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ATOMIC MAPS AND THE CHOGOSHVILI-PONTRJAGIN CLAIM

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Abstract. It is proved that all spaces of dimension three or more disobey the Chogoshvili-Pontrjagin claim. This is of particular interest in view of the recent proof (in *Certain 2-stable embeddings*, by Dobrowolski, Levin, and Rubin, Topology Appl. **80** (1997), 81–90) that two-dimensional ANRs obey the claim.

The construction utilizes the properties of atomic maps which are maps whose fibers (=point inverses) are atoms (=hereditarily indecomposable continua).

A construction of M. Brown is applied to prove that every finite dimensional compact space admits an atomic map with a one-dimensional range.

1. The Chogoshvili-Pontrjagin claim

Let X and H be subsets of m-dimensional Euclidean space $\mathbb{R}^m$. X is removable from H if for every $\epsilon > 0$ there exist a map $f : X \longrightarrow \mathbb{R}^m$ with $\| x - f(x) \| < \epsilon$ for all x in X such that $f(X) \cap H = \emptyset$.

In the late 1920's P.S. Alexandroff [11] proved that if X is a compact n– dimensional subset of $\mathbb{R}^m$ then there exists an $(m-n)$– dimensional piecewise linear polyhedron H in $\mathbb{R}^m$ from which X is unremovable.

Naturally this Alexandroff result raises the question of whether X must also be unremovable from one of the $(m-n)$–dimensional simplices of H. In the late 30's, in an attempt to resolve this question G. Chogoshvili [10] published a paper in which he claimed to prove the following result which is now known as the Chogoshvili claim.

Claim 1.1 (Chogoshvili). *An n-dimensional subset of $\mathbb{R}^m$ is unremovable from some $(m-n)$–dimensional affine plane.*

The following reformulation offers a different viewpoint on this claim. Recall that a point $y \in Y$ is a stable value of a map $f : X \to Y$ if $y \in g(X)$ for every map $g : X \to Y$ which is sufficiently close to f. Let $L \subset \mathbb{R}^m$ be an $(m-n)$–dimensional affine subspace. Let $L^\perp = \mathbb{R}^n$ be the orthogonal complement of L in $\mathbb{R}^m$. Set $\{y\} = L \cap L^\perp$, and let $P : \mathbb{R}^m \to L^\perp = \mathbb{R}^n$ be the orthogonal projection. (Thus $P(L) = y$.) It is easy to verify that $X \subset \mathbb{R}^m$ is unremovable from L if and only if y is a stable value of $P|_X : X \to \mathbb{R}^n$. Thus the Chogoshvili claim 1.1 is equivalent to the following:

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Claim 1.2 (Chogoshvili)**.** *Let $X \subset \mathbb{R}^m$ be n–dimensional. Then some orthogonal projections P of $\mathbb{R}^m$ onto one of its n-dimensional linear subspaces has a stable value on X.*

In a statement attributed to Pontrjagin in [10] it is further claimed that the subspace in 1.1 can actually be chosen to be a coordinate subspace or equivalently that the projection in 1.2 can be chosen to be a coordinate projection, namely a projection of the form $P(x_1, x_2, \ldots, x_m) = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ for some $1 \leq i_1 < i_2 < \cdots < i_n \leq m$. This "coordinate version" is referred to as the Chogoshvili-Pontrjagin claim (see [12]).

Claim 1.3 (Chogoshvili-Pontrjagin)**.** *An n-dimensional subset of $\mathbb{R}^m$ is unremovable from some $(m-n)$–dimensional coordinate affine plane. Equivalently, some n–dimensional coordinate projection has a stable value on it.*

Clearly, 1.3 is stronger than 1.1 and 1.2.

