DEGENERATE PRINCIPAL SERIES AND LOCAL THETA CORRESPONDENCE

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ABSTRACT. In this paper we determine the structure of the natural module \( \Omega^{p,q}(l) \) which is the Howe quotient corresponding to the determinant character \( \det^l \) of \( U(p,q) \). We first give a description of the tempered distributions on \( M_{p+q,n}(C) \) which transform according to the character \( \det^{-1} \) under the linear action of \( U(p,q) \). We then show that after tensoring with a character, \( \Omega^{p,q}(l) \) can be embedded into one of the degenerate series representations of \( U(n,n) \). This allows us to determine the module structure of \( \Omega^{p,q}(l) \). Moreover we show that certain irreducible constituents in the degenerate series can be identified with some of these representations \( \Omega^{p,q}(l) \) or their irreducible quotients. We also compute the Gelfand-Kirillov dimensions of the irreducible constituents of the degenerate series.

1. Introduction

In his thesis ([L]), the first named author studied the module structure and unitarity of the following degenerate principal series representations of \( U(n,n) \). Let \( \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}\} \) be the standard basis of \( C^{2n} \) and \( B \) be the basis \( \{\varepsilon_1 + \varepsilon_{n+1}, \ldots, \varepsilon_n + \varepsilon_{2n}, \varepsilon_1 - \varepsilon_{n+1}, \ldots, \varepsilon_n - \varepsilon_{2n}\} \). Let \( SH(n) \) be the additive group of \( n \times n \) skew-Hermitian matrices. For each \( a \in GL(n, C) \) and \( b \in SH(n) \), we let \( m_a \) and \( n_b \) be the elements of \( U(n,n) \) with matrix representations \( \left( \begin{array}{cc} a & 0 \\ 0 & (\sigma^t)^{-1} \end{array} \right) \) and \( \left( \begin{array}{cc} I_n & b \\ 0 & I_n \end{array} \right) \) with respect to \( B \) respectively. Let

\[ M = \{ m_a : a \in GL(n, C) \}, \quad N = \{ n_b : b \in SH(n) \}. \]

Then \( P = MN \) is a parabolic subgroup of \( U(n,n) \). For \( s \in C \) and \( \nu \in Z \), let \( \chi_{s,\nu} \) be the character of \( P \) given by

\[ \chi_{s,\nu}(p) = \chi_{s,\nu}(m_a n_b) = |\det a|^s \left( \frac{\det a}{|\det a|} \right)^\nu, \quad p = m_a n_b \in P. \]

Let \( I(s;\nu) = \text{Ind}_P^{U(n,n)} \chi_{s,\nu} \) be the corresponding (normalized) induced representation of \( U(n,n) \). The representation space for \( I(s;\nu) \) is

\[ \{ f \in C^\infty(U(n,n)) : f(pg) = \chi_{s,\nu}(p)\Delta(p)^{\frac{1}{2}} f(g), \forall g \in U(n,n), p \in P \} \]

where

\[ \Delta(p) = \Delta(m_a n_b) = |\det a|^{2n}, \quad p = m_a n_b \in P, \]

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In section 3, we show that $\Omega$ is irreducible if and only if $k+n+n \in \mathbb{Z}$. Moreover in the case when $I(s;\nu)$ is reducible, he obtained all its irreducible constituents.

Several other authors ([S1],[S2],[J1],[J2],[Zha]) have studied these representations in more general settings.

The purpose of this paper is two-fold. Firstly, we want to describe the irreducible constituents of $I(s;\nu)$ more carefully in two ways. We consider the reductive dual pair

$$(H,G) = (U(p,q), U(n,n)) \subseteq Sp(4(p+q)n, \mathbb{R}).$$

Let $\tilde{Sp}(4(p+q)n, \mathbb{R})$ be the unique nontrivial double cover of $Sp(4(p+q)n, \mathbb{R})$. For a subgroup $E$ of $Sp(4(p+q)n, \mathbb{R})$, $\tilde{E}$ shall denote the preimage of $E$ under the projection $\tilde{Sp}(4(p+q)n, \mathbb{R}) \rightarrow Sp(4(p+q)n, \mathbb{R})$. Let $V = \mathbb{C}^{p+q}$ and $V^n$ be the direct sum of $n$ copies of $V$. Then $\tilde{Sp}(4(p+q)n, \mathbb{R})$ acts on $L^2(V^n)$ via the oscillator representation $\omega$. We shall twist $\omega$ by a character so that it will factor through the standard linear action of $H$ on $L^2(V^n)$. Let $S(V^n)$ be the Schwartz space of $V^n$. Let $S \subseteq S(V^n)$ be the space of Schwartz functions which correspond to polynomials in the Fock model of $\omega$. For $l \in \mathbb{Z}$, let $D(l)$ be the irreducible $(u(p,q), U(p) \times U(q))$ module which corresponds to the determinant character $g \rightarrow (\det g)^l$ of $U(p,q)$, and let $\Omega^{p,q}(l)$ be the maximal quotient of $S$ on which $(u(p,q), U(p) \times U(q))$ acts by a representation of class $D(l)$. They shall be referred to as the Howe quotients. We shall examine the relationship between the irreducible constituents in $I(s;\nu)$ and these Howe quotients $\Omega^{p,q}(l)$. In particular, we shall show that some of the irreducible unitary submodules in $I(s;\nu)$ can be identified with $\Omega^{p,q}(l)$. On the other hand we shall also measure the size of the irreducible constituents in $I(s;\nu)$ by computing their Gelfand-Kirillov dimensions. In particular our results show that the Gelfand-Kirillov dimensions of the irreducible constituents can be read off from their position in the module diagram (or Hasse diagram, see [L] or [Al]) of $I(s;\nu)$. Our second purpose is to use the identification above and the results in [L] on the structure of the degenerate series to describe the module structure of the Howe quotients $\Omega^{p,q}(l)$. These results often play a role in the study of certain Eisenstein series and of $L$-functions attached to cuspidal automorphic representation for split groups ([KRS]).

This paper is arranged as follows. In section 2, we shall determine a set of all possible $\tilde{K}$-types in $\Omega^{p,q}(l)$ and show that each of them occur with multiplicity at most one. Here $\tilde{K} = U(n) \times U(n)$ is a maximal compact subgroup of $U(n,n)$. In section 3, we show that $\Omega^{p,q}(l)$ indeed contains these $\tilde{K}$-types by considering the space $(\Omega^*)^{p,q}(l)$ of tempered distributions on $V^n$ which transform according to the character $\det^{-l}$ of $U(p,q)$. We show that as a $U(n,n)$ module, $(\Omega^*)^{p,q}(l)$ is generated by a distinguished element $\mathcal{D}$ of $(\Omega^*)^{p,q}(l)$, and $\mathcal{D}$ has a non-zero projection to each of those $\tilde{K}$-types which are the contragradient modules of the possible $\tilde{K}$-types in $\Omega^{p,q}(l)$. This generalizes the results of the second named author on invariant tempered distributions [Zhu]. In section 4, we shall twist $\Omega^{p,q}(l)$ by a character $\xi$ of $U(n,n)$ and consider $\Omega_{\xi}^{p,q}(l) = \xi^{-1} \otimes \Omega^{p,q}(l)$. It turns out that $\Omega_{\xi}^{p,q}(l)$ factors through $U(n,n)$. We then use the element $\mathcal{D}$ to construct an embedding of
\(\Omega_{\xi}^{p,q}(l)\) into the degenerate series \(\{I(s;\nu) : s \in \mathbb{C}, \nu \in \mathbb{Z}\}\) of \(U(n,n)\) described at the beginning of this section.

**Theorem 4.1.** Let \(m\) be a positive integer and let \(p + q = m\). Assume either (i) \(l = 0\), or (ii) \(l \neq 0\) and \(m = n\). Then we have a \(G\)-equivariant embedding

\[
\lambda : \Omega_{\xi}^{p,q}(l) \rightarrow \begin{cases} 
I(m - n; 0), & \text{if } l = 0 \text{ and } m \text{ is even,} \\
I(m - n; -m), & \text{if } l = 0 \text{ and } m \text{ is odd,} \\
I([l]_1; -l), & \text{if } l \neq 0 \text{ and } m = n \text{ is even,} \\
I([l]_1; -n - l), & \text{if } l \neq 0 \text{ and } m = n \text{ is odd.}
\end{cases}
\]

When \(l \neq 0\), the \(U(n,n)\) module \(\Omega_{\xi}^{p,q}(l)\) is non-zero if and only if \(m \leq n\), and in this case one can show that \(\Omega_{\xi}^{p,q}(l)\) is irreducible and unitary. Also when \(m < n\), it is impossible to embed \(\Omega_{\xi}^{p,q}(l)\) into any \(I(s;\nu)\) (by comparing their \(K\)-types).

In section 5, we use the results in sections 3 and 4 to identify the image of \(\Omega_{\xi}^{p,q}(l)\) in \(I(s;\nu)\) under the above embedding. We then use the results of [L] to deduce the module structure of \(\Omega_{\xi}^{p,q}(l)\). If \(l = 0\), we shall simply write \(\Omega_{\xi}^{p,q}\) for \(\Omega_{\xi}^{p,q}(0)\). Let \(Q_{\xi}^{p,q}\) be the unique irreducible quotient of \(\Omega_{\xi}^{p,q}\). The following theorem summarizes the results given in section 5.

**Theorem.** Let 

\[(s,\nu) = \begin{cases} 
(m - n, 0), & \text{if } m \text{ is even,} \\
(m - n, -m), & \text{if } m \text{ is odd.}
\end{cases}\]

(a) If \(1 \leq m \leq n\), then the set \(\{\lambda(\Omega_{\xi}^{p,q}) : p + q = m\}\) exhausts all the irreducible unitary submodules of \(I(s;\nu)\). Moreover, if \(m = n\), then

\[I(s;\nu) = \bigoplus_{p + q = n} \lambda(\Omega_{\xi}^{p,q}).\]

(b) If \(m \geq n + 1\), then

\[\lambda(\Omega_{\xi}^{p,q}) = M(d_1, d_2),\]

and

\[Q_{\xi}^{p,q} \cong R_{a(d_1, d_2)},\]

where \(d_1 = \max\{0, n - q\}\) and \(d_2 = \max\{0, n - p\}\). Here \(R_{a(d_1, d_2)}\) is an irreducible constituent of \(I(s;\nu)\) (see section 5 for its definition) and \(M(d_1, d_2)\) is the submodule of \(I(s;\nu)\) generated by \(R_{a(d_1, d_2)}\).

(c) \(Q_{\xi}^{p,q}\) is unitary if and only if either \(p \leq n\) and \(q \leq n\), or \(pq = 0\). It is finite dimensional if and only if \(p \geq n\) and \(q \geq n\).

We also show that in the case when \(l = 0\) and \(p + q \geq n\), all the constituents in the corresponding representation \(I(s;\nu)\) are determined by the images of the various Howe quotients \(\Omega_{\xi}^{p,q}\). Since \(I(s;\nu)\) and \(I(s; -\nu)\) are contragradient, it follows that the images of Howe quotients determine all constituents of \(I(s;\nu)\) in general. Finally in section 6, we compute the Gelfand-Kirillov dimensions of the irreducible constituents in \(I(s;\nu)\).
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2. HOWE’S QUOTIENT \(\Omega^{p,q}(l)\)

Consider the reductive dual pair \((H, G) = (U(p, q), U(n, n)) \subseteq Sp(4(p + q)n, \mathbb{R})\). Let \(L = U(p) \times U(q)\) and \(K = U(n) \times U(n)\). Then \(L\) and \(K\) are maximal compact subgroups of \(H\) and \(K\), respectively. We shall choose a maximal compact subgroup \(U \cong U(2(p + q)n)\) of \(Sp = Sp(4(p + q)n, \mathbb{R})\) in such a way that \(L = U \cap H\) and \(K = U \cap G\). We also let \(\mathfrak{h} = \text{Lie}(H)_{\mathbb{C}}\) and \(\mathfrak{g} = \text{Lie}(G)_{\mathbb{C}}\) be the complexified Lie algebras of the groups \(H\) and \(G\), respectively.

Let \(V = \mathbb{C}^{p+q}\) and let \(V^n\) be the direct sum of \(n\) copies of \(V\). Then \(Sp(4(p + q)n, \mathbb{R})\) acts on \(L^2(V^n)\) via the Oscillator representation \(\omega\). As mentioned in the introduction, we can twist \(\omega\) by a character so that it will factor through the following standard linear action of \(H\) on \(L^2(V^n)\):

\[
h \cdot f(v_1, ..., v_n) = f(h^{-1} \cdot v_1, ..., h^{-1} \cdot v_n), \quad h \in H, \ (v_1, ..., v_n) \in V^n.
\]

We shall assume that this has been done from now on. Let \(S(V^n)\) be the Schwartz space of \(V^n\) and let \(S\) be the space consisting of those Schwartz functions which correspond to polynomials in a Fock model of \(\omega\). Since \(S\) is the space of \(\tilde{U}\)-finite vectors of \(\omega\), it is naturally a \((\mathfrak{h}, L) \times (\mathfrak{g}, K))\)-module.

For any integer \(l\), we let \(D(l)\) be the irreducible \((\mathfrak{h}, L)\) module corresponding to the character \(h \mapsto (\det h)^l\) of \(U(p, q)\). Let \(\Omega^{p,q}(l)\) be the maximal quotient of \(S\) on which \((\mathfrak{h}, L)\) acts by a representation of class \(D(l)\). By the results in [H2], \(\Omega^{p,q}(l)\) is a quasi-simple \((\mathfrak{g}, K))\)-module of finite length and has a unique irreducible quotient \(Q^{p,q}(l)\). Note that \(Q^{p,q}(l)\) is the representation of \((\mathfrak{g}, K)\) which corresponds to the character \(\det\) of \(U(p, q)\) under Howe’s quotient correspondence.

