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EIGENFUNCTIONS OF THE LAPLACIAN ON ROTATIONALLY SYMMETRIC MANIFOLDS

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Abstract. Eigenfunctions of the Laplacian on a negatively curved, rotationally symmetric manifold $M = (\mathbf{R}^n, ds^2)$, $ds^2 = dr^2 + f(r)^2 d\theta^2$, are constructed explicitly under the assumption that an integral of $f(r)$ converges. This integral is the same one which gives the existence of nonconstant harmonic functions on M.

1. Introduction

Let us equip $\mathbf{R}^n$, $n \geq 2$, with a Riemannian metric which can be written in polar coordinates as

$$\begin{gathered} ds^2 = dr^2 + f(r)^2 d\theta^2, \\ f(0) = 0,\ f'(0) = 1,\ \text{and } f(r) > 0,\ \forall r > 0. \end{gathered} \tag{1.1}$$

$\mathbf{R}^n$ with the metric (1.1) becomes a rotationally symmetric manifold M. M is called a weak model in [9] and a Ricci model in [1]. The Laplace operator in these coordinates is

$$\Delta = f(r)^2 \Delta_\theta + \partial_r^2 + (n-1) f(r)^{-1} f'(r) \partial_r,$$

where Δ_θ is the Laplace operator of the unit sphere $S_{n-1}(0,1) = S$. In this paper we assume that the radial curvature $k(r) = -f''(r)f(r)^{-1}$ is negative.

In this paper we give a method of constructing eigenfunctions of the Laplacian Δ with eigenvalue $\lambda > 0$, provided that

$$J(f) := \int_1^\infty f(r)^{n-3} dr \int_r^\infty f(\rho)^{1-n} d\rho < +\infty. \tag{1.2}$$

The integral $J(f)$ is the same which gives the existence of nonconstant bounded harmonic functions on M. In [12], March, using a probabilistic approach, proved the following alternative: If $J(f) < \infty$, then there exist nonconstant bounded harmonic functions on M, while if $J(f) = \infty$, there are none such.

The integral $J(f)$ is closely related to the radial curvature. Indeed, if $c_2 = 1$, $c_n = 1/2$, $n \geq 3$, then, [12], under the assumption that $k(r)$ is nonpositive, it follows that

$$J(f) < +\infty, \text{ if } k(r) \leq \frac{-c}{r^2 \log r}, \text{ for } c > c_n \text{ and large } r,$$

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while

$$J(f) = +\infty, \text{ if } k(r) \geq \frac{-c}{r^2 \log r}, \text{ for } c < c_n \text{ and large } r.$$

This alternative had been observed before March by Milnor, [14], in dimension 2 and in part by Choi, [3], in dimension 3. In [13], using generalised Poisson kernels, it is proved that $J(f) < +\infty$ is a sufficient condition for the existence of nonconstant bounded harmonic functions on M.

Under the assumption (1.2), eigenfunctions W_λ, $\lambda > 0$, of Δ with eigenvalue λ, are constructed explicitly in Section 3 by means of the heat kernel K_S of the unit sphere S and a subordination measure $\mu(\lambda, r, t)dt$, $t > 0$. More precisely, in Section 2, we show that there exists a positive and nondecreasing θ-independent eigenfunction $g(\lambda, r)$, $r > R$, with eigenvalue $\lambda > 0$, such that $g(\lambda, R) = 1$. The measure $\mu(\lambda, r, t)$ is a solution of the parabolic equation $\partial_t \mu = B_{r,\lambda}\mu$, where

$$B_{r,\lambda} = f(r)^2 \partial_r^2 + \left\{(n-1)f(r)f'(r) + 2f(r)^2 g'(\lambda, r) g(\lambda, r)^{-1}\right\} \partial_r.$$

The measure $\mu(\lambda, r, t)$ is in fact the law of the first hitting time of the boundaries of (R, ∞) by the diffusion associated with the operator $B_{r,\lambda}$. This gives the splitting $\mu = \mu_1 + \mu_2$, where μ_1 (resp. μ_2) is related to the first hitting time of ∞ (resp. R).

