

THE DIRICHLET PROBLEM FOR MONGE-AMPÈRE  
EQUATIONS IN NON-CONVEX DOMAINS AND SPACELIKE  
HYPERSURFACES OF CONSTANT GAUSS CURVATURE

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ABSTRACT. In this paper we extend the well known results on the existence and regularity of solutions of the Dirichlet problem for Monge-Ampère equations in a strictly convex domain to an arbitrary smooth bounded domain in  $\mathbb{R}^n$  as well as in a general Riemannian manifold. We prove for the nondegenerate case that a sufficient (and necessary) condition for the classical solvability is the existence of a subsolution. For the totally degenerate case we show that the solution is in  $C^{1,1}(\bar{\Omega})$  if the given boundary data extends to a locally strictly convex  $C^2$  function on  $\bar{\Omega}$ . As an application we prove some existence results for spacelike hypersurfaces of constant Gauss-Kronecker curvature in Minkowski space spanning a prescribed boundary.

1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with  $C^\infty$  boundary  $\partial\Omega$ . In this paper we consider the Dirichlet problem for Monge-Ampère equations

$$(1.1) \quad \det(u_{ij}) = \psi(x, u, Du) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where  $\varphi \in C^\infty(\partial\Omega)$ ,  $\psi \in C^\infty(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ ,  $\psi \geq 0$ ,  $Du = (u_1, \dots, u_n)$  denotes the gradient of  $u$ ,  $u_i = \partial u / \partial x_i$  and  $u_{ij} = \partial^2 u / \partial x_i \partial x_j$ .

When  $\Omega$  is a strictly convex domain, this problem has received considerable study both in the non-degenerate case ( $\psi > 0$ ) and in the degenerate case ( $\psi = 0$  somewhere). A well known theorem (see Caffarelli, Nirenberg and Spruck [6], Ivochkina [17] and Krylov [19]) states that in the non-degenerate case  $\psi > 0$ , (1.1) has a strictly convex solution in  $C^\infty(\bar{\Omega})$ , provided  $\Omega$  is strictly convex and there exists a strictly convex subsolution in  $C^2(\bar{\Omega})$ . (Please see, for example, [6], [12] and [22] for further references, including the earlier work of, among others, Pogorelov, Cheng and Yau, and P. L. Lions.) For the degenerate case ( $\psi \geq 0$ ), counterexamples have been found showing that the Dirichlet problem, in general, does not have a solution in  $C^2(\bar{\Omega})$ ; whether or not the weak solutions belong to  $C^{1,1}(\bar{\Omega})$  has attracted a lot of attention. In the totally degenerate case  $\psi \equiv 0$ , the  $C^{1,1}$  regularity was established by Caffarelli, Nirenberg and Spruck [7], who proved that if  $\Omega$  is a strictly convex domain with  $\partial\Omega \in C^{3,1}$  and  $\varphi \in C^{3,1}(\partial\Omega)$ , then the unique convex solution to the degenerate problem

$$(1.2) \quad \det(u_{ij}) = 0 \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega$$

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belongs to  $C^{1,1}(\overline{\Omega})$ . Earlier Trudinger and Urbas [26] obtained local  $C^{1,1}$  regularity under the weaker hypotheses that  $\partial\Omega \in C^{1,1}$  and  $\varphi \in C^{1,1}(\partial\Omega)$ . The recent work of Krylov [20], [21] provides a unified treatment of the non-degenerate and totally degenerate cases. The main purpose of the present paper is to extend the above mentioned results to non-convex domains.

The Monge-Ampère equations are closely related to problems involving Gauss-Kronecker curvature in differential geometry, such as the Minkowski and Weyl problems. From the viewpoint of geometric applications, it is of interest to study the Dirichlet problem for Monge-Ampère equations in non-convex domains. In his book [2], T. Aubin also raised the question of whether one can remove the hypothesis of convexity of the domain for a problem. Recently, an effort to extend the results of [6] to non-convex domains was made by J. Spruck et al. in [16] and [15]. It was proved in [15] that for  $\psi > 0$  the Dirichlet problem (1.1) in an arbitrary smooth domain  $\Omega$  admits a locally strictly convex solution in  $C^\infty(\overline{\Omega})$  provided that  $(\psi(x, z, p))^{1/n}$  is a convex function with respect to  $p$  and that there exists a locally strictly convex *strict* subsolution  $\underline{u} \in C^2(\overline{\Omega})$  (i.e., assuming  $\underline{u}$  satisfies the strict inequality in (1.4) below). This result applies to, for example, the Gauss curvature equation

$$(1.3) \quad \det(u_{ij}) = K(x, u)(1 + |Du|^2)^{\frac{n+2}{2}}$$

for hypersurfaces in Euclidean space and has interesting geometric consequences (see, for example, [15], [23]). In this paper we will prove

**Theorem 1.1.** *Let  $\varphi \in C^\infty(\partial\Omega)$ ,  $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ ,  $\psi > 0$ . Assume there exists a locally strictly convex subsolution  $\underline{u} \in C^2(\overline{\Omega})$  satisfying*

$$(1.4) \quad \det(\underline{u}_{ij}) \geq \psi(x, \underline{u}, D\underline{u}) \quad \text{in } \Omega, \quad \underline{u} = \varphi \quad \text{on } \partial\Omega.$$

*Then there exists a locally strictly convex solution  $u \in C^\infty(\overline{\Omega})$  of (1.1) with  $u \geq \underline{u}$ . The solution is unique if  $\psi_u \geq 0$ .*

Here a function  $v \in C^2(\overline{\Omega})$  is said to be *locally strictly convex* if its Hessian matrix  $\{v_{ij}\}$  is positive definite everywhere in  $\overline{\Omega}$ . Obviously, condition (1.4) in Theorem 1.1 cannot be removed even when  $\Omega$  is strictly convex. In case  $\psi \equiv \psi(x)$  or, more generally (due to P. L. Lions; see [6]), when  $\psi$  satisfies

$$0 < \psi(x, z, p) \leq C(1 + |p|^2)^{n/2} \quad \text{for } x \in \overline{\Omega}, \quad z \leq \max \varphi, \quad p \in \mathbb{R}^n,$$

one can construct a strictly convex subsolution if  $\Omega$  is strictly convex; this fails for non-convex  $\Omega$ .

There has been some recent work on the Dirichlet problem for Monge-Ampère equations on general smooth Riemannian manifolds. In [14], Y. Y. Li and the author established an analogue of the result of Guan-Spruck [15] cited above. Some of the result was also obtained independently by A. Atallah and C. Zuily [1]. Building on [14] we will, in Section 5, extend Theorem 1.1 to general Riemannian manifolds.

The totally degenerate problem (1.2) is important to understanding hypersurfaces of vanishing Gauss-Kronecker curvature. It is of interest to study the regularity of its solution in non-convex domains. Applying Theorem 1.1 to the problem

$$(1.5) \quad \det(u_{ij}) = \varepsilon \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega$$

for sufficiently small  $\varepsilon > 0$ , by approximation we find that (1.2) has a unique locally convex weak solution (in the sense of Alexandrov, which is the same as weak solution

in the viscosity sense; see [4]) in  $C^{0,1}(\bar{\Omega})$ , provided  $\varphi$  extends to a locally strictly convex function in  $C^2(\bar{\Omega})$ . We will prove that this solution actually belongs to  $C^{1,1}(\bar{\Omega})$ . More precisely, we have the following extension of the regularity theorem in [7].

