -module of order prime to p . Since $(G_n \oplus R)_p \cong D_p$, $Q(G_n \oplus R) \cong QD$ where Q is the field of rational numbers. It follows directly from the definition of G_n , that this lattice is ZC -free. Therefore $(G_n \oplus R)_q$ is Z_qC -projective for all primes q dividing $p - 1$, and [B1, Theorem 1.2] implies that so is D_q . Since the first 2 terms of the sequence are Z_qC -projective for all primes $q \neq p$, Y is cohomologically trivial. Therefore we have a ZN -exact sequence

$$0 \rightarrow Pr \rightarrow Fr \rightarrow Y \rightarrow 0$$

with Fr , free. Since Y is cohomologically trivial, so is Pr , hence Pr is ZN -projective by [BK, Theorem 8.10].

Forming a commutative diagram with the above sequences we obtain

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & G_n \oplus R & \longrightarrow & D & \longrightarrow & Y \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & G_n \oplus R & \longrightarrow & M & \longrightarrow & Fr \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & Pr & \longrightarrow & Pr & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

This implies that $G_n \oplus R \oplus Fr \cong D \oplus Pr$ since Pr and Fr are projective. Inducing up to G we obtain

$$ZG \otimes_{ZN} (G_n \oplus R \oplus Fr) \cong ZG \otimes_{ZN} (D \oplus Pr).$$

Since Pr and Fr are ZN -projective, the induced modules are ZG -projective and by [EM, Theorem 3.3] they are stably permutation. The result now follows by Lemma 2.1 and Theorem 2.6.

The following lemma will be needed later.

Lemma 2.8. *There exist a ZG -lattice W such that $G_n \oplus W \cong V_n \oplus Z(G/H)$. As ZN -lattices we have $G_n \oplus W \cong ZN^k \oplus Z(N/H)$ for some positive integer k .*

Proof. Since H is cyclic and $A \cong ZH(h-1)$ as a ZH -module, we have the ZH -exact sequence

$$0 \rightarrow Z \rightarrow ZH \rightarrow A \rightarrow 0,$$

tensoring by ZG over ZH we get

$$0 \rightarrow Z(G/H) \rightarrow ZG \rightarrow Z(G/H) \otimes A \rightarrow 0.$$

From this sequence we form the following diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 & & & I \otimes A & \longrightarrow & I \otimes A & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & Z(G/H) & \longrightarrow & ZG & \longrightarrow & Z(G/H) \otimes A \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & Z(G/H) & \longrightarrow & W & \longrightarrow & A \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow & \\
 & & & & 0 & & 0 &
 \end{array}$$

where $I = I_{G/H}$.

For all primes q different from p , dividing the order of G , $W_q \cong (A \oplus Z(G/H))_q$, since A_q is Z_qG -invertible by the proof of Proposition 1.1. For the prime p , we have

$$Z_p(G/H) \cong Z_p \oplus I_p.$$

So

$$Z_p(G/H) \otimes A_p \cong A_p \oplus I_p \otimes A_p.$$

Since $H^0(H, ZG/H \otimes A) \cong H^1(H, ZG/H) = 0$, we have

$$H^1(H, W) \cong H^0(H, I \otimes A) = 0,$$

and thus W_p is Z_pH -invertible since H is cyclic by [CT, Corollary 2 and Proposition 2, section 1]. By [B, Theorem 2.1], W is invertible. We have the following

commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & G_n & \longrightarrow & V_n & \longrightarrow & A \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 0 & \longrightarrow & G_n & \longrightarrow & M & \longrightarrow & W \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & ZG/H & \longrightarrow & ZG/H \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array}$$

Since V_n, G_n, W and ZG/H are invertible, $V_n \oplus ZG/H \cong W \oplus G_n$ by [CT, Lemma 1, section 1]. The second statement follows from the fact that $V_n \cong ZN$ and $Z(G/H) \cong Z(N/H) \oplus ZN^{k-1}$ as ZN -lattices.

