

THE DIRICHLET PROBLEM FOR MONGE-AMPÈRE
EQUATIONS IN NON-CONVEX DOMAINS AND SPACELIKE
HYPERSURFACES OF CONSTANT GAUSS CURVATURE

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ABSTRACT. In this paper we extend the well known results on the existence and regularity of solutions of the Dirichlet problem for Monge-Ampère equations in a strictly convex domain to an arbitrary smooth bounded domain in \mathbb{R}^n as well as in a general Riemannian manifold. We prove for the nondegenerate case that a sufficient (and necessary) condition for the classical solvability is the existence of a subsolution. For the totally degenerate case we show that the solution is in $C^{1,1}(\bar{\Omega})$ if the given boundary data extends to a locally strictly convex C^2 function on $\bar{\Omega}$. As an application we prove some existence results for spacelike hypersurfaces of constant Gauss-Kronecker curvature in Minkowski space spanning a prescribed boundary.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^n with C^∞ boundary $\partial\Omega$. In this paper we consider the Dirichlet problem for Monge-Ampère equations

$$(1.1) \quad \det(u_{ij}) = \psi(x, u, Du) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where $\varphi \in C^\infty(\partial\Omega)$, $\psi \in C^\infty(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\psi \geq 0$, $Du = (u_1, \dots, u_n)$ denotes the gradient of u , $u_i = \partial u / \partial x_i$ and $u_{ij} = \partial^2 u / \partial x_i \partial x_j$.

When Ω is a strictly convex domain, this problem has received considerable study both in the non-degenerate case ($\psi > 0$) and in the degenerate case ($\psi = 0$ somewhere). A well known theorem (see Caffarelli, Nirenberg and Spruck [6], Ivochkina [17] and Krylov [19]) states that in the non-degenerate case $\psi > 0$, (1.1) has a strictly convex solution in $C^\infty(\bar{\Omega})$, provided Ω is strictly convex and there exists a strictly convex subsolution in $C^2(\bar{\Omega})$. (Please see, for example, [6], [12] and [22] for further references, including the earlier work of, among others, Pogorelov, Cheng and Yau, and P. L. Lions.) For the degenerate case ($\psi \geq 0$), counterexamples have been found showing that the Dirichlet problem, in general, does not have a solution in $C^2(\bar{\Omega})$; whether or not the weak solutions belong to $C^{1,1}(\bar{\Omega})$ has attracted a lot of attention. In the totally degenerate case $\psi \equiv 0$, the $C^{1,1}$ regularity was established by Caffarelli, Nirenberg and Spruck [7], who proved that if Ω is a strictly convex domain with $\partial\Omega \in C^{3,1}$ and $\varphi \in C^{3,1}(\partial\Omega)$, then the unique convex solution to the degenerate problem

$$(1.2) \quad \det(u_{ij}) = 0 \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega$$

Received by the editors August 11, 1995 and, in revised form, November 11, 1996.
1991 *Mathematics Subject Classification*. Primary 35J65, 35J70; Secondary 58G20.

belongs to $C^{1,1}(\overline{\Omega})$. Earlier Trudinger and Urbas [26] obtained local $C^{1,1}$ regularity under the weaker hypotheses that $\partial\Omega \in C^{1,1}$ and $\varphi \in C^{1,1}(\partial\Omega)$. The recent work of Krylov [20], [21] provides a unified treatment of the non-degenerate and totally degenerate cases. The main purpose of the present paper is to extend the above mentioned results to non-convex domains.

The Monge-Ampère equations are closely related to problems involving Gauss-Kronecker curvature in differential geometry, such as the Minkowski and Weyl problems. From the viewpoint of geometric applications, it is of interest to study the Dirichlet problem for Monge-Ampère equations in non-convex domains. In his book [2], T. Aubin also raised the question of whether one can remove the hypothesis of convexity of the domain for a problem. Recently, an effort to extend the results of [6] to non-convex domains was made by J. Spruck et al. in [16] and [15]. It was proved in [15] that for $\psi > 0$ the Dirichlet problem (1.1) in an arbitrary smooth domain Ω admits a locally strictly convex solution in $C^\infty(\overline{\Omega})$ provided that $(\psi(x, z, p))^{1/n}$ is a convex function with respect to p and that there exists a locally strictly convex *strict* subsolution $\underline{u} \in C^2(\overline{\Omega})$ (i.e., assuming \underline{u} satisfies the strict inequality in (1.4) below). This result applies to, for example, the Gauss curvature equation

$$(1.3) \quad \det(u_{ij}) = K(x, u)(1 + |Du|^2)^{\frac{n+2}{2}}$$

for hypersurfaces in Euclidean space and has interesting geometric consequences (see, for example, [15], [23]). In this paper we will prove