For non-compact spaces Sitnikov [13] disproved 1.1 in 1953. But for compact spaces 1.3 was assumed to be true for many years and there were some good reasons to support that. In 1933 Nobeling [14] proved that if $X \subset \mathbb{R}^m$ is n-dimensional and compact then some n–dimensional coordinate projection maps it onto an n–dimensional subset of $\mathbb{R}^n$. (See also [15]; a stronger result can be found in [17] Th. 4.9 and [18] Th.3.d.2.) Moreover, it can be shown that there exist coordinate subspaces of all dimensions $n \leq k \leq m$ of $\mathbb{R}^m$ such that the range of X under the orthogonal projections on them is at least n–dimensional. This of course does not guarantee the existence of stable values but is an indication.

Clearly every piecewise linear or smooth manifold in $\mathbb{R}^m$ obeys the Chogoshvili-Pontrjagin claim. Also if dim X is $1, m$ or $m-1$ (see [2]), then the claim holds. Some other regularity conditions on the embedding of X in $\mathbb{R}^m$ also guarantee it obeys 1.3. (See [2] Th.4.6. and also [16] and [19].)

In the late 1980's R. Pol and independently A. Dranishnikov and R. J. Daverman noticed a gap in Chogoshvili's argument which went unnoticed for many years. In 1993 Sternfeld [1] constructed a counterexample to the Chogoshvili-Pontrjagin claim; he has shown that every n–dimensional atom (=hereditarily indecomposable continuum) X admits an embedding $e : X \longrightarrow \mathbb{R}^m$, where m depends exponentially on n, such that eX disobeys 1.3.

In 1995 Dranishnikov [5] constructed counterexamples to the Chogoshvili claim. He has constructed a two-dimensional compact subset of $\mathbb{R}^4$ which refutes 1.1 and 1.2, and extended it to higher dimensions.

Knowing that 1.1 and 1.3 are false in general it is of interest to study the case of specific spaces.

Definition 1.4. An n–dimensional compact space X obeys the Chogoshvili claim (the Chogoshvili-Pontrjagin claim respectively) if for every embedding $e : X \longrightarrow \mathbb{R}^m$ some n–dimensional projection (coordinate projection respectively) has a stable value on eX, i.e. if $eX \subset \mathbb{R}^m$ obeys 1.1 and 1.2 (1.3 respectively).

In [9] Dobrowolski, Levin and Rubin proved that several classes of 2-dimensional compacta including all 2-dimensional ANRs obey the Chogoshvili-Pontrjagin claim. Their results strongly suggested that "nice" n-dimensional spaces and in particular the n–cube I^n, $n \geq 3$, would also satisfy the claim.

Therefore it took us by surprise to find out that no space of dimension three or more obeys the claim. This is the main result of this article. Actually we prove the following slightly stronger result.

Theorem 1.5. *Let X be an n–dimensional compact space ($3 \leq n < \infty$). There exists an embedding $e : X \to \mathbb{R}^m$, $m = \binom{3n-2}{2n-1}(12n-7)$, such that for every 3-dimensional coordinate projection P of $\mathbb{R}^m$, $P|_{eX}$ factors through some 2-dimensional space and hence has only unstable values in $\mathbb{R}^3$. Thus also all n–dimensional coordinate projections of $\mathbb{R}^m$ have only unstable values on eX.*

The construction of the embedding e applies the methods of [1] as well as the Atomic Map Theorem 3.3 which says that every compact space admits an atomic map (which is a map with atomic point inverses) with a one-dimensional range. The Atomic Map Theorem is a simple corollary of Brown's Decomposition Theorem 3.1.

We present the necessary background in the next three sections and prove Theorem 1.5 at section 5. In section 6, we present a simple observation on compact ANRs in $\mathbb{R}^m$.

Theorem 1.5 leaves two problems open.

Problem 1.6. Are there spaces of dimension ≥ 3 which obey the Chogoshvili claim? In particular does I^n, $n \geq 3$, obey it?

It seems as if the methods of this article are applicable to "coordinate versions" only; Dranishnikov's construction [5] on the other hand leaves some degrees of freedom, but it is not clear how to modify it in order to end it up with a specific space such as I^3.

The Euclidean dimension in Theorem 1.5 depends exponentially on n.