3.

Recall that N was defined to be the normalizer of a p -sylog subgroup H of G . So $N = N_G(H) = H \rtimes C$, where C is cyclic of order $p - 1$, and if we let h and c , generate H and C respectively then $c.h = h^a$, where a is primitive $(p - 1)$ st root of 1 mod p .

Now

$$\widehat{Z}_p N/H = \bigoplus_{k=1}^{p-1} Z_k$$

where Z_k is the $\widehat{Z}_p N$ -module of rank 1 with trivial H -action, and such that c acts on 1 as θ^k , where θ was chosen to be the primitive $(p - 1)$ -st root of 1 in \widehat{Z}_p for which $\theta \equiv a \pmod{p}$.

We let $X_k = Z_k/pZ_k$ for $k = 1, \dots, p-1$. So $X = X_{p-1}$ is the trivial $Z_p N$ -module Z/pZ .

Lemma 3.1. *Keeping the definitions of A and A^* . The $Z_p N$ -modules A_p and A_p^* are isomorphic as $Z_p H$ -modules. Therefore $Z_p N \otimes_{Z_p H} A \cong Z_p N \otimes_{Z_p H} A^*$.*

Proof. As a ZN -module $U_n \cong ZN/C \cong ZH$ where $u_i \rightarrow h^i$ and $c.h = h^a$. Thus $A \cong ZH(h - 1)$. Since H is cyclic we have a ZH -exact sequence

$$0 \rightarrow Z \rightarrow ZH \rightarrow ZH(h - 1) \rightarrow 0.$$

Equivalently

$$0 \rightarrow Z \rightarrow ZH \rightarrow A \rightarrow 0.$$

From this sequence and the sequence

$$0 \rightarrow Z \rightarrow ZH \rightarrow A^* \rightarrow 0.$$

We obtain the commutative diagram of ZH -modules.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 0 & \longrightarrow & Z & \longrightarrow & ZH & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & ZH & \longrightarrow & M & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^* & \longrightarrow & A^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
 \end{array}$$

Thus $A^* \oplus ZH \cong A \oplus ZH$. By the Krull-Schmidt-Azumaya (K-S-A) $\widehat{A}_p \cong \widehat{A}_p^*$, and by [CR1, Theorem 30.17] A_p and A_p^* . \square

Notation. For each m dividing $p-1$, we denote by C_m the subgroup of C generated by c^m , thus C_m has order $(p-1)/m$, and index m in C .

We will denote by A_k the $\widehat{Z}_p N$ -lattice $\widehat{A}_p^* \otimes Z_k$, for $k = 1, \dots, p-1$.

For each k , we will denote by G_k the $\widehat{Z}_p N$ -lattice $\widehat{A}_p^* \otimes A_k$.

Theorem 3.2. *We have the following isomorphisms of $\widehat{Z}_p N$ -lattices.*

- (1) $A^* \otimes Z_1 \cong \widehat{A}_p$ as $Z_p N$ -lattices.
- (2) $(\widehat{G}_N)_p \cong \widehat{A}_p^* \otimes A_2 \cong G_2$.
- (3) $\widehat{B}_p \cong G_1$.

Proof. By Lemma 3.1 we have $Z_p N \otimes_{Z_p H} A \cong Z_p N \otimes_{Z_p H} A^*$. Thus

$$\bigoplus_{k=1}^{p-1} A_k \cong \bigoplus_{k=1}^{p-1} \widehat{A}_p^* \otimes Z_k.$$