Theorem 1.1. *Let $\varphi \in C^\infty(\partial\Omega)$, $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\psi > 0$. Assume there exists a locally strictly convex subsolution $\underline{u} \in C^2(\overline{\Omega})$ satisfying*

$$(1.4) \quad \det(\underline{u}_{ij}) \geq \psi(x, \underline{u}, D\underline{u}) \quad \text{in } \Omega, \quad \underline{u} = \varphi \quad \text{on } \partial\Omega.$$

Then there exists a locally strictly convex solution $u \in C^\infty(\overline{\Omega})$ of (1.1) with $u \geq \underline{u}$. The solution is unique if $\psi_u \geq 0$.

Here a function $v \in C^2(\overline{\Omega})$ is said to be *locally strictly convex* if its Hessian matrix $\{v_{ij}\}$ is positive definite everywhere in $\overline{\Omega}$. Obviously, condition (1.4) in Theorem 1.1 cannot be removed even when Ω is strictly convex. In case $\psi \equiv \psi(x)$ or, more generally (due to P. L. Lions; see [6]), when ψ satisfies

$$0 < \psi(x, z, p) \leq C(1 + |p|^2)^{n/2} \quad \text{for } x \in \overline{\Omega}, \quad z \leq \max \varphi, \quad p \in \mathbb{R}^n,$$

one can construct a strictly convex subsolution if Ω is strictly convex; this fails for non-convex Ω .

There has been some recent work on the Dirichlet problem for Monge-Ampère equations on general smooth Riemannian manifolds. In [14], Y. Y. Li and the author established an analogue of the result of Guan-Spruck [15] cited above. Some of the result was also obtained independently by A. Atallah and C. Zuily [1]. Building on [14] we will, in Section 5, extend Theorem 1.1 to general Riemannian manifolds.

The totally degenerate problem (1.2) is important to understanding hypersurfaces of vanishing Gauss-Kronecker curvature. It is of interest to study the regularity of its solution in non-convex domains. Applying Theorem 1.1 to the problem

$$(1.5) \quad \det(u_{ij}) = \varepsilon \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega$$

for sufficiently small $\varepsilon > 0$, by approximation we find that (1.2) has a unique locally convex weak solution (in the sense of Alexandrov, which is the same as weak solution

in the viscosity sense; see [4]) in $C^{0,1}(\bar{\Omega})$, provided φ extends to a locally strictly convex function in $C^2(\bar{\Omega})$. We will prove that this solution actually belongs to $C^{1,1}(\bar{\Omega})$. More precisely, we have the following extension of the regularity theorem in [7].

Theorem 1.2. *Assume $\partial\Omega$ is in $C^{3,1}$ and $\varphi \in C^{3,1}(\partial\Omega)$. Suppose there exists a locally strictly convex function $\underline{u} \in C^2(\bar{\Omega})$ with $\underline{u} = \varphi$ on $\partial\Omega$. Then there is a unique locally convex weak solution of (1.2) in $C^{1,1}(\bar{\Omega})$.*