Problem 1.7. Do the Chogoshvili or the Chogoshvili-Pontrjagin claims hold for "nice" spaces (such as I^n) if the Euclidean dimension m is close to n ($m \leq 2n$ say)?

The recent developments with the general case suggest that the answer to both problems is negative; but it also suggests that we may expect some surprises.

We conclude this introduction with a remark concerning terminology.

Hereditarily indecomposable continua were studied quite often in research articles in recent years. Moreover, in many cases (as in the case of this article) they were applied to solve problems related to general, not necessarily indecomposable, spaces (such examples are [20], [21], [22], [23], [24]). A comprehensive textbook [8] on these spaces is in preparation. We feel that in these circumstances the traditional term "Hereditarily indecomposable continuum" is improper. We use neither "Finite subcoverable spaces" for compacta nor "Joinable by segments" for convex sets. So why use such a lengthy and complicated term as "Hereditarily indecomposable continua"? We feel that the term "atom" expresses the indecomposability property properly, and by the Atomic Map Theorem 3.3, atoms are indeed the building blocks of every compact space. We hope the readers, including those used to the traditional term, will find it suitable. We thank V.M. for the idea.

We are grateful to Wayne Lewis for his help. All spaces in this article are assumed to be separable metric and all maps are continuous.

2. The lattice $D(X)$ of upper semicontinuous decompositions

Let $\sim$ denote the equivalence relation defined on the class of all maps on some compact space X by $f \sim g$ if and only if f and g have the same fibers in X,

i.e. for all x in X, $f^{-1}f(x) = g^{-1}g(x)$. This occurs if and only if there exists a homeomorphism $h : f(X) \longrightarrow g(X)$ such that $h \circ f = g$. Let $D(X)$ denote the class of all maps on X modulo this equivalence relation. The elements of $D(X)$ may be regarded as upper semicontinuous decompositions $\{f^{-1}f(x) : x \in X\}$ of X. Still we shall avoid formality and use functional notation.

$D(X)$ has a natural order relation, namely $f \leq g$ if (the decomposition) f refines g, i.e. if for all $x \in X$, $f^{-1}f(x) \subset g^{-1}g(x)$. This occurs if and only if $g = h \circ f$ for some $h : f(X) \longrightarrow g(X)$. Clearly $D(X)$ has a smallest (=identity map) and a largest (=constant map) element, and it is a complete lattice: if T is any subset of $D(X)$ then $\inf T = \bigwedge T = h$ is the element whose fiber $h^{-1}h(x)$ at $x \in X$ is $\bigcap\{f^{-1}f(x) : f \in T\}$. Note that h can be regarded as the product map $h : X \longrightarrow \prod\{f(X) : f \in T\}$ whose f coordinate, $f \in T$, is f. We shall use the notation $f \wedge g = \min\{f, g\}$ and $f \vee g = \max\{f, g\} = \inf\{h : h \geq f \text{ and } h \geq g\}$.

Note that while the fiber of $f \wedge g$ at $x \in X$ is $f^{-1}f(x) \cap g^{-1}g(x)$, the structure of the fibers of $f \vee g$ is not clear at all. Later in Section 4, we shall see that in the case of monotone decompositions f and g dominated by some atomic map, there is a complete symmetry between the fiber structure of $f \wedge g$ and $f \vee g$. More information on $D(X)$ can be found in [1], [2] and [3].

3. Atoms and atomic maps

A continuum is a connected compact space. A continuum X is decomposable if it is representable as $X = X_1 \cup X_2$ with X_i, $i = 1, 2$, proper subcontinua of X. An atom (or an Atomic space) is a hereditarily indecomposable continuum, namely, a continuum every subcontinuum of which is indecomposable. Note that a continuum X is an atom if and only if for every two subcontinua A, B of X with $A \cap B \neq \emptyset$ either $A \subset B$ or $B \subset A$ (since otherwise $A \cup B$ would be a decomposable continuum). Atoms were first constructed by Knaster in [26] (see also [8] for a comprehensive study of atoms). In ([4], 1951) Bing proved the existence of n–dimensional atoms for all $n \geq 0$. In ([6], 1958) (see also [8]) M. Brown proved the following result which is essential for this article.