**Theorem 1.2.** *Assume  $\partial\Omega$  is in  $C^{3,1}$  and  $\varphi \in C^{3,1}(\partial\Omega)$ . Suppose there exists a locally strictly convex function  $\underline{u} \in C^2(\bar{\Omega})$  with  $\underline{u} = \varphi$  on  $\partial\Omega$ . Then there is a unique locally convex weak solution of (1.2) in  $C^{1,1}(\bar{\Omega})$ .*

As we have observed, a major motivation for our studying the Dirichlet problem (1.1) in non-convex domains comes from its close connection with geometric problems. In this paper we consider one such problem, which concerns existence of spacelike hypersurfaces of constant Gauss-Kronecker curvature with specified boundary in Minkowski space  $\mathbb{R}^{n,1}$ . (Please see [15], [13] for related results for hypersurfaces in Euclidean space, and [23] in hyperbolic space.) We are interested in the following question: given a disjoint collection  $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$  of codimension-two closed smooth submanifolds of  $\mathbb{R}^{n,1}$ , decide whether there exists a spacelike hypersurface  $M$  of constant Gauss-Kronecker curvature with boundary  $\partial M = \Gamma$ . Locally  $M$  is given as the graph of a function  $x_{n+1} = u(x)$ ,  $x \in \mathbb{R}^n$ , satisfying the spacelike condition  $|Du| < 1$  and the Monge-Ampère equation (1.1) with

$$(1.6) \quad \psi(x, u, Du) = \mathcal{K}(1 - |Du|^2)^{\frac{n+2}{2}},$$

where  $\mathcal{K}$  is the Gauss-Kronecker curvature of  $M$ . We note that the right hand side of (1.6), in contrast with that of (1.3), is not a convex function with respect to the gradient  $Du$ .

**Theorem 1.3.** *Suppose  $\Gamma$  bounds a compact  $C^2$  spacelike locally strictly convex hypersurface  $\tilde{M}$ . Then for any constant  $0 \leq K \leq \min_{q \in \tilde{M}} \mathcal{K}[\tilde{M}](q)$ , where  $\mathcal{K}[\tilde{M}]$  denotes the Gauss-Kronecker curvature of  $\tilde{M}$ , there exists a compact spacelike hypersurface  $M_K$  of constant Gauss-Kronecker curvature  $K$  with boundary  $\partial M_K = \Gamma$ . Moreover,  $M_K$  is  $C^\infty$  for  $K > 0$  and  $M_0$  is  $C^{1,1}$ .*

The corresponding problem for spacelike hypersurfaces with prescribed boundary value and mean curvature was treated by R. Bartnik and L. Simon [3]; see also [10] and references therein for related results.

This paper is organized as follows. In Section 2 we first derive *a priori* estimates for the  $C^2$  norms of the desired solutions of (1.1) for the non-degenerate case. Then Theorem 1.1 is proved using the continuity method and degree theory based on these *a priori* estimates. Section 3 contains the proof of Theorem 1.2. Theorem 1.3 is proved in Section 4 as a consequence of more general theorems presented there. We note that since in general the hypersurface  $M_K$  is not globally a graph over a domain in  $\mathbb{R}^n$ , Theorem 1.3 does not follow directly from Theorems 1.1 and 1.2. To overcome this difficulty, we will reformulate the problem in a more general setting and appeal for its proof to Theorem 5.1, the extension of Theorem 1.1 to general Riemannian manifolds, which is proved in Section 5.

## 2. THE NON-DEGENERATE MONGE-AMPÈRE EQUATIONS

In this section we prove Theorem 1.1 using the method of continuity and degree theory. As usual, the proof is based on the establishment of global  $C^{2,\alpha}$  *a priori* estimates for prospective solutions. A somewhat surprising fact to us is that in

deriving these estimates one only needs the assumption that  $\underline{u}$  is a locally strictly convex  $C^2$  function; it does not necessarily satisfy (1.4). As we shall see, condition (1.4) is needed in the proof of Theorem 1.1 only to guarantee that, when  $\psi_u \geq 0$ , the unique solution  $u$  of (1.1) satisfies  $u \geq \underline{u}$  in  $\Omega$  by the maximum principle.

**2.1. A priori estimates.** In this subsection we only assume  $\underline{u} \in C^2(\overline{\Omega})$  is a locally strictly convex function; thus there exists a constant  $\varepsilon > 0$  (without loss of generality, we may assume  $\varepsilon \leq 1$ ) such that

$$(2.1) \quad \{\underline{u}_{ij}\} \geq \varepsilon \{\delta_{ij}\} \quad \text{on } \overline{\Omega}.$$

Set

$$\mathcal{A} = \{w \in C^\infty(\overline{\Omega}) : \{w_{ij}\} > 0, w \geq \underline{u}, w|_{\partial\Omega} = \varphi\}.$$

As in [15] it is easy to see that

$$(2.2) \quad |w| + |Dw| \leq K_1, \quad \text{for any } w \in \mathcal{A},$$

where the constant  $K_1$  depends only on  $\Omega$ ,  $n$ , and  $\|\underline{u}\|_{C^1(\overline{\Omega})}$ . From (2.2) we have

$$(2.3) \quad 0 < \psi_0 \equiv \inf_{x \in \overline{\Omega}, w \in \mathcal{A}} \psi(x, w(x), Dw(x)) \leq \sup_{x \in \overline{\Omega}, w \in \mathcal{A}} \psi(x, w(x), Dw(x)) \equiv \psi_1 < \infty.$$

**Theorem 2.1.** *Let  $u \in \mathcal{A}$  be a solution of (1.1). Then*

$$(2.4) \quad |D^2u| \leq C \quad \text{on } \overline{\Omega}.$$

Here the constant  $C$  depends on  $\Omega$ ,  $n$ ,  $\varepsilon$ ,  $K_1$ ,  $\psi_0$ ,  $\psi_1$ ,  $\|\varphi\|_{C^{3,1}(\overline{\Omega})}$ ,  $\|\psi\|_{C^2(\overline{\Omega})}$  and  $\|\underline{u}\|_{C^2(\overline{\Omega})}$ .

*Proof.* It is shown in [6] how to derive (2.4) from  $C^2$  estimates on the boundary. Thus we need only estimate  $D^2u$  on  $\partial\Omega$ . Consider any point 0 on  $\partial\Omega$ ; we may assume it is the origin of  $\mathbb{R}^n$  and choose the coordinates so that the positive  $x_n$  axis is the interior normal to  $\partial\Omega$  at 0. Near the origin,  $\partial\Omega$  can be represented as a graph

$$(2.5) \quad x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}).$$

Since  $u - \underline{u} = 0$  on  $\partial\Omega$ ,

$$(2.6) \quad (u - \underline{u})_{\alpha\beta}(0) = -(u - \underline{u})_n(0) B_{\alpha\beta}, \quad \alpha, \beta < n.$$

It follows that

$$(2.7) \quad |u_{\alpha\beta}(0)| \leq C, \quad \alpha, \beta < n.$$

Next we estimate the mixed normal-tangential derivative  $u_{\alpha n}(0)$ . Rewrite equation (1.1) in the form

$$(2.8) \quad \log \det(u_{ij}) = \log \psi(x, u, Du) \equiv f(x, u, Du)$$

and let  $\mathcal{L}$  denote the linear operator defined by

$$(2.9) \quad \mathcal{L}w = u^{ij} w_{ij} - f_{p_i}(x, u, Du) w_i \quad \text{for } w \in C^2(\overline{\Omega}),$$

where  $\{u^{ij}\}$  is the inverse matrix of  $\{u_{ij}\}$  and  $f_{p_i} = f_{p_i}(x, u, p)$ . For fixed  $\alpha < n$  consider the operator

$$T = \partial_\alpha + \sum_{\beta < n} B_{\alpha\beta} (x_\beta \partial_n - x_n \partial_\beta).$$

As in [6] we have

$$(2.10) \quad |\mathcal{L}T(u - \underline{u})| \leq C(1 + \sum u^{ii}).$$

Since  $|Du| \leq K_1$ ,  $|T(u - \underline{u})| \leq C$  in  $\bar{\Omega}$ . Moreover, on  $\partial\Omega$  near the origin

$$(2.11) \quad |T(u - \underline{u})| \leq C|x|^2.$$

We will employ a barrier function of the form

$$(2.12) \quad v = (u - \underline{u}) + t(h - \underline{u}) - Nd^2,$$

where  $h$  is the harmonic function in  $\Omega$  with  $h|_{\partial\Omega} = \varphi$ ,  $d$  is the distance function from  $\partial\Omega$ , and  $t, N$  are positive constants to be determined. We may take  $\delta > 0$  small enough so that  $d$  is smooth in  $\Omega_\delta = \Omega \cap B_\delta(0)$ . The key ingredient is the following:

**Lemma 2.2.** *For  $N$  sufficiently large and  $t, \delta$  sufficiently small,*

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \text{ in } \Omega_\delta, \quad v \geq 0 \text{ on } \partial\Omega_\delta.$$

*Proof.* By (2.1) we have  $u^{ij}(u_{ij} - \underline{u}_{ij}) \leq n - \varepsilon \sum u^{ii}$ . It follows that

$$(2.13) \quad \mathcal{L}(u - \underline{u}) \leq C_0 - \varepsilon \sum u^{ii}.$$

Next, since  $\Delta \underline{u} \geq n\varepsilon > 0$ ,

$$(h - \underline{u})(x) \geq c_0d(x), \quad \text{for } x \in \Omega$$

for some uniform constant  $c_0 > 0$ . Moreover, we have

$$\mathcal{L}(h - \underline{u}) \leq C_1(1 + \sum u^{ii}),$$

for some constant  $C_1 > 0$  under control. Thus

$$\mathcal{L}v \leq C_0 + tC_1 + (tC_1 - \varepsilon) \sum u^{ii} - 2N(d\mathcal{L}d + u^{ij}d_id_j) \text{ in } \Omega_\delta.$$

It is easy to see that

$$\mathcal{L}d \geq -C_2(1 + \sum u^{ii}).$$

Furthermore, since  $\{u^{ij}\}$  is positive definite and  $d_n(0) = 1$ ,  $d_\beta(0) = 0$  for all  $\beta < n$ , we have, for  $\delta$  sufficiently small,

$$(2.14) \quad u^{ij}d_id_j \geq u^{nn}d_n^2 + 2 \sum_{\beta < n} u^{n\beta}d_nd_\beta \geq \frac{u^{nn}}{2} - C_3\delta \sum u^{ii} \text{ in } \Omega_\delta.$$

Let  $\lambda_1 \leq \dots \leq \lambda_n$  be the eigenvalues of  $\{u_{ij}\}$ . We have  $\sum u^{ii} = \sum \lambda_i^{-1}$  and  $u^{nn} \geq \lambda_n^{-1}$ . By the inequality for arithmetic and geometric means,

$$\frac{\varepsilon}{4} \sum u^{ii} + Nu^{nn} \geq \frac{n\varepsilon}{4} (N\lambda_1^{-1} \dots \lambda_n^{-1})^{\frac{1}{n}} \geq \frac{n\varepsilon}{4(\psi_1)^{1/n}} N^{\frac{1}{n}} \equiv c_1N^{\frac{1}{n}}.$$

Now we fix  $t > 0$  sufficiently small so that  $tC_1 \leq \frac{\varepsilon}{4}$  and fix  $N$  so that  $c_1N^{1/n} \geq C_0 + \varepsilon$ . We obtain

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \text{ in } \Omega_\delta$$

if we require  $\delta$  to satisfy  $2(C_2 + C_3)N\delta \leq \frac{\varepsilon}{4}$  in  $\Omega_\delta$ .

It remains to examine the value of  $v$  on  $\partial\Omega_\delta$ . On  $\partial\Omega \cap B_\delta(0)$  we have  $v = 0$ . On  $\Omega \cap \partial B_\delta(0)$ ,

$$v \geq tc_0d - Nd^2 \geq (tc_0 - N\delta)d \geq 0,$$

if we require, in addition,  $N\delta \leq tc_0$ . Now we can fix  $\delta$  sufficiently small to complete the proof of Lemma 2.2.  $\square$

Using Lemma 2.2 we can choose  $A \gg B \gg 1$  so that

$$\mathcal{L}(Av + B|x|^2 \pm T(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta$$

and

$$Av + B|x|^2 \pm T(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta$$

by (2.10) and (2.11). It thus follows from the maximum principle that

$$(2.15) \quad |u_{\alpha n}(0)| \leq C.$$

Finally, the tangential strict convexity of  $u$  has been established in [15]; i.e.

$$(2.16) \quad \sum_{\alpha, \beta < n} w_{\alpha\beta}(0) \xi_\alpha \xi_\beta \geq c_0 > 0$$

for any unit vector  $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ . Thus it follows as in [6] that

$$(2.17) \quad |u_{nn}(0)| \leq C.$$

The proof of Theorem 2.1 is complete.  $\square$

From the Evans-Krylov theory (see [11], [24], [18] and [5]) we thus have an *a priori* bound for the  $C^{2,\alpha}$  norm of  $u$ ,

$$(2.18) \quad \|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \quad 0 < \alpha < 1,$$

with constant  $C$  depending only on  $\Omega$ ,  $n$ ,  $\varepsilon$ ,  $\psi_0$ ,  $\psi_1$ ,  $\|\varphi\|_{C^{3,1}(\overline{\Omega})}$ ,  $\|\psi\|_{C^3(\overline{\Omega})}$  and  $\|\underline{u}\|_{C^2(\overline{\Omega})}$ . By the standard Schauder theory, we thus obtain the *a priori* bound for each integer  $k \geq 3$ :

$$(2.19) \quad \|u\|_{C^{k,\alpha}(\overline{\Omega})} \leq C.$$

With the aid of such estimates we can apply the continuity method and degree theory to prove Theorem 1.1 as in [6] with some modifications.

**2.2. Proof of Theorem 1.1.** We first assume that the subsolution  $\underline{u}$  is in  $C^\infty(\overline{\Omega})$  and prove the existence of a solution to (1.1) in  $\mathcal{A}$  in two steps as follows.

(a) The special case:  $\psi_u \geq 0$ . For each fixed  $t \in [0, 1]$  consider the Dirichlet problem

$$(2.20) \quad \det(u_{ij}) = t\psi(x, u, Du) + (1-t)\det(\underline{u}_{ij}) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega.$$

Note that since  $\underline{u}$  is a subsolution of (2.20), it follows from the maximum principle that any locally strictly convex solution  $u \in C^\infty(\overline{\Omega})$  of (2.20) satisfies  $u \geq \underline{u}$  and hence (2.19), independent of  $t$ . Thus we can utilize the continuity method to show that for each  $t \in [0, 1]$  there exists a locally strictly convex solution of (2.20) in  $C^\infty(\overline{\Omega})$ . The uniqueness follows from the maximum principle.

(b) Turning to the general case, we assume  $\underline{u}$  is not a solution of (1.1) and let  $u^0 \in \mathcal{A}$  be the unique solution of

$$(2.21) \quad \det(u_{ij}) = \psi_0 \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where  $\psi_0$  is as in (2.3). The existence of  $u^0$  follows from part (a).

For  $r > 0$  we consider the open convex set of functions in  $C^5(\overline{\Omega})$

$$\mathcal{C}_r = \left\{ v \in C^5(\overline{\Omega}) : \|v\|_{C^5(\overline{\Omega})} < r, v > 0 \text{ in } \Omega, v|_{\partial\Omega} = 0 \text{ and } v_\nu > 0 \text{ on } \partial\Omega \right\}.$$

where  $\nu$  is the unit interior normal to  $\partial\Omega$ . We want to prove that for each  $t \in [0, 1]$  there exists a solution in  $\mathcal{A}$  of the form

$$(2.22) \quad u = \underline{u} + v, \quad v \in \mathcal{C}_r \text{ with } r \text{ sufficiently large,}$$

to the Dirichlet problem

$$(2.23) \quad \det(u_{ij}) = t\psi(x, u, Du) + (1 - t)\psi_0 \equiv \psi^t(x, u, Du) \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

As we observed, any solution  $u \in \mathcal{A}$  of (2.23) satisfies the *a priori* bound

$$(2.24) \quad \|u\|_{C^5(\overline{\Omega})} \leq C \text{ independent of } t.$$

Moreover, by the maximum principle and the Hopf lemma we have

$$(2.25) \quad u > \underline{u} \text{ in } \Omega \text{ and } (u - \underline{u})_\nu > 0 \text{ on } \partial\Omega.$$

Thus we can choose  $r$  sufficiently large so that (2.23) has no solution in  $\mathcal{A}$  of the form (2.22) with  $v \in \partial\mathcal{C}_r$ , the boundary of  $\mathcal{C}_r$ .