In Section 3, Theorem 9, we show that if $J(f) < \infty$, then for any $h_1, h_2 \in C^\infty(S)$ and $\lambda > 0$,

$$W_\lambda(r, \theta) := g(\lambda, r) \sum_{j=1,2} \int_0^\infty \int_S K_S(t, \theta, \theta') h_j(\theta') \mu_j(\lambda, r, t) d\theta' dt, \tag{1.3}$$

is an eigenfunction of Δ with eigenvalue λ, outside the ball $B(0, R)$, such that

$$\lim_{r \to \infty} g(\lambda, r)^{-1} W_\lambda(r, \theta) = h_1(\theta) \quad \text{and} \quad \lim_{r \to R} g(\lambda, r)^{-1} W_\lambda(r, \theta) = h_2(\theta). \tag{1.4}$$

In a second step, using the probabilistic interpretation of harmonic functions, it is proved, Theorem 11, that an eigenfunction on $B(0, R)^c$ which satisfies (1.4) is given by (1.3). This allow us to extend uniquely to the whole of M the eigenfunction $W_\lambda(r, \theta)$ given by (1.3), Theorem 12.

Finally, in Section 4, we treat the case of the Euclidean upper-half space $\mathbf{R}_+^{n+1}$. The Poisson kernels which give rise to the eigenfunctions with eigenvalue $\lambda > 0$ are computed explicitly and reduce to the classical one as $\lambda \to 0$.

2. The subordination measures

In this section we construct a pair of measures $\mu_j(\lambda, r, t)dt$, $j = 1, 2$, which enable one to write down explicitly eigenfunctions of the Laplacian with eigenvalue $\lambda > 0$.

We start with the existence of a θ-independent eigenfunction $g(\lambda, r)$.

Lemma 1. *For any $\lambda > 0$, there exists a positive and nondecreasing θ-independent eigenfunction $g(\lambda, r)$, $r > 0$, with eigenvalue $\lambda > 0$, such that $g(\lambda, 0) = 1$.*

Proof. The radial part of Δ is given by $\partial_r^2 + (n-1)f(r)^{-1}f'(r)\partial_r$ and $f(r) \sim r$, for r small enough. So, if

$$B(r) = \int_1^r (n-1)f'(s)f(s)^{-1}\, ds = (n-1)\log f(r)f(1)^{-1},$$

then

$$e^{B(r)} = cf(r)^{n-1} \sim r^{n-1}, \quad \text{for } r \text{ small enough.}$$

The integral tests as in [5], p. 408, give that 0 is an entrance boundary and the result follows. $\square$

Let $W_\lambda(r,\theta)$ be an eigenfunction of Δ with eigenvalue $\lambda > 0$. If we set

$$w_\lambda(r,\theta) = g(\lambda, r)^{-1} W_\lambda(r,\theta),$$

then it is straightforward to check that $w_\lambda(r,\theta)$ is a harmonic function for the operator $\Delta_\theta + B_{r,\lambda}$, where

$$\begin{aligned} B_{r,\lambda} &= f(r)^2 \partial_r^2 + \left\{(n-1) f(r) f'(r) + 2 f(r)^2 g'(\lambda, r) g(\lambda, r)^{-1}\right\} \partial_r \\ &= a(r)\partial_r^2 + b(r)\partial_r. \end{aligned}$$

Lemma 2. *If $J(f) < \infty$, then for any $\lambda > 0$, the origin is an entrance boundary for the operator $B_{r,\lambda}$, while $+\infty$ is an exit boundary.*

Proof. As in Lemma 1 we set

$$W(r) = \int_1^r b(s) a(s)^{-1} ds = (n-1)\log f(r) f(1)^{-1} + 2\log g(\lambda, r) g(\lambda, 1)^{-1}.$$

So,

$$e^{W(r)} = c f(r)^{n-1} g(\lambda, r)^2 \sim r^{n-1},$$

for r small enough, since $g(\lambda, r) \sim 1$, as $r \to 0$, by Lemma 1. It follows ([7], p.515), that 0 is an entrance boundary.