Theorem 3.1 (Brown's Decomposition Theorem)**.** *For every $n \geq 2$ there is a continuous decomposition g of $\mathbb{R}^n - \{0\}$ such that each fiber of g is an atomic irreducible separator of $\mathbb{R}^n - \{0\}$ between 0 and ∞ which separates it into two components.*

Note that the quotient space $g(\mathbb{R}^n - \{0\})$ is naturally ordered by $y_1 < y_2$ if $g^{-1}(y_2)$ separates $g^{-1}(y_1)$ from ∞ and this order induces a homeomorphism of the real line $\mathbb{R}$ onto $g(\mathbb{R}^n - \{0\})$. Hence $g(\mathbb{R}^n - \{0\})$ is homeomorphic to $\mathbb{R}$.

Definition 3.2. A map $g : X \longrightarrow Y$ is atomic if for all y in $g(X)$ $g^{-1}(y)$ is an atom.

Theorem 3.3 (Atomic Map Theorem)**.** *Every finite dimensional compact space X admits an atomic map $g : X \longrightarrow Y$ with $\dim Y \leq 1$.*

The atomic map theorem is a corollary of Brown's decomposition theorem. Before proving the general case we discuss some trivial cases.

Case 1. X is an atom. In that case g may be taken to be a constant map.

Case 2. X is a Bing space, i.e. every component of X is an atom (see [3]). In that case let g be the decomposition of X into components. The quotient space $g(X)$ is 0-dimensional.

Case 3. dim $X \leq 1$. Take $Y = X$ and g =identity map. (Note that a single point is an atom.)

Proof of the Atomic Map Theorem. Let n =dim$X$. We may assume $X \subset \mathbb{R}^{2n+1} - \{0\}$. Let g_1 denote the Brown decomposition of $\mathbb{R}^{2n+1} - \{0\}$. Let $g_2 = g_1|_X$. Recall that $g_1(\mathbb{R}^{2n+1} - \{0\})$ is homeomorphic to $\mathbb{R}$; hence $g_2(X)$ is a subset of $\mathbb{R}$ and in particular dim$g_2(X) \leq 1$. Consider the monotone-light decomposition (see [7] p.184) of g_2:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "g_2"'] & & Y \arrow[dl, "h"] \\ & g_2(X) & \end{tikzcd}$$

$g_2 = h \circ g$ with g monotone (i.e. $g^{-1}(y)$ is connected for all y in Y) and h light (=0-dimensional). g is an atomic map and dim $Y \leq 1$. Indeed, a fiber $g^{-1}(y)$, $y \in Y$, of g is a component of a fiber of $g_2 = g_1|X$. In particular $g^{-1}(y)$ is a subcontinuum of some fiber of g_1 which is an atom, and hence $g^{-1}(y)$ is an atom too. By the Hurewicz theorem on mappings which lower dimension, dim$Y \leq$ dim$g_2(X)$ + dim$h$ = dim$g_2(X) \leq 1$ as dim $h = 0$. □

Remark 3.4. Theorem 3.3 holds for every compact space X (not necessarily finite dimensional). This follows from the following result (see [25]).

Theorem 3.5. *Let X be a compactum. There exists a Bing map (=a map with fibers which are Bing spaces) from X to $\mathbb{R}$. Moreover, the set of all Bing maps from X to $\mathbb{R}$ forms a dense G_δ-subset of the function space $C(X, \mathbb{R})$.*

Indeed, let $f : X \longrightarrow \mathbb{R}$ be a Bing map and let $f = hg$ be the monotone-light decomposition of f with g monotone and h light. Then g is atomic and $\dim g(X) \leq 1$ as $h : g(X) \longrightarrow \mathbb{R}$ is 0-dimensional and by the Hurewicz theorem $\dim g(X) \leq \dim h + \dim \mathbb{R} = 1$.