In [KR1], Kudla and Rallis consider the dual pair

\[
(O(p, q), Sp(2n, \mathbb{R})) \subseteq Sp(2(p + q)n, \mathbb{R})
\]

and study the Howe’s quotient \(R\) which corresponds to the trivial representation of \(O(p, q)\). In particular, they give the \(\tilde{U}(n)\)-spectrum of \(R\), which is multiplicity free. In the present case, we shall show that the \(K\)-spectrum of \(\Omega^{p,q}(l)\) is also multiplicity free.

We need to introduce some notation. Let \(\Lambda^+_n\) be the set of all dominant integral weights for the unitary group \(U(n)\) with respect to the Borel subalgebra \(\mathfrak{b}^+_n\) of upper triangular matrices. \(\Lambda^+_n\) can be identified in the usual way with the set of all \(n\)-tuples of integers \(\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)\) such that \(\lambda_1 \geq \lambda_2 \geq \cdot \cdot \cdot \geq \lambda_n\). For each \(\lambda \in \Lambda^+_n\), \(\rho^\lambda\) shall denote a copy of the irreducible representation of \(U(n)\) with highest weight \(\lambda\). Recall that if \(\rho = \rho^\lambda\), then the contragradient of \(\rho\) is given by \(\rho^* = \rho^{\lambda^*}\), where \(\lambda^* = (-\lambda_n, ..., -\lambda_1)\). For convenience, we also let

\[
1_n = (1, ..., 1).
\]
Note that $1_n$ is the highest weight of the determinant character of $U(n)$. In a similar way, every irreducible representation of $\hat{U}(n)$ is of the form $\rho^\lambda$, but the components of $\lambda$ can be half integers.

**Proposition 2.1.** Every $\tilde{K}$-type $\tau$ in $\Omega^{p,q}(0)$ is of the form

$$\tau \cong \rho^\lambda \otimes \rho^{\lambda^\ast},$$

where

$$\lambda = \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_t, 0, \ldots, 0, -\gamma_s, \ldots, -\gamma_l),$$

and $\alpha_1 \geq \ldots \geq \alpha_t \geq 0$, $\gamma_1 \geq \ldots \geq \gamma_s \geq 0$ are integers with $t \leq \min(p,n)$ and $s \leq \min(q,n)$. Moreover each such $\tilde{K}$-type occurs with multiplicity at most one.

**Proposition 2.2.** For $l \neq 0$, $\Omega^{p,q}(l)$ is nontrivial only if $p+q \leq n$. If $p+q \leq n$, then every $\tilde{K}$-type $\tau$ in $\Omega^{p,q}(l)$ is of the form

$$\tau \cong \rho^\lambda \otimes \rho^{\lambda^\ast + (-l, \ldots, -l, 0, \ldots, 0)}$$

where

$$\lambda = \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\gamma_q, \ldots, -\gamma_l),$$

and $\alpha_i, \gamma_i$ are integers satisfying

$$\alpha_1 \geq \ldots \geq \alpha_p \geq 0, \gamma_1 \geq \ldots \geq \gamma_q \geq 0, \quad \text{if } l > 0,$$

and

$$\alpha_1 \geq \ldots \geq \alpha_p \geq -l, \gamma_1 \geq \ldots \geq \gamma_q \geq 0, \quad \text{if } l < 0.$$

Moreover each such $\tilde{K}$-type occurs with multiplicity at most one.

**Proof of Propositions 2.1 and 2.2.** The argument given below is similar to those of [KR1] in the case of dual pair $(O(p,q), Sp(2n, \mathbb{R}))$. We consider the seesaw dual pair ([Ku1])

$$U(p,q) \cap U(n) \cap U(p,q) \times U(p,q) \cap U(n,n).$$

Assume that $\tau = \rho^\lambda \otimes \rho^\mu$ is a $\tilde{K}$-type occurring in $\Omega^{p,q}(l)$. As usual we use a subscript $\tau$ to denote the $\tau$-isotypic component. By the standard result of Howe ([H2]) or Kashiwara and Vergne ([KV]), we have

$$S_\tau \cong \pi(\tau) \otimes \tau$$

for some irreducible $(u(p,q) \oplus u(p,q), K')$-module $\pi(\tau)$, where $K' = U(p) \times U(q) \times U(p) \times U(q)$. Let $\mathcal{H}(K)$ be the space of $K$-harmonics of $S$. Then we have $\mathcal{H}(K)_\tau \cong \sigma(\tau) \otimes \tau$, where $\sigma(\tau) = \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_4 \in \tilde{K'}$, $\sigma_1, \sigma_2 \in \hat{U}(p)$ and $\sigma_3, \sigma_4 \in \hat{U}(q)$, and $\sigma(\tau)$ occurs in $\pi(\tau)$ with multiplicity one.

The arguments in [KR1] give

$$\Omega^{p,q}(l)_\tau = pr(S_\tau) = pr(\mathcal{H}(K)_\tau),$$
where \( pr : S \rightarrow \Omega^{p,q}(l) \) is the natural projection map. Now the projection \( pr : \mathcal{H}(K)_r \rightarrow \Omega^{p,q}(l)_r \) factors through the projection \( \mathcal{H}(K)_r \rightarrow \mathcal{H}(K)_{r,l} \), where \( \mathcal{H}(K)_{r,l} \) consists of those \( v \in \mathcal{H}(K)_r \) such that
\[
h \cdot v = (\det h_1)^l (\det h_2)^p v, \quad h = (h_1, h_2) \in U(p) \times U(q).
\]

Recall that we have twisted the oscillator representation \( \omega \) by a character so that \( H = U(p, q) \) acts linearly. Thus \( U(p) \times U(q) \) also acts linearly on \( \mathcal{H}(K) \). Taking this into consideration, the results of [KV] imply that
\[
\lambda = \frac{p-q}{2} \mathbf{1}_n + (\alpha_1, \ldots, \alpha_t, 0, \ldots, 0, -\gamma_s, \ldots, -\gamma_l),
\]
\[
\mu = -\frac{p-q}{2} \mathbf{1}_n + (\delta_1, \ldots, \delta_s, 0, \ldots, 0, -\beta_t, \ldots, -\beta_l),
\]
where \( t \leq \min(p, n) \), \( s \leq \min(q, n) \), \( \alpha_i, \beta_i, \gamma_i, \delta_i \) are non-negative integers, and
\[
\sigma_1 \otimes \sigma_2 = \rho^{(0, \ldots, 0, -\alpha_1, \ldots, -\alpha_1)} \otimes \rho^{(\beta_1, \ldots, \beta_t, 0, \ldots, 0)} \in \tilde{K}_1\text{,}
\]
\[
\sigma_3 \otimes \sigma_4 = \rho^{(\gamma_1, \ldots, -\gamma_s, 0, \ldots, 0)} \otimes \rho^{(0, \ldots, 0, -\delta_t, \ldots, -\delta_t)} \in \tilde{K}_2\text{,}
\]
where \( \tilde{K}_1 = U(p) \times U(p) \), \( \tilde{K}_2 = U(q) \times U(q) \).

Hence for \( \Omega^{p,q}(l)_r \neq 0 \), the representation \( \sigma(\tau) = \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_4 \) must contain the character \( \det_{p,l}^l \otimes \det_{q,l}^l \) of \( U(p) \times U(q) \). This occurs when \( \sigma_2 \cong \sigma_1^l \otimes \det_{p,l}^l \) and \( \sigma_4 \cong \sigma_3^l \otimes \det_{q,l}^l \); i.e.,
\[
\beta_i = \alpha_i, \quad \delta_i = \gamma_i, \quad \text{if } l = 0,
\]
\[
\beta_i = \alpha_i + l, \quad \delta_i = \gamma_i - l, \quad t = p, s = q, \quad \text{if } l \neq 0.
\]

Moreover when the above conditions are satisfied, \( \sigma(\tau) \) contains the character \( \det_{p,l}^l \otimes \det_{q,l}^l \) exactly once. For \( l \neq 0 \), the second condition above implies that \( p + q \leq n \), \( \gamma_i \geq l \) if \( l > 0 \), and \( \alpha_i \geq -l \) if \( l < 0 \). Further we have
\[
\mu = -\frac{p-q}{2} \mathbf{1}_n + (\gamma_1, \ldots, \gamma_s, 0, \ldots, 0, -\alpha_t, \ldots, -\alpha_1) + (\underbrace{-l, \ldots, -l}_{q \text{ times}}, \underbrace{0, \ldots, 0, -l, \ldots, -l}_{p \text{ times}}).
\]

Thus \( \mu = \lambda^* + (l, \ldots, -l, 0, \ldots, 0, -l, \ldots, -l) \).

We shall show in the next section that these \( \tilde{K} \)-types indeed occur in \( \Omega^{p,q}(l) \) by considering a certain space \( (\Omega^*)^{p,q}(l) \) of tempered distributions.

\section{3. \( U(p, q) \)-equivariant tempered distributions}

Recall that \( S(V^n) \) is the Schwartz space of \( V^n \), where \( V \) is the standard module of \( H = U(p, q) \). Let \( S^*(V^n) \) be the space of tempered distributions on \( V^n \). Then \( Sp(4(p + q)n, \mathbb{R}) \) acts on \( S(V^n) \) by the restriction of the oscillator representation \( \omega \) and this action induces an action of \( Sp(4(p + q)n, \mathbb{R}) \) on \( S^*(V^n) \) in the usual way. We shall denote this action on \( S^*(V^n) \) also by \( \omega \).

Now for each integer \( l \), let \( (\Omega^*)^{p,q}(l) \) be the subspace of tempered distributions consisting of those \( \Phi \in S^*(V^n) \) such that
\[
h \cdot \Phi = (\det h)^{-l} \Phi, \quad h \in H,
\]
where \( h \cdot \Phi = \omega(h)\Phi \) is the linear action of \( H \) on \( S^*(V^n) \).
Since the action of $\tilde{G}$ commutes with the action of $H$, $(\Omega^*)^{p,q}(l)$ is a $\tilde{G}$ module. When $l = 0$, $(\Omega^*)^{p,q}(0) = S^*(V^n)^H$ is the space of $H$-invariant tempered distributions on $V^n$. A precise description of this representation is given in [Zhu] and we shall recall it in Theorem 3.1 below for the convenience of the readers. The purpose of this section is to show that for $l \neq 0$, the $\tilde{G}$ module $(\Omega^*)^{p,q}(l)$ is generated by a certain tempered distribution $\mathcal{D}$ and to determine the $\hat{K}$-types in $(\Omega^*)^{p,q}(l)$.

As usual, $\delta$ shall denote the Dirac distribution at the origin of $V^n$. It is $H$-invariant.

**Theorem 3.1** (Theorem, [Zhu]). (a) $S^*(V^n)^H$ is the closed span of the set $\{\omega(g)\delta : g \in G\}$.

(b) For any $\sigma \in \hat{K}$, the multiplicity of $\sigma$ in $S^*(V^n)^H$ is at most one. It is equal to one if and only if

$$\sigma^* \cong \rho^\lambda \otimes \rho^{\lambda^*},$$

where

$$\lambda = \frac{p-q}{2} \mathbf{1}_n + (\alpha_1, ..., \alpha_t, 0, ..., 0, -\gamma_s, ..., -\gamma_l),$$

and $\alpha_1 \geq ... \geq \alpha_t \geq 0$, $\gamma_1 \geq ... \geq \gamma_s \geq 0$

are integers and $t \leq \min(p, n)$ and $s \leq \min(q, n)$.

**Remark 3.2.** Theorem I of [Zhu] states that the $\hat{K}$-types in $S^*(V^n)$ are of the form $\sigma \cong \rho^\lambda \otimes \rho^{\lambda^*}$, where $\lambda$ is given above. This is incorrect (see Theorem 3.3 below). A proof for the corrected version can be obtained by making only a minor change in the original proof (see the proof of Theorem 3.3 below).