As we have observed, a major motivation for our studying the Dirichlet problem (1.1) in non-convex domains comes from its close connection with geometric problems. In this paper we consider one such problem, which concerns existence of spacelike hypersurfaces of constant Gauss-Kronecker curvature with specified boundary in Minkowski space $\mathbb{R}^{n,1}$. (Please see [15], [13] for related results for hypersurfaces in Euclidean space, and [23] in hyperbolic space.) We are interested in the following question: given a disjoint collection $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$ of codimension-two closed smooth submanifolds of $\mathbb{R}^{n,1}$, decide whether there exists a spacelike hypersurface M of constant Gauss-Kronecker curvature with boundary $\partial M = \Gamma$. Locally M is given as the graph of a function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, satisfying the spacelike condition $|Du| < 1$ and the Monge-Ampère equation (1.1) with

$$(1.6) \quad \psi(x, u, Du) = \mathcal{K}(1 - |Du|^2)^{\frac{n+2}{2}},$$

where \mathcal{K} is the Gauss-Kronecker curvature of M . We note that the right hand side of (1.6), in contrast with that of (1.3), is not a convex function with respect to the gradient Du .

Theorem 1.3. *Suppose Γ bounds a compact C^2 spacelike locally strictly convex hypersurface \tilde{M} . Then for any constant $0 \leq K \leq \min_{q \in \tilde{M}} \mathcal{K}[\tilde{M}](q)$, where $\mathcal{K}[\tilde{M}]$ denotes the Gauss-Kronecker curvature of \tilde{M} , there exists a compact spacelike hypersurface M_K of constant Gauss-Kronecker curvature K with boundary $\partial M_K = \Gamma$. Moreover, M_K is C^∞ for $K > 0$ and M_0 is $C^{1,1}$.*

The corresponding problem for spacelike hypersurfaces with prescribed boundary value and mean curvature was treated by R. Bartnik and L. Simon [3]; see also [10] and references therein for related results.

This paper is organized as follows. In Section 2 we first derive *a priori* estimates for the C^2 norms of the desired solutions of (1.1) for the non-degenerate case. Then Theorem 1.1 is proved using the continuity method and degree theory based on these *a priori* estimates. Section 3 contains the proof of Theorem 1.2. Theorem 1.3 is proved in Section 4 as a consequence of more general theorems presented there. We note that since in general the hypersurface M_K is not globally a graph over a domain in \mathbb{R}^n , Theorem 1.3 does not follow directly from Theorems 1.1 and 1.2. To overcome this difficulty, we will reformulate the problem in a more general setting and appeal for its proof to Theorem 5.1, the extension of Theorem 1.1 to general Riemannian manifolds, which is proved in Section 5.

2. THE NON-DEGENERATE MONGE-AMPÈRE EQUATIONS

In this section we prove Theorem 1.1 using the method of continuity and degree theory. As usual, the proof is based on the establishment of global $C^{2,\alpha}$ *a priori* estimates for prospective solutions. A somewhat surprising fact to us is that in

deriving these estimates one only needs the assumption that \underline{u} is a locally strictly convex C^2 function; it does not necessarily satisfy (1.4). As we shall see, condition (1.4) is needed in the proof of Theorem 1.1 only to guarantee that, when $\psi_u \geq 0$, the unique solution u of (1.1) satisfies $u \geq \underline{u}$ in Ω by the maximum principle.

2.1. A priori estimates. In this subsection we only assume $\underline{u} \in C^2(\overline{\Omega})$ is a locally strictly convex function; thus there exists a constant $\varepsilon > 0$ (without loss of generality, we may assume $\varepsilon \leq 1$) such that

$$(2.1) \quad \{\underline{u}_{ij}\} \geq \varepsilon \{\delta_{ij}\} \quad \text{on } \overline{\Omega}.$$

Set

$$\mathcal{A} = \{w \in C^\infty(\overline{\Omega}) : \{w_{ij}\} > 0, w \geq \underline{u}, w|_{\partial\Omega} = \varphi\}.$$

As in [15] it is easy to see that

$$(2.2) \quad |w| + |Dw| \leq K_1, \quad \text{for any } w \in \mathcal{A},$$

where the constant K_1 depends only on Ω , n , and $\|\underline{u}\|_{C^1(\overline{\Omega})}$. From (2.2) we have

$$(2.3) \quad 0 < \psi_0 \equiv \inf_{x \in \overline{\Omega}, w \in \mathcal{A}} \psi(x, w(x), Dw(x)) \leq \sup_{x \in \overline{\Omega}, w \in \mathcal{A}} \psi(x, w(x), Dw(x)) \equiv \psi_1 < \infty.$$

Theorem 2.1. *Let $u \in \mathcal{A}$ be a solution of (1.1). Then*

$$(2.4) \quad |D^2u| \leq C \quad \text{on } \overline{\Omega}.$$

Here the constant C depends on Ω , n , ε , K_1 , ψ_0 , ψ_1 , $\|\varphi\|_{C^{3,1}(\overline{\Omega})}$, $\|\psi\|_{C^2(\overline{\Omega})}$ and $\|\underline{u}\|_{C^2(\overline{\Omega})}$.

