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# ON THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVES

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Abstract. If $p(z)$ is univariate polynomial with complex coefficients having all its zeros inside the closed unit disk, then the Gauss-Lucas theorem states that all zeros of $p'(z)$ lie in the same disk. We study the following question: what is the maximum distance from the arithmetic mean of all zeros of $p(z)$ to a nearest zero of $p'(z)$? We obtain bounds for this distance depending on degree. We also show that this distance is equal to $\frac{1}{3}$ for polynomials of degree 3 and polynomials with real zeros.

## 1. Introduction

Let $\mathcal{P}_n$ denote the family of all complex polynomials of exact degree $n$, having all zeros in the closed unit disk, i.e.

$$\mathcal{P}_n := \left\{ p(z) = a_n \prod_{k=1}^{n} (z - z_k), \qquad |z_k| \le 1 \quad (1 \le k \le n,\ a_n \ne 0) \right\} \tag{1.1}$$

and let

$$\mathcal{P} := \bigcup_{n \ge 1} \mathcal{P}_n. \tag{1.2}$$

Let $p \in \mathcal{P}$. From the Gauss-Lucas theorem we know that $p' \in \mathcal{P}$. A well-known attempt to further characterize the location of the zeros of $p'$, which are also called the critical points of $p$, is the Sendov Conjecture.

**Conjecture 1.1** (Sendov)**.** If $p(z) \in \mathcal{P}$ then each of the disks $z \in \mathbb{C} : |z - z_k| \le 1$, $k = 1, \dots, n$, contains at least one zero of $p'(z)$.

The Sendov Conjecture has been verified in a number of special cases [2]. Let $H(p)$ denote the convex hull of the zeros of a polynomial $p$. Schmeisser [10] posed the following generalization of the Sendov Conjecture.

**Conjecture 1.2** (Schmeisser)**.** If $p(z) \in \mathcal{P}$, then for every $\zeta \in H(p)$, the disk $\{z : |z - \zeta| \le 1\}, k = 1, \dots, n$, contains at least one zero of $p'(z)$.

Schmeisser [10] showed that this conjecture is true if $H(p)$ is a triangular region. It is also shown to be true if $H(p)$ has all its vertices on the unit circle [2]. We investigate a special case of the Schmeisser problem: we concentrate on one point in $H(p)$ - the average of zeros of $p$. Although in this case the Schmeisser Conjecture follows immediately from the Gauss-Lucas Theorem, we show that the disk $\{z :$

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$|z-\zeta| \le 1\}$ in Conjecture (1.2) can be replaced by a smaller concentric disk with radius $r$. We give bounds for $r$ depending on $n$ for any $p \in \mathcal{P}_n$. We also obtain sharp estimates of $r$ for polynomials of degree 3 and for polynomials having only real zeros.

Let

$$p(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k) \in \mathcal{P}_n. \tag{1.3}$$

Then the *centroid*, $A_p$, of the polynomial $p(z)$ is the arithmetic average of its zeros:

$$A_p := \frac{1}{n}(z_1 + \cdots + z_n) = -\frac{1}{n}\frac{a_{n-1}}{a_n}. \tag{1.4}$$

It is easy to check that the centroid is differentiation invariant. Namely, for $n > 1$

$$p'(z) = \sum_{k=0}^{n-1} (k+1) a_{k+1} z^k = n a_n \prod_{k=1}^{n-1} (z - \zeta_k), \tag{1.5}$$

so that

$$A_{p'} = \frac{1}{n-1}(\zeta_1 + \cdots + \zeta_{n-1}) = -\frac{1}{n-1}\frac{(n-1)a_{n-1}}{n a_n} = -\frac{1}{n}\frac{a_{n-1}}{a_n} = A_p, \tag{1.6}$$

which gives

$$A_p = A_{p^{(k)}}, \qquad 0 \le k \le n-1. \tag{1.7}$$

We also define

$$J(p) := \min_{1 \le k \le n-1} |A_p - \zeta_k| \quad \text{and} \quad \gamma_n := \sup_{p \in \mathcal{P}_n} J(p). \tag{1.8}$$

*Remark* 1.3. There exist results that characterize the location of zeros of $p^{(k)}$ for $k \ge 1$ for polynomials having only real zeros. Milovanovic et al. [6] give an account of many results of this type. For instance, if $p$ is a polynomial with zeros $x_1 \le x_2 \le \cdots \le x_n$, then we define $span(p) := x_n - x_1$. Clearly,

$$span(p^{(k)}) \le span(p), \qquad 0 \le k \le n-1. \tag{1.9}$$

by the Gauss-Lucas Theorem. Robinson [8], [9] has shown that the strict inequality holds in (1.9) if k is sufficiently large. He also gave estimates on $span(p^{(k)})$, depending on $span(p), n$ and $k$. Since the centroid of a polynomial is invariant under differentiation, the zeros of $p^{(k)}$ must be getting closer to $A_p$ as $k$ grows big. Although $span(p^{(k)})$ may not decrease if $k < n/2$, we will show in section 4, that not all zeros of $p'$ may stay away from $A_p$.