On the other hand, since the radial curvature is negative, we get that $f(r) \geq cr$ ([9], p.36). This yields

$$\int_1^\infty e^{W(r)}\, dr = c \int_1^\infty f(r)^{n-1} g(\lambda, r)^2\, dr \geq c g(\lambda, 1)^2 \int_1^\infty r^{n-1}\, dr = +\infty,$$

since $g(\lambda, r)$ is nondecreasing by Lemma 1. Therefore, $+\infty$ is not a regular boundary. But

$$\begin{aligned} &\int_1^\infty e^{-W(r)}\, dr \int_1^r e^{W(s)} a(s)^{-1}\, ds \\ &= c \int_1^\infty f(r)^{1-n} g(\lambda, r)^{-2}\, dr \int_1^r f(s)^{n-3} g(\lambda, s)^2\, ds \\ &\leq c \int_1^\infty f(r)^{1-n}\, dr \int_1^r f(s)^{n-3}\, ds := I(f), \end{aligned}$$

since $g(\lambda, r)$ is nondecreasing. From Fubini's theorem we get that $I(f) = J(f) < +\infty$, so $+\infty$ is an exit boundary. $\square$

From now on we fix a positive R. As in Lemma 2, one can show that R is a regular boundary for $B_{r,\lambda}$. Therefore the situation is the same as in [11], Section 3. Following the method developed in [11], one can prove the statements below, which we give without proof.

Lemma 3. *If $J(f) < \infty$, then for any $\lambda > 0$ and $k > 0$, there exists a positive and nondecreasing solution $\varphi(\lambda, k, r)$ of the equation*

$$B_{r,\lambda}\varphi(\lambda, k, r) = k\varphi(\lambda, k, r), \ \ r \in (R, \infty), \tag{2.1}$$

such that $\varphi(\lambda, k, R) = 0$ and $\varphi(\lambda, k, r) \longrightarrow 1$, as $r \to \infty$.

The function $\varphi(\lambda, k, r)$ appearing in Lemma 3 has the following probabilistic interpretation. Let us denote by r_t^λ the diffusion on (R, ∞) associated with the operator $B_{r,\lambda}$, and let σ (resp. τ) be the first hitting time of ∞ (resp. R) by r_t^λ starting at $r > R$. The explosion time of r_t^λ is then given by $\zeta = \sigma \wedge \tau$. Let us also denote by $P_{\lambda,r}$ the probability attached to the motion r_t^λ and by $E_{\lambda,r}$ the corresponding expectation.

Lemma 4. *If $J(f) < \infty$, then for any $\lambda > 0$,*

$$P_{\lambda,r}\left\{\zeta < +\infty\right\} = 1, \ \forall r \in (R, \infty),$$

and

$$\varphi(\lambda, k, r) = E'_{\lambda,r}\left(e^{-k\sigma}\right), \ \forall r \in (R, \infty), \tag{2.2}$$

where $P'_{\lambda,r}$ is the restriction of $P_{\lambda,r}$ to the set Ω_1 of the paths r_t^λ such that $r_\sigma^\lambda = \infty$.

As in [11], Section 3, using (2.2), one can show the following

Proposition 5. *If $J(f) < \infty$, then for any $\lambda > 0$ and $r \in (R, \infty)$, there exists a positive, bounded and C^∞ function $t \to \mu_1(\lambda, r, t)$, $t > 0$, such that*

$$\varphi(\lambda, k, r) = \int_0^\infty e^{-kt}\mu_1(\lambda, r, t)\, dt, \tag{2.3}$$

$$\partial_t \mu_1 = B_{r,\lambda}\mu_1, \tag{2.4}$$

$$\mu_1(\lambda, r, t) = o(t^m), \ \forall m > 0, \ \text{as } t \to 0, \ \text{and } \lim_{t\to\infty} \mu_1(\lambda, r, t) = 0, \tag{2.5}$$

$$\int_0^\infty \mu_1(\lambda, r, t)dt < 1, \ \text{for any } \lambda > 0, \ \text{and } r \in (R, \infty). \tag{2.6}$$