As an $Z_p H$ -module $A_p^* \cong Z_p[w]$ where w is a primitive p -th root of 1 on Z_p . Therefore \widehat{A}_p^* is indecomposable as a $\widehat{Z}_p H$ -module and a fortiori as a $Z_p N$ -module. By the K-S-A, this implies that $\widehat{A}_p \cong \widehat{A}_p^* \otimes Z_j$ for some j . We will show that $j = 1$. Since $\text{Ext}_{\widehat{Z}_p N}^1(\widehat{A}_p, \widehat{Z}_p N/H) \cong \text{Ext}_{\widehat{Z}_p N}^1(\widehat{Z}_p N/H, A_p^*) \cong H^1(H, A_p^*) \cong Z/p$ we have $\text{Ext}_{\widehat{Z}_p N}^1(\widehat{A}_p, Z_k) = 0$ for all k except for a unique k for which $\text{Ext}_{\widehat{Z}_p N}^1(\widehat{A}_p, Z_k) = Z/p$. It is easily seen that $Z_k^* = \text{Hom}(Z_k, Z_p) \cong Z_{p-1-k}$, for $k = 1, \dots, p-1$.

Now taking the cohomology of the sequence

$$0 \rightarrow \widehat{A}_p \rightarrow (\widehat{U}_n)_p \rightarrow \widehat{Z}_p \rightarrow 0$$

we have $0 \rightarrow \text{Hom}_N(\widehat{Z}_p, Z_k) \rightarrow \text{Hom}_N((\widehat{U}_n)_p, Z_k) \rightarrow \text{Hom}_N(\widehat{A}_p, Z_k) \rightarrow \text{Ext}_N^1(\widehat{Z}_p, Z_k) = 0$ since Z_k is invertible. Now if $k \neq p-1$ then $\text{Hom}_N(\widehat{Z}_p, Z_k) \cong Z_k^N = 0$, and $\text{Hom}_N((\widehat{U}_n)_p, Z_k) \cong Z_k^C \cong 0$, thus $\text{Hom}_N(\widehat{A}_p, Z_k) \cong 0$. If $k = p-1$, then $\text{Hom}_N(\widehat{Z}_p, Z_k) \cong \widehat{Z}_p$, $\text{Hom}_N((\widehat{U}_n)_p, Z_k) \cong \widehat{Z}_p$. Since $\text{Hom}_N(\widehat{A}_p, Z_k)$ is torsion free it must be 0.

Now consider the exact sequence

$$0 \rightarrow Z_k \rightarrow Z_k \rightarrow X_k \rightarrow 0.$$

From this we have $0 \rightarrow \text{Hom}_N(\widehat{A}_p, Z_k) \rightarrow \text{Hom}_N(\widehat{A}_p, Z_k) \rightarrow \text{Hom}_N(\widehat{A}_p, X_k) \rightarrow \text{Ext}_N^1(\widehat{A}_p, Z_k) \rightarrow 0$. Therefore the unique k for which $\text{Ext}_N^1(\widehat{A}_p, Z_k) \cong Z/p$ is the k for which $\text{Hom}_N(A_p, X_k) = Z/p$. Now we have a nonzero $Z_p N$ -homomorphism $A_p \rightarrow X_1$ given by $h - 1 \rightarrow 1$. So $k = 1$.

Now tensoring the sequence

$$0 \rightarrow \widehat{Z}_p \rightarrow (\widehat{U}_n)_p \rightarrow A_p^* \rightarrow 0$$

by Z_j , and using the fact that $\widehat{A}_p \cong \widehat{A}_p^* \otimes Z_j$ we obtain

$$0 \rightarrow Z_j \rightarrow (\widehat{U}_n)_p \otimes Z_j \rightarrow \widehat{A}_p \rightarrow 0.$$

This sequence is not split, hence $j = 1$. This proves (1).