## 2. A lower bound for the constant $\gamma_n$

We first give simple upper and lower bounds for $\gamma_n$.

**Proposition 2.1.**

$$\text{For any } n > 1,\ n^{-\frac{1}{n-1}} \le \gamma_n \le 1. \tag{2.1}$$

*Proof.* The second inequality follows from the fact that the centroid of a polynomial is differentiation invariant, so the centroid of $p(z)$ is also the centroid of $p'(z)$. Since all of the zeros of $p'(z)$ are inside the unit circle, at least one of them lies within distance 1 from the centroid. The first inequality follows from the special choice of

$p(z) = z^n - z$. Here, we have $p'(z) = nz^{n-1} - 1$ and for every critical point $\zeta_k$ of $p(z)$, we have

$$|\zeta_k| = \frac{1}{n^{\frac{1}{n-1}}} \text{ so } J(p) = \frac{1}{n^{\frac{1}{n-1}}}. \tag{2.2}$$

□

*Remark* 2.2. $1 - n^{-\frac{1}{n-1}}$ is asymptotically equal to $\dfrac{\log n}{n}$, i.e.

$$\lim_{n\to\infty} \frac{1 - n^{-\frac{1}{n-1}}}{\frac{\log n}{n}} = 1. \tag{2.3}$$

We can show it by taking Taylor series expansion of $1 - n^{-\frac{1}{n-1}}$:

$$\begin{aligned} 1 - n^{-\frac{1}{n-1}} &= 1 - \exp\left(-\frac{\log n}{n-1}\right) = \frac{\log n}{n-1} - \frac{1}{2!}\left(\frac{\log n}{n-1}\right)^2 + \frac{1}{3!}\left(\frac{\log n}{n-1}\right)^3 - \dots \\ &= \frac{\log n}{n-1} + o\left(\frac{\log n}{n-1}\right). \end{aligned}$$

Therefore, there exist constants $c_1, c_2$ such that

$$1 - c_1\frac{\log n}{n} < n^{-\frac{1}{n-1}} < 1 - c_2\frac{\log n}{n}. \tag{2.4}$$

## 3. An upper bound for constant $\gamma_n$

If $p$ is monic and $A_p = 0$, the product of all critical points is $\prod_{k=1}^{n-1} \zeta_k = |a_1|/n$, so $J(p)$ is bounded by $\sqrt[n-1]{|a_1|/n}$. We can use a good upper bound on $|a_1|$ given by the following lemma:

**Lemma 3.1.**

(3.1) *If* $p(z) = z^n + \cdots + a_1 z + a_0 \in \mathcal{P}_n$ *and* $A_p = 0$, *then* $|a_1| = |p'(0)| \le 1$.

*Proof.* If $A_p = 0$, then

$$\begin{aligned} |a_1| &= \left|\sum_{k=1}^{n}\prod_{j\neq k} z_j\right| = \left|\sum_{k=1}^{n}\Big(\prod_{j\neq k} z_j - \overline{z_k}\prod_{j=1}^{n} z_j\Big)\right| \\ &= \left|\sum_{k=1}^{n}(1-|z_k|^2)\prod_{j\neq k} z_j\right| \le \sum_{k=1}^{n}(1-|z_k|^2)\prod_{j\neq k}|z_j|. \end{aligned} \tag{3.2}$$

Let $|z_k| = r_k,\ 0 \le r_k \le 1,\ 1 \le k \le n$. and let

$$B_n := \sum_{k=1}^{n}(1-r_k^2)\prod_{j\neq k} r_j, \qquad n \ge 1. \tag{3.3}$$

We will show that

$$B_n \le 1 \text{ for every } n \ge 1. \tag{3.4}$$

First, we will show that $B_n$ attains its maximum on the boundary of the set $[0,1]^n$. If $n = 1$, then the maximum of $B_1 = 1 - r_1$ is attained at $r_1 = 0$. If $n \ge 2$ and $B_n$

attains its maximum on the set $[0,1]^n$ at some point $(r_1,\dots,r_n)\in(0,1)^n$. Then we have

$$\frac{\partial B_n}{\partial r_k}=-2r_k\prod_{j\neq k}r_j+\sum_{j\neq k}(1-r_j^2)\prod_{i\neq k,j}r_i=0,\qquad 1\le k\le n. \tag{3.5}$$