Proof. It is shown in [6] how to derive (2.4) from C^2 estimates on the boundary. Thus we need only estimate D^2u on $\partial\Omega$. Consider any point 0 on $\partial\Omega$; we may assume it is the origin of \mathbb{R}^n and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at 0. Near the origin, $\partial\Omega$ can be represented as a graph

$$(2.5) \quad x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}).$$

Since $u - \underline{u} = 0$ on $\partial\Omega$,

$$(2.6) \quad (u - \underline{u})_{\alpha\beta}(0) = -(u - \underline{u})_n(0) B_{\alpha\beta}, \quad \alpha, \beta < n.$$

It follows that

$$(2.7) \quad |u_{\alpha\beta}(0)| \leq C, \quad \alpha, \beta < n.$$

Next we estimate the mixed normal-tangential derivative $u_{\alpha n}(0)$. Rewrite equation (1.1) in the form

$$(2.8) \quad \log \det(u_{ij}) = \log \psi(x, u, Du) \equiv f(x, u, Du)$$

and let \mathcal{L} denote the linear operator defined by

$$(2.9) \quad \mathcal{L}w = u^{ij} w_{ij} - f_{p_i}(x, u, Du) w_i \quad \text{for } w \in C^2(\overline{\Omega}),$$

where $\{u^{ij}\}$ is the inverse matrix of $\{u_{ij}\}$ and $f_{p_i} = f_{p_i}(x, u, p)$. For fixed $\alpha < n$ consider the operator

$$T = \partial_\alpha + \sum_{\beta < n} B_{\alpha\beta} (x_\beta \partial_n - x_n \partial_\beta).$$

As in [6] we have

$$(2.10) \quad |\mathcal{L}T(u - \underline{u})| \leq C(1 + \sum u^{ii}).$$

Since $|Du| \leq K_1$, $|T(u - \underline{u})| \leq C$ in $\bar{\Omega}$. Moreover, on $\partial\Omega$ near the origin

$$(2.11) \quad |T(u - \underline{u})| \leq C|x|^2.$$

We will employ a barrier function of the form

$$(2.12) \quad v = (u - \underline{u}) + t(h - \underline{u}) - Nd^2,$$

where h is the harmonic function in Ω with $h|_{\partial\Omega} = \varphi$, d is the distance function from $\partial\Omega$, and t, N are positive constants to be determined. We may take $\delta > 0$ small enough so that d is smooth in $\Omega_\delta = \Omega \cap B_\delta(0)$. The key ingredient is the following:

Lemma 2.2. *For N sufficiently large and t, δ sufficiently small,*

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \text{ in } \Omega_\delta, \quad v \geq 0 \text{ on } \partial\Omega_\delta.$$

Proof. By (2.1) we have $u^{ij}(u_{ij} - \underline{u}_{ij}) \leq n - \varepsilon \sum u^{ii}$. It follows that

$$(2.13) \quad \mathcal{L}(u - \underline{u}) \leq C_0 - \varepsilon \sum u^{ii}.$$

Next, since $\Delta \underline{u} \geq n\varepsilon > 0$,

$$(h - \underline{u})(x) \geq c_0d(x), \quad \text{for } x \in \Omega$$

for some uniform constant $c_0 > 0$. Moreover, we have

$$\mathcal{L}(h - \underline{u}) \leq C_1(1 + \sum u^{ii}),$$

for some constant $C_1 > 0$ under control. Thus

$$\mathcal{L}v \leq C_0 + tC_1 + (tC_1 - \varepsilon) \sum u^{ii} - 2N(d\mathcal{L}d + u^{ij}d_id_j) \text{ in } \Omega_\delta.$$