Now for  $0 \leq t \leq 1$  and fixed  $v \in \overline{\mathcal{C}_r}$  consider the Dirichlet problem

$$(2.26) \quad \det(u_{ij}) = \psi^t(x, u, Du)e^{t\Lambda(u - \underline{u} - v)} \equiv \eta^t(x, u, Du) \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

where

$$\Lambda = \frac{1}{\psi_0} \sup_{x \in \overline{\Omega}} \sup_{u \in \mathcal{A}} \psi_u(x, u, Du) < \infty, \quad \psi_0 \text{ as in (2.3).}$$

We observe that  $\underline{u}$  is a subsolution of (2.26) and  $\eta^t_u \geq 0$ . Thus by part (a) there exists a unique solution  $u^t \in \mathcal{A}$  for each  $t \in [0, 1]$ . For  $t = 0$ , this solution is our  $u^0$ .

From elliptic theory, the map  $T^t v = u^t - \underline{u}$  is compact in  $C^5$ . On the other hand, we have seen that there are no solutions of

$$(2.27) \quad v - T^t v = 0$$

on the boundary of  $\mathcal{C}_r$ . Thus the degree

$$(2.28) \quad \deg(I - T^t, \mathcal{C}_r, 0) = \gamma$$

is well defined and independent of  $t$ . For  $t = 0$ , (2.27) has a unique solution  $v^0 = u^0 - \underline{u}$ . By the maximum principle, when  $t = 0$  the linearized operator of (2.23), linearized at  $u^0$ , is invertible. Thus  $v^0$  is a regular point of  $I - T^0$ . Consequently  $\gamma = \pm 1$ , and (2.27) has a solution  $v^t \in \mathcal{C}_r$  for all  $0 \leq t \leq 1$ . The function  $u^1 = \underline{u} + v^1$  is then a solution of (1.1). The elliptic regularity theory implies that  $u^1 \in C^\infty(\overline{\Omega})$ .

To finish the proof, we have to consider the case  $\underline{u} \in C^2(\overline{\Omega})$ . We may take a sequence of locally strictly convex functions  $\underline{u}^m \in C^\infty(\overline{\Omega})$  converging to  $\underline{u}$  in  $C^2(\overline{\Omega})$ , such that

$$\det(\underline{u}^m_{ij}) \geq \frac{m}{m+1} \psi(x, \underline{u}^m, D\underline{u}^m) \text{ in } \overline{\Omega}.$$

For each integer  $m \geq 1$ , set

$$\psi^m = \frac{m}{m+1} \psi, \quad \varphi^m = \underline{u}^m|_{\partial\Omega}$$

and consider the Dirichlet problem

$$(2.29) \quad \det(u_{i\bar{j}}) = \psi^m \quad \text{in } \bar{\Omega}, \quad u = \varphi^m \quad \text{on } \partial\Omega.$$

Note that  $\underline{u}^m$  is a subsolution of (2.29). Consequently, there exists a locally strictly convex solution  $u^m \in C^\infty(\bar{\Omega})$  of (2.29) satisfying the *a priori* estimates

$$\|u^m\|_{C^k(\bar{\Omega})} \leq C(m, k) \quad \text{for every } k \geq 1,$$

where the constant  $C(m, k)$  only depends on, besides other known data,  $\underline{u}^m$  and its derivatives up to second order. It follows that a subsequence of  $u^m$  converges to a solution of (1.1) in  $C^\infty(\bar{\Omega})$ . This completes the proof of Theorem 1.1.

### 3. THE TOTALLY DEGENERATE MONGE-AMPÈRE EQUATION

The main purpose of this section is to prove Theorem 1.2. We will first obtain a weak solution in  $C^{0,1}(\bar{\Omega})$  of the totally degenerate Monge-Ampère equation by approximation. The major part of the section is devoted to the proof of the  $C^{1,1}$  regularity of this weak solution.

For each small constant  $\lambda > 0$ , we may first take smooth approximations of  $\Omega$  and  $\underline{u}$  and apply Theorem 1.1, then pass to the limit to obtain, under the hypothesis of Theorem 1.2, a locally strictly convex function  $u^\lambda \in C^2(\bar{\Omega})$  satisfying

$$(3.1) \quad \det(u^\lambda_{i\bar{j}}) = \lambda \quad \text{in } \Omega, \quad u^\lambda = \varphi \quad \text{on } \partial\Omega$$

and the  $C^1$  estimate

$$(3.2) \quad \|u^\lambda\|_{C^1(\bar{\Omega})} \leq C_0$$

for some constant  $C_0 > 0$  independent of  $\lambda$ . Therefore, there exists a sequence  $u^{\lambda_k}$  that converges to a locally convex function  $u \in C^{0,1}(\bar{\Omega})$ , and

$$(3.3) \quad \|u\|_{C^{0,1}(\bar{\Omega})} \leq C_0.$$

By the maximum principle,  $u$  is the unique locally convex solution of (1.2). In the rest of this section we will modify the proof of [7] to show that  $u$  is in  $C^{1,1}(\bar{\Omega})$ .

We say a subset  $U$  of  $\bar{\Omega}$  is relatively convex in  $\bar{\Omega}$  if any segment contained in  $\bar{\Omega}$  with endpoints in  $U$  lies completely in  $U$ . The relative convex hull, denoted by  $\Gamma_{\bar{\Omega}}(U)$ , of a set  $U \subset \bar{\Omega}$  is the smallest relatively convex set in  $\bar{\Omega}$  containing  $U$ .

**Lemma 3.1.** *Assume  $x^0 \in \Omega$  is such that  $u(x^0) = 0$  and  $u \geq 0$  near  $x^0$ . Let  $S_{x^0}$  be the component of  $\{x \in \bar{\Omega} : u(x) = 0\}$  containing  $x^0$ . Then  $S_{x^0} = \Gamma_{\bar{\Omega}}(S_{x^0} \cap \partial\Omega)$ .*

*Proof.* By the local convexity of  $u$ , we see that  $S_{x^0}$  is relatively convex in  $\bar{\Omega}$ . Thus we only have to show that  $S_{x^0} \subset \Gamma_{\bar{\Omega}}(S_{x^0} \cap \partial\Omega)$ . If not, there is a point, which we may assume to be the origin after translation and rotation of coordinates, in  $\Omega \cap S_{x^0}$  such that 0 is the only point in  $S_{x^0} \cap B_{\delta_0}(0)$  that lies in the half space  $\{x_n \geq 0\}$ , for some small  $\delta_0 > 0$ . Therefore, there is a constant  $\delta_1 > 0$  small such that  $u \geq a$  on  $\partial B_{\delta_0}(0) \cap \{x_n \geq -\delta_1\}$  for some constant  $a > 0$ . But then the function

$$v = \delta_2(\delta_1 + 2x_n + \delta_3|x|^2) \quad \text{for } \delta_2, \delta_3 > 0 \text{ small}$$

satisfies

$$\det(v_{i\bar{j}}) = 2\delta_2\delta_3 > 0 \quad \text{in } U, \quad v \leq u \quad \text{on } \partial U,$$

where  $U \equiv B_{\delta_0}(x^0) \cap \{x_n > -\delta_1\}$ . Consequently,  $u(0) \geq v(0) = \delta_1\delta_2 > 0$  by the maximum principle. This contradicts the fact that  $u(0) = 0$ .  $\square$

In this section a constant is said to be under control if it depends only on  $\Omega$ ,  $\|\varphi\|_{C^{3,1}(\partial\Omega)}$  and  $\underline{u}$  (up to its second derivatives).