Then

$$\begin{aligned} r_k\frac{\partial B_n}{\partial r_k} &= -2r_k^2\prod_{j\neq k}r_j+\sum_{j\neq k}(1-r_j^2)\prod_{i\neq j}r_i \\ &= (-1-r_k^2)\prod_{j\neq k}r_j+\sum_{j=1}^{n}(1-r_j^2)\prod_{i\neq j}r_i \\ &= \left(-r_k-\frac{1}{r_k}\right)\prod_{j=1}^{n}r_j+\sum_{j=1}^{n}(1-r_j^2)\prod_{i\neq j}r_i=0,\qquad 1\le k\le n. \end{aligned} \tag{3.6}$$

For $i\neq j$ we get

$$r_i\frac{\partial B_n}{\partial r_i}-r_j\frac{\partial B_n}{\partial r_j}=\left(r_j+\frac{1}{r_j}-r_i-\frac{1}{r_i}\right)\prod_{k=1}^{n}r_k=0. \tag{3.7}$$

Hence

$$r_j+\frac{1}{r_j}=r_i+\frac{1}{r_i}. \tag{3.8}$$

Since the function $f(x)=x+\frac{1}{x}$ is one to one on $(0,1)$, $r_i=r_j$. Thus the local maximum $M$ of $B_n$ can be attained at $(r_1,\dots,r_n)\in(0,1)^n$ only if all $r_k$ are equal. Let $r:=r_1=\cdots=r_n$. Then

$$M\le\sup_{0<r<1}\left(nr^{n-1}(1-r^2)\right). \tag{3.9}$$

The above supremum is attained at $r=\sqrt{\frac{n-1}{n+1}}$ and

$$M\le\frac{2n}{n+1}\left(1+\frac{2}{n-1}\right)^{-\frac{n-1}{2}},\quad n\ge 2. \tag{3.10}$$

It is well known that the sequence $e_n:=(1+\frac{2}{n-1})^{-\frac{n-1}{2}}\to\frac{1}{e}$ as $n\to\infty$ and $e_n$ is decreasing. Thus, for $n\ge 3$, $e_n\le e_3=1/2$. Therefore, from (3.10) we have

$$M\le\begin{cases}\frac{4}{3\sqrt{3}}<1 & \text{if } n=2,\\ \frac{n}{n+1}<1 & \text{if } n\ge 3.\end{cases} \tag{3.11}$$

Since $B_n=1$ if $(r_1,\dots,r_n)=(0,1,\dots,1)$, $B_n$ must attain its maximum on the boundary of $[0,1]^n$. □

Now, we will prove (3.4) by induction. If $n=1$, then $B_n=1-r_1\le 1$. Let $N$ be fixed and assume that (3.4) holds for $n=1,\dots,N-1$. If $B_N$ attains its maximum on the set $[0,1]^n$ at the point $(r_1,\dots,r_n)$, then $r_k=0$ or $r_k=1$ for some $k, 1\le k\le N$. If $r_k=0$, then $B_N=\prod_{j\neq k}r_j\le 1$. Otherwise, if $r_k=1$, then $B_N=B_{N-1}\le 1$ by induction.

*Remark* 3.2. From Lemma 3.1 we get an estimate on $J(p)$ in the special case when $A_p=0$. Then we have $J(p)\le(|p'(0)|/n)^{\frac{1}{n-1}}\le(1/n)^{\frac{1}{n-1}}$. This estimate is sharp as shown by $p(z)=z^n-z$.

Using Lemma 3.1 we can find an upper bound on $\gamma_n$ that is smaller than 1.

**Theorem 3.3.**

$$\gamma_n \leq \frac{2n^{\frac{1}{n-1}}}{n^{\frac{2}{n+1}}+1}. \tag{3.12}$$

*Proof.* If $p(z) \in \mathcal{P}_n$ and $|A_p| = r$, then

$$q(z) = \frac{1}{(r+1)^n} p\left(A_p + (r+1)z\right) \in \mathcal{P}_n \text{ and } A_q = 0. \tag{3.13}$$

The polynomial $q(z)$ is monic, so by Lemma 3.1, there exist $w$ such that $q'(w) = 0$ and

$$|w| \leq \left(\frac{|q'(0)|}{n}\right)^{\frac{1}{n-1}} \leq \frac{1}{n^{\frac{1}{n-1}}}. \tag{3.14}$$

If $\xi = (r+1)w + A_p$, then $p'(\xi) = 0$ and

$$J(p) \leq |\xi - A_p| = (r+1)|w| \leq \frac{r+1}{n^{\frac{1}{n-1}}}. \tag{3.15}$$

On the other hand, there must be a zero of $p'(z)$ in the half-plane $\mathrm{Re}\big(z\overline{A_p}\big) \geq r$, so we have

$$J(p) \leq \sqrt{1-r^2}. \tag{3.16}$$

Therefore, from (3.15) and (3.16)

$$J(p) \leq \sup_{0\leq r\leq 1} \min\left\{\frac{r+1}{n^{\frac{1}{n-1}}}, \sqrt{1-r^2}\right\}. \tag{3.17}$$