Now we shall examine the space $(\Omega^*)^{p,q}(l)$ for $l \neq 0$. We shall introduce a distinguished tempered distribution in $(\Omega^*)^{p,q}(l)$, which will take the place of $\delta$ in $\Omega^*(0)$. Let $Z = (z_{ij})_{1 \leq i \leq p+q, 1 \leq j \leq n}$ be the natural complex coordinates of $V^n \cong M_{p+q,n}(C)$. For $1 \leq l \leq \min(p+q, n)$, let

$$d_l(Z) = \det \begin{pmatrix} z_{11} & ... & z_{1l} \\ ... & ... & ... \\ z_{tl} & ... & z_{tt} \end{pmatrix},$$

(3.1)

$$\overline{d}_l(Z) = \det \begin{pmatrix} \overline{z}_{11} & ... & \overline{z}_{1l} \\ ... & ... & ... \\ \overline{z}_{tl} & ... & \overline{z}_{tt} \end{pmatrix},$$

(3.2)

$$\partial_t = \partial_l(Z) = \det \begin{pmatrix} \frac{\partial}{\partial z_{11}} & ... & \frac{\partial}{\partial z_{1l}} \\ ... & ... & ... \\ \frac{\partial}{\partial z_{tl}} & ... & \frac{\partial}{\partial z_{tt}} \end{pmatrix},$$

$$\overline{\partial}_l = \overline{\partial}_l(Z) = \det \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & ... & \frac{\partial}{\partial \overline{z}_{1l}} \\ ... & ... & ... \\ \frac{\partial}{\partial \overline{z}_{tl}} & ... & \frac{\partial}{\partial \overline{z}_{tt}} \end{pmatrix}.$$ Assume that $p+q \leq n$ and $d > 0$. Then one can verify that for $h \in H$, we have

$$h \cdot (\partial^d_{p+q} \delta) = (\det h)^{d}(\partial^d_{p+q} \delta),$$

$$h \cdot (\overline{\partial}^d_{p+q} \delta) = (\det h)^{-d}(\overline{\partial}^d_{p+q} \delta).$$

Consequently the tempered distribution $\mathcal{D}$ given by

$$\mathcal{D} = \begin{cases} \overline{\partial}_{p+q}^d \delta, & \text{if } l > 0, \\ \partial_{p+q}^d \delta, & \text{if } l < 0, \end{cases}$$

is in the space $(\Omega^*)^{p,q}(l)$.
**Theorem 3.3.** For \( l \neq 0 \), \((\Omega^*)^{p,q}(l)\) is nontrivial if and only if \( p + q \leq n \). If \( p + q \leq n \), then

(a) \((\Omega^*)^{p,q}(l)\) is the closed span of the set \{\(\omega(g)D|g \in \tilde{G}\)\}.

(b) For any \( \sigma \in \tilde{K} \), the multiplicity of \( \sigma \) in \((\Omega^*)^{p,q}(l)\) is at most one. It is equal to one if and only if

\[
\sigma^* \cong p^\lambda \otimes p^{\lambda^* + (-1,\ldots,-1,0,\ldots,0,\ldots,-1,\ldots,-1)},
\]

where

\[
\lambda = \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\gamma_q, \ldots, -\gamma_1),
\]

and \( \alpha_i, \gamma_i \) are integers satisfying

\[
\alpha_1 \geq \ldots \geq \alpha_p \geq 0, \ \gamma_1 \geq \ldots \geq \gamma_q \geq l, \quad \text{if} \ l > 0,
\]

and

\[
\alpha_1 \geq \ldots \geq \alpha_p \geq -l, \ \gamma_1 \geq \ldots \geq \gamma_q \geq 0, \quad \text{if} \ l < 0.
\]

We outline our strategy for proving Theorems 3.1 and 3.3, and we thank the referee for helping us to clarify this. Let \( R^{p,q}(l) \) be the maximal quotient of \( S(V^n) \) on which \( U(p,q) \) acts by the character \( g \rightarrow (\det g)^l \). Clearly, \( R^{p,q}(l) \) and \((\Omega^*)^{p,q}(l)\) are duals under the natural pairing \( (, ) \) between \( S(V^n) \) and \( S^*(V^n) \). There is also a natural map \( \eta : \Omega^{p,q}(l) \rightarrow R^{p,q}(l) \), which makes the following diagram commutative:

\[
\begin{array}{ccc}
S & \hookrightarrow & S(V^n) \\
\downarrow & & \downarrow \\
\Omega^{p,q}(l) & \rightarrow & R^{p,q}(l)
\end{array}
\]

Here the vertical maps are the natural quotient maps. If \( \tau \) is a possible \( \tilde{K} \)-type in \( \Omega^{p,q}(l) \) (as given in Propositions 2.1 and 2.2), we shall prove that there exists a vector \( \phi \in S_\tau \) such that \((D, \phi) \neq 0 \). Since \( D \in (\Omega^*)^{p,q}(l) \), we see that the image of \( \phi \) is non-zero in \( R^{p,q}(l) \), and since the map of \( \phi \) to this space factors through \( \Omega^{p,q}(l) \), the image of \( \phi \) in \( \Omega^{p,q}(l) \) is also non-zero. Thus (i) all such \( \tau \)-s occur in \( \Omega^{p,q}(l) \), and (ii) since \( \Omega^{p,q}(l) \) is \( \tilde{K} \)-isotypic, the map \( \eta \) is injective. Now, since the space \( S \) is dense in the Fréchet space \( S(V^n) \), we see \( \eta(\Omega^{p,q}(l)) \) is dense in \( R^{p,q}(l) \), and by the injectivity of \( \eta \), we obtain a non-degenerate pairing between \( \Omega^{p,q}(l) \) and \((\Omega^*)^{p,q}(l)\), which is \( \tilde{K} \)-equivariant. We can then conclude that \( \tilde{K} \)-types of \((\Omega^*)^{p,q}(l) \) must be contragradient to the \( \tilde{K} \)-types of \( \Omega^{p,q}(l) \), and are thus of the form given in Theorems 3.1 and 3.3, with their \( \tilde{K} \)-multicities equal to one, and each \( \tilde{K} \)-isotypic component of \((\Omega^*)^{p,q}(l) \) is generated as a \( \tilde{K} \)-module by the corresponding \( \tilde{K} \)-isotypic component of \( D \). This will imply Theorems 3.1 and 3.3.

Let \( \omega^{-\infty} \) be the space of formal vectors on \( \omega \) ([HT], [Zhu]). It is the space of generalized functions on \( V^n \) with \( \tilde{U} \) Fourier components, where \( \tilde{U} \cong U(2(p+q)n) \) is a maximal compact subgroup of \( Sp(4(p+q)n, \mathbb{R}) \). Moreover, \( S(V^n) \) (resp. \( S^*(V^n) \)) can be characterized as the subspace of \( \omega^{-\infty} \) such that their \( \tilde{U} \) Fourier components decay rapidly (resp. grow at most polynomially). Notice that \( Sp(4(p+q)n, \mathbb{R}) \) acts on \( \omega^{-\infty} \) by linear extension from the action on \( S(V^n) \) and thus it is contragradient.
to the action on $S^*(V^n)$. The main purpose in introducing the space $\omega^{-\infty}$ is that the distribution $\mathcal{D}$ turns out to have a simple expression as an formal vector, and that simple expression enables us to compute its inner product with certain lowest highest weight vectors. We shall first examine the compact case.

**Proposition 3.4.** (a) Suppose that $H = U(p, 0) = U(p)$ and $p \leq n$. Let $\tau \in \widehat{K}$ be such that

$$\lambda^* + (0, \ldots, 0, -l, \ldots, -l)$$

where $\lambda = \frac{q}{2} I_n + (\alpha_1, \alpha_2, \ldots, \alpha_p, 0, \ldots, 0)$, and $\alpha_i$ are integers satisfying

$$\alpha_1 \geq \ldots \geq \alpha_p \geq 0,$$

and

$$\alpha_1 \geq \ldots \geq \alpha_p \geq -l,$$  \quad if $l > 0$,

$\alpha_1 \geq \ldots \geq \alpha_p \geq l,$  \quad if $l < 0$.

Then there exists a vector $\phi \in S_\tau$ such that $(\mathcal{D}, \phi) \neq 0$. In particular, $\mathcal{D}$ has a non-zero $\sigma$-isotypic component for $\sigma \cong \tau^*$.

(b) Suppose $H = U(0, q)$ and $q \leq n$. Let $\tau \in \widehat{K}$ be such that

$$\lambda^* + (-l, \ldots, -l, 0, \ldots, 0)$$

where $\lambda = -\frac{q}{2} I_n + (0, \ldots, 0, -\gamma_1, \ldots, -\gamma_l)$, and $\gamma_i$ are integers satisfying

$$\gamma_1 \geq \ldots \geq \gamma_l \geq 0,$$

and

$$\gamma_1 \geq \ldots \geq \gamma_l \geq l,$$  \quad if $l > 0$,

$\gamma_1 \geq \ldots \geq \gamma_l \geq l,$  \quad if $l < 0$.

Then there exists a vector $\phi \in S_\tau$ such that $(\mathcal{D}, \phi) \neq 0$. In particular, $\mathcal{D}$ has a non-zero $\sigma$-isotypic component for $\sigma \cong \tau^*$.

**Proof.** We shall only prove part (a). The proof for part (b) is similar. Recall $Z = (z_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$ are the standard coordinates of $V^n \cong M_{p,n}(\mathbb{C})$. Let $Q = (q_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$ and $\overline{Q} = (\overline{q}_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$ be the complex coordinates in the Fock model of the oscillator representation of $\omega$. The isometric isomorphism of the Schrödinger model with the Fock model is such that (cf. [Fo])

$$z_{ij} \rightarrow \frac{1}{\sqrt{2}} \left(2 \frac{\partial}{\partial q_{ij}} + q_{ij}\right), \quad \frac{\partial}{\partial z_{ij}} \rightarrow \frac{1}{2\sqrt{2}} \left(2 \frac{\partial}{\partial q_{ij}} - \overline{q}_{ij}\right),$$

$$\overline{z}_{ij} \rightarrow \frac{1}{\sqrt{2}} \left(2 \frac{\partial}{\partial \overline{q}_{ij}} + \overline{q}_{ij}\right), \quad \frac{\partial}{\partial \overline{z}_{ij}} \rightarrow \frac{1}{2\sqrt{2}} \left(2 \frac{\partial}{\partial \overline{q}_{ij}} - q_{ij}\right).$$

Let $\Delta$ be the image of $\delta$, the Dirac distribution at the origin of $V^n$. Since $\delta$ satisfies

$$z_{ij} \cdot \delta = \overline{z}_{ij} \cdot \delta = 0,$$

we have

$$\Delta = \exp \left(-\frac{\sum_{ij} q_{ij}\overline{q}_{ij}}{2}\right) \quad \text{(up to a scalar)}$$

as an element in $\omega^{-\infty}$.

Recall the definitions of $d_t(Z)$, $\overline{d}_t(Z)$, $\partial_t(Z)$, $\overline{\partial}_t(Z)$ as given in eqs. (3.1), (3.2). We define $d_t(Q)$, $\overline{d}_t(Q)$, $\partial_t(Q)$, $\overline{\partial}_t(Q)$ by replacing $z_{ij}$ with $q_{ij}$ in $d_t(Z)$, $\overline{d}_t(Z)$, $\partial_t(Z)$ and $\overline{\partial}_t(Z)$ respectively.
For any operator $A$, let $A^*$ be its adjoint with respect to the Hilbert space structure in the Fock model. We have (cf. [Ba])

\[(3.3)\quad \left(\frac{\partial}{\partial \eta_{ij}}\right)^* = \frac{1}{2} q_{ij}, \quad \left(\frac{\partial}{\partial \eta_{ij}}\right)^* = \frac{1}{2} \bar{q}_{ij}.\]

Thus

\[(3.4)\quad \partial_t (Q) = \frac{1}{2t} d_t (Q), \quad \bar{\partial}_t (Q) = \frac{1}{2t} \bar{d}_t (Q).\]

Under the isomorphism from the Schrödinger model to the Fock model, a tempered distribution $\Phi \in S'(V^n)$ is identified with a formal vector $F_\Phi \in \omega^{-\infty}$. If a polynomial coefficient differential operator $B$ in $z_{ij}, \bar{z}_{ij}$ is identified with a polynomial coefficient differential operator $F_B$ in $q_{ij}, \bar{q}_{ij}$, then the distribution $B \cdot \Phi$ with be identified with the formal vector $-(F_B)^* \cdot F_\Phi$, in view of the fact that $Sp(4(p + q)n, \mathbb{R}) \times H$ acts on $\omega^{-\infty}$ by linear extension from the action on $S(V^n)$, which is contragradient to the action on $S^*(V^n)$. Here $H$ is the standard Heisenberg group associated to the symplectic group $Sp(4(p + q)n, \mathbb{R})$. This implies that

\[
\begin{align*}
\frac{\partial}{\partial z_{ij}} \delta &\longrightarrow -\frac{1}{2\sqrt{2}} (2 \frac{\partial}{\partial q_{ij}} - q_{ij}) \exp(-\frac{\sum_{i<j} q_{ij} \bar{q}_{ij}}{2}) = \frac{1}{\sqrt{2}} \frac{\partial}{\partial q_{ij}} \exp(-\frac{\sum_{i<j} q_{ij} \bar{q}_{ij}}{2}), \\
\frac{\partial}{\partial \bar{z}_{ij}} \delta &\longrightarrow -\frac{1}{2\sqrt{2}} (2 \frac{\partial}{\partial \bar{q}_{ij}} - q_{ij}) \exp(-\frac{\sum_{i<j} q_{ij} \bar{q}_{ij}}{2}) = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \bar{q}_{ij}} \exp(-\frac{\sum_{i<j} q_{ij} \bar{q}_{ij}}{2}).
\end{align*}
\]

Thus we have

\[
\begin{align*}
\partial_t (Z) &\longrightarrow \left(\frac{1}{\sqrt{2}}\right)^t \overline{\partial}_t (Q) \Delta, \\
\bar{\partial}_t (Z) &\longrightarrow \left(\frac{1}{\sqrt{2}}\right)^t \partial_t (Q) \Delta,
\end{align*}
\]

and so the formal vector $F_D$ corresponding to $\mathcal{D}$ is

\[(3.5)\quad F_D = \left\{ \begin{array}{ll} (\frac{1}{\sqrt{2}})^l \partial_l (Q) \Delta, & l > 0, \\
(\frac{1}{\sqrt{2}})^l \bar{\partial}_l (Q) \Delta, & l < 0. \end{array} \right.\]

In the Fock model, a highest weight vector in the $\tilde{K}$-type

\[\tau = \rho^\lambda \otimes \rho^\mu = \rho^{\frac{1}{2} \lambda_1 + (\alpha_1, ..., \alpha_t, 0, ..., 0)} \otimes \rho^{-\frac{1}{2} \lambda_1 + (0, ..., -\beta_t, ..., -\beta_1)} = \rho^{\lambda^* + (0, ..., 0, \underbrace{-l, ..., -l}_{p \text{ times}})} \quad t \leq p,\]

is given by

\[d_1 (Q)^{a_1} d_2 (Q)^{a_2} ... d_t (Q)^{a_t} \bar{d}_1 (Q)^{b_1} \bar{d}_2 (Q)^{b_2} ... \bar{d}_t (Q)^{b_t}\]

where

\[
\alpha_i = \sum_{j=i}^t a_j, \quad \beta_i = \sum_{j=i}^t b_j.
\]

We may assume that $t = p$ by allowing $\alpha_i, \beta_i$ to be zero.