4. The lattice $M_g(X)$ of monotone upper semicontinuous decompositions dominated by some atomic map g

In this section X is assumed to be a compact space and $g : X \longrightarrow Y = g(X)$ is an atomic map on X. Denote $M(X) = \{f : f \in D(X), f$ monotone$\}$ and $M_g(X) = \{f : f \in M(X), f \leq g\}$.

In general $M(X)$ is not a sublattice of $D(X)$, but for $M_g(X)$ we have:

Proposition 4.1. *$M_g(X)$ is a sublattice of $D(X)$. For $f, h \in M_g(X)$ the fiber of $f \vee h$ at $x \in X$ is $f^{-1}f(x) \cup h^{-1}h(X)$.*

Proof. Let $f \in M_g(X)$. A fiber of f is a continuum (as f is monotone) and is contained in some fiber of g (as $f \leq g$) which is an atom. Hence a fiber of f is an atom. For f, h in $M_g(X)$ and $x \in X$, $h^{-1}h(x)$ and $f^{-1}f(x)$ are subatoms of the atom $g^{-1}g(x)$. As $h^{-1}h(x)$ and $f^{-1}f(x)$ intersect (at x) it follows that either $f^{-1}f(x) \subset h^{-1}h(x)$ or $h^{-1}h(x) \subset f^{-1}f(x)$. It follows that $f^{-1}f(x) \cap h^{-1}h(x)$ is one of the intersecting sets and hence an atom contained in $g^{-1}g(x)$. Therefore $f \wedge g \in M_g(X)$. Let l denote the decomposition of X which at $x \in X$ is $f^{-1}f(x) \cup h^{-1}h(x)$. The reader may verify (or else check at [1] Proposition 4.3 p.248) that l is an upper semicontinuous decomposition, i.e. $l \in D(X)$ and that $l = f \vee h$. Clearly $l \leq g$ and l is monotone. Hence $l \in M_g(X)$. Thus $M_g(X)$ is a sublattice of $D(X)$. □

For $f \in D(X)$ define $S_f = \{x : x \in X, f^{-1}f(x) = \{x\}\}$

Proposition 4.2. *Let $f, h \in M_g(X)$ then $S_{f\wedge h} = S_f \cup S_h$ and $S_{f\vee h} = S_f \cap S_h$.*

Proof. $(f \wedge h)^{-1}(f \wedge h)(x) = f^{-1}f(x) \cap h^{-1}h(x)$ is an intersection of subatoms of $g^{-1}g(x)$ which contain one another and contain x. This intersection is equal to $\{x\}$ if and only if one of the sets $f^{-1}f(x)$ or $h^{-1}h(x)$ equals $\{x\}$. This proves that $S_{f\wedge h} = S_f \cup S_h$.

$(f \vee h)^{-1}(f \vee h)(x) = f^{-1}f(x) \cup h^{-1}h(x)$ and this union is $\{x\}$ if and only if $f^{-1}f(x) = \{x\} = h^{-1}h(x)$ □

Proposition 4.3. *Let g be an atomic map on a compact space X. The correspondence $f \to \pi f = \pi_g f =$ the monotone part of $(g \wedge f)$ defines an order preserving projection $\pi = \pi_g : D(X) \to M_g(X)$ which satisfies* $\dim \pi f(X) \leq \dim f(X) + \dim g(X)$.

We call π the canonical projection of $D(X)$ onto $M_g(X)$. This estimate of $\dim \pi_g f(x)$ cannot be improved in general. See Remark 5.3.

Proof. By definition we have

$$\begin{tikzcd} X \arrow[rr, "\pi f"] \arrow[dr, "g\wedge f"'] & & \pi f(X) \arrow[dl, "h"] \\ & (g \wedge f)(X) & \end{tikzcd}$$

i.e. $g \wedge f = h \circ \pi f$ where h is light. Hence $\pi f \leq g \wedge f \leq g$ and as πf is monotone $\pi f \in M_g(X)$. If $f \in M_g(X)$, then as $f \leq g$, $g \wedge f = f = \pi f$ since f is monotone. Clearly if $f_1 \leq f_2$, then $\pi f_1 \leq \pi f_2$.