As functions of $r$, the bounds in (3.15) and (3.16) are respectively increasing and decreasing on the interval $[0,1]$, so the supremum in (3.17) is attained if $r$ is the unique solution of the equation

$$\frac{r+1}{n^{\frac{1}{n-1}}} = \sqrt{1-r^2}, \tag{3.18}$$

which gives

$$r = \frac{n^{\frac{2}{n-1}}-1}{n^{\frac{2}{n-1}}+1} \text{ and } J(p) \leq \frac{2n^{\frac{1}{n-1}}}{n^{\frac{2}{n-1}}+1}. \tag{3.19}$$

□

*Remark* 3.4. Theorem 1.1 leads to the following estimate

$$\gamma_n \leq \frac{2n^{\frac{1}{n-1}}}{n^{\frac{2}{n-1}}+1} = 1 - \frac{\left(1-n^{\frac{1}{n-1}}\right)^2}{n^{\frac{2}{n-1}}+1} < 1 - c\left(\frac{\log n}{n}\right)^2, \tag{3.20}$$

where $c$ is some constant, because as we have already noted in (2.3) $1 - n^{-\frac{1}{n-1}}$ is asymptotically equal to $\log n/n$. However, the estimate (3.20) doesn't seem to be sharp. Our conjecture is that a sharp inequality would have the form $\gamma_n \leq 1 - c\frac{\log n}{n}$, as in the example $p(z) = z^n - z$.

## 4. Extremal polynomials of low degrees

In this section we compute $\gamma_3$ exactly. In order to do this we restrict our attention to extremal polynomials, i.e. the subset of $\mathcal{P}_n$ consisting of polynomials for which $J(p) = \gamma_n$. Such polynomials exist since $J(p)$ is continuous in the roots of $p'$, which are continuous in the coefficients of $p$, which are continuous in the roots of $p$ (being elementary symmetric functions of those roots). The roots of $p$ are contained in the compact set $D$, so $J(p)$ achieves its supremum on $D$. For extremal problems of this type, Phelps and Rodriguez ([7]) have shown that every extremal polynomial has at least 2 zeros on the unit circle and that every sub-arc of the unit circle of length at least $\pi$ must contain at least one zero. Otherwise, it would be possible to inscribe all zeros of the polynomial in the circle with radius smaller than 1, which would contradict extremality. We can conclude that unless the extremal polynomial has 2 zeros at the endpoints of diameter of the unit circle, it must have at least 3 zeros on the unit circle. This is also valid in our problem.

**Proposition 4.1.** $\gamma_3 = \frac{2}{3}$, and all extremal polynomials are of the form $p(z) = q(e^{it}z)$ where $q(z) = z^3 - z^2 - z + 1$

*Proof.* Let $p(z) \in \mathcal{P}_3$. Without loss of generality we can assume that 1 is one of the zeros of $p(z)$, i.e.

$$p(z) = (z-1)(z-z_1)(z-z_2), \tag{4.1}$$

and

$$A_p = \frac{1+z_1+z_2}{3}. \tag{4.2}$$

Since $p'(z)/3$ is monic, the distance between $A_p$ and the nearest critical point is bounded by $\sqrt{|p'(A_p)/3|}$. We have

$$\frac{|p'(A_p)|}{3} = \frac{1}{9}\left|-z_1^2 - z_2^2 + z_1 + z_2 + z_1 z_2 - 1\right|. \tag{4.3}$$

By the maximum modulus principle $|p'(A_p)/3|$ attains its maximum on $D$ when both $z_1$ and $z_2$ are on the unit circle. We can also assume that $p(z)$ is extremal and $z_1 = e^{i\phi_1}, 0 \le \phi_1 \le \pi$, and $z_2 = e^{i\phi_2}, \pi \le \phi_2 \le 2\pi$, and $\phi_2 - \phi_1 \le \pi$. Then

$$\begin{aligned} \frac{|p'(A_p)|}{3} &\le \frac{1}{9}\left(|z_1|^2 + |z_2|^2 + |z_1 + z_2 + z_1 z_2 - 1|\right) \\ &\le \frac{2}{9} + \frac{1}{9}\sqrt{4 + 4\sin\phi_1 \sin\phi_2} \le \frac{4}{9}, \end{aligned} \tag{4.4}$$

where the last inequality follows from the fact that $\sin\phi_1 \sin\phi_2 \le 0$. Thus we have

$$J(p) \le \sqrt{\frac{4}{9}} = \frac{2}{3}, \tag{4.5}$$

with equality only if $\sin\phi_1 \sin\phi_2 = 0$. In this case either $z_1 = z_2 = -1$ and $p(z) = (z+1)^2(z-1)$ or $z_1 = 1, z_2 = -1$ and $p(z) = (z-1)^2(z+1)$. These are the only extremal polynomials for which $p(1) = 0$. In both cases $J(p) = \frac{2}{3}$, so $\gamma_3 = \frac{2}{3}$. □