**Theorem 3.2.** *There is a constant  $C$  under control such that for every  $x^0 \in \Omega$  there exist  $\delta(x^0) > 0$  and  $V(x^0) \in \mathbb{R}^n$  so that for every  $x \in \Omega$  with  $|x - x^0| \leq \delta(x^0)$ , we have*

$$(3.4) \quad |u(x) - u(x^0) - (x - x^0) \cdot V(x^0)| \leq C|x - x^0|^2.$$

We note that Theorem 3.2 implies that  $u$  is differentiable and  $Du(x^0) = V(x^0)$ . Thus it follows from (3.3) that

$$(3.5) \quad |Du| \leq C_0 \quad \text{on } \overline{\Omega}.$$

According to [7], Theorem 3.2 then implies that  $u \in C^{1,1}(\overline{\Omega})$ ; namely, for some constant  $C$  under control,

$$(3.6) \quad |Du(x) - Du(y)| \leq C|x - y| \quad \text{for all } x, y \in \Omega.$$

*Proof of Theorem 3.2.* For any fixed point  $x^0 \in \Omega$ , since  $u$  is locally convex, there exists an affine function  $p$  such that  $p(x^0) = u(x^0)$  and  $p \leq u$  near  $x^0$ . Let  $V(x^0) = Dp(x^0)$ ; (3.4) is then equivalent to

$$(3.7) \quad |u(x) - p(x)| \leq C|x - x^0|^2.$$

Without loss of generality, we may suppose  $p \equiv 0$  and, therefore,  $u(x^0) = 0$  and  $u \geq 0$  near  $x^0$ . By Lemma 3.1,  $x^0$  then lies in a simplex  $S \subset \{x \in \overline{\Omega} | u(x) = 0\}$  of dimension  $k \leq n$  with vertices on  $\partial\Omega$ . According to [7], we only have to consider the case  $k = 1$ . So we assume  $S$  is a segment with end points  $x^1, x^2 \in \partial\Omega$ . By the local convexity of  $u$  we have  $u \geq 0$  in a neighborhood of  $S$ .

Of the two end points of  $S$ , suppose  $x^2$  is closer to  $x^0$ . We may assume  $x^2 = 0$ . After rotation of coordinates, we suppose the positive  $x_n$ -axis is interior normal to  $\partial\Omega$  at 0 and  $x^0 = (x_1^0, 0, \dots, 0, x_n^0)$  with  $x_1^0 \geq 0$ ,  $x_n^0 = x_1^0 \tan \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$ . Near the origin,  $\partial\Omega$  is represented as a graph

$$(3.8) \quad x_n = \rho(x') = \frac{1}{2} \sum_{i,j < n} \rho_{ij}(0)x_i x_j + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}).$$

Lemma 3.3 below implies that if  $\theta$  is sufficiently small,  $x^1$  falls in that piece of  $\partial\Omega$  given by (3.8). Set  $\xi = (\cos \theta, 0, \dots, 0, \sin \theta) \in \mathbb{R}^n$ .

**Lemma 3.3.** *For some constants  $c_1, C_1$  under control,*

$$(3.9) \quad 0 < c_1 \theta \leq |x^1| \leq C_1 \theta.$$

*Proof.* We have

$$\underline{u}(x^1) = \underline{u}(0) + \underline{u}_\xi(0)|x^1| + \underline{u}_{\xi\xi}(tx^1)|x^1|^2, \quad \text{for some } 0 < t < 1.$$

But  $\underline{u}(0) = \underline{u}(x^1) = 0$  and  $\underline{u}_\xi(0) = \underline{u}_n(0) \sin \theta$ , since  $\underline{u}_i(0) = 0$  for  $1 \leq i \leq n - 1$ ; thus

$$\underline{u}_{\xi\xi}(tx^1)|x^1| = -\underline{u}_n(0) \sin \theta,$$

and (3.9) follows from the local strict convexity of  $\underline{u}$  and the comparison principle. □

Lemma 3.3 also implies that  $\theta > 0$ ; that is,  $S$  cannot be tangential to  $\partial\Omega$  at 0.

**Lemma 3.4.** *There exist uniform positive constants  $\varepsilon_0$  and  $\theta_0$  sufficiently small so that if  $\theta \leq \theta_0$  then*

$$\rho_{11}(0) \geq \varepsilon_0 > 0.$$

*Proof.* For any  $\mu > 0$ , there exists  $u^\lambda \in C^{2,1}(\overline{\Omega})$  satisfying (3.1) for some  $\lambda > 0$  such that

$$\|u^\lambda - u\|_{C^{0,1}(\overline{\Omega})} \leq \mu.$$

Since  $u(0) = u^\lambda(0) = 0$ , it follows that

$$0 \leq u(0, t) - u^\lambda(0, t) \leq t\mu, \quad \text{for any } t > 0 \text{ with } (0, t) \in \Omega.$$

But  $u(0, t) \geq 0$  for all  $t$  sufficiently small; thus

$$(3.10) \quad u_n^\lambda(0) \geq -\mu.$$

As in the proof of Lemma 3.3, we have

$$u_{\xi\xi}^\lambda(tx^1)|x^1| = -u_n^\lambda(0) \sin \theta, \quad \text{for some } 0 < t < 1,$$

since  $u^\lambda(x^1) = u^\lambda(0) = 0$  and  $u_\xi^\lambda(0) = u_n^\lambda(0) \sin \theta$ . Since  $u^\lambda \in C^2(\overline{\Omega})$ , it follows from Lemma 3.3 that

$$|u_{\xi\xi}^\lambda(tx^1) - u_{\xi\xi}^\lambda(0)| < \mu, \quad \text{when } \theta \text{ is sufficiently small.}$$

By (3.10) and Lemma 3.3 we thus obtain

$$(3.11) \quad u_{\xi\xi}^\lambda(0) \leq C_2\mu, \quad \text{for } \theta \text{ sufficiently small.}$$

Next,

$$u_{\xi\xi}^\lambda(0) = u_{11}^\lambda(0) \cos^2 \theta + u_{nn}^\lambda(0) \sin^2 \theta + 2u_{1n}^\lambda(0) \sin \theta \cos \theta.$$

We have  $u_{nn}^\lambda(0) > 0$  and, from the proof of (2.15) in Section 2,  $|u_{1n}^\lambda(0)| \leq C$  for some constant  $C$  independent of  $\lambda$ . Thus, by (3.11),

$$(3.12) \quad u_{11}^\lambda(0) \leq C_3\mu + C_4\theta, \quad \text{for } \theta \text{ sufficiently small.}$$

Finally, it follows from (see (2.6))  $\underline{u}_{11}(0) - u_{11}^\lambda(0) = (u_n^\lambda(0) - \underline{u}_n(0))\rho_{11}(0)$  and (3.2), (3.12) that

$$\rho_{11}(0) \geq c_0(\underline{u}_{11}(0) - C_3\mu - C_4\theta),$$

where  $c_0 > 0$  is a uniform constant. By the strict convexity of  $\underline{u}$  we can first fix  $\mu$  small, then choose  $\theta_0$  sufficiently small to complete the proof of Lemma 3.4.  $\square$

Returning to the proof of Theorem 3.2, we first consider the case  $\theta \leq \theta_0$ , where  $\theta_0$  is fixed such that Lemma 3.4 holds for some  $\varepsilon_0 > 0$ . To set up notation we fix a positive constant  $r_0$  depending only on  $\Omega$  such that  $\Gamma \equiv \{(x', \rho(x')) : |x'| \leq r_0\} \subset \partial\Omega$ ; by Lemma 3.3 we may assume  $\theta_0$  is sufficiently small so that  $x^1 \in \Gamma$ . As in [7], using Lemma 3.3 and the hypothesis that  $\varphi \in C^{3,1}(\partial\Omega)$ , one can prove that

$$(3.13) \quad |\varphi_{11}(0)| \leq A\theta^2, \quad |\varphi_{1j}(0)| \leq A\theta, \quad |\varphi_{ij}(0)| \leq A, \quad 1 < i, j \leq n - 1,$$

where  $A$  is a constant under control.