Remark 6. *From (2.3) of Proposition 5 and the boundary values of $\varphi(\lambda, k, r)$, we deduce that*

$$\mu_1(\lambda, r, t) \longrightarrow \delta_0(t), \ \text{as } r \to \infty, \tag{2.7}$$

in the sense of distributions, and

$$\mu_1(\lambda, r, t) \longrightarrow 0, \ \text{as } r \to R. \tag{2.8}$$

Remark 7. *The subordination measure μ_1 admits a companion μ_2 which, like μ_1, is related to the first hitting time τ of R. Indeed, since ∞ is an exit boundary and R is a regular one, there exists a positive and nonincreasing solution $\psi(\lambda, k, r)$ of (2.1) such that $\psi(\lambda, k, R) = 1$ and $\psi(\lambda, k, r) \longrightarrow 0$, as $r \to \infty$. As in the case of φ, the subordination measure μ_2 associated with ψ satisfies the properties stated for μ_1 in Proposition 5. From the boundary values of ψ we get that*

$$\mu_1(\lambda, r, t) \longrightarrow \delta_0(t), \ \text{as } r \to R, \tag{2.9}$$

and

$$\mu_1(\lambda, r, t) \longrightarrow 0, \ \text{as } r \to \infty. \tag{2.10}$$

Remark 8. *For any $\lambda > 0$ and $r \in (R, \infty)$, we set $\mu = \mu_1 + \mu_2$. The measure μ is the law of the explosion time $\zeta = \sigma \wedge \tau$ of the diffusion r_t^λ. Indeed, as in Lemma 4, we have*

$$\begin{aligned}E_{\lambda,r}\left(e^{-k\zeta}\right) &= E_{\lambda,r}\left(e^{-k\zeta}\chi_{\Omega_1}\right) + E_{\lambda,r}\left(e^{-k\zeta}\chi_{\Omega_1^c}\right)\\ = E_{\lambda,r}\left(e^{-k\sigma}\chi_{\Omega_1}\right) &+ E_{\lambda,r}\left(e^{-k\tau}\chi_{\Omega_1^c}\right) = \varphi(\lambda,k,r)+\psi(\lambda,k,r)\\ &= \int_0^\infty e^{-kt}\mu(\lambda,r,t)dt.\end{aligned}$$

3. A class of eigenfunctions

In this section, using the subordination measures, we construct, in a first step, eigenfunctions of Δ outside the ball $B(0,R)$, with eigenvalue $\lambda > 0$. In a second step, we show that they can be extended uniquely to the whole of M.

Theorem 9. *If $J(f) < \infty$, then for any $\lambda > 0$ and $h_1, h_2 \in C^\infty(S)$,*

$$W_\lambda(r,\theta) := g(\lambda,r)\sum_{j=1,2}\int_0^\infty \int_S K_S(t,\theta,\theta')h_j(\theta')\mu_j(\lambda,r,t)\,d\theta' dt, \tag{3.1}$$

is an eigenfunction of Δ outside the ball $B(0,R)$, with eigenvalue $\lambda > 0$, such that

$$\lim_{r\to\infty} g(\lambda,r)^{-1}W_\lambda(r,\theta) = h_1(\theta) \quad \textit{and} \quad \lim_{r\to R} g(\lambda,r)^{-1}W_\lambda(r,\theta) = h_2(\theta). \tag{3.2}$$

Let us set

$$W_\lambda^j(r,\theta) := g(\lambda,r)\int_0^\infty \int_S K_S(t,\theta,\theta')h_j(\theta')\mu_j(\lambda,r,t)\,dt,\ j=1,2. \tag{3.3}$$