Now the condition that $\mu = \lambda^* + (0, ..., 0, \underbrace{-l, ..., -l}_{p \text{ times}})$ amounts to $\beta_i = \alpha_i + l,$

\[\forall i \leq p, \text{ i.e.,} \]

\[
\begin{align*}
b_i &= a_i, & 1 \leq i \leq p - 1, \\
b_p &= a_p + 1.
\end{align*}
\]
We first consider the case \( l > 0 \). We compute the following inner product of \( F_D \) with a \( \widetilde{K} \) highest weight vector (using the adjoint relation (3.4)):
\[
(\partial_p(Q)^l \Delta, d_1(Q)^{a_1} d_2(Q)^{a_2} \ldots d_p(Q)^{a_p} \overline{\eta}_1(Q)^{b_1} \overline{\eta}_2(Q)^{b_2} \ldots \overline{\eta}_p(Q)^{b_p})
= (1/2^p)^l (\Delta, d_1(Q)^{a_1} d_2(Q)^{a_2} \ldots d_p(Q)^{a_p} \overline{\eta}_1(Q)^{b_1} \overline{\eta}_2(Q)^{b_2} \ldots \overline{\eta}_p(Q)^{b_p})
= (1/2^p)^l (\Delta, d_1(Q)^{b_1} d_2(Q)^{b_2} \ldots d_p(Q)^{b_p} \overline{\eta}_1(Q)^{b_1} \overline{\eta}_2(Q)^{b_2} \ldots \overline{\eta}_p(Q)^{b_p})
= (1/2^p)^l \prod_{1 \leq t \leq p \leq 1 \leq b_t \leq b_i} (-2)^{c_1 + 2c_2 + \ldots + tc_r} B(c_1, c_2, \ldots, c_l)(\Delta, 1) \neq 0,
\]
where
\[
B(c_1, c_2, \ldots, c_l) = \prod_{i=1}^{l} \left( \sum_{s=1}^{l} (c_s + t - i) \right).
\]

See [Zhu, pp. 106 and 116] for the last equality in the above computation.

Similarly for \( l < 0 \), we have
\[
(\overline{\partial}_p(Q)^{-l} \Delta, d_1(Q)^{a_1} d_2(Q)^{a_2} \ldots d_p(Q)^{a_p} \overline{\eta}_1(Q)^{b_1} \overline{\eta}_2(Q)^{b_2} \ldots \overline{\eta}_p(Q)^{b_p})
= (1/2^p)^{-l} \prod_{1 \leq t \leq p \leq 1 \leq c_t \leq a_i} \prod_{1 \leq t \leq p \leq 1 \leq c_t \leq a_i} (-2)^{c_1 + 2c_2 + \ldots + tc_r} B(c_1, c_2, \ldots, c_l)(\Delta, 1) \neq 0.
\]

We now examine the general (non-compact) case; i.e., \( H = U(p, q) \).

**Proposition 3.5.** Assume that \( p + q \leq n \). Let \( \tau \in \hat{K} \) be such that
\[
\tau^* \cong \rho^\lambda \otimes \rho^\rho \equiv \rho^\lambda \otimes \rho^\rho \quad \text{with } \rho = \frac{p-2}{2} \mathbf{1}_n + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\gamma_q, \ldots, -\gamma_1),
\]
where \( \alpha_i, \gamma_i \) are integers satisfying
\[
\alpha_1 \geq \ldots \geq \alpha_p \geq 0, \quad \gamma_1 \geq \ldots \geq \gamma_q \geq 0, \quad \text{if } l > 0,
\]
and
\[
\alpha_1 \geq \ldots \geq \alpha_p \geq -l, \quad \gamma_1 \geq \ldots \geq \gamma_q \geq 0, \quad \text{if } l < 0.
\]
Then there exists a vector \( \phi \in S_\tau \) such that \( (D, \phi) \neq 0 \). In particular, \( D \) has a non-zero \( \sigma \)-isotypic component for \( \sigma \cong \tau^* \).

Since \( D \in (\Omega^*)^{p-q}(l) \), its \( \sigma \)-isotypic component \( D_\sigma \) clearly belongs to \( (\Omega^*)^{p-q}(l)_{\sigma} \). The discussion on possible \( \widetilde{K} \)-types of \( (\Omega^*)^{p-q}(l) \) before Proposition 3.4 together with Proposition 3.5 will therefore imply Theorem 3.3.

**Proof of Proposition 3.5.** We consider the following subgroups of \( Sp(4(p+q)n, \mathbb{R}) \):
\[
L = H_1 \times H_2 = U(p) \times U(q),
E = E_1 \times E_2 = (U(p) \times U(p)) \times (U(q) \times U(q)),
N = N_1 \times N_2 = (U(n) \times U(n)) \times (U(n) \times U(n)),
\]
which fit into the following diamond dual pairs ([H2]):
i.e. the pairs of Lie groups similarly placed in the two diamonds are reductive dual pairs. Let $\omega_1$ (resp. $\omega_2$) be the oscillator representation associated to the dual pair $(H_1, G) = (U(p), U(n, n)) \subseteq Sp(4n, \mathbb{R})$ (resp. $(H_2, G) = (U(q), U(n, n)) \subseteq Sp(4n_q, \mathbb{R})$). Then we have the functorial property ([H5]):

$$\omega|_{\tilde{E}_1 \times \tilde{N}_1} \cong \omega_1|_{\tilde{E}_1 \times \tilde{N}_1} \otimes \omega_2|_{\tilde{E}_2 \times \tilde{N}_2},$$

where $\otimes$ here denotes outer tensor product. Let $Q = (q_{ij}), Q = (\overline{q}_{ij})$ (1 $\leq i \leq p, 1 \leq j \leq n$) be the complex coordinates in the Fock model of $\omega_1$ and let $W = (w_{ij}), W = (\overline{w}_{ij})$ (1 $\leq i \leq q, 1 \leq j \leq n$) be the complex coordinates in the Fock model of $\omega_2$. Let

$$h_1(Q) = d_1(Q)^{a_1}d_2(Q)^{a_2}...d_p(Q)^{a_p}\tilde{d}_1(Q)^{b_1}\tilde{d}_2(Q)^{b_2}...\tilde{d}_p(Q)^{b_p}$$

be a highest weight vector under the action of $\tilde{E}_1 \times \tilde{N}_1$. Similarly let

$$h_2(W) = \tilde{d}_1(W)^{c_1}\tilde{d}_2(W)^{c_2}...\tilde{d}_q(W)^{c_q}\overline{d}_1(W)^{\overline{c}_1}\overline{d}_2(W)^{\overline{c}_2}...\overline{d}_q(W)^{\overline{c}_q}$$

be a highest weight vector under the action of $\tilde{E}_2 \times \tilde{N}_2$, where

$$d_t(Q) = \det \begin{pmatrix} q_{11} & \ldots & q_{1t} \\ \vdots & \ddots & \vdots \\ q_{1t} & \ldots & q_{tt} \end{pmatrix}, \quad 1 \leq t \leq \min(p, n),$$

and

$$\tilde{d}_t(W) = \det \begin{pmatrix} w_{1,1-t+1} & \ldots & w_{1,n} \\ \vdots & \ddots & \vdots \\ w_{t,1-t+1} & \ldots & w_{tn} \end{pmatrix}, \quad 1 \leq t \leq \min(q, n),$$

and likewise for $d_t(q)$ and $\tilde{d}_t(W)$.

As usual, we denote the space of $K$-harmonics by $\mathcal{H}(K)$. For $\tau \in \hat{K}$, let $H(K)_\tau$ be the $\tau$-isotypic component of $\mathcal{H}(K)$. Then each $\mathcal{H}(K)_\tau$ is an irreducible $K' \times K$ module, where $K' = U(p) \times U(q) \times U(p) \times U(q)$ ([H2]). Let $\tau \in \hat{K}$ be given as in the hypothesis. The results of Kashiwara and Vergne (Proposition 6.1 of [KV]) tells us that with appropriate choices of Borel subgroups, a $K' \times K$ joint highest
weight vector \( h \) in \( \mathcal{H}(K)_\tau \) can be written as the product of two simultaneous highest weight vectors for the dual pairs \((E_i, N_i), i = 1, 2\); that is, \( h \left( \frac{Q}{W} \right) = h_1(Q)h_2(W) \).

Moreover the highest weight of \( h_1 \) (resp. \( h_2 \)) is given precisely as in Proposition 3.4 (a) (resp., (b)).

We shall only consider the case \( l > 0 \). The case of \( l < 0 \) is similar. Note that \( \mathcal{D} = \bar{\partial}_{p+q}(Z)\delta \) is mapped to the formal vector \( \left( \frac{1}{\sqrt{2}}(p+q)^{\frac{1}{2}}\partial_{p+q} \left( \frac{Q}{W} \right) \Delta \right) \) in the Fock model (see eq. (3.5)). We claim that

\[
\left( \partial_{p+q} \left( \frac{Q}{W} \right) \Delta, h_1(Q)h_2(W) \right) = \left( \begin{pmatrix} \partial_{p}^{\ell}(Q) & 0 \\ 0 & \partial_{q}^{\ell}(W) \end{pmatrix} \Delta, h_1(Q)h_2(W) \right).
\]

The reason is as follows. We have \( \frac{\partial}{\partial q_{ij}} \Delta = -\frac{1}{2} q_{ij} \Delta \), and \( (\bar{q}_{ij})^* = 2 \frac{\partial}{\partial \bar{q}_{ij}} \). Note that for \( 1 \leq i \leq p, p + 1 \leq j \leq n \), \( q_{ij} \) is an entry in the upper right corner of \( \left( \frac{Q}{W} \right) \), and in this case, we have \( \frac{\partial}{\partial q_{ij}} \Delta \) for some \( \Delta \).

Clearly we have \( \Delta = \Delta_1 \otimes \Delta_2 \), where \( \Delta_i \) is the image (in the Fock model of \( \omega_i \)) of the Dirac distribution at the origin of \( V^n_i \), the direct sum of \( n \)-copies of the standard module for \( H_i, i = 1, 2 \), and \( \otimes \) denotes the tensor product given in equation (3.6).

If \( h_1(Q) \) (resp. \( h_2(W) \)) is the highest weight vector in one of those \( \tilde{K} \)-types given in Proposition 3.4 (a) (resp., (b)), then

\[
(\partial_{p+q} \left( \frac{Q}{W} \right) \Delta, h_1(Q)h_2(W)) = (\partial_{p}^{\ell}(Q)\Delta_1, h_1(Q)) \neq 0,
\]

by our computations in Proposition 3.4.

\[\square\]

**Corollary 3.6.** All the \( \tilde{K} \)-types given in Propositions 2.1 and 2.2 occur in \( \Omega^{p,q}(l) \) with multiplicity one.

4. **Embedding of \( \Omega^{p,q}(l) \) into degenerate principal series**

In this section, we show that after twisting by a character of \( \tilde{U}(n, n) \), the Howe quotients \( \Omega^{p,q}(0) \) for all \( p \) and \( q \) and \( \Omega^{p,q}(l) \) \( (l \neq 0) \) for \( p + q = n \) can be embedded into the degenerate series \( I(s; \nu) \) of \( U(n, n) \) studied in [L] (see the introductory section for a description of \( I(s; \nu) \)). Thus we can use the results in [L] to deduce the structure of these Howe quotients by identifying their images in the degenerate series.

Recall the reductive dual pair \((H, G) = (U(p, q), U(n, n)) \subseteq Sp(4(p + q)n, \mathbb{R}), \) and the action of \( \tilde{Sp}(4(p + q)n, \mathbb{R}) \) on \( L^2(V^n) \) via the oscillator representation \( \omega \).

As before, we let \( \mathcal{D} \) be the following tempered distribution on \( V^n \) given by

\[
\mathcal{D} = \begin{cases} \delta, & \text{if } l = 0, \\
\partial_{p+q}^{l} \delta, & \text{if } l > 0, p + q \leq n, \\
\partial_{p+q}^{-l} \delta, & \text{if } l < 0, p + q \leq n. 
\end{cases}
\]

We have \( \mathcal{D} \in (\Omega^*)^{p,q}(l) \); i.e, \( h \cdot \mathcal{D} = (\det h)^{-l} \mathcal{D} \) for \( h \in U(p, q) \).