It is easy to see that

$$\mathcal{L}d \geq -C_2(1 + \sum u^{ii}).$$

Furthermore, since $\{u^{ij}\}$ is positive definite and $d_n(0) = 1$, $d_\beta(0) = 0$ for all $\beta < n$, we have, for δ sufficiently small,

$$(2.14) \quad u^{ij}d_id_j \geq u^{nn}d_n^2 + 2 \sum_{\beta < n} u^{n\beta}d_nd_\beta \geq \frac{u^{nn}}{2} - C_3\delta \sum u^{ii} \text{ in } \Omega_\delta.$$

Let $\lambda_1 \leq \dots \leq \lambda_n$ be the eigenvalues of $\{u_{ij}\}$. We have $\sum u^{ii} = \sum \lambda_i^{-1}$ and $u^{nn} \geq \lambda_n^{-1}$. By the inequality for arithmetic and geometric means,

$$\frac{\varepsilon}{4} \sum u^{ii} + Nu^{nn} \geq \frac{n\varepsilon}{4} (N\lambda_1^{-1} \dots \lambda_n^{-1})^{\frac{1}{n}} \geq \frac{n\varepsilon}{4(\psi_1)^{1/n}} N^{\frac{1}{n}} \equiv c_1N^{\frac{1}{n}}.$$

Now we fix $t > 0$ sufficiently small so that $tC_1 \leq \frac{\varepsilon}{4}$ and fix N so that $c_1N^{1/n} \geq C_0 + \varepsilon$. We obtain

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \text{ in } \Omega_\delta$$

if we require δ to satisfy $2(C_2 + C_3)N\delta \leq \frac{\varepsilon}{4}$ in Ω_δ .

It remains to examine the value of v on $\partial\Omega_\delta$. On $\partial\Omega \cap B_\delta(0)$ we have $v = 0$. On $\Omega \cap \partial B_\delta(0)$,

$$v \geq tc_0d - Nd^2 \geq (tc_0 - N\delta)d \geq 0,$$

if we require, in addition, $N\delta \leq tc_0$. Now we can fix δ sufficiently small to complete the proof of Lemma 2.2. \square

Using Lemma 2.2 we can choose $A \gg B \gg 1$ so that

$$\mathcal{L}(Av + B|x|^2 \pm T(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta$$

and

$$Av + B|x|^2 \pm T(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta$$

by (2.10) and (2.11). It thus follows from the maximum principle that

$$(2.15) \quad |u_{\alpha n}(0)| \leq C.$$

Finally, the tangential strict convexity of u has been established in [15]; i.e.

$$(2.16) \quad \sum_{\alpha, \beta < n} w_{\alpha\beta}(0) \xi_\alpha \xi_\beta \geq c_0 > 0$$

for any unit vector $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Thus it follows as in [6] that

$$(2.17) \quad |u_{nn}(0)| \leq C.$$

The proof of Theorem 2.1 is complete. \square

From the Evans-Krylov theory (see [11], [24], [18] and [5]) we thus have an *a priori* bound for the $C^{2,\alpha}$ norm of u ,

$$(2.18) \quad \|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \quad 0 < \alpha < 1,$$

with constant C depending only on Ω , n , ε , ψ_0 , ψ_1 , $\|\varphi\|_{C^{3,1}(\overline{\Omega})}$, $\|\psi\|_{C^3(\overline{\Omega})}$ and $\|\underline{u}\|_{C^2(\overline{\Omega})}$. By the standard Schauder theory, we thus obtain the *a priori* bound for each integer $k \geq 3$:

$$(2.19) \quad \|u\|_{C^{k,\alpha}(\overline{\Omega})} \leq C.$$

With the aid of such estimates we can apply the continuity method and degree theory to prove Theorem 1.1 as in [6] with some modifications.

2.2. Proof of Theorem 1.1. We first assume that the subsolution \underline{u} is in $C^\infty(\overline{\Omega})$ and prove the existence of a solution to (1.1) in \mathcal{A} in two steps as follows.