According to the boundary behaviour of the measures μ_1, μ_2, given in Remark 6 and Remark 7 of Section 2, for the proof of the Theorem 9, it suffices to show that $W_\lambda^j(r,\theta)$ are eigenfunctions of Δ outside the ball $B(0,R)$, with eigenvalue $\lambda > 0$, such that

$$\lim_{r\to\infty} g(\lambda,r)^{-1}W_\lambda^1(r,\theta) = h_1(\theta) \quad \text{and} \quad \lim_{r\to R} g(\lambda,r)^{-1}W_\lambda^1(r,\theta) = 0, \tag{3.4}$$

and

$$\lim_{r\to\infty} g(\lambda,r)^{-1}W_\lambda^2(r,\theta) = 0 \quad \text{and} \quad \lim_{r\to R} g(\lambda,r)^{-1}W_\lambda^2(r,\theta) = h_2(\theta). \tag{3.5}$$

Below, we treat the case of $W_\lambda^1(r,\theta)$. The treatment of $W_\lambda^2(r,\theta)$ is similar.

Let us set $w_\lambda^1(r,\theta) = g(\lambda,r)^{-1}W_\lambda^1(r,\theta)$. For the proof of Theorem 9, it suffices to show that w_λ^1 satisfies the following Dirichlet problem:

$$\begin{aligned}&(\Delta_\theta + B_{r,\lambda})\,w_\lambda^1(r,\theta) = 0,\ r\in(R,\infty),\ \theta\in S,\\ &\lim_{r\to\infty} w_\lambda^1(r,\theta) = h_1(\theta),\ \text{and}\ \lim_{r\to R} w_\lambda^1(r,\theta) = 0.\end{aligned} \tag{3.6}$$

In general, the proof of (3.6) is probabilistic, [12], [11]. Here, we give an analytic proof since estimates of $\partial_t K_S$ are available. Indeed, since S is a complete manifold with strictly positive Ricci curvature, then

$$\left|\partial_t^l K_S(t,\theta,\theta')\right| \le cV(\theta,\sqrt{t})^{-1}t^{-l}\left(1+d^2/t\right)^{l+\frac{n}{2}} e^{-d^2/4t}, \tag{3.7}$$

for all $t > 0$, $\forall\ \theta,\theta' \in S$, where $V(\theta,r)$ is the volume of the ball $B(\theta,r)$ and $d = d(\theta,\theta')$ is the distance of θ, θ'. (See for instance the survey article [15], Theorem 4.2 combined with Example 1.)

Using the volume estimates of [2]:

$$ct^n V(\theta,1) \leq V(\theta,t) \leq V(\theta,1),\ \forall\theta \in S,\ \forall t \leq 1,$$

we get from (3.7) that for $t \leq 1$

$$\left|\partial_t^l K_S(t,\theta,\theta')\right| \leq cV(\theta,1)^{-1} t^{-l-\frac{n}{2}} \left(1+d^2/t\right)^{l+\frac{n}{2}} e^{-d^2/4t}. \tag{3.8}$$

On the other hand, from (3.7) it follows that for $t > 1$,

$$\left|\partial_t^l K_S(t,\theta,\theta')\right| \leq cVol(S)^{-1} t^{-l} \left(1+d^2/t\right)^{l+\frac{n}{2}} e^{-d^2/4t}. \tag{3.9}$$

Lemma 10. *If $J(f) < \infty$, then for any $\lambda > 0$ and $h_1 \in C^\infty(S)$:*
(i) The integral

$$w_\lambda^1(r,\theta) := \int_0^\infty \int_S K_S(t,\theta,\theta') h_1(\theta') \mu_1(\lambda,r,t)\, d\theta'\, dt$$

is absolutely convergent and satisfies

$$\lim_{r\to\infty} w_\lambda^1(r,\theta) = h_1(\theta),\ \textit{and}\ \lim_{r\to R} w_\lambda^1(r,\theta) = 0.$$

(ii) The integral

$$\int_0^\infty \int_S \partial_t K_S(t,\theta,\theta') h_1(\theta') \mu_1(\lambda,r,t)\, d\theta'\, dt,$$

is absolutely convergent.