It follows from Lemma 3.4 that for any point  $x \in \Omega$  with  $|x - x^0| \leq \delta$  sufficiently small (depending on  $\varepsilon_0$  and possibly on  $x^0$ ), the ray from  $x^1$  to  $x$  strikes  $\Gamma$  at a point  $\bar{x} = (\bar{x}', \bar{x}_n)$  with  $\bar{x}' = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1})$  satisfying

$$(3.14) \quad \theta^2(\bar{x}_1)^2 + (\bar{x}_2)^2 + \dots + (\bar{x}_{n-1})^2 \leq C|x - x^0|^2$$

for a constant  $C$  under control (here we also use the fact that  $|x^1 - x^0| \geq |x^0|$ ). Since  $u(x^1) = 0$  it follows from the local convexity of  $u$  that

$$(3.15) \quad u(x) \leq u(\bar{x}) = \varphi(\bar{x}') = \sum_{i,j=1}^{n-1} \varphi_{ij} \bar{x}_i \bar{x}_j + O(|\bar{x}'|^3).$$

Consequently by (3.13) and (3.14),

$$u(x) \leq C \left( A + \frac{\delta}{\theta^3} \right) |x - x^0|^2,$$

with  $C$  under control. Now we may fix  $\delta = \delta(x^0) \leq \theta^3$  to obtain (3.7) for  $\theta \leq \theta_0$ .

The case  $\theta > \theta_0$  is simple, and we refer the reader to [7] for the details.

The proof of Theorem 3.2 is complete, and thus so is that of Theorem 1.2.  $\square$

#### 4. SPACELIKE HYPERSURFACES OF PRESCRIBED GAUSS CURVATURE

Recall that Minkowski space  $\mathbb{R}^{n,1}$  is the space  $\mathbb{R}^n \times \mathbb{R}$  endowed with the Lorentz metric  $ds^2 = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2$ , where  $x = (x_1, \dots, x_n)$  and  $x_{n+1}$  are the coordinates in  $\mathbb{R}^n$  and  $\mathbb{R}$ . A spacelike hypersurface of  $\mathbb{R}^{n,1}$  is a codimension-one submanifold whose induced metric is Riemannian. Locally a spacelike hypersurface  $M$  is given as the graph of a function  $x_{n+1} = u(x)$  satisfying the spacelike condition  $|Du| < 1$ . (We will also denote the hypersurface by  $u$  when it is globally given as the graph of  $u$ ). The first and second fundamental forms of  $M$  are given respectively by

$$g_{ij} = \delta_{ij} - u_i u_j, \quad A_{ij} = \frac{u_{ij}}{(1 - |Du|^2)^{\frac{1}{2}}}.$$

We say  $M$  is a *locally strictly convex* hypersurface if its second fundamental form is positive definite everywhere. The Gauss-Kronecker curvature of  $M$  has the expression

$$\mathcal{K}[M] = \frac{\det(u_{ij})}{(1 - |Du|^2)^{\frac{n+2}{2}}}.$$

Thus the equation

$$(4.1) \quad \det(u_{ij}(x)) = K(x, u(x))(1 - |Du(x)|^2)^{\frac{n+2}{2}}$$

locally describes spacelike hypersurfaces with prescribed Gauss-Kronecker curvature  $K$ . As an immediate consequence of Theorem 1.1, we first state an existence result for spacelike graphs with prescribed boundary value and Gauss-Kronecker curvature. (By a graph in  $\mathbb{R}^{n,1}$ , we always mean a submanifold, with or without boundary, that can be represented globally as the graph of a function  $x_{n+1} = u(x)$  defined in a subset of  $\mathbb{R}^n$ .)

**Theorem 4.1.** *Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$ . For given  $\varphi \in C^\infty(\partial\Omega)$  and  $K \in C^\infty(\bar{\Omega} \times \mathbb{R})$ ,  $K > 0$ , suppose there exists a spacelike locally strictly convex hypersurface  $\underline{u} \in C^2(\bar{\Omega})$  with  $\mathcal{K}[\underline{u}](x) \geq K(x, \underline{u}(x))$  for  $x \in \bar{\Omega}$  and  $\underline{u}|_{\partial\Omega} = \varphi$ . Then there exists a spacelike locally strictly convex hypersurface  $u \in C^\infty(\bar{\Omega})$  with prescribed boundary value  $u|_{\partial\Omega} = \varphi$  and Gauss-Kronecker curvature  $\mathcal{K}[u](x) = K(x, u(x))$  for  $x \in \bar{\Omega}$ , i.e.,  $u$  satisfies (4.1) in  $\Omega$  and the gradient bound*

$$(4.2) \quad |Du| < 1 \quad \text{on } \bar{\Omega}.$$

*Proof.* Note that since  $\underline{u}$  is a locally strictly convex subsolution of (4.1), Theorem 4.1 will follow from Theorem 1.1 once the *a priori* gradient bound (4.2) is established. By the local convexity of  $u$  we see that  $|Du|$  attains its maximum value on  $\partial\Omega$ . Next, we want to show that

$$(4.3) \quad \max_{\partial\Omega} |Du| \leq \max_{\partial\Omega} |D\underline{u}|,$$

which implies (4.2) since  $\underline{u}$  is a spacelike hypersurface.

Let  $\gamma$  denote the interior normal vector to  $\partial\Omega$  and let  $\xi \in \mathbb{R}^n$  be a unit vector. Consider an arbitrary point  $\bar{x} \in \partial\Omega$ . If  $\xi \cdot \gamma(\bar{x}) \leq 0$ , then

$$(4.4) \quad u_\xi(\bar{x}) \leq \underline{u}_\xi(\bar{x}) \leq |D\underline{u}(\bar{x})|,$$

since  $u \geq \underline{u}$  in  $\Omega$  and  $u = \underline{u}$  on  $\partial\Omega$ . Now suppose  $\xi \cdot \gamma(\bar{x}) > 0$ , and let  $y \in \partial\Omega$  be the first point where the ray  $\bar{x} + t\xi$ ,  $t > 0$ , touches  $\partial\Omega$ . Then we have

$$(4.5) \quad u_\xi(\bar{x}) \leq u_\xi(y) \leq \underline{u}_\xi(y) \leq |D\underline{u}(y)|.$$

The first inequality follows from the local convexity of  $u$ , the second from (4.4) since  $\xi \cdot \gamma(y) \leq 0$ . Finally, suppose  $|Du(\bar{x})| \neq 0$  and take  $\xi = Du(\bar{x})/|Du(\bar{x})|$ . From (4.4) and (4.5) it follows that

$$|Du(\bar{x})| = u_\xi(\bar{x}) \leq \max_{\partial\Omega} |D\underline{u}|.$$

This proves (4.3). □

We note that if  $M$  is a compact spacelike hypersurface and  $\partial M$  is a graph over the boundary of a domain  $\Omega \subset \mathbb{R}^n$ , then  $M$  is necessarily a graph over  $\Omega$ . Thus Theorem 1.3 follows from Theorem 4.1 and Theorem 1.2 when  $\Gamma$  is a graph. To prove Theorem 1.3 in the general situation, we formulate an extension of Theorem 4.1 as follows: Let  $U$  be a compact domain that immerses into  $\mathbb{R}^n$  with smooth boundary  $\partial U$ , and let  $\pi : U \rightarrow \mathbb{R}^n$  denote this immersion. Given a function  $u : U \rightarrow \mathbb{R}$ , one obtains a hypersurface of  $\mathbb{R}^{n,1}$  defined by

$$(4.6) \quad X : U \rightarrow \mathbb{R}^{n,1}, \quad X(q) = (\pi(q), u(q)) \quad \text{for } q \in U.$$