(a) The special case: $\psi_u \geq 0$. For each fixed $t \in [0, 1]$ consider the Dirichlet problem

$$(2.20) \quad \det(u_{ij}) = t\psi(x, u, Du) + (1-t)\det(\underline{u}_{ij}) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega.$$

Note that since \underline{u} is a subsolution of (2.20), it follows from the maximum principle that any locally strictly convex solution $u \in C^\infty(\overline{\Omega})$ of (2.20) satisfies $u \geq \underline{u}$ and hence (2.19), independent of t . Thus we can utilize the continuity method to show that for each $t \in [0, 1]$ there exists a locally strictly convex solution of (2.20) in $C^\infty(\overline{\Omega})$. The uniqueness follows from the maximum principle.

(b) Turning to the general case, we assume \underline{u} is not a solution of (1.1) and let $u^0 \in \mathcal{A}$ be the unique solution of

$$(2.21) \quad \det(u_{ij}) = \psi_0 \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where ψ_0 is as in (2.3). The existence of u^0 follows from part (a).

For $r > 0$ we consider the open convex set of functions in $C^5(\overline{\Omega})$

$$\mathcal{C}_r = \left\{ v \in C^5(\overline{\Omega}) : \|v\|_{C^5(\overline{\Omega})} < r, v > 0 \text{ in } \Omega, v|_{\partial\Omega} = 0 \text{ and } v_\nu > 0 \text{ on } \partial\Omega \right\}.$$

where ν is the unit interior normal to $\partial\Omega$. We want to prove that for each $t \in [0, 1]$ there exists a solution in \mathcal{A} of the form

$$(2.22) \quad u = \underline{u} + v, \quad v \in \mathcal{C}_r \text{ with } r \text{ sufficiently large,}$$

to the Dirichlet problem

$$(2.23) \quad \det(u_{ij}) = t\psi(x, u, Du) + (1 - t)\psi_0 \equiv \psi^t(x, u, Du) \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

As we observed, any solution $u \in \mathcal{A}$ of (2.23) satisfies the *a priori* bound

$$(2.24) \quad \|u\|_{C^5(\overline{\Omega})} \leq C \text{ independent of } t.$$

Moreover, by the maximum principle and the Hopf lemma we have

$$(2.25) \quad u > \underline{u} \text{ in } \Omega \text{ and } (u - \underline{u})_\nu > 0 \text{ on } \partial\Omega.$$

Thus we can choose r sufficiently large so that (2.23) has no solution in \mathcal{A} of the form (2.22) with $v \in \partial\mathcal{C}_r$, the boundary of \mathcal{C}_r .

Now for $0 \leq t \leq 1$ and fixed $v \in \overline{\mathcal{C}_r}$ consider the Dirichlet problem

$$(2.26) \quad \det(u_{ij}) = \psi^t(x, u, Du)e^{t\Lambda(u - \underline{u} - v)} \equiv \eta^t(x, u, Du) \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

where

$$\Lambda = \frac{1}{\psi_0} \sup_{x \in \overline{\Omega}} \sup_{u \in \mathcal{A}} \psi_u(x, u, Du) < \infty, \quad \psi_0 \text{ as in (2.3).}$$

We observe that \underline{u} is a subsolution of (2.26) and $\eta^t_u \geq 0$. Thus by part (a) there exists a unique solution $u^t \in \mathcal{A}$ for each $t \in [0, 1]$. For $t = 0$, this solution is our u^0 .

From elliptic theory, the map $T^t v = u^t - \underline{u}$ is compact in C^5 . On the other hand, we have seen that there are no solutions of

$$(2.27) \quad v - T^t v = 0$$

on the boundary of \mathcal{C}_r . Thus the degree

$$(2.28) \quad \deg(I - T^t, \mathcal{C}_r, 0) = \gamma$$

is well defined and independent of t . For $t = 0$, (2.27) has a unique solution $v^0 = u^0 - \underline{u}$. By the maximum principle, when $t = 0$ the linearized operator of (2.23), linearized at u^0 , is invertible. Thus v^0 is a regular point of $I - T^0$. Consequently $\gamma = \pm 1$, and (2.27) has a solution $v^t \in \mathcal{C}_r$ for all $0 \leq t \leq 1$. The function $u^1 = \underline{u} + v^1$ is then a solution of (1.1). The elliptic regularity theory implies that $u^1 \in C^\infty(\overline{\Omega})$.