*Remark* 4.2. Numerical computations using nonlinear optimization algorithms suggest that $\gamma_4 = 2/3$ and an example of extremal polynomial

$$p_4(z) = z^4 - \frac{4}{3}z^3 + \frac{2}{3}z^2 - \frac{4}{3}z + 1. \tag{4.6}$$

Zeros of $p_4$ are

$$z_1 = z_2 = 1,\ z_3 = -1/3 + \frac{2\sqrt{2}}{3}i \text{ and } z_4 = -1/3 - \frac{2\sqrt{2}}{3}i. \tag{4.7}$$

Zeros of $p_4'$ are

$$w_1 = 1,\ w_2 = \frac{\sqrt{3}}{3}i \text{ and } w_3 = -\frac{\sqrt{3}}{3}i. \tag{4.8}$$

$\gamma_5 = 3/4$ and an example of extremal polynomial is

$$p_5(z) = z^5 - \frac{5}{4}z^4 - \frac{5}{4}z^3 + \frac{5}{4}z^2 + \frac{5}{4}z + 1. \tag{4.9}$$

Zeros of $p_5$ are

$$z_1 = z_2 = z_3 = 1,\ z_4 = -\frac{7}{8} + \frac{\sqrt{15}}{8}i \text{ and } z_5 = -\frac{7}{8} - \frac{\sqrt{15}}{8}i. \tag{4.10}$$

Zeros of $p_5'$ are

$$w_1 = w_2 = 1 \text{ and } w_3 = w_4 = -1/2. \tag{4.11}$$

Both the Sendov Conjecture and our problem are special cases of extremal problems considered by Miller ([4],[5]). He defined $S(n, \beta)$ to be the set of complex polynomials of degree $n$ that have all their zeros in the unit disk and at least one zero at $\beta$ and $|p|_\alpha$ to be the distance between $\alpha$ and the closest root of $p'$. Polynomial $p \in S(n, \beta)$ is said to be **maximal with respect to** $\alpha$ if for every $q \in S(n, \beta)$ we have $|p|_\alpha \geq |q|_\alpha$. He also defined **critical circle** to be the circle with center $\alpha$ and radius $|p|_\alpha$. Our problem can be formulated as follows: find maximum of $|p|_{A_p}$ for $p \in S(n, \beta)$. Miller ([4]) made the following conjecture:

**Conjecture 4.3** (Miller)**.** Let $p \in S(n, \beta)$ be maximal with respect to $\alpha$, let $s$ be the number of zeros of $p$ on the unit circle, and let $r$ be the number of zeros of $p'$ on the critical circle. Then

$$r = n - 1, \quad \textit{and} \tag{4.12}$$

$$\text{given that all zeros of } p' \text{ lie on a circle, } s \text{ is as large as possible} \tag{4.13}$$

He also proved that under the hypothesis of Conjecture 1.3, $2r + s \geq n + 1$. In our problem, if $n = 3, 4$ or $5$, it seems that all zeros of a maximal polynomial $p$ lie on the unit circle and all zeros of $p'$ lie on the critical circle. $p \in S(n, \beta)$ is said to be **maximal with respect to** $\alpha$ if for every $q \in S(n, \beta)$ we have $|p|_\alpha \geq |q|_\alpha$. He also defined **critical circle** to be the circle with center $\alpha$ and radius $|p|_\alpha$. Our problem can be formulated as follows:

## 5. Polynomials with real zeros

The polynomial $p(z) = z^n - z$ with high $J(p)$ has all its zeros, except one, spread evenly around the unit circle. If this situation cannot happen, for instance for polynomials with real zeros, we may expect $J(p)$ to be lower. Let's consider the subclass of $\mathcal{P}_n$ consisting of polynomials having only real zeros. In this case we get the following theorem:

**Theorem 5.1.** *If $p(x) \in \mathcal{P}_n$ is a polynomial having only real zeros, then*

$$J(p) \leq \frac{2}{3}. \tag{5.1}$$

*Equality in (5.1) is attained if and only if $n$ is a multiplicity of 3 and* $p(x)=(x-1)^{\frac{2n}{3}}(x+1)^{\frac{n}{3}}$ *or* $p(x)=(x-1)^{\frac{n}{3}}(x+1)^{\frac{2n}{3}}$

*Proof.* It's enough to show that (5.1) is true if $A_p$ is non-negative; otherwise we can consider $p(-x)$ instead of $p(x)$. If $A_p > 1/3$, then obviously $J(p) < 2/3$, so let us assume that $A_p \in [0, 1/3]$. Let $I = (A_p - 2/3, A_p + 2/3)$. We want to show that the interval $\overline{I} \subset [-1, 1]$ contains at least one zero of $p'(x)$. By Rolle's Theorem, this is true if $I$ contains at least two zeros of $p(x)$, counting multiplicities. Let us consider the remaining two cases.