**Theorem 4.2.** *Let  $\varphi \in C^\infty(\partial U)$  and  $K \in C^\infty(\mathbb{R}^{n+1})$ ,  $K > 0$ . Suppose there exists a spacelike locally strictly convex hypersurface  $\tilde{M}$  of  $\mathbb{R}^{n,1}$  represented by*

$$(4.7) \quad q \in U \mapsto (\pi(q), \underline{u}(q)) \in \mathbb{R}^{n,1}$$

*with  $\underline{u} \in C^2(\bar{U})$  and  $\underline{u}|_{\partial U} = \varphi$ , such that  $\mathcal{K}[\tilde{M}](q) \geq K(\pi(q), \underline{u}(q))$  for  $q \in U$ . Then there exists a spacelike locally strictly convex hypersurface  $M$  given by (4.6) with  $u \in C^\infty(\bar{U})$  satisfying*

$$(4.8) \quad \mathcal{K}[M](q) = K(\pi(q), u(q)) \quad \text{for } q \in U, \quad u|_{\partial U} = \varphi.$$

*Proof.* We observe that it suffices to prove that, with respect to the metric on  $U$  induced by the immersion  $\pi : U \rightarrow \mathbb{R}^n$ , the Monge-Ampère equation

$$(4.9) \quad \det(u_{i,j}) = K(1 - |Du|^2)^{\frac{n+2}{2}} \quad \text{in } U, \quad u = \varphi \quad \text{on } \partial U$$

has a locally strictly convex solution in  $C^\infty(\bar{U})$  that satisfies the spacelike condition

$$(4.10) \quad |Du| < 1 \quad \text{in } U.$$

The existence of such a solution will follow from Theorem 5.1 in Section 5 once (4.10) is derived. To complete the proof, one observes that (4.10) can be derived as in the proof of Theorem 4.1 with some slight modification. □

The case  $K \equiv 0$  leads to the degenerate Monge-Ampère equation.

**Theorem 4.3.** *Let  $\varphi \in C^{3,1}(\partial U)$  and suppose there exists a spacelike locally strictly convex hypersurface  $\tilde{M}$  of  $\mathbb{R}^{n,1}$  given by (4.7) with  $\underline{u} \in C^2(\bar{U})$  and  $\underline{u}|_{\partial U} = \varphi$ . Then there exists a locally convex spacelike hypersurface  $M$  given by (4.6) with  $u \in C^{1,1}(\bar{U})$  and  $u|_{\partial U} = \varphi$ , whose Gauss-Kronecker curvature vanishes everywhere.*

*Proof.* The existence of  $M$  of the form (4.6) with  $u \in C^{0,1}(\bar{U})$  follows from Theorem 4.2 by approximation. In order to obtain the desired  $C^{1,1}$  regularity, we observe that for an arbitrary point  $q \in U$ , since it is locally convex,  $M$  has a supporting hyperplane,  $T$ , at  $Q \equiv (\pi(q), u(q)) \in M$ . By Lemma 3.1, there is a simplex  $S \subset M \cap T$ , containing  $Q$ , of dimension  $k \leq n$  with vertices on  $\partial M$ . In a neighborhood of  $S$ ,  $M$  lies above  $T$  and is given as a graph  $x_{n+1} = \tilde{u}(x)$  which solves  $\det(\tilde{u}_{i,j}) = 0$  weakly. Now we can repeat the proof of Theorem 3.2 to show that  $\tilde{u}$  satisfies an inequality of the form (3.4), which implies that  $u \in C^{1,1}(\bar{U})$  since, clearly,  $\tilde{u} = u \circ \pi$  locally.  $\square$

Finally, in order to see that Theorem 1.3 is a consequence of Theorems 4.2 and 4.3, we take  $U = \tilde{M}$  and let  $\pi : U \rightarrow \mathbb{R}^n$  be the orthogonal projection from  $\tilde{M} \subset \mathbb{R}^{n,1}$  to  $\mathbb{R}^n$ . By the spacelike condition, we see  $\pi$  is an immersion and  $\tilde{M}$  can be represented in the form (4.7) with  $\underline{u} \in C^2(\bar{U})$ . Theorem 1.3 thus follows.

Entire spacelike hypersurfaces with constant or prescribed mean curvature have also been studied in Minkowski space and in more general Lorentzian manifolds as well; for references please see, for example, [10]. In [9], Cheng and Yau proved a Bernstein type theorem for entire maximal spacelike hypersurfaces. It seems of interest to study entire spacelike hypersurfaces of constant Gauss-Kronecker curvature.

### 5. MONGE-AMPÈRE EQUATIONS ON MANIFOLDS

In this section we extend Theorem 1.1 to Monge-Ampère equations on Riemannian manifolds. Let  $M^n$  be a smooth Riemannian manifold of dimension  $n \geq 2$  and  $\Omega \subset M^n$  a smooth domain with compact closure  $\bar{\Omega}$ . We consider the Dirichlet problem

$$(5.1) \quad g^{-1} \det(\nabla_{i,j} u) = \psi(x, u, \nabla u) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where  $g = \det(g_{i,j}) > 0$ ,  $g_{i,j}$  denotes the metric of  $M^n$ ,  $\nabla$  is the Levi-Civita connection, and  $\nabla_{i,j} u$  denotes the Hessian of  $u$  (with respect to the metric  $g_{i,j}$ ). We assume  $\varphi \in C^\infty(\partial\Omega)$  and  $\psi > 0$  is  $C^\infty$  with respect to  $(x, z, p) \in \bar{\Omega} \times \mathbb{R} \times T_x M$ ; here  $T_x M$  denotes the tangent space at  $x \in M$ . The main result of this section is the following analogue of Theorem 1.1, which extends some of the results in [14].

**Theorem 5.1.** *There exists a locally strictly convex solution of (5.1) in  $C^\infty(\bar{\Omega})$ , provided that there exists a locally strictly convex subsolution  $\underline{u} \in C^2(\bar{\Omega})$  to (5.1). Furthermore, the solution is unique if  $\psi_z \geq 0$ .*

*Proof.* Let  $\mathcal{A}$  be as in Section 2. We shall prove the existence of a solution of (5.1) in  $\mathcal{A}$ . It is clear that the proof of Theorem 1.1 in Section 2 still works in this general case once  $C^2$  a priori estimates are established for such solutions. According to [14], we need only estimate the second derivatives at the boundary.

About a point  $x_0 \in \partial\Omega$ , let  $e_1, \dots, e_n$  be a local orthonormal frame on  $M^n$  obtained by parallel translation of a local orthonormal frame on  $\partial\Omega$  and the interior

unit normal vector field to  $\partial\Omega$  along the geodesics perpendicular to  $\partial\Omega$  on  $M^n$ . We assume  $e_n$  is the parallel translation of the unit normal field on  $\partial\Omega$ .

Let  $u \in \mathcal{A}$  be a solution of (5.1). Since  $u - \underline{u} = 0$  on  $\partial\Omega$ , it is straightforward to bound the pure tangential second derivatives

$$(5.2) \quad |\nabla_{\alpha\beta}u| \leq C \quad \text{on } \partial\Omega \text{ for } \alpha, \beta < n.$$

Next, we note that Lemma 2.2 readily extends to the present general case. For completeness we restate the lemma. As in (2.9), set

$$\mathcal{L} = u^{ij}\nabla_{ij} - f_{p_i}(x, u, \nabla u)\nabla_i,$$

where  $\{u^{ij}\}$  is the inverse matrix of the Hessian  $\{\nabla_{ij}u\}$ ,  $f = \log \psi$ . Let  $v$  be the function as defined in (2.12).