To prove (2), we tensor both sides of $\widehat{A}_p \cong \widehat{A}_p^* \otimes Z_1$ by \widehat{A}_p and we get

$$(\widehat{G}_n)_p \cong \widehat{A}_p^* \otimes Z_1 \otimes \widehat{A}_p^* \otimes Z_1 \cong \widehat{A}_p^* \otimes \widehat{A}_p^* \otimes Z_2 \cong G_2.$$

To prove (3), we tensor $\widehat{A}_p \cong \widehat{A}_p^* \otimes Z_1$ by A_p^* and we get

$$B_p \cong A_p^* \otimes \widehat{A}_p \cong \widehat{A}_p^* \otimes \widehat{A}_p^* \otimes Z_1 \cong G_1.$$

Remark. The lattice A_2 is an important lattice for the following reason. Consider the exact $Z_p N$ -sequence

$$0 \rightarrow \widehat{Z}_p \rightarrow \widehat{Z}_p H \rightarrow A_p^* \rightarrow 0.$$

Tensoring by A_2 we get

$$0 \rightarrow A_2 \rightarrow \widehat{Z}_p H \otimes A_2 \rightarrow G_2 \rightarrow 0.$$

Since $ZH \cong Z(N/C)$, $\text{Res}_C^N U_n \cong Z \oplus ZC$ by Mackey's subgroup theorem [CR1, Theorem 10.13]. As in the proof of Proposition 1.1, $(\widehat{U}_n)_q \cong A_q^* \oplus Z_q \cong A_q \oplus Z_q$ for all primes q dividing $p - 1$. By the K-S-A and by [CR1, Theorem 30.17] this implies that $A_q^* \cong A_q \cong Z_q C$. If we let Q be the rationals this implies that $QA \cong QA^*$ as QC -modules. Since $Z_p C$ is a maximal order in QC we have $A_p \cong A_p^* \cong Z_p C$ as $Z_p C$ -modules by [CR1, Proposition 31.2]. Thus $A_k \cong \widehat{Z}_p C$ as $\widehat{Z}_p C$ -modules for all $k = 1, \dots, p - 1$. Hence $\widehat{Z}_p H \otimes A_2 \cong \widehat{Z}_p N$, so $\phi(A_2) = [G_2]$. By the same argument we have $\phi(A_k) = [G_k]$ for all $k = 1, \dots, p - 1$.

By the above $(G_n)_q, (G_n^*)_q$ and B_q are $Z_q C$ -free for all primes q dividing $p - 1$. This fact will be needed later.

We will now use the machinery developed here to give new and very simple proofs for $p = 5$ and $p = 7$.

Theorem 3.3. *The center of the ring of 5×5 generic matrices is stably rational over C .*

Proof. Let $p = 5$; then

$$\widehat{Z}_p N/H = \bigoplus_{k=1}^4 Z_k$$

and so

$$\widehat{Z}_p N \otimes_{\widehat{Z}_p H} A_p^* = A_p^* \oplus A_1 \oplus A_2 \oplus A_3.$$

Now $A_2 \cong \widehat{A}_p^* \otimes Z_2 \cong \widehat{A}_p^{*-}$, since $Z_2 \cong \widehat{Z}_p^-$. Tensoring both sides of $A_2 \cong \widehat{A}_p^* \otimes Z_2$ by \widehat{A}_p^* we get by Theorem 3.2

$$(\widehat{G}_N)_p \cong (\widehat{G}_N^{*-})_p.$$

Now

$$\widehat{Z}_p N \otimes_{Z_p HC_2} A_p^* \cong \widehat{A}_p^* \oplus A_2.$$

Tensoring for \widehat{A}_p^* we get

$$\widehat{Z}_p N \otimes_{Z_p HC_2} (\widehat{G}_N^*)_p \cong (\widehat{G}_N^*)_p \oplus (\widehat{G}_N^{*-})_p.$$

Therefore

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p \cong (G_n^*)_p \oplus (G_n)_p,$$

and by [CR1, Theorem 30.17]

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p \cong (G_n^*)_p \oplus (G_n)_p.$$

By [CT, Proposition 3, section 1], $\phi((A_n^*)) = 0$ as a ZHC_2 -module since C_2 is of order 2. Since G_n^* is invertible, this implies that $ZN \otimes_{ZHC_2} G_n^*$ is stably permutation as a ZN -module. Now adding W_p^* to both sides of