*Case 1:* $I$ doesn't contain any zeros of $p(x)$. Let $A_p \in [0, 1/3]$ be fixed and let $n_1$ and $n_2 = n - n_1$ be the numbers of zeros of $p(x)$ in the intervals $I_1 = [-1, A_p - 2/3]$ and $I_2 = [A_p + 2/3, 1]$ respectively. The condition $x_1 + \cdots + x_n = nA_p$ imposes some constraints on $n_1$ and $n_2$. Namely, $n_1$ is maximal and $n_2$ is minimal if we move all zeros to the right as far as possible. In the extreme situation we would have $n_1$ zeros at $A_p - 2/3$ and $n_2$ zeros at 1. We then have a linear system

$$\begin{cases} n_1(A_p - 2/3) + n_2 &= nA_p, \\ n_1 + n_2 &= n. \end{cases} \tag{5.2}$$

Although the solution of (5.2) is not a pair of integers in general, they give an upper bound for $n_1$ and a lower bound for $n_2$. Similarly, if we put $n_1$ zeros at $-1$ and $n_2$ zeros at $A_p + 2/3$ we get a linear system

$$\begin{cases} -n_1 + n_2(A_p + 2/3) &= nA_p, \\ n_1 + n_2 &= n \end{cases} \tag{5.3}$$

that gives a lower bound for $n_1$ and an upper bound for $n_2$. Combining (5.2) and (5.3) we obtain range of $n_1$ and $n_2$ depending on $A_p$

$$\frac{2n}{5 + 3A_p} \le n_1 \le \frac{3n(1 - A_p)}{5 - 3A_p}, \tag{5.4}$$

$$\frac{2n}{5 - 3A_p} \le n_2 \le \frac{3n(1 + A_p)}{5 + 3A_p}. \tag{5.5}$$

Since all zeros of $p(x)$ lie in 2 disjoint intervals, we can apply Walsh's two circle theorem ([2, p. 89]). □

**Theorem 5.2** (Walsh)**.** *If all the zeros of an $n_1$-degree polynomial $f_1(z)$ are contained in the closed interior of the circle $C_1$ with center $c_1$ and radius $r_1$ and all the zeros of an $n_2$-degree polynomial $f_2(z)$ are contained in the closed interior of the circle $C_2$ with center $c_2$ and radius $r_2$, then all zeros of the derivative of the product $f(z) = f_1(z)f_2(z)$ are contained in the union of $C_1$, if $n_1 > 1$, and $C_2$, if $n_2 > 1$, and a third circle $C$ with center $c$ and radius $r$ where*

$$c = \frac{n_1c_2 + n_2c_1}{n_1 + n_2} \text{ and } r = \frac{n_1r_2 + n_2r_1}{n_1 + n_2}. \tag{5.6}$$

*Moreover, if the closed interiors of the circles $C_1, C_2$ and $C$ have no point in common, the number of zeros of $f'(z)$ which they contain is respectively $n_1 - 1, n_2 - 1$ and 1.*

By (5.6), all zeros of $p'(x)$ are contained in the union of the intervals $I_1, I_2$ and $I_3$, where

$$I_3 = \left[\frac{n_1(A_p + 2/3) - n_2}{n}, \frac{n_1 + n_2(A_p - 2/3)}{n}\right] \tag{5.7}$$

Applying inequalities (5.4) and (5.5) we obtain

$$\frac{n_1(A_p+2/3)-n_2}{n}-\left(A_p-\frac{2}{3}\right)\geq\frac{1}{3}-A_p\geq 0 \tag{5.8}$$

and

$$\left(A_p+\frac{2}{3}\right)-\frac{n_1+n_2(A_p-2/3)}{n}\geq A_p+\frac{1}{3}>0. \tag{5.9}$$

If $\frac{1}{3}-A_p>0$ in (5.8), then the intervals $I_1, I_2$ and $I_3$ are disjoint and Walsh's theorem guarantees that there is a unique zero of $p'$ in $I_3\subset I$ and therefore $J(p)<\dfrac{2}{3}$. The equality in (5.1) holds only if $A_p=\frac{1}{3}$ in (5.8). Then, from (5.2) and (5.3) we must have $n_1=\frac{n}{3}$ and $n_2=\frac{2n}{3}$. Therefore, the only extremal polynomial for $A_p>0$ is $p(x)=(x-1)^{\frac{2n}{3}}(x+1)^{\frac{n}{3}}$.