To finish the proof, we have to consider the case $\underline{u} \in C^2(\overline{\Omega})$. We may take a sequence of locally strictly convex functions $\underline{u}^m \in C^\infty(\overline{\Omega})$ converging to \underline{u} in $C^2(\overline{\Omega})$, such that

$$\det(\underline{u}^m_{ij}) \geq \frac{m}{m+1} \psi(x, \underline{u}^m, D\underline{u}^m) \text{ in } \overline{\Omega}.$$

For each integer $m \geq 1$, set

$$\psi^m = \frac{m}{m+1} \psi, \quad \varphi^m = \underline{u}^m|_{\partial\Omega}$$

and consider the Dirichlet problem

$$(2.29) \quad \det(u_{i\bar{j}}) = \psi^m \quad \text{in } \bar{\Omega}, \quad u = \varphi^m \quad \text{on } \partial\Omega.$$

Note that \underline{u}^m is a subsolution of (2.29). Consequently, there exists a locally strictly convex solution $u^m \in C^\infty(\bar{\Omega})$ of (2.29) satisfying the *a priori* estimates

$$\|u^m\|_{C^k(\bar{\Omega})} \leq C(m, k) \quad \text{for every } k \geq 1,$$

where the constant $C(m, k)$ only depends on, besides other known data, \underline{u}^m and its derivatives up to second order. It follows that a subsequence of u^m converges to a solution of (1.1) in $C^\infty(\bar{\Omega})$. This completes the proof of Theorem 1.1.

3. THE TOTALLY DEGENERATE MONGE-AMPÈRE EQUATION

The main purpose of this section is to prove Theorem 1.2. We will first obtain a weak solution in $C^{0,1}(\bar{\Omega})$ of the totally degenerate Monge-Ampère equation by approximation. The major part of the section is devoted to the proof of the $C^{1,1}$ regularity of this weak solution.

For each small constant $\lambda > 0$, we may first take smooth approximations of Ω and \underline{u} and apply Theorem 1.1, then pass to the limit to obtain, under the hypothesis of Theorem 1.2, a locally strictly convex function $u^\lambda \in C^2(\bar{\Omega})$ satisfying

$$(3.1) \quad \det(u_{i\bar{j}}^\lambda) = \lambda \quad \text{in } \Omega, \quad u^\lambda = \varphi \quad \text{on } \partial\Omega$$

and the C^1 estimate

$$(3.2) \quad \|u^\lambda\|_{C^1(\bar{\Omega})} \leq C_0$$

for some constant $C_0 > 0$ independent of λ . Therefore, there exists a sequence u^{λ_k} that converges to a locally convex function $u \in C^{0,1}(\bar{\Omega})$, and

$$(3.3) \quad \|u\|_{C^{0,1}(\bar{\Omega})} \leq C_0.$$

By the maximum principle, u is the unique locally convex solution of (1.2). In the rest of this section we will modify the proof of [7] to show that u is in $C^{1,1}(\bar{\Omega})$.

We say a subset U of $\bar{\Omega}$ is relatively convex in $\bar{\Omega}$ if any segment contained in $\bar{\Omega}$ with endpoints in U lies completely in U . The relative convex hull, denoted by $\Gamma_{\bar{\Omega}}(U)$, of a set $U \subset \bar{\Omega}$ is the smallest relatively convex set in $\bar{\Omega}$ containing U .