Proof. (i) For any $h_1 \in C^\infty(S)$,

$$\int_0^\infty \int_S K_S(t,\theta,\theta') \left|h_1(\theta')\right| \mu_1(\lambda,r,t)\, d\theta'\, dt$$
$$\leq \left\|h_1\right\|_\infty \int_0^\infty \mu_1(\lambda,r,t)\, dt \leq \left\|h_1\right\|_\infty,$$

since $\int_S K_S(t,\theta,\theta')\, d\theta' = 1$, by [16], and $\int_0^\infty \mu_1(\lambda,r,t)\, dt < 1$, by (2.6).

So, $t \to \int_S K_S(t,\theta,\theta') h_1(\theta') d\theta'$ belongs in $L^1(\mu_1)$. Thus $\lim_{r\to\infty} w_\lambda^1(r,\theta) = h_1(\theta)$ by (2.7) since C_0^∞ is dense in $L^1(\mu_1)$. From (2.8) we get that $\lim_{r\to R} w_\lambda^1(r,\theta) = 0$.

(ii) Suppose for instance that $d \leq 1$. Bearing in mind that $V(\theta,T) > c$, $\forall\theta \in S$, $\forall T \geq 1$, the estimates (3.7), (3.8) and (3.9) of $\partial_t K_S$ give

$$\int_0^\infty \left|\partial_t K_S(t,\theta,\theta')\right| \mu_1(\lambda,r,t)\, dt$$
$$\leq c\int_0^{d^2} t^{-1-\frac{n}{2}} \left(1+d^2/t\right)^{1+\frac{n}{2}} e^{-d^2/4t} \mu_1(\lambda,r,t)\, dt$$
$$+c\int_{d^2}^1 t^{-1-\frac{n}{2}} e^{-d^2/4t} \mu_1(\lambda,r,t)\, dt + cvol(S)^{-1} \int_1^\infty t^{-1} \mu_1(\lambda,r,t)\, dt$$
$$\leq cd^{n+2} \int_0^1 t^{-n-2} \mu_1(\lambda,r,t)\, dt$$
$$+c\int_0^1 t^{-\frac{n}{2}-1} \mu_1(\lambda,r,t)\, dt + c\int_1^\infty t^{-1} \mu_1(\lambda,r,t)\, dt \leq c,$$

since $t \to \mu_1(\lambda,r,t)$ is infinitely flat near $t = 0$ by (2.5), and

$$\int_1^\infty t^{-1}\mu_1(\lambda,r,t)\,dt \le \left(\int_1^\infty t^{-2}\,dt\right)^{\frac{1}{2}}\left(\int_1^\infty \mu_1(\lambda,r,t)^2\,dt\right)^{\frac{1}{2}}$$

$$\le c\,\|\mu\|_\infty^{\frac{1}{2}}\left(\int_1^\infty \mu_1(\lambda,r,t)\,dt\right)^{\frac{1}{2}} \le c,$$

by (2.6). This gives that

$$\int_0^\infty\int_S |\partial_t K_S(t,\theta,\theta')h_1(\theta')|\,\mu_1(\lambda,r,t)d\theta' dt \le c\,\|h_1\|_1\,,$$

if $d \le 1$. The case $d > 1$ is similar. □

End of the proof of Theorem 9. It remains to show that

$$(\Delta_\theta + B_{r,\lambda})\,w_\lambda^1(r,\theta) = 0,\ r\in(R,\infty),\ \theta\in S.$$