Let \( m = p + q \), as before. If \( m \) is even, then \( \tilde{U}(n, n) \) splits over \( U(n, n) \), and if \( m \) is odd, then \( \tilde{U}(n, n) \) is isomorphic to the cover defined by the character \( \det \tilde{\omega} \) of \( \tilde{U}(n, n) \) ([Ad]). The image of the covering group \( \tilde{M} \) in \( \tilde{U}(n, n) \) can then be identified as a set with \( \{ (m_a, \lambda) : a \in GL(n, \mathbb{C}), \lambda = \pm (\frac{\det a}{|\det a|})^m \} \).
Let \( \xi \) be the following character of \( U(n, n) \):

\[
(4.1) \quad \xi = \begin{cases} 
\text{trivial,} & \text{if } m \text{ is even,} \\
\det \frac{a}{b}, & \text{if } m \text{ is odd.}
\end{cases}
\]

We have \( \xi((m, \lambda)) = \lambda \), for \( (m, \lambda) \in \tilde{M} \). Let

\[
\omega_\xi = \xi^{-1} \otimes \omega, \quad \Omega^{p,q}_\xi(l) = \xi^{-1} \otimes \Omega^{p,q}(l).
\]

Then \( \omega_\xi \) factors through \( U(n, n) \), and \( \Omega^{p,q}_\xi(l) \) is a \( U(n, n) \) module.

We shall identify the maximal parabolic subgroup \( P = MN \) with the subgroup of \( U(n, n) \) fixing \( V^n \cong M_{p+q, n}(\mathbb{C}) \). Recall that the modular function of \( P \) is given by

\[
\Delta(p) = \Delta(m_a n_b) = |\det a|^{2n}, \quad p = m_a n_b \in P.
\]

Now we have

\[
[\omega((m_a, \lambda)) f](x) = |\det a|^m f(xa),
\]

\[
[\omega(n_b) f](x) = e^{\frac{1}{2} \text{tr}(f, x^T x)} f(x),
\]

for \( (m_a, \lambda) \in \tilde{M} \), \( n_b \in N \), \( f \in L^2(M_{p+q, n}(\mathbb{C})) \), \( x \in M_{p+q, n}(\mathbb{C}) \). Thus

\[
[\omega_\xi(m_a) f](x) = \begin{cases} 
\chi_{m,0}(m_a)f(xa), & \text{if } m \text{ is even,} \\
\chi_{m,-m}(m_a)f(xa), & \text{if } m \text{ is odd,}
\end{cases}
\]

and therefore

\[
\omega_\xi(y) \cdot \delta = \begin{cases} 
\chi_{m,0}^{-1}(y)\delta, & \text{if } m \text{ is even,} \\
\chi_{m,-m}^{-1}(y)\delta, & \text{if } m \text{ is odd,}
\end{cases}
\]

where \( y = m_a n_b \in P \) and we extend the characters of \( M \) to \( P \) by letting \( N \) act trivially.

For any \( \phi \in \mathcal{S}(V^n) \), we consider the function

\[
\mathcal{D}_\phi(g) = \mathcal{D}(\omega_\xi(g)\phi), \quad g \in G = U(n, n).
\]

For \( l = 0 \), we compute

\[
\mathcal{D}_\phi(n_b m_a g) = \delta(\omega_\xi(n_b m_a g)\phi) = (\omega_\xi(m_a^{-1} n_b^{-1}) \cdot \delta)(\omega_\xi(g)\phi)
\]

\[
= \begin{cases} 
\chi_{m,0}(n_b m_a)\mathcal{D}_\phi(g) = \chi_{m,-m,0}(n_b m_a)\Delta(n_b m_a)^{\frac{1}{2}}\mathcal{D}_\phi(g), & \text{if } m \text{ is even,} \\
\chi_{m,-m}(n_b m_a)\mathcal{D}_\phi(g) = \chi_{m,-m,-m}(n_b m_a)\Delta(n_b m_a)^{\frac{1}{2}}\mathcal{D}_\phi(g), & \text{if } m \text{ is odd,}
\end{cases}
\]

where \( \Delta \) is the modular function of \( P \). Thus the function \( \mathcal{D}_\phi \) is in the space of the induced representation \( I(m - n; 0) \) or \( I(m - n; -m) \) depending on whether \( m \) is even or odd. Consequently we obtain a map

\[
\lambda: \phi \longrightarrow \mathcal{D}_\phi
\]

from \( \mathcal{S}(V^n) \) to certain induced representation \( I(s; \nu) \). We now restrict the map to the space \( \mathcal{S} \subseteq \mathcal{S}(V^n) \). Because of the way Howe’s quotient \( \Omega^{p,q}_\xi \) is defined and the transformation property of \( \mathcal{D} \), this restriction map factors through \( \Omega^{p,q}_\xi \). To summarize, we have

\[
(4.2) \quad \lambda: \Omega^{p,q}_\xi \longrightarrow I(s; \nu) = \begin{cases} 
I(m - n; 0), & \text{if } m \text{ is even,} \\
I(m - n; -m), & \text{if } m \text{ is odd.}
\end{cases}
\]
For \( l \neq 0 \), if we further assume that \( m = n \), then a similar computation shows that for \( n \) even,  
\begin{equation}
\lambda : \Omega_{\xi}^{p,q}(l) \longrightarrow I(s;\nu) = I(|l|; -l),
\end{equation}
and for \( n \) odd,  
\begin{equation}
\lambda : \Omega_{\xi}^{p,q}(l) \longrightarrow I(s;\nu) = I(|l|; -n - l).
\end{equation}

**Theorem 4.1.** Assume either (i) \( l = 0 \), or (ii) \( l \neq 0 \) and \( m = n \). Then we have a \( G \)-equivariant embedding:  
\begin{equation}
\lambda : \Omega_{\xi}^{p,q}(l) \hookrightarrow I(s;\nu).
\end{equation}
Here \( s \) and \( \nu \) are as given in equations (4.2), (4.3) and (4.4).

**Proof.** \( \lambda \) is easily checked to be \( G \)-equivariant. We only need to show injectivity. We fix an element \( \phi \) of \( S \) with nonzero image \( \phi_{\tau_{\xi}} \) in the quotient \( \Omega_{\xi}^{p,q}(l) \). We may assume that some isotypic component of \( \phi_{\tau_{\xi}} \), denoted by \( \phi_{\tau_{\xi}} \), is not zero, where \( \tau_{\xi} = \xi^{-1} \otimes \tau \), and \( \tau \) is one of the \( \tilde{K} \)-types given in Propositions 2.1 and 2.2.

Suppose that \( \lambda(\phi_{\tau_{\xi}}) = 0 \), then since \( \lambda \) is \( G \)-equivariant, we have \( \lambda(\phi_{\tau_{\xi}}) = 0 \). We replace \( \omega \) by \( \omega_{\xi} \) in the definition of \( (\Omega_{\xi}^{*})^{p,q}(l) \) to get \( (\Omega_{\xi}^{*})^{p,q}(l) \). Let \( \phi_{\tau_{\xi}} \in (\Omega_{\xi}^{*})^{p,q}(l)_{\tau_{\xi}} \) be the corresponding \( \tau_{\xi}^{*} \)-isotypic component of \( \phi \), which we know to be non-zero. For any \( k \in K \), we have  
\[
0 = \lambda(\phi_{\tau_{\xi}})(k) = D(\omega_{\xi}(k))\phi_{\tau_{\xi}} = D_{\tau_{\xi}}(\omega_{\xi}(k)\phi_{\tau_{\xi}}) = (D_{\tau_{\xi}}, \omega_{\xi}(k)\phi_{\tau_{\xi}}).
\]
Since \( \Omega_{\xi}^{p,q}(l) \) is multiplicity one, \( \{\omega_{\xi}(k)\phi_{\tau_{\xi}} | k \in K\} \) spans \( \Omega_{\xi}^{p,q}(l)_{\tau_{\xi}} \). Since \( 0 \neq D_{\tau_{\xi}} \in (\Omega_{\xi}^{*})^{p,q}(l)_{\tau_{\xi}} \), and since the pairing between \( \Omega_{\xi}^{p,q}(l) \) and \( (\Omega_{\xi}^{*})^{p,q}(l) \) is non-degenerate (see the discussion before Proposition 3.4), we see that the last expression cannot be identically zero. We have a contradiction. This proves that \( \lambda : \Omega_{\xi}^{p,q}(l) \rightarrow I(s;\nu) \) is injective. \( \square \)

### 5. Identification of Constituents through Local Theta Correspondence

In this section, we shall describe the image of \( \Omega_{\xi}^{p,q}(l) \) under the embedding \( \lambda : \Omega_{\xi}^{p,q}(l) \hookrightarrow I(s;\nu) \) given in Theorem 4.1. Since \( I(s;\nu) \) is multiplicity free as a \( K \)-module, the image of \( \Omega_{\xi}^{p,q}(l) \) in \( I(s;\nu) \) is completely determined by the \( K \)-types in \( \Omega_{\xi}^{p,q}(l) \). Since the detailed structure of \( I(s;\nu) \) is described in [L], we can deduce the reducibility, composition series and unitarity of \( \Omega_{\xi}^{p,q}(l) \). On the other hand, for different values of \( p \) and \( q \), \( \Omega_{\xi}^{p,q}(l) \) is embedded into the same induced representation \( I(s;\nu) \), provided that \( p + q \) is fixed. We shall show that in the case \( l = 0 \) and \( p + q \geq n \), every constituent in \( I(p + q - n; 0) \) with \( p + q \) even and in \( I(p + q - n; -p - q) \) with \( p + q \) odd is isomorphic to a quotient of submodules which are intersections of the images of some of the \( \Omega_{\xi}^{p,q} \), in a very precise way.

We shall now summarize some results in [L]. Recall that the induced representation \( I(s;\nu) \) can be identified with a function space \( S^{\alpha,\beta}(X^{\omega}) \), where \( \alpha = -\frac{p + q - \nu}{2} \) and \( \beta = -\frac{p + q + \nu}{2} \). We shall now describe this space. Let \( U(n,n) \) act on the space \( M_{2n,n}(\mathbb{C}) \) of \( 2n \times n \) complex matrices by  
\[
g.x = (g^{-1})^t x, \quad g \in U(n,n), \ x \in M_{2n,n}(\mathbb{C}),
\]
and let \( X^{oo} \) be the \( U(n,n) \) orbit of \( x_o = (I_n^g \, I_n). \) Here \( g^t \) denotes the transpose of \( g \) and \( I_n \) denotes the \( n \times n \) identity matrix. For \( \alpha, \beta \in \mathbb{C} \) such that \( \alpha - \beta \in \mathbb{Z} \), let

\[
S^{\alpha, \beta}(X^{oo}) = \{ f \in C^\infty(X^{oo}) : f(a) = (\det a)^\alpha (\det a)^\beta f(x), \forall x \in X^{oo}, a \in GL(n, \mathbb{C}) \},
\]

and let \( U(n,n) \) act on it by

\[
(g.f)(x) = f(g^t x), \quad g \in U(n,n), \quad x \in X^{oo}.
\]

\( S^{\alpha, \beta}(X^{oo}) \) admits the following decomposition into a sum of \( K \)-types:

\[
S^{\alpha, \beta}(X^{oo}) = \bigoplus_{\lambda \in \Lambda_n^+} V_\lambda,
\]

where \( V_\lambda \) is isomorphic to \( \rho^\lambda \otimes \rho^{\lambda^\vee + (\alpha-\beta)1_n} \) as a representation of \( K \). For \( 1 \leq j \leq n \), let

\[
e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{Z}^n.
\]

It is proved in [L] (see Corollary 6.1 of [L]) that a \( K \)-type \( V_\lambda \) can be transformed to the \( K \)-type \( V_{\lambda+e_j} \) (respectively \( V_{\lambda-e_j} \) \( (1 \leq j \leq n) \) if and only if the transition coefficient \( \alpha - \lambda_j + j - 1 \) (respectively \( \beta + \lambda_j + n - j \)) is nonzero. Thus \( S^{\alpha, \beta}(X^{oo}) \) is irreducible if and only if both \( \alpha, \beta \in \mathbb{Z} \). In the case when \( S^{\alpha, \beta}(X^{oo}) \) is reducible, i.e., when both \( \alpha, \beta \in \mathbb{Z} \), the hyperplanes

\[
\ell_j^\pm : \lambda_j = \alpha + j - 1, \quad \ell_j : \lambda_j = -(\beta + n - j), \quad (1 \leq j \leq n)
\]

divide \( S^{\alpha, \beta}(X^{oo}) \) into a number of irreducible constituents. Each of these constituents is a subquotient of \( S^{\alpha, \beta}(X^{oo}) \). We can then construct a graph which captures the most important information on the module structure of \( S^{\alpha, \beta}(X^{oo}) \). This graph is called the module diagram of \( S^{\alpha, \beta}(X^{oo}) \). The nodes of this graph are the irreducible constituents of \( S^{\alpha, \beta}(X^{oo}) \), and if there is an edge joining two constituents \( R_1 \) and \( R_2 \) with \( R_1 \) placed at a higher position, then the vectors in \( R_1 \) can be transformed to the vectors in \( R_2 \). The readers are advised to refer either to [Al] or section 7 of [L] for a more precise definition of module diagram. It turns out that the module diagram for \( S^{p,q}(X^{oo}) \) is a (complete or incomplete) triangle, and its pattern depends on the number \( i = \alpha + \beta + n - 1 \). The module diagrams of all the \( S^{p,q}(X^{oo}) \) are also given in [L].