**Lemma 5.2.** *For  $N$  sufficiently large and  $t, \delta$  sufficiently small,*

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \quad \text{in } \Omega_\delta, \quad v \geq 0 \quad \text{on } \partial\Omega_\delta,$$

where  $\Omega_\delta = \Omega \cap B_\delta(x_0)$ ; here  $B_\delta(x_0)$  denotes the geodesic ball of radius  $\delta$  about  $x_0$ .

*Proof.* It is the same as that of Lemma 2.2 except that (2.14) takes a simpler form:

$$u^{ij}\nabla_i d \nabla_j d \geq u^{nn}(\nabla_n d)^2,$$

since  $\nabla_\beta d = 0$  for all  $\beta < n$ . □

Using Lemma 5.2 one can estimate  $\nabla_{n\alpha}u$  on  $\partial\Omega$  for  $\alpha \leq n$  as in [14]. For any fixed  $\alpha \leq n$ , differentiate equation (5.1) and use the formula for commuting the covariant derivatives

$$\nabla_{ijk}w - \nabla_{jik}w = R^l_{kji}\nabla_l w,$$

to find

$$(5.3) \quad |\mathcal{L}\nabla_\alpha(u - \underline{u})| \leq C(1 + \sum u^{ii}).$$

The mixed normal tangential derivatives  $\nabla_{\alpha n}u(x_0)$ ,  $\alpha < n$ , can be estimated the same way as in Section 2. Namely, by (5.3), we may choose  $A \gg B \gg 1$  such that

$$\mathcal{L}(Av + B|x|^2 \pm \nabla_\alpha(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta,$$

where  $|x|$  denotes the (geodesic) distance between  $x$  and  $x_0$ , and

$$Av + B|x|^2 \pm \nabla_\alpha(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

since  $\nabla_\alpha(u - \underline{u}) = 0$  on  $\partial\Omega \cap B_\delta(x_0)$ , and  $|\nabla_\alpha(u - \underline{u})| \leq C$  in  $\Omega$ . It follows from the maximum principle that  $Av + B|x|^2 \geq |\nabla_\alpha(u - \underline{u})|$  in  $\Omega_\delta$ . Consequently,

$$(5.4) \quad |\nabla_{n\alpha}u(x_0)| \leq A\nabla_n v(x_0) + |\nabla_{n\alpha}\underline{u}(x_0)| \leq C, \quad \alpha < n.$$

For the double normal derivative  $\nabla_{nn}u$ , since  $u$  is locally convex, it suffices to derive an upper bound

$$(5.5) \quad \nabla_{nn}u \leq C \quad \text{on } \partial\Omega.$$

We use an idea of Trudinger [25]. For  $x \in \partial\Omega$  let

$$\lambda(x) = \min_{|\xi|=1, \xi \in T_x(\partial\Omega)} \nabla_{\xi\xi}u(x),$$

and assume that  $\lambda(x)$  is minimized at  $x_0 \in \partial\Omega$  with  $\xi = e_1(x_0)$ , that is,  $\nabla_{11}u(x_0) \leq \nabla_{\xi\xi}u(x)$  for all  $x \in \partial\Omega$  and any unit vector  $\xi \in T_x(\partial\Omega)$ . As in [6], (5.5) will follow from

$$(5.6) \quad \nabla_{11}u(x_0) \geq c_0 > 0.$$

To show (5.6), we may assume  $\nabla_{11}u(x_0) < \frac{1}{2}\nabla_{11}\underline{u}(x_0)$ , since otherwise we are done as  $\nabla_{11}\underline{u}(x_0) \geq c_1 > 0$  for some uniform  $c_1 > 0$ . We have

$$(5.7) \quad \nabla_{11}u = \nabla_{11}\underline{u} - B_{11}\nabla_n(u - \underline{u}) \quad \text{on } \partial\Omega,$$

where  $B_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$ ,  $1 \leq \alpha, \beta \leq n - 1$ . It follows that

$$B_{11}(x_0)\nabla_n(u - \underline{u})(x_0) \geq \frac{1}{2}\nabla_{11}\underline{u}(x_0) \geq \frac{c_1}{2},$$

and for  $x \in \partial\Omega$  near  $x_0$ , since  $\nabla_{11}u|_{\partial\Omega}$  is minimized at  $x_0$ ,

$$B_{11}(x)\nabla_n(u - \underline{u})(x) \leq \nabla_{11}\underline{u}(x) - \nabla_{11}\underline{u}(x_0) + B_{11}(x_0)\nabla_n(u - \underline{u})(x_0).$$

Because  $B_{11}$  is smooth near  $\partial\Omega$  and  $0 < \nabla_n(u - \underline{u}) \leq C$ , we must have  $B_{11} \geq c_2 > 0$  on  $\Omega_\delta$  for some uniform  $c_2 > 0$ , if  $\delta$  is chosen sufficiently small. Therefore,

$$(5.8) \quad \nabla_n(u - \underline{u})(x) \leq \Psi(x) \quad \text{for } x \in \Omega_\delta \cap \partial\Omega \quad \text{and } \nabla_n(u - \underline{u})(x_0) = \Psi(x_0)$$

where  $\Psi(x) = B_{11}^{-1}(x)[\nabla_{11}\underline{u}(x) - \nabla_{11}\underline{u}(x_0) + B_{11}(x_0)\nabla_n(u - \underline{u})(x_0)]$ .

We observe that since  $\Psi$  is smooth in  $\Omega_\delta$ , by (5.3), (5.8) and Lemma 5.2 we may choose  $A \gg B \gg 1$  such that

$$Av + B|x|^2 + \Psi - \nabla_n(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

$$\mathcal{L}(Av + B|x|^2 + \Psi - \nabla_n(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta.$$

By the maximum principle,  $v + \Psi - \nabla_n(u - \underline{u}) \geq 0$  in  $\Omega_\delta$ , and therefore

$$\nabla_{nn}u(x_0) \leq C.$$

This shows that the eigenvalues of  $\{\nabla_{ij}u(x_0)\}$  are all bounded (and all positive). On the other hand, equation (5.1) says the product of these eigenvalues is bounded below away from zero by a uniform positive constant ( $\psi_0$  as in (2.3)). Thus each of them must be bounded below away from zero. In particular, we obtain the estimate (5.6), which in turn implies (5.5).  $\square$

We conclude this section by a remark on the following equation of Monge-Ampère type on  $\mathbb{S}^n$ :

$$(5.9) \quad g^{-1} \det(ug_{ij} + \nabla_{ij}u) = \psi(x, u, \nabla u).$$

This equation arises in various geometric problems related to Gauss curvature such as the Minkowski problem (see for example Cheng and Yau [8] and the references therein). The Dirichlet problem was studied, in connection with the boundary value problem of finding hypersurfaces in  $\mathbb{R}^{n+1}$  of prescribed Gauss-Kronecker curvature, by J. Spruck and the author in [15] and [13] under the hypothesis that  $\psi^{1/n}$  is a convex function with respect to the gradient  $\nabla u$ . By employing a better barrier similar to that in Lemma 5.2, we may refine the argument in [13] to prove

**Theorem 5.3.** *Let  $\Omega \subset \mathbb{S}^n$  be a smooth domain that does not contain any hemisphere. Let  $\varphi \in C^\infty(\partial\Omega)$ , and let  $\psi > 0$  be a smooth function. Then (5.9) has a solution  $u \in C^\infty(\bar{\Omega})$  satisfying  $\{ug_{ij} + \nabla_{ij}u\} > 0$  in  $\bar{\Omega}$  and  $u = \varphi$  on  $\partial\Omega$ , provided that there exists a subsolution  $\underline{u} \in C^2(\bar{\Omega})$  with  $\{\underline{u}g_{ij} + \nabla_{ij}\underline{u}\} > 0$  in  $\bar{\Omega}$  and  $\underline{u} = \varphi$  on  $\partial\Omega$ .*

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