From Lemma 10 we get

$$\Delta_\theta w_\lambda^1(r,\theta) = \int_0^\infty\int_S \Delta_\theta K_S(t,\theta,\theta')h_1(\theta')\mu_1(\lambda,r,t)\,d\theta'\,dt$$

$$= -\int_0^\infty\int_S K_S(t,\theta,\theta')h_1(\theta')\partial_t\mu_1(\lambda,r,t)\,d\theta'\,dt = -B_{r,\lambda}w_\lambda^1(r,\theta),$$

since $\partial_t\mu_1 = B_{r,\lambda}\mu_1$, by (2.4). □

Theorem 11. *If $J(f) < \infty$, then for any $\lambda > 0$ the eigenfunction $W_\lambda(r,\theta)$ defined by (3.1) is the unique eigenfunction satisfying (3.2).*

Proof. The proof of the theorem is probabilistic and is based on the integral representation of bounded harmonic functions.

Let $U_\lambda(r,\theta)$ be an eigenfunction of Δ outside the ball $B(0,R)$, with eigenvalue $\lambda > 0$, satisfying (3.2). Then $u_\lambda(r,\theta) = g(\lambda,r)^{-1}U_\lambda(r,\theta)$ satisfies the Dirichlet problem (3.6).

Let us denote by B_t the Brownian motion on S. The diffusion on $B(0,R)^c$ associated with the operator $\Delta_\theta + B_{r,\lambda}$ is $\left(B_t, r_t^\lambda\right)$. We choose r_t^λ independent of B_t. Let us also denote by P_θ the probability attached to the Brownian motion B_t. From the independence of the motions B_t, r_t^λ, it follows that the probability $P_{(r,\theta)}^\lambda$ attached to the motion $\left(B_t, r_t^\lambda\right)$ splits to the product $P_\theta P_r^\lambda$. Finally, from the fact that B_t does not explode, we get that the explosion time of the product motion $\left(B_t, r_t^\lambda\right)$ is in fact the explosion time $\zeta = \sigma\wedge\tau$ of r_t^λ, which satisfies

$$P_{(r,\theta)}^\lambda\left\{\zeta < +\infty\right\} = P_r^\lambda\left\{\zeta < +\infty\right\} = 1, \tag{3.10}$$

by Lemma 4 of Section 2.

From (3.10) and [8], Theorem 2.1, p.127, and Remark 2, p.130, it follows that every bounded harmonic function $u_\lambda(r,\theta)$ for the operator $\Delta_\theta + B_{r,\lambda}$ on $B(0,R)^c$ has the following integral representation:

$$u_\lambda(r,\theta) = E_\theta E_r^\lambda\left\{\Psi\left(B_\zeta, r_\zeta^\lambda\right)\right\}, \tag{3.11}$$

where Ψ is the boundary value of $u_\lambda(r,\theta)$. But, the boundary of $B(0,R)^c$ is $S(0,R)\cup S_\infty$, where the "sphere at infinity" S_∞ is isomorphic to the unit sphere S, [4].

Therefore

$$\Psi(r,\theta)=\begin{cases}h_1(\theta), & \text{if } r=+\infty,\\ h_2(\theta), & \text{if } r=R.\end{cases} \tag{3.12}$$

From (3.11), (3.12) and the fact that $\mu(\lambda,r,t)$ is the law of the explosion time ζ, we obtain that

$$u_\lambda(r,\theta)=\sum_{j=1,2}\int_0^\infty\int_S K_S(t,\theta,\theta')h_j(\theta')\mu_j(\lambda,r,t)d\theta'\,dt,$$

and the result follows. $\square$

Theorem 12. *The eigenfunction $W_\lambda(r,\theta)$ defined by (3.1) extends uniquely to the whole of M.*

Proof. By (1.1) and Lemma 1, the operator $\Delta_\theta+B_{r,\lambda}$ has bounded coefficients on the bounded domain $B(0,R)$. Further, its boundary $S(0,R)$, is regular. So, by [8], Theorem 3.1, p.142, the Dirichlet problem

$$\{\Delta_\theta+B_{r,\lambda}\}\,u=0,\ \forall(r,\theta)\in B(0,R),$$
$$u(r,\theta)\longrightarrow h_2(\theta),\ \text{as } r\to R,$$

admits a unique solution which is given by

$$u(r,\theta)=E_\theta E_r^\lambda\left\{h_2\left(r_T^\lambda\right)\right\},$$

where $T=\inf\left\{t>0|\ r_t^\lambda=R\right\}$ is the first hitting time of $S(0,R)$. $\square$