*Case 2:* $I$ contains one zero of $p(x)$. Let $t$ be the unique zero of $p(x)$ in $I$. Let $n_1$ and $n_2=n-1-n_1$ be the numbers of zeros of $p(x)$ in the intervals $I_1=[-1,A_p-2/3]$ and $I_2=[A_p+2/3,1]$ respectively. As in the first case, when we fix $t$ and $A_p$, we can get bounds for $n_1$ and $n_2$. If we move all other zeros to the right as far as possible, i.e. put $n_1$ zeros at $A_p-2/3$ and $n_2$ zeros at 1, then the solution of the linear system

$$\begin{cases} n_1(A_p-2/3)+n_2+t &= nA_p,\\ n_1+n_2+1 &= n\end{cases} \tag{5.10}$$

gives an upper bound for $n_1$ and a lower bound for $n_2$. Similarly, if we move all other zeros to the left as far as possible, i.e. put $n_1$ zeros at $-1$ and $n_2$ zeros at $A_p+2/3$ we get a linear system

$$\begin{cases} -n_1+n_2(A_p+2/3)+t &= nA_p,\\ n_1+n_2+1 &= n\end{cases} \tag{5.11}$$

that gives a lower bound for $n_1$ and an upper bound for $n_2$. Combining these two we obtain that the range of $n_1$ and $n_2$ depending on $A_p$ and $t$ is given by

$$\frac{2n+3t+3A_p-2}{5+3A_p}\leq n_1\leq 3\frac{n(1-A_p)+t-1}{5-3A_p} \tag{5.12}$$

and

$$\frac{2n-3t+3A_p-2}{5-3A_p}\leq n_2\leq 3\frac{n(1+A_p)-t-1}{5+3A_p}. \tag{5.13}$$

We will see that by keeping $t$ and $A_p$ fixed and moving the other zeros of $p(x)$ we can get not only the range of $n_1$ and $n_2$, but also estimate the distance from $A_p$ to a nearest zero of $p'(x)$. In order to do that we apply so called "Root Dragging Theorem" ([1]) which states that if we move any of the roots of the real-root polynomial to the right (left) then all of the critical points other than multiple roots move to the right (left). Since $t$ is the only zero of $p(x)$ in $I$, $I$ can contain at most two zeros of $p'(x)$ and they must be those that are closest to $t$ from the left and right. Let's call them $w$ and $v$, $w<t<v$. We will show that at least one of $w$ and $v$ is in $I$. $A_p-w$ will attain maximum when we move all zeros as far as possible to the left and so $A_p-w\leq A_p-w'$ where $w'$ is the zero of derivative of

$$(x+1)^{n_1}(x-(A_p+2/3))^{n_2}(x-t) \tag{5.14}$$

that is closest to $t$ from the left, i.e. $w'$ is the leftmost zero of the quadratic equation

$$nx^2 - \left((n-1)t + n_1\left(A_p + \frac{2}{3}\right) - n_2 + A_p - \frac{1}{3}\right)x$$
$$+ t\left(n_1\left(A_p + \frac{2}{3}\right) - n_2\right) - A_p - \frac{2}{3} = 0.$$

From this and the bounds on $n_1$ and $n_2$,

(5.15)
$$A_p - w \le F(t) := \frac{1}{2}\left(2A_p - t + \frac{1}{3} + \sqrt{\left(t + \frac{1}{3}\right)^2 + \frac{4(t+1)(A_p - t + \frac{2}{3})}{n}}\right).$$

We need to show that $F(t) < 2/3$ for every $t \in I$. $F(t) - 2/3$ has at most 2 zeros, which are the zeros of the quadratic function

$$\begin{aligned}\overline{F}(t) &:= \frac{4(t+1)(A_p - t + \frac{2}{3})}{n} + \left(t + \frac{1}{3}\right)^2 - (t + 1 - 2A_p)^2 \\ &= -\frac{4}{n}t^2 + 4\frac{n+1}{n}\left(A_p - \frac{1}{3}\right)t - 4\left(A_p^2 - A_p - \frac{A_p}{n} + 2\frac{n-3}{3n}\right).\end{aligned}$$

However, $F(t) - 2/3$ has at most one zero in the interval $I$, since

$$\begin{aligned}F\left(A_p - \frac{2}{3}\right) - \frac{2}{3} &= \frac{1}{2}\left(A_p - \frac{1}{3} + \sqrt{\left(A_p - \frac{1}{3}\right)^2 + \frac{16}{3n}\left(\frac{1}{3} + A_p\right)}\right) \\ &> \frac{1}{2}\left(A_p - \frac{1}{3} + \left|A_p - \frac{1}{3}\right|\right) = 0\end{aligned}$$

and

$$F\left(A_p + \frac{2}{3}\right) - \frac{2}{3} = \frac{1}{2}\left(A_p + |A_p + 1| - \frac{5}{3}\right) = A_p - \frac{1}{3} \le 0. \tag{5.16}$$