By the Hurewicz theorem on mappings which lower dimension we have $\dim \pi f(X) \leq \dim(g \wedge f)(X) + \dim h = \dim(g \wedge f)(X)$ since $\dim h = 0$. But $g \wedge f$ maps X into $g(X) \times f(X)$ hence $\dim \pi f(X) \leq \dim g(X) + \dim f(X)$. □

5. An embedding of an n–dimensional compact space in Euclidean space and counterexamples to the Chogoshvili-Pontrjagin claim

In this section X is assumed to be an n–dimensional compact space $(2 \leq n < \infty)$ and $g : X \to Y$ is an atomic map with $\dim Y \leq 1$ which exists by the Atomic Map Theorem.

Let $s \geq 3n - 2$ be an integer and let k be the greatest integer $\leq \frac{s-1}{n-1}$. So $k \geq 3$.

Let $A = \{a : a \subset \{1, 2, \ldots, s\}, |a| =$ cardinality of $a = s - n + 1\}$. Then $|A| = \binom{s}{s-n+1}$ and we claim that

(i) All k elements of A have a nonempty intersection.

Proof of (i):

$$\left|\{1, 2, \ldots, s\} \setminus \bigcap_{i=1}^{k} a_i\right| = \left|\bigcup_{i=1}^{k} (\{1, 2, \ldots, s\} \setminus a_i)\right| \leq \sum_{i=1}^{k} |\{1, 2, \ldots, s\} \setminus a_i|$$
$$= k(n-1) \leq \frac{s-1}{n-1}(n-1) = s - 1.$$

Hence $\bigcap_{i=1}^{k} a_i$ cannot be empty. □

By a theorem of Hurewicz ([7] p.125) the light maps form a dense G_δ set in the function space $C(X, \mathbb{R}^n)$. Also every projection $\mathbb{R}^s \longrightarrow \mathbb{R}^n$ induces an open map $C(X, \mathbb{R}^s) \longrightarrow C(X, \mathbb{R}^n)$ under which the light maps in $C(X, \mathbb{R}^n)$ pull back to a dense G_δ subset of $C(X, \mathbb{R}^s)$. Applying these results we can find maps $f_i^1 : X \longrightarrow \mathbb{R}$, $1 \leq i \leq s$, such that every n-tuple when regarded as a map $(f_{i_1}^1, f_{i_2}^1, \cdots, f_{i_n}^1) : X \longrightarrow \mathbb{R}^n$ is a light map. Let $\pi = \pi_g$ be the canonical projection of $D(X)$ onto $M_g(X)$ as defined in Proposition 4.3. Set $f_i = \pi f_i^1$, $1 \leq i \leq s$. Then $f_i \in M_g(X)$. It follows from 4.3 that

(ii) $\dim f_i(X) = \dim \pi_g f_i^1(X) \leq \dim f_i^1(X) + \dim g(X) \leq 1 + 1 = 2$.

Also, $f_i \leq f_i^1$ in $D(X)$. Hence, for every n-tuple $i_1 < i_2 < \cdots < i_n$ of indices $\bigwedge_{1 \leq j \leq n} f_{i_j} \leq \bigwedge_{1 \leq j \leq n} f_{i_j}^1$. As the latter map is light so is the first. Thus $\bigwedge_{1 \leq j \leq n} f_{i_j}$ is both monotone (since $Mg(X)$ is a lattice by 4.1) and light and hence is an embedding.

From Proposition 4.2 it follows that

(iii) $X = S_{\wedge_{1 \leq j \leq n} f_{i_j}} = \bigcup_{1 \leq j \leq n} S_{f_{i_j}}$.

Let $S_i = S_{f_i}$. We proved:

(iv) The union of every n of the sets S_i is X.

It follows that

(v) Every point x in X belongs to S_i for at least $s - n + 1$ values of i.