We shall first study \( \Omega_{\xi}^{p,q} = \Omega_{\xi}^{p,q}(0) \). We let \( m \) be a positive integer and consider all pairs of nonnegative integers \( (p,q) \) such that \( p + q = m \). Then by Theorem 4.1, we have \( U(n,n) \) embeddings

\[
\lambda : \Omega_{\xi}^{p,q} \rightarrow \begin{cases} I(m-n;0) \cong S^{-\frac{m}{2},-\frac{m}{2}}(X^{oo}) & \text{if } m \text{ even,} \\
I(m-n;0) \cong S^{-m,0}(X^{oo}) & \text{if } m \text{ odd.}
\end{cases}
\]

Recall that the pattern of the module diagram of \( S^{\alpha, \beta}(X^{oo}) \) depends on the number \( i = \alpha + \beta + n - 1 \) and its sign. Let

\[
(\alpha, \beta) = \begin{cases} \left( -\frac{m}{2}, -\frac{m}{2} \right), & \text{if } m \text{ is even,} \\
(0,0), & \text{if } m \text{ is odd.}
\end{cases}
\]

Then we have \( i = -m + n - 1 \). Hence we shall consider 2 cases separately, i.e., \( 1 \leq m \leq n - 1 \) and \( m \geq n \).
Case 1. $1 \leq m \leq n-1$. In this case, we have $0 \leq i \leq n-2$. We shall first recall a description of the irreducible constituents in $S^{\alpha,\beta}(X^{oo})$ when $0 \leq i = \alpha + \beta + n - 1 \leq n - 2$, as given in [L].

The hyperplanes $\ell_j^\pm$ (1 $\leq j \leq n$) divide $\Lambda_n^+$ into the following disjoint subsets:

\[
\begin{align*}
A_1^1 &= \{ \lambda \in \Lambda_n^+ : \lambda_j < - (\beta + n - j) \}, \\
A_2^1 &= \{ \lambda \in \Lambda_n^+ : - (\beta + n - j) \leq \lambda_j \leq \alpha + j - 1 \}, \\
A_3^1 &= \{ \lambda \in \Lambda_n^+ : \lambda_j > \alpha + j - 1 \}.
\end{align*}
\]

For $0 \leq s, t \leq n$ with $s + t \leq n$, let $R_{a(s,t)}$ be the direct sum of all the $K$-types $V_\lambda$ with $\lambda$ in the set

\[
(5.1) \quad (A_1^1 \cap \cdots \cap A_3^s) \cap (A_2^{s+1} \cap \cdots \cap A_2^{n-t}) \cap (A_1^{n-t+1} \cap \cdots \cap A_1^n).
\]

If $R_{a(s,t)}$ is nonempty, then it forms an irreducible constituent in $S^{\alpha,\beta}(X^{oo})$. We shall frequently abuse notation and identify $R_{a(s,t)}$ with the subset of $\Lambda_n^+$ given in eq. (5.1). We shall do the same for all other irreducible constituents defined later.

The module diagram of $S^{\alpha,\beta}(X^{oo})$ is a “incomplete inverted triangle” with $i + 2$ rows (i.e. the lowest $n - i - 1$ rows in the complete triangle has been removed). For example, the module diagram for $S^{-\frac{2}{p}}(X^{oo})$ in the case when $n = 5$ and $m = 2$ is given in Fig. 1. Here a blacken circle represents a unitary constituent.

For $S^{-\frac{2}{p}}(X^{oo})$ and $S^{-m,0}(X^{oo})$, the constituents at the lowest row of the diagram are

\[
R_{a(m,0)}, \quad R_{a(m-1,1)}, \quad R_{a(m-2,2)}, \quad \ldots, \quad R_{a(0,m)}.
\]

By Theorem 9.18 of [L], each of these constituents is an irreducible unitary sub-module of $S^{-\frac{2}{p}}(X^{oo})$ or $S^{-m,0}(X^{oo})$.

Proposition 5.1. If $p + q = m$ and $m$ is an integer such that $1 \leq m \leq n - 1$, then

\[
\lambda(\Omega^{p,q}_\xi) = R_{a(p,q)} \subseteq \left\{ \begin{array}{ll}
I(m - n; 0), & \text{if } m \text{ is even}, \\
I(m - n; -m), & \text{if } m \text{ is odd}.
\end{array} \right.
\]

Hence $\Omega^{p,q}$ is irreducible and unitary.
Proof. We first assume that $m$ is even. By Proposition 2.1 and Corollary 3.6, the $K$-types in $\Omega^{p,q}_\xi$ are exactly those of the form $\rho^\lambda \otimes \rho^{\lambda^*}$ and

$$\lambda = \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\gamma_q, \ldots, -\gamma_1)$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_p \geq 0$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_q \geq 0$. On the other hand the $K$-types in $R_{a(p,q)} \subseteq S^{-\frac{p+q}{2}}(X^{oo})$ are of the form $\rho^\lambda \otimes \rho^{\lambda^*}$ where $\lambda$ satisfies

(i) $\lambda_j \geq \alpha + j, \ 1 \leq j \leq p$,
(ii) $- (\beta + n - j) \leq \lambda_j \leq \alpha + j - 1, \ p + 1 \leq j \leq n - q$,
(iii) $\lambda_j \leq - (\beta + n - j) - 1, \ n - q + 1 \leq j \leq n$,

where $\alpha = \beta = -\frac{p+q}{2}$. Since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, (i)-(iii) are equivalent to

(i)' $\lambda_j \geq \alpha + p, \ 1 \leq j \leq p$,
(ii)' $- (\beta + q) \leq \lambda_j \leq \alpha + p, \ p + 1 \leq j \leq n - q$,
(iii)' $\lambda_j \leq - (\beta + q), \ n - q + 1 \leq j \leq n$.

Observe since $\alpha = \beta = -\frac{p+q}{2}$, $\alpha + p = - (\beta + q) = -\frac{p+q}{2}$. Thus (i)'-(iii)' are equivalent to

$$\lambda = \frac{p-q}{2} 1_n$$

Next we assume that $m$ is odd. In this case the $K$-types in $\Omega^{p,q}_\xi$ are exactly those of the form $\rho^\lambda \otimes \rho^{\lambda^*} \otimes \rho^\lambda \otimes \rho^{\lambda^*}$ where $\lambda$ satisfies the condition (5.2). If we let $\mu = \lambda - \frac{m}{2} 1_n$, then $\rho^\mu \otimes \rho^{\mu^*} \otimes \rho^\mu \otimes \rho^{\mu^*}$ and

$$\mu = -q + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0 - \gamma_q, \ldots, -\gamma_1)$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_p \geq 0$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_q \geq 0$. It can be checked in a similar way that these are exactly the $K$-types which occur in $R_{a(p,q)} \subseteq S^{-m,0}(X^{oo})$.

We shall denote the unique irreducible quotient of $\Omega^{p,q}_\xi$ by $Q^{p,q}_\xi$. Then by the above proposition, if $1 \leq p + q \leq n - 1$, then $Q^{p,q}_\xi = \Omega^{p,q}_\xi$. In particular $Q^{p,q}_\xi$ is unitary.

Case 2. $m \geq n$. In this case, we have $i \leq -1$ where $i = \alpha + \beta + n - 1$. The hyperplanes $\ell_j^\pm (1 \leq j \leq n)$ divide $\Lambda^+_n$ into the following subsets:

$$B_1^j = \{ \lambda \in \Lambda^+_n : \lambda_j \leq \alpha + j - 1 \}$$
$$B_2^j = \{ \lambda \in \Lambda^+_n : \alpha + j \leq \lambda_j \leq - (\beta + n - j) - 1 \}$$
$$B_3^j = \{ \lambda \in \Lambda^+_n : \lambda_j \geq - (\beta + n - j) \}$$

Again for $0 \leq s, t \leq n$ with $s + t \leq n$, we let $R_{a(s,t)}$ be the direct sum of all the $K$-types $V_\chi$ with $\lambda$ in the set

$$(B_3^s \cap \cdots \cap B_3^t) \cap (B_2^{s+1} \cap \cdots \cap B_2^{n-t}) \cap (B_{n-t+1} \cap \cdots \cap B_n^n),$$

If this set is nonempty, then $R_{a(s,t)}$ forms an irreducible constituent of $S^{\alpha,\beta}(X^{oo})$

We now assume that $m = n$, so we have

$$(\alpha, \beta) = \left\{ \begin{array}{ll} (-\frac{n}{2}, -\frac{n}{2}) & \text{if } n \text{ is even,} \\ (-n, 0) & \text{if } n \text{ is odd.} \end{array} \right.$$ 

Then $S^{\alpha,\beta}(X^{oo})$ is on the unitary axis (see Proposition 9.1 of [L]) and

$$S^{\alpha,\beta}(X^{oo}) = \bigoplus_{j=0}^n R_{a(j,n-j)}$$
is a decomposition of $S^{\alpha,\beta}(X^{oo})$ into a direct sum of $n + 1$ irreducible unitary submodules. Hence the module diagram of $S^{\alpha,\beta}(X^{oo})$ consist of only one row of $n + 1$ nodes with no edge.

If $n$ is even, then by Proposition 2.1 and Corollary 3.6, the $K$-types of $\Omega_\xi^{p,q}$ are exactly those of the form $\rho_\lambda \otimes \rho_\lambda^{\gamma}$ where

$$\lambda = \frac{p - q}{2} n + (\alpha_1, \alpha_2, ..., \alpha_p, -\gamma_q, ..., -\gamma_1)$$

and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_p \geq 0$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_q \geq 0$. Now for $\alpha = \beta = -\frac{n}{2}$, we have $\frac{p - q}{2} = -(\beta + n - p) = \alpha + (p + 1) - 1$. Hence $\rho_\lambda \otimes \rho_\lambda^{\gamma}$ occurs in $\Omega_\xi^{p,q}$ if and only if $\lambda_p \geq -(\beta + n - p)$ and $\lambda_{p+1} \leq \alpha + (p + 1) - 1$; that is $\lambda \in R_{a(p,q)} \subseteq S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$.

Consequently $\lambda(\Omega_\xi^{p,q}) = R_{a(p,q)} \subseteq S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$.

If $n$ is odd, then the $K$-types of $\Omega_\xi^{p,q}$ are exactly those of the form $\rho_\lambda \otimes \rho_\lambda^{\gamma - m,1}$ where

$$\lambda = -q + (\alpha_1, \alpha_2, ..., \alpha_p, -\gamma_q, ..., -\gamma_1).$$

For $\alpha = -n$ and $\beta = 0$, we have $\alpha + (p + 1) - 1 = -(\beta + n - p) = -q$. Thus as before, $\lambda \geq -(\beta + n - p)$ and $\lambda_{p+1} \leq \alpha + (p + 1) - 1$ which shows that $\lambda \in R_{a(p,q)} \subseteq S^{-n,0}(X^{oo})$. Hence we obtain the following proposition.

**Proposition 5.2.** Assume that $m = n$. Let

$$(s, \nu) = \begin{cases} (0,0) & \text{if } n \text{ is even,} \\ (0,-n) & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\lambda(\Omega_\xi^{p,q}) = R_{a(p,q)} \subseteq I(s; \nu).$$

Hence $\Omega_\xi^{p,q}$ is irreducible and unitary. Moreover,

$$I(s; \nu) = \bigoplus_{p+q=n} \lambda \left( \Omega_\xi^{p,q} \right).$$

It again follows that in this case $Q_\xi^{p,q} = \Omega_\xi^{p,q}$ and that $Q_\xi^{p,q}$ is unitary.

We remark that the fact $Q_\xi^{p,q}$ is unitary when $p+q \leq n$ also follows from general results on theta liftings for stable dual pairs ([Li]).

**Corollary 5.3.** Let $1 \leq m \leq n$.

(i) If $m$ is even, then every irreducible unitary submodule of $I(m-n;0)$ is the image of some $\Omega_\xi^{p,q}$ where $p+q = m$.

(ii) If $m$ is odd, then every irreducible unitary submodule of $I(m-n;-m)$ is the image of some $\Omega_\xi^{p,q}$ where $p+q = m$.

Next we assume that $m \geq n+1$. Since in this case, we have $i = -m+n-1 \leq -2$, the module diagram for $S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$ or $S^{-m,0}(X^{oo})$ is a (upright) complete or incomplete triangle. If $m \geq 2n$ (so that $i \leq -(n+1)$), then the module diagram for $S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$ or $S^{-m,0}(X^{oo})$ is a complete triangle with $n+1$ rows (see Fig. 2). If $n+1 \leq m \leq 2n-1$, then $-n \leq i \leq -2$ and the diagram for $S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$ or $S^{-m,0}(X^{oo})$ consists of only the lowest $|i|$ rows of the complete triangle.

If $R_{a(d_1,d_2)}$ is an irreducible constituent in $S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$ (resp. $S^{-m,0}(X^{oo})$), we let $M(d_1,d_2)$ be the submodule of $S^{-\frac{q}{2},-\frac{p}{2}}(X^{oo})$ (resp. $S^{-m,0}(X^{oo})$) generated
by $R_a(d_1, d_2)$. Specifically,

$$M(d_1, d_2) = \bigoplus \{ R_a(s, t) : s \geq d_1, t \geq d_2 \}.$$  

Note that the module diagram for $M(d_1, d_2)$ is a “subtriangle” in the module diagram for $S^{-m,0}(X^oo)$ (resp. $S^{-m,0}(X^oo)$) (see Fig. 3).

**Proposition 5.4.** Let $m$ be integer such that $m \geq n+1$ and let $p + q = m$.

(i) We have

$$\lambda(Ω^p,q_\xi) = M(d_1, d_2),$$

and

$$Q^p,q_\xi \cong R_a(d_1, d_2),$$

where $d_1 = \max\{0, n - q\}$ and $d_2 = \max\{0, n - p\}$.

(ii) If $n + 1 \leq m \leq 2n$, then $Q^p,q_\xi$ is unitary if and only if $p \leq n$ and $q \leq n$ or $pq = 0$.