4. The Euclidean upper half-space

In this section we treat the case of the Euclidean upper half-space $\mathbf{R}_+^{n+1}$. The Poisson kernels which give rise to the eigenfunctions with eigenvalue $\lambda>0$ are computed explicitly and reduce to the classical one as $\lambda\to 0$.

For any $\lambda>0$, $g(\lambda,y)=e^{-y\sqrt{\lambda}}$, $y>0$, is an x-independent eigenfunction of the Laplacian $\Delta=\Delta_x+\partial_y^2$ with eigenvalue λ. So, if $U_\lambda(x,y)$ is an eigenfunction of Δ with eigenvalue λ, then $u_\lambda(x,y)=g(\lambda,y)^{-1}U_\lambda(x,y)$ is a harmonic function for the operator $\Delta_x+\partial_y^2-2\sqrt{\lambda}\partial_y$. Therefore, the subordination measure $\nu(\lambda,y,t)dt$ is a solution of the parabolic equation

$$\partial_t\nu=\partial_y^2\nu-2\sqrt{\lambda}\partial_y\nu,\ y>0,\ t>0,$$

which satisfies $\nu(\lambda,y,t)\longrightarrow\delta_0(t)$, as $y\to 0$, and $\nu(\lambda,y,t)\longrightarrow 0$, as $y\to\infty$.

Passing to the Laplace transform variables, we get that $\nu(\lambda,y,t)$ is the Laplace transform of the function

$$\varphi(\lambda,y,k)=e^{-y\sqrt{\lambda}}e^{-y\sqrt{\lambda+k}}.$$

But, $e^{-y\sqrt{k}}$ is the Laplace transform of

$$(4\pi)^{-1/2}yt^{-3/2}e^{-y^2/4t},$$

so $k\to\varphi(\lambda,y,k)$ is the Laplace transform of

$$\nu(\lambda,y,t)=\frac{ye^{-y\sqrt{\lambda}}}{2\sqrt{\pi}}\frac{e^{-\lambda t}e^{-y^2/4t}}{t^{3/2}}.$$

The Poisson kernels $Q_\lambda(x,y,x')$ which give rise to the eigenfunctions with eigenvalue $\lambda > 0$ are obtained by

$$\begin{aligned} Q_\lambda(x,y,x') &= g(\lambda,y)\int_0^\infty K_{R^n}(t,x,x')\nu(\lambda,y,t)dt \\ &= \frac{ye^{-2y\sqrt{\lambda}}}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\left\|x-x'\right\|^2/4t}}{(4\pi t)^{n/2}}\frac{e^{-\lambda t}e^{-y^2/4t}}{t^{3/2}}\,dt \\ &= 2\frac{y\lambda^{\frac{n+1}{4}}e^{-2y\sqrt{\lambda}}}{\pi^{\frac{n+1}{2}}}\left(\left\|x-x'\right\|^2+y^2\right)^{-\frac{n+1}{4}}K_{\frac{n+1}{2}}\left(\sqrt{\lambda}\sqrt{\left\|x-x'\right\|^2+y^2}\right), \end{aligned}$$

[6], p.146, formula (29), where K_m is the modified Bessel function.

We note that, since $K_m(x)\sim\frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m$, $m>0$, as $x\to 0$, we get

$$Q_\lambda(x,y,x')\longrightarrow\left(\frac{2}{\pi}\right)^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)\frac{y}{\left(\left\|x-x'\right\|^2+y^2\right)^{\frac{n+1}{2}}},\ \text{as } \lambda\to 0,$$

i.e. the classical Poisson kernel of the Euclidean upper-half space.

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