Indeed, if some $x \in X$ belongs only to $s - n$ S_i's, then the remaining n S_i's would not cover X violating (iv).

(v) is equivalent to

(vi) $X = \bigcup_{a \in A} \bigcap_{i \in a} S_i$.

For $a \in A$ let $f_a = \bigvee_{i \in a} f_i \in M_g(X)$, and $f = \bigwedge_{a \in A} f_a$. (See Proposition 4.1.)

(vii) f is an embedding.

Proof of (vii): $f = \bigwedge_{a \in A}(\bigvee_{i \in a} f_i)$. By Proposition 4.2 followed by (vi) $S_f = \bigcup_{a \in A} \bigcap_{i \in a} S_i = X$. □

(viii) Let $a_1, a_2, \ldots, a_k$ be k elements of A. Then $\bigwedge_{1 \leq j \leq k} f_{a_j}$ factors through some 2-dimensional space.

Proof of (viii): By (i) $\bigcap_{1 \leq j \leq k} a_j \neq \emptyset$. Let $i \in \bigcap_{1 \leq j \leq k} a_j$. Then for all $1 \leq j \leq k$, $f_i \leq f_{a_j}$ (in $M_g(X)$) and hence also $f_i \leq \bigwedge_{1 \leq j \leq k} f_{a_j}$. It follows that there exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f_i"] \arrow[dr, "\bigwedge_{1 \leq j \leq k} f_{a_j}"'] & & f_i(X) \arrow[dl, "h_i"] \\ & (\bigwedge_{1 \leq j \leq k} f_{a_j}(X) & \end{tikzcd}$$

By (ii) dim $f_i(X) \leq 2$ and we are done. □

(ix) For all a in A, $\dim f_a(X) \leq 3(s - n) + 2$.

Proof of (ix): For $i \in a$ let $W_i = \{y \in f_a(X) : f_a^{-1}(y)$ is a fiber of $f_i\}$. Recall that $f_a = \bigvee_{i \in a} f_i$ so each fiber of f_a is a fiber of some f_i, $i \in a$. Hence $f_a(X) = \bigcup_{i \in a} W_i$. The two maps f_i and f_a induce the same decomposition on $f_a^{-1}(W_i)$. Thus $W_i = f_a f_a^{-1}(W_i)$ and $f_i f_a^{-1}(W_i)$ are homeomorphic. But $f_i f_a^{-1}(W_i) \subset f_i(X)$ and by (ii) $\dim f_i(X) \leq 2$.

It follows that $f_a(X)$ is the union of $s-n+1$ two-dimensional subsets so $\dim f_a(X) \leq 2(s-n+1)+(s-n+1)-1 = 3(s-n)+2$. □

The following theorem summarizes the current stage of the construction.

Theorem 5.1. *Let $s \geq 3n-2$ be an integer and let k=the greatest integer $\leq \frac{s-1}{n-1}$. Let X be an n-dimensional compact space, $3 \leq n < \infty$. Let $A = \{a \subset \{1,2,\ldots,s\} : |a| = s-n+1\}$. There exist maps $f_a : X \longrightarrow f_a(X)$, $a \in A$, with $dim f_a(X) \leq 3(s-n)+2$, such that $f = (f_a)_{a\in A} : X \to \prod_{a\in A} f_a(X)$ is an embedding and, for all k indices $a_1, a_2, \ldots, a_k$ the map $(fa_j)_{i\leq j\leq k} : X \to \prod_{1\leq j\leq k} f_{a_j}(X)$ factors through some two-dimensional space.* □

Theorem 1.5 is a simple corollary of Theorem 5.1.

Proof of Theorem 1.5. Let X be an n-dimensional compact space, $3 \leq n < \infty$. Set $s = 3n-2$ and $k = 3$. Consider Theorem 5.1 with these values of n, s and k. For each $a \in A$ embed $f_a(X)$ in $(12n-7)$-dimensional Euclidean space E_a with coordinate maps $\{e_{a,l}\}_{l=1}^{12n-7}$. Then $\{e_{a,l} \circ f_a\}$, $a \in A, 1 \leq l \leq 12n-7$, defines an embedding of X into m-dimensional Euclidean space $\mathbb{R}^m = \sum_{a\in A} \oplus E_a$ with $m = \binom{3n-2}{2n-1}(12n-7)$.