(iii) If $m \geq 2n + 1$, then $Q^p,q_\xi$ is unitary if and only if $pq = 0$.

**Proof.** We shall only prove the case when $m$ is even. We first consider (i). Assume that $n + 1 \leq m \leq 2n$. In this case we observe that a $K$-type $\rho^\lambda \otimes \rho^{\lambda'}$ occurs in $Ω^p,q_\xi$ if and only if $\lambda$ is of the form

$$\lambda = \begin{cases} \frac{p-q}{2} 1_n + (\alpha_1 - \gamma_n, \ldots, \alpha_p - \gamma_{n-p+1}, -\gamma_{n-p}, \ldots, -\gamma_1), & \text{if } 0 \leq p \leq m - n, \\ \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_{n-q}, \alpha_{n-q+1} - \gamma_{q}, \ldots, \alpha_p - \gamma_{n-p+1}, -\gamma_{n-p}, \ldots, -\gamma_1), & \text{if } m - n + 1 \leq p \leq n - 1, \\ \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_{n-q}, \alpha_{n-q+1} - \gamma_q, \ldots, \alpha_n - \gamma_1), & \text{if } n \leq p \leq m, \end{cases}$$

$$\iff \begin{cases} \lambda_{p+1} \leq \frac{p-q}{2}, & \text{if } 0 \leq p \leq m - n, \\ \lambda_{n-q} \geq \frac{p-q}{2} \text{ and } \lambda_{p+1} \leq \frac{p-q}{2}, & \text{if } m - n + 1 \leq p \leq n - 1, \\ \lambda_{n-q} \geq \frac{p-q}{2}, & \text{if } n \leq p \leq m. \end{cases}$$
Consequently,}

\[
\lambda(\Omega^{p,q}_x) = \begin{cases} 
B^{p+1}_1 = \bigoplus_{t \geq n-p} R_{\alpha(s,t)} & \text{if } 0 \leq p \leq m - n, \\
B^{p+1}_1 \cap B^{n-q}_3 = \bigoplus_{t \geq n-p} R_{\alpha(s,t)} & \text{if } m - n + 1 \leq p \leq n - 1, \\
B^{n-q}_3 = \bigoplus_{s \geq n-q} R_{\alpha(s,t)} & \text{if } n \leq p \leq m.
\end{cases}
\]

\[
= \begin{cases} 
M(0, n-p), & \text{if } 0 \leq p \leq m - n, \\
M(n-q, n-p), & \text{if } m - n + 1 \leq p \leq n - 1, \\
M(n-q, 0), & \text{if } n \leq p \leq m.
\end{cases}
\]

The proof for the case \( m > 2n \) is similar.

Next we prove (ii). Assume that \( m \) is even and \( n + 1 \leq m \leq 2n \). We recall that by Theorem 9.18 of [L], an irreducible constituent \( R_{\alpha(s,t)} \) of \( S^{-\frac{n}{2}} \cdot \tau(X^{\infty}) \)
is unitary if and only if it is on the “highest level” or the “lowest level” of the module diagram; i.e., when \( s + t = 2n - m \) or \( s + t = n \). If \( pq = 0 \), then \( Q^{pq} \) corresponds to either \( R_{a(n,0)} \) or \( R_{a(0,n)} \), hence is unitary. If \( m - n \leq p \leq n \), then \( Q^{pq} \equiv R_{a(n-q,n-p)} \) is unitary since \( (n - q) + (n - p) = 2n - m \). It is easy to see these are all the \( Q^{pq} \) which are unitary.

The proof for (iii) is similar.

We shall identify the images of \( \Omega_{\xi}^{p,q} \) in the module diagram of \( S^{m,0}(X^{oo}) \) or \( S^{-m,0}(X^{oo}) \). The diagrams for the cases \( n + 1 \leq m \leq 2n - 1 \) are given in Fig. 4. In the case \( m \geq 2n \), the module diagram for \( S^{-m,0}(X^{oo}) \) or \( S^{m,0}(X^{oo}) \) is a full triangle and the images of \( \Omega_{\xi}^{p,q} \) are given in Fig. 5.

We shall now summarize the module structure of \( \Omega_{\xi}^{p,q} \) in the following theorem. Recall that if \( M \) is a module for a group or algebra, then the socle of \( M \) is the sum of all irreducible submodules of \( M \), and is written \( \text{Soc}(M) \) (see [GW]). The socle series of \( M \) is the ascending chain

\[
\text{Soc}^0(M) \subseteq \text{Soc}^1(M) \subseteq \text{Soc}^2(M) \subseteq \cdots
\]

of submodules of \( M \) defined inductively by setting \( \text{Soc}^0(M) = 0 \) and \( \text{Soc}^{r+1}(M)/\text{Soc}^r(M) = \text{Soc}(M/\text{Soc}^r(M)) \) for any nonnegative integer \( r \).

**Theorem 5.5.** Let \( m \) be a positive integer and \( p + q = m \).

(a) If \( 1 \leq m \leq n \), then \( \Omega_{\xi}^{p,q} \) is irreducible and unitary. In this case we have \( \Omega_{\xi}^{p,q} = Q_{\xi}^{p,q} \) so that \( Q_{\xi}^{p,q} \) is also unitary.

(b) If \( m \geq n + 1 \), then \( \Omega_{\xi}^{p,q} \) and \( Q_{\xi}^{p,q} \) has the following structure.

(i) If we denote the preimage of a constituent \( R_{a(s,t)} \) in \( \lambda(\Omega_{\xi}^{p,q}) \) also by \( R_{a(s,t)} \), then

\[
\Omega_{\xi}^{p,q} = \sum \{ R_{a(s,t)} : s + t \leq n, s \geq d_1, t \geq d_2 \},
\]

where \( d_1 = \max \{ 0, n - q \} \) and \( d_2 = \max \{ 0, n - p \} \).
The module diagram of $\Omega_{p,q}^{\xi}$ is a (upright) triangle with $(n - d_1 - d_2 + 1)$ rows. Consequently the socle series of $\Omega_{p,q}^{\xi}$ is given by

$$\text{Soc}^j(\Omega_{p,q}^{\xi}) = \sum \{ R_{a(s,t)} : n - j + 1 \leq s + t \leq n, \ s \geq d_1, \ t \geq d_2 \}$$

for $1 \leq j \leq n - d_1 - d_2$ and $\text{Soc}^j(\Omega_{p,q}^{\xi}) = \Omega_{p,q}^{\xi}$ for $j \geq n - d_1 - d_2 + 1$.

Hence $\Omega_{p,q}^{\xi}$ has a socle length $n - d_1 - d_2 + 1$.

(iii) For $0 \leq i \leq n - d_1 - d_2$ and $0 \leq j \leq n - d_1 - d_2 - i$, set

$$M_{i,j} = \text{Soc}^j(\Omega_{p,q}^{\xi}) \oplus \sum_{k=0}^{j} R_{a(n-d_2-i-k,d_2+k)}.$$

Then

$$0 \subseteq M_{0,0} \subseteq M_{0,1} \subseteq \cdots \subseteq M_{0,n-d_1-d_2}$$

$$\subseteq M_{1,0} \subseteq M_{1,1} \subseteq \cdots \subseteq M_{1,n-d_1-d_2-1}$$

$$\subseteq \cdots$$

$$\subseteq M_{n-d_1-d_2,0} = \Omega_{p,q}^{\xi}$$

is a composition series for $\Omega_{p,q}^{\xi}$.

(iv) $Q_{p,q}^{\xi}$ is isomorphic to $R_{a(d_1,d_2)}$. It is unitary if and only if either $n + 1 \leq m \leq 2n$ and $p, q \leq n$, or $pq = 0$. It is finite dimensional if and only if $p \geq n$ and $q \geq n$, and in this case, $Q_{p,q}^{\xi} \cong R_{a(0,0)}$. 
The positions of the constituents in the module diagram for \( \Omega_{p,q}^\xi \) in the case \( m \geq n + 1 \) is given in Fig. 6.

We now consider the modules \( \Omega_{p,q}^\xi(l) \) for \( l \neq 0 \) and \( p + q = n \).

**Proposition 5.6.** Let \( l \) be a nonzero integer and let \( m = n \). Then

\[
\lambda(\Omega_{p,q}^\xi(l)) = R_{a(p,q)} \subseteq \begin{cases} I(|l|,-l) & \text{if } n \text{ is even,} \\ I(|l|,-n-l) & \text{if } n \text{ is odd.} \end{cases}
\]

Hence as a \( U(n,n) \) module, \( \Omega_{p,q}^\xi(l) \) is irreducible and unitary.

**Proof.** We shall only prove the case when \( n \) is even and \( l > 0 \) as the proofs for the other cases are similar. By Theorem 4.1, there is an embedding \( \lambda : \Omega_{p,q}^\xi(l) \rightarrow I(l; -l) \cong S^{\alpha,\beta}(X^{\infty}) \) where \( \alpha = -(\frac{n}{2} + l) \) and \( \beta = -\frac{n}{2} \). Now by Proposition 2.2 and Corollary 3.6, the \( K \)-types in \( \Omega_{p,q}^\xi(l) \) are of the form \( \rho^\lambda \otimes \rho^{\lambda - l} \), where

\[
\lambda = \frac{p-q}{2} 1_n + (\alpha_1, \ldots, \alpha_p, -\gamma_q, \ldots, -\gamma_1),
\]

and \( \alpha_1 \geq \cdots \geq \alpha_p \geq 0 \) and \( \gamma_1 \geq \cdots \geq \gamma_q \geq 1 \). Observe that \( \frac{p-q}{2} = -(\beta + n - p) \) and \( \frac{p-q}{2} - l = \alpha + (p + 1) - 1 \). Hence the condition on \( \lambda \) is equivalent to the conditions

\[
\lambda_p \geq \frac{p-q}{2} = -(\beta + n - p), \quad \lambda_{p+1} \leq \frac{p-q}{2} - l = \alpha + (p + 1) - 1.
\]

Note that these are the conditions which define the constituent \( R_{a(p,q)} \) in the space \( S^{\frac{1}{2}-l,-\frac{1}{2}}(X^{\infty}) \). This shows that \( \lambda(\Omega_{p,q}^\xi(l)) \) and \( R_{a(p,q)} \) have the same collection of \( K \)-types. Consequently \( \lambda(\Omega_{p,q}^\xi(l)) = R_{a(p,q)} \).

**Remark 5.7.** By using results from [Li], one can show in general that \( \Omega_{p,q}^\xi(l) \) is an irreducible and unitary module of \( U(n,n) \) if \( p + q \leq n \) (see also Proposition 3.1 of [KR2]).
We have seen in Theorem 5.5 that the structure of $\Omega_\xi^{p,q}$ is determined by the representation $I(m - n; 0)$ or $I(m - n; -m)$. Howe and Kudla have raised the following question: To what extent is the structure of $\Omega_\xi^{p,q}$ determined by the images of the various $\Omega_\xi^{p,q}$ for $p + q = m$? This question now has a neat answer.

**Proposition 5.8.** Let $m$ be an integer such that $m \geq n + 1$. For $s, t \geq 0$ with $s + t \leq n$, we consider the constituent $R_{\alpha(s,t)}$ in $I(m - n; 0) \cong S^{-\frac{m}{2}}(X_{oo}^m)$ if $m$ is even and in $I(m - n; -m) \cong S^{m,0}(X_{oo}^m)$ if $m$ is odd.

(i) For $s + t = n$, we have

$$R_{\alpha(s,t)} = \lambda(\Omega^{m-n+s,n-s}) \cap \lambda(\Omega^{n-t,m-n+t}).$$

(ii) For $s + t < n$, $R_{\alpha(s,t)}$ is isomorphic to the quotient of

$$\lambda(\Omega^{m-n+s,n-s}) \cap \lambda(\Omega^{n-t,m-n+t})$$

by

$$[\lambda(\Omega^{m-n+s+1,n-s-1}) \cap \lambda(\Omega^{n-t,m-n+t})] + [\lambda(\Omega^{m-n+s,n-s}) \cap \lambda(\Omega^{n-t-1,m-n+t+1})].$$

**Proof.** We shall only prove part (ii) in the case when $m$ is even. Recall for $0 \leq s, t \leq n$ and $s + t \geq 2n - m$, $M(s,t)$ is a submodule of $S^{-\frac{m}{2}}(X_{oo}^m)$ and $R_{\alpha(s,t)}$ is the constituent in $S^{-\frac{m}{2}}(X_{oo}^m)$ which occupies the tip of its module diagram. In fact,

$$R_{\alpha(s,t)} \cong M(s,t)/[M(s + 1,t) + M(s,t + 1)].$$

On the other hand, one observes that

$$M(s,t) = M(s,0) \cap M(0,t).$$

Hence by Proposition 5.4,

$$M(s,t) = \lambda(\Omega^{m-n+s,n-s}) \cap \lambda(\Omega^{n-t,m-n+t}).$$

Consequently, $R_{\alpha(s,t)}$ is isomorphic to the quotient of

$$\lambda(\Omega^{m-n+s,n-s}) \cap \lambda(\Omega^{n-t,m-n+t})$$

by

$$[\lambda(\Omega^{m-n+s+1,n-s-1}) \cap \lambda(\Omega^{n-t,m-n+t})] + [\lambda(\Omega^{m-n+s,n-s}) \cap \lambda(\Omega^{n-t-1,m-n+t+1})].$$

\[ \square \]

6. **Gelfand-Kirillov dimensions of irreducible constituents**

Recall that each irreducible constituent $R_{\alpha(s,t)}$ of $S^{a,b}(X_{oo}^m)$ corresponds to a subquotient of $S^{a,b}(X_{oo}^m) \cong I(s;\nu)$. We shall abuse notation and denote this subquotient also by $R_{\alpha(s,t)}$. The purpose of this section is to determine the Gelfand-Kirillov dimension of the subquotient $R_{\alpha(s,t)}$.