Also the leading coefficient in $\overline{F}(t)$ is negative and

(5.17)
$$\overline{F}\left(A_p - \frac{2}{3}\right) = \frac{16}{3n}\left(\frac{1}{3} + A_p\right) > 0 \text{ and } \overline{F}\left(A_p + \frac{2}{3}\right) = \frac{16}{3}\left(A_p - \frac{1}{3}\right) \le 0,$$

so the only zero of $F(t) - 2/3$ in $I$ can be $\eta$, the rightmost zero of $\overline{F}(t)$ and $F(t) \le \frac{2}{3}$ in $I$ if $t \ge \eta$. Also $v - A_p$ will attain maximum when we move all zeros as far as possible to the right and $v - A_p \le v' - A_p$ where $v'$ is the zero of derivative of

$$(x - (A_p - 2/3))^{n_1}(x-1)^{n_2}(x-t) \tag{5.18}$$

that is closest to $t$ from the right, i.e. $v'$ is the rightmost zero of the quadratic equation

$$nx^2 - \left((n-1)t + n_2\left(A_p - \frac{2}{3}\right) + n_1 + A_p + \frac{1}{3}\right)x$$
$$+ t\left(n_2\left(A_p - \frac{2}{3}\right) + n_1\right) + A_p - \frac{2}{3} = 0.$$

From this and the bounds on $n_1$ and $n_2$,

(5.19)
$$v - A_p \le G(t) := \frac{1}{2}\left(-2A_p + t + \frac{1}{3} + \sqrt{\left(-t+\frac{1}{3}\right)^2 + \frac{4(-t+1)(-A_p+t+\frac{2}{3})}{n}}\right).$$

We need to show that $G(t) < 2/3$ for every $t \in I$. $G(t) - 2/3$ has at most 2 zeros, which are the zeros of the quadratic function

$$\begin{aligned}\overline{G}(t) &:= \frac{4(-t+1)(-A_p+t+\frac{2}{3})}{n} + \left(-t+\frac{1}{3}\right)^2 - (t-1-2A_p)^2 \\ &= -\frac{4}{n}t^2 + 4\frac{n+1}{n}\left(A_p+\frac{1}{3}\right)t - 4\left(A_p^2 + A_p + \frac{A_p}{n} + 2\frac{n-3}{3n}\right).\end{aligned}$$

However, $G(t) - 2/3$ has at most one zero in the interval $I$, because

$$G\left(A_p - \frac{2}{3}\right) - \frac{2}{3} = \frac{1}{2}\left(-A_p + |A_p - 1| - \frac{5}{3}\right) = -A_p - \frac{1}{3} < 0 \tag{5.20}$$

and

$$\begin{aligned}G\left(A_p + \frac{2}{3}\right) - \frac{2}{3} &= \frac{1}{2}\left(-A_p - \frac{1}{3} + \sqrt{\left(A_p+\frac{1}{3}\right)^2 + \frac{16}{3n}\left(\frac{1}{3} - A_p\right)}\right) \\ &\ge \frac{1}{2}\left(-A_p - \frac{1}{3} + \left|A_p + \frac{1}{3}\right|\right) = 0.\end{aligned}$$

Also the leading coefficient in $\overline{G}(t)$ is negative and

(5.21)
$$\overline{G}\left(A_p - \frac{2}{3}\right) = -\frac{16}{3}\left(\frac{1}{3} + A_p\right) < 0 \text{ and } \overline{G}\left(A_p + \frac{2}{3}\right) = \frac{16}{3n}\left(\frac{1}{3} - A_p\right) \ge 0,$$

so the only zero of $G(t) - 2/3$ in $I$ can be $\xi$, the leftmost zero of $\overline{G}(t)$ and $G(t) \le \frac{2}{3}$ in $I$ if $t \le \xi$. We want to show that $\xi \ge \eta$. The equation $\overline{F}(t) = \overline{G}(t)$ has the unique solution $t = 3A_p$ so it's enough to show that $\overline{F}(3A_p) = \overline{G}(3A_p) < 0$. But

$$\overline{F}(3A_p) = 8\frac{n-3}{n}\left(A_p - \frac{1}{3}\right)\left(A_p + \frac{1}{3}\right) \le 0 \text{ for } 0 \le A_p \le \frac{1}{3} \tag{5.22}$$

and equality is attained only if $A_p = \frac{1}{3}$ and $t = 1$, but this is case $I$, and we must have $p(x) = (x-1)^{\frac{2n}{3}}(x+1)^{\frac{n}{3}}$ with $J(p) = \frac{2}{3}$. Otherwise $\xi > \eta$ and either $\xi$ or $\eta$ lie in $I$ and $J(p) < \frac{2}{3}$. If $A_p < 0$ the above argument applied to polynomial $p(-x)$ shows that $J(p) = \frac{2}{3}$ only if $p(x) = (x+1)^{\frac{2n}{3}}(x-1)^{\frac{n}{3}}$.

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