The restriction to eX of any 3-dimensional coordinate projection of $\mathbb{R}^m$ is of the form

$$P|_{eX} = (e_{a_1,l_1} \circ f_{a_1}, e_{a_2,l_2} \circ f_{a_2}, e_{a_3,l_3} \circ f_{a_3}) = (e_{a_1,l_1}, e_{a_2,l_2}, e_{a_3,l_3}) \circ (f_{a_1}, f_{a_2}, f_{a_3}).$$

Since $(f_{a_1}, f_{a_2}, f_{a_3})$ factors through some 2–dimensional space so does $P|_{eX}$. This implies that $P|_{eX}$ has no stable values in $\mathbb{R}^3$ and that on eX every n–dimensional coordinate projection of $\mathbb{R}^m$ has only unstable values in $\mathbb{R}^n$. □

Remark 5.2. The indices a_1, a_2, a_3 which appear in the definition of the projection P need not be different.

Remark 5.3. The estimates $\dim \pi_g f(X) \leq \dim g(X) + \dim f(X)$ in Proposition 4.3 and $\dim f_i(X) \leq 2$ in (ii) cannot be improved in general. Indeed $\dim f_i(X) \leq 1$ in (ii) would enable us to embed every 2-dimensional space in some Euclidean space as in Theorem 1.5 so that it does not satisfy the Chogoshvili-Pontrjagin claim, in contrast to the results of [9].

Remark 5.4. The construction in Theorem 1.5 applies also for 2-dimensional spaces but the conclusion (that the projections factor through some 2–dimensional space) is trivial in that case.

6. Stable points of a space and an observation concerning ANRs

We will say that a point $a \in X$ is a stable point of X if a is a stable value of the identity map of X in itself. Thus, $a \in X$ is a stable point of X if for some $\epsilon > 0$ and all $f : X \longrightarrow X$ with $d(x, f(x)) < \epsilon$ for all $x \in X, a \in f(X)$.

Denote $X^{\#} = \{a \in X : a \text{ is a stable point of } X\}$. For example: for the Cantor set $\triangle, \triangle^{\#} = \emptyset$; $([-1,1]^n)^{\#} = (-1,1)^n$; $S_n^{\#} = S_n$.

Proposition 6.1. *Let $X \subset U \subset \mathbb{R}^m$ where U is an open subset of $\mathbb{R}^m$ and X is compact. Let $a \in X^{\#}$ and let $r : U \longrightarrow X$ be a retraction. Then X is unremovable from $r^{-1}(a)$.*

Proof. Let $\epsilon > 0$ be so small that for all $f : X \longrightarrow X$, $\| x - f(x) \| < \epsilon$ for all x in X implies $a \in f(X)$. Let $\delta > 0$ be so small that for $g : X \longrightarrow \mathbb{R}^m$, $\| x - g(x) \| < \delta$ for all $x \in X$ implies that $g(X) \subset U$ and $\| x - rg(x) \| < \epsilon$ for all $x \in X$.

Then $a \in rg(X)$, i.e. $g(X) \cap r^{-1}(a) \neq \emptyset$. □

Corollary 6.2. *Let X be an n-dimensional compact ANR, $n \geq 3$. Assume X is embedded in $\mathbb{R}^m$ by the embedding constructed in Theorem 1.5. Then for every neighborhood U of X in $\mathbb{R}^m$, for every retraction $r : U \to X$ and every $a \in X^{\#}$, $r^{-1}(a)$ does not contain any coordinate $(m-n)$–ball centered at a (i.e. any neighborhood of a in some $(m-n)$–dimensional coordinate plane). Indeed, Proposition 6.1 would otherwise imply that X is unremovable from that $(m-n)$–coordinate plane.*

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