We first recall the definition of Gelfand-Kirillov dimension for a finitely generated $\mathfrak{U}(\mathfrak{g})$ module $W$, where $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ and $\mathfrak{U}(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$ (see [V]). Choose a finite dimensional subspace $W_0$ so that $W = \mathfrak{U}(\mathfrak{g})W_0$. For each positive integer $k$, let $\mathfrak{U}(\mathfrak{g})_k$ be the subspace of $\mathfrak{U}(\mathfrak{g})$ spanned by products of at most $k$ elements in $\mathfrak{g}$. We set $d_{W,W_0}(k) = \dim \mathfrak{U}(\mathfrak{g})_k W_0$. Then there is a polynomial $\phi$ of degree at most $\dim \mathfrak{g}$ such that $d_{W,W_0}(k) = \phi(k)$ for large $k$. Moreover the
where \( P \) and \( \Lambda \) then the lemma follows from the identity transition formulas (Propositions 5.7 and 5.15 of \([L]\)), we see that \( R \) be one of the subquotient \( R_{\omega(s,t)} \) of \( S^{\alpha,\beta}(X^{\infty}) \). Then \( V \) admits the \( K \)-isotypic decomposition

\[
V|_K \simeq \sum_{\lambda \in R} V_\lambda,
\]

where \( R \) is the subset of \( \Lambda_+^n \) given in (5.1) and (5.3). Now for each \( \lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_+^n \), we let \( |\lambda| = |\lambda_1| + ... + |\lambda_n| \). Let \( \lambda_0 \) be such that \( |\lambda_0| \) is minimum among all \( \lambda \in R \). Let

\[
V_0 = \sum_{\lambda \in R, |\lambda| = |\lambda_0|} V_\lambda
\]

and

\[
V_k = \sum_{\lambda \in R, |\lambda| = |\lambda_0| + k} V_\lambda,
\]

for a positive integer \( k \). Now \( \mathfrak{u}(n,n) \) admits a Cartan decomposition \( \mathfrak{u}(n,n) = \mathfrak{t} \oplus \mathfrak{p} \), where \( \mathfrak{t} \) is the Lie algebra of \( K \), and \( \mathfrak{p} = \{ \begin{pmatrix} 0 & a \\ \pi(1)^t & 0 \end{pmatrix} : a \in \mathfrak{gl}_n(\mathbb{C}) \} \). From the transition formulas (Propositions 5.7 and 5.15 of \([L]\)), we see that

\[
\mathfrak{p}(V_0) = V_1, \quad \text{and} \quad \mathfrak{p}(V_j) = V_{j-1} \oplus V_{j+1}, \quad (j > 1).
\]

It follows from this that \( \mathfrak{U}_k(\mathfrak{g})V_0 = \sum_{j \leq k} V_j \). Therefore if \( \dim V_k \) is a polynomial in \( k \) of degree \( d - 1 \), then the Gelfand-Kirillov dimension of \( V \) is equal to \( d \).

We now discuss some general properties of generating functions.

**Lemma 6.1.** Let \( \phi \) be a polynomial of degree \( d \), then

\[
\sum_{k \geq 0} \phi(k)t^k = \frac{F(t)}{(1-t)^{d+1}},
\]

where \( F \) is a polynomial with \( F(1) \neq 0 \).

**Proof.** If we write \( \phi(x) = a_0 + a_1(x+1) + ... + a_d(x+1) \cdots (x+d) \), where \( a_d \neq 0 \), then the lemma follows from the identity

\[
\sum_{k \geq 0} (k+1) \cdots (k+d)t^k = \frac{d!}{(1-t)^{d+1}}.
\]

\( \square \)

Note that in fact, \( F(1) = a_d \cdot d! \). In particular, \( F(1) \) is positive for \( \phi(k) = k^d \).

**Lemma 6.2.** Let \( \phi(\alpha_1, ..., \alpha_p) \) be a polynomial of total degree \( d \). Write \( \phi = \phi_h + \phi' \), where \( \phi_h \) is homogeneous of total degree \( d \) and \( \phi' \) is of total degree less than or equal to \( d - 1 \). Suppose that the coefficients of \( \phi_h \) are non-negative. Then for any given positive integers \( c_1, ..., c_p \), we have

\[
\sum_{\alpha_1, ..., \alpha_p \geq 0} \phi(\alpha_1, ..., \alpha_p)t^{c_1\alpha_1 + ... + c_p\alpha_p} = \frac{P(t)}{(1-t)^{d+p}Q(t)},
\]

where \( P \) and \( Q \) are polynomials with \( P(1) > 0 \), \( Q(1) > 0 \).
Proof. Note that
\begin{equation}
\sum_{\alpha_1, \ldots, \alpha_p \geq 0} \alpha_1 c_1 \cdots \alpha_p c_p t_1^{\alpha_1} \cdots t_p^{\alpha_p} = \prod_{1 \leq l \leq p} \left( \sum_{\alpha_l \geq 0} \alpha_l c_l t_l^{\alpha_l} \right).
\end{equation}
By Lemma 6.1, for each $1 \leq l \leq p$, we have
\[ \sum_{\alpha_l \geq 0} \alpha_l c_l t_l^{\alpha_l} = F_l(t_c) = \frac{F_l(t^c)}{(1 - t^c)(k_l + 1)}, \]
where $F_l$ is a polynomial with $F_l(1) > 0$. Hence if we put $t_1 = \ldots = t_p = t$, then the expression in (6.1) is equal to
\begin{equation}
\sum_{\alpha_1, \ldots, \alpha_p \geq 0} \alpha_1 c_1 \cdots \alpha_p c_p t_1^{\alpha_1} \cdots t_p^{\alpha_p} = \frac{F_1(t^c) \cdots F_p(t^c)}{(1 - t_1)(1 - t_p)(k_1 + 1) \cdots (k_p + 1)},
\end{equation}
where $F_l(1)$ is positive, for $1 \leq l \leq p$. The lemma clearly follows from this.

Lemma 6.3. Under the hypothesis of Lemma 6.2,
\[ \psi(k) = \sum_{\alpha_1, \ldots, \alpha_p \geq 0, c_1 \alpha_1 + \ldots + c_p \alpha_p = k} \phi(\alpha_1, \ldots, \alpha_p) \]
is a polynomial in $k$ of degree $d + p - 1$.

Proof. First note that the sum defining $\psi(k)$ is a finite sum, so $\psi(k)$ is a polynomial in $k$. Let $e$ be its degree. We know by Lemma 6.1 that
\[ \sum_{k \geq 0} \psi(k) t^k = \frac{F(t)}{(1 - t)^{e+1}}, \]
where $F$ is a polynomial with $F(1) \neq 0$. On the other hand, by Lemma 6.2, we have
\[ \sum_{k \geq 0} \psi(k) t^k = \sum_{\alpha_1, \ldots, \alpha_p \geq 0} \phi(\alpha_1, \ldots, \alpha_p) t_1^{\alpha_1} \cdots t_p^{\alpha_p} = \frac{P(t)}{(1 - t)^{d+p} Q(t)}, \]
where $P$ and $Q$ are polynomials with $P(1) > 0, Q(1) > 0$. By comparing the order of the pole at $t = 1$, we obtain $e + 1 = d + p$.

We are now ready to compute the Gelfand-Kirillov dimensions of $R_{\alpha(s,t)}$. 

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Theorem 6.4. The Gelfand-Kirillov dimension of \( R_{a(s,t)} \) is equal to \((s+t)(2n-s-t)\).

Proof. We shall only examine the case when \(\alpha + \beta + n - 1 < 0\). The case when \(\alpha + \beta + n - 1 > 0\) is similar. We know that \( V_\lambda \subseteq R_{a(s,t)} \) if and only if

\[
\lambda_1 \geq \ldots \geq \lambda_s \geq \alpha + s,
\]

\[
\alpha + s \geq \lambda_{s+1} \geq \ldots \geq \lambda_{n-t} \geq -(\beta + t),
\]

\[-(\beta + t) \geq \lambda_{n-t} \geq \ldots \geq \lambda_n.
\]

We make the following change of variables:

\[
\mu_j = \lambda_j - \lambda_{j+1}, \quad 1 \leq j \leq s - 1,
\]

\[
\mu_s = \lambda_s - (\alpha + s),
\]

\[
\mu_j = \lambda_j, \quad s + 1 \leq j \leq n - t,
\]

\[
\mu_j = \lambda_{2n-1-j+t} - \lambda_{2n-1-j+t+1}, \quad n - t + 1 \leq j \leq n - 1,
\]

\[
\mu_n = -(\beta + t) - \lambda_{n-t+1}.
\]

We shall now write \( V(\mu) \) for \( V_\lambda \). Observe that \(\lambda \in R_{a(s,t)}\) and \(|\lambda| = k\) imply that

\[
\mu_1, \ldots, \mu_s, \mu_1, \ldots, \mu_t \geq 0,
\]

\[
\alpha + s \geq \mu_{s+1} \geq \ldots \geq \mu_{n-t} \geq -(\beta + t),
\]

\[
k - k_0 \leq \mu_1 + 2\mu_2 + \ldots + s\mu_s + \mu_{n-t+1} + 2\mu_{n-t+2} + \ldots + t\mu_n \leq k + k_0,
\]

for some fixed integer \( k_0 \). Conversely,

\[
\mu_1, \ldots, \mu_s, \mu_1, \ldots, \mu_t \geq 0,
\]

\[
\alpha + s \geq \mu_{s+1} \geq \ldots \geq \mu_{n-t} \geq -(\beta + t),
\]

\[
\mu_1 + 2\mu_2 + \ldots + s\mu_s + \mu_{n-t+1} + 2\mu_{n-t+2} + \ldots + t\mu_n = k
\]

will imply that \(\lambda \in R_{a(s,t)}\) and \( k - k_1 \leq |\lambda| \leq k + k_1 \) for some fixed integer \( k_1 \).

Therefore when \( k \) is sufficiently large,

\[
\sum_{\lambda \in R_{a(s,t)} \atop |\lambda| = k} \dim V_\lambda
\]

will have the same degree as

\[
\sum_{\mu_1, \ldots, \mu_s, \mu_1, \ldots, \mu_t \geq 0} \dim V(\mu)
\]

as a polynomial in \( k \).

By the Weyl dimension formula, we have

\[
\dim V_\lambda = (\dim \rho)^2 = \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)^2}{(j - i)^2}.
\]

In the new variables, we see that \(\dim V(\mu)\) is a polynomial in \(\mu_1, \ldots, \mu_s, \mu_{n-t+1}, \ldots, \mu_n\) of total degree

\[
2[(n-1) + \ldots + (n-s) + t + \ldots + t + (t+1) + \ldots + 1]
\]

\[
= (s+t)[2n-(s+t)-1].
\]
Moreover the relationship between $\lambda$ and $\mu$ ($\lambda_1 - \lambda_3 = \mu_1 + \mu_2$, etc.) implies that the condition on the coefficients of the homogeneous part of $\dim V(\mu)$ specified in Lemma 6.2 is satisfied. Therefore by Lemma 6.3, we see that

$$\sum_{\alpha + \beta \geq 0, \mu_1 + \ldots + \mu_n = k} \dim V(\mu)$$

is a polynomial in $k$ of degree

$$(s + t)[2n - (s + t) - 1] + (s + t) - 1 = (s + t)[2n - (s + t)] - 1.$$

Thus

$$\sum_{\lambda \in R_{\alpha(s,t)} \atop |\lambda| = k} \dim V_{\lambda}$$

is a polynomial in $k$ of the same degree. It follows from this and the discussion before Lemma 6.1 that the Gelfand-Kirillov dimension of $R_{\alpha(s,t)}$ is equal to $(s + t)[2n - (s + t)]$.

We recall that all the constituents $R_{\alpha(s,t)}$ in $S^{\alpha,\beta}(X^{\infty})$ with $s + t$ fixed are on the same “level” of the module diagram of $S^{\alpha,\beta}(X^{\infty})$. Now by Theorem 6.4, we see that the Gelfand-Kirillov dimension of $R_{\alpha(s,t)}$ is a function of $s + t$. Hence all the constituents occupying the same level in the module diagram have the same Gelfand-Kirillov dimension. Here we consider an example. Assume that $\alpha + \beta + n - 1 \leq -(n + 1)$. Then the module diagram of $S^{\alpha,\beta}(X^{\infty})$ is a (complete) upright triangle with $n + 1$ rows (see Fig. 5 or Fig. 11 of [L]). The top row consists of only the constituent $R_{\alpha(0,0)}$ which has Gelfand-Kirillov dimension 0; that is, $R_{\alpha(0,0)}$ is finite dimensional. The next row consists of the constituents $R_{\alpha(1,0)}$ and $R_{\alpha(0,1)}$. Each of them has Gelfand-Kirillov dimension $2n - 1$. The Gelfand-Kirillov dimensions of the constituents increase when we move from one level of the module diagram to the next lower level. The lowest level consists of the constituents $R_{\alpha(n,0)}$, $R_{\alpha(n-1,1)}$, \ldots, $R_{\alpha(0,n)}$. They have the largest possible Gelfand-Kirillov dimension among the $R_{\alpha(s,t)}$’s, which is $n^2$.

**References**


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