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p = (G_n^*)_p \oplus (G_n)_p,$$

we get by Lemma 2.8

$$Z_p N \otimes_{Z_p HC_2} (G_n^*)_p \oplus W_p^* \cong (G_n)_p \oplus Z_p N^k \oplus Z_p N/H.$$

Since $ZN^k \oplus ZN/H$ and $ZN \otimes_{ZHC_2} (G_n^*)_p \oplus W_p^*$ are ZC -free by Lemma 2.8, Theorem 2.7 implies that $\mathbf{C}(G_n \oplus ZG^k \oplus ZG/H)$ and $\mathbf{C}(ZG/N \otimes W^* \oplus ZG/HC_2 \otimes G_n^*)$ are stably equivalent as G -fields. By Lemma 2.1, $\mathbf{C}(G_n)$ is stably equivalent to $\mathbf{C}(ZG/N \otimes W^*)$ as G -fields. Now $\phi(A^*) = [G_n^*] = \Phi(W^*)$, hence $\mathbf{C}(ZG/N \otimes W^*)$ and $\mathbf{C}(ZG/N \otimes A^*)$ are stably equivalent as G -fields by Lemma 2.1. By Corollary 2.5 these fields are stably equivalent to $\mathbf{C}(A^*)$. Therefore by Theorem 2.6, C_n is stably isomorphic to $\mathbf{C}(A^*)^G$ which is rational over \mathbf{C} , generated by the $(n-1)$ -st elementary symmetric function [BL].

Theorem 3.4. *The center of the ring of 7×7 generic matrices is stably rational over \mathbf{C} .*

Proof. The proof is similar to that of $p = 5$. Here $p - 1 = 6$.

$$\widehat{Z}_p N/NC_3 \cong \widehat{Z}_p \oplus Z_2 \oplus Z_4$$

so

$$\widehat{Z}_p N/HC_3 \otimes \widehat{A}_p^* \oplus A_2 \oplus A_4.$$

Tensoring by \widehat{A}_p^* we get by Theorem 3.2 and [CR1, Theorem 30.17]

$$Z_p N/HC_3 \otimes G_n^* \cong (G_n^* \oplus G_n \oplus B^-)_p$$

since $A_4 \cong \widehat{A}_p^-$. Now consider the exact sequence

$$0 \rightarrow B^- \rightarrow ZN/HC_2 \otimes B \rightarrow B \rightarrow 0.$$

This sequence splits at all primes different from 2, and for the prime 2, $\text{Ext}_N^1(B_2, B_2^-)$ injects into $\text{Ext}_{ZC_3}^1(B_2, B_2^-) = 0$ since B_2 is Z_2C -free. So $ZN/HC_2 \otimes B \cong B \oplus B^-$,

and B^- is stably permutation. Also since C_3 is of order 2, $ZN/HC_3 \otimes G_n^*$ is stably permutation by [CT, Proposition 3, section 1]. Now adding W_p^* to both sides of

$$Z_p N \otimes_{Z_p HC_3} (G_n^*)_p = (G_n^*)_p \oplus (G_n)_p \oplus B_p^-$$

we get by Lemma 2.8

$$Z_p N \otimes_{Z_p HC_3} (G_n^*)_p \oplus W_p^* \cong (G_n)_p \oplus Z_p N^k \oplus Z_p N/H \oplus B_p^-.$$

Since $ZN^k \oplus ZN/H \oplus B^-$ and $ZN \otimes_{ZHC_3} (G_n^*) \oplus W^*$ are ZC -free, Theorem 2.7 implies that $\mathbf{C}(G_n \oplus ZG^k \oplus ZG/H \oplus ZG/H \otimes B^-)$ and $\mathbf{C}(ZG/N \otimes W^* \oplus ZG/HC_3 \otimes G_n^*)$ are stably equivalent as G -fields. The argument is now the same as for $p = 5$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-PARKSIDE, KENOSHA, WISCONSIN 53141-2000