Lemma 3.1. *Assume $x^0 \in \Omega$ is such that $u(x^0) = 0$ and $u \geq 0$ near x^0 . Let S_{x^0} be the component of $\{x \in \bar{\Omega} : u(x) = 0\}$ containing x^0 . Then $S_{x^0} = \Gamma_{\bar{\Omega}}(S_{x^0} \cap \partial\Omega)$.*

Proof. By the local convexity of u , we see that S_{x^0} is relatively convex in $\bar{\Omega}$. Thus we only have to show that $S_{x^0} \subset \Gamma_{\bar{\Omega}}(S_{x^0} \cap \partial\Omega)$. If not, there is a point, which we may assume to be the origin after translation and rotation of coordinates, in $\Omega \cap S_{x^0}$ such that 0 is the only point in $S_{x^0} \cap B_{\delta_0}(0)$ that lies in the half space $\{x_n \geq 0\}$, for some small $\delta_0 > 0$. Therefore, there is a constant $\delta_1 > 0$ small such that $u \geq a$ on $\partial B_{\delta_0}(0) \cap \{x_n \geq -\delta_1\}$ for some constant $a > 0$. But then the function

$$v = \delta_2(\delta_1 + 2x_n + \delta_3|x|^2) \quad \text{for } \delta_2, \delta_3 > 0 \text{ small}$$

satisfies

$$\det(v_{i\bar{j}}) = 2\delta_2\delta_3 > 0 \quad \text{in } U, \quad v \leq u \quad \text{on } \partial U,$$

where $U \equiv B_{\delta_0}(x^0) \cap \{x_n > -\delta_1\}$. Consequently, $u(0) \geq v(0) = \delta_1\delta_2 > 0$ by the maximum principle. This contradicts the fact that $u(0) = 0$. \square

In this section a constant is said to be under control if it depends only on Ω , $\|\varphi\|_{C^{3,1}(\partial\Omega)}$ and \underline{u} (up to its second derivatives).

Theorem 3.2. *There is a constant C under control such that for every $x^0 \in \Omega$ there exist $\delta(x^0) > 0$ and $V(x^0) \in \mathbb{R}^n$ so that for every $x \in \Omega$ with $|x - x^0| \leq \delta(x^0)$, we have*

$$(3.4) \quad |u(x) - u(x^0) - (x - x^0) \cdot V(x^0)| \leq C|x - x^0|^2.$$

We note that Theorem 3.2 implies that u is differentiable and $Du(x^0) = V(x^0)$. Thus it follows from (3.3) that

$$(3.5) \quad |Du| \leq C_0 \quad \text{on } \overline{\Omega}.$$

According to [7], Theorem 3.2 then implies that $u \in C^{1,1}(\overline{\Omega})$; namely, for some constant C under control,

$$(3.6) \quad |Du(x) - Du(y)| \leq C|x - y| \quad \text{for all } x, y \in \Omega.$$

Proof of Theorem 3.2. For any fixed point $x^0 \in \Omega$, since u is locally convex, there exists an affine function p such that $p(x^0) = u(x^0)$ and $p \leq u$ near x^0 . Let $V(x^0) = Dp(x^0)$; (3.4) is then equivalent to

$$(3.7) \quad |u(x) - p(x)| \leq C|x - x^0|^2.$$

Without loss of generality, we may suppose $p \equiv 0$ and, therefore, $u(x^0) = 0$ and $u \geq 0$ near x^0 . By Lemma 3.1, x^0 then lies in a simplex $S \subset \{x \in \overline{\Omega} | u(x) = 0\}$ of dimension $k \leq n$ with vertices on $\partial\Omega$. According to [7], we only have to consider the case $k = 1$. So we assume S is a segment with end points $x^1, x^2 \in \partial\Omega$. By the local convexity of u we have $u \geq 0$ in a neighborhood of S .

Of the two end points of S , suppose x^2 is closer to x^0 . We may assume $x^2 = 0$. After rotation of coordinates, we suppose the positive x_n -axis is interior normal to $\partial\Omega$ at 0 and $x^0 = (x_1^0, 0, \dots, 0, x_n^0)$ with $x_1^0 \geq 0$, $x_n^0 = x_1^0 \tan \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Near the origin, $\partial\Omega$ is represented as a graph

$$(3.8) \quad x_n = \rho(x') = \frac{1}{2} \sum_{i,j < n} \rho_{ij}(0)x_i x_j + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}).$$

Lemma 3.3 below implies that if θ is sufficiently small, x^1 falls in that piece of $\partial\Omega$ given by (3.8). Set $\xi = (\cos \theta, 0, \dots, 0, \sin \theta) \in \mathbb{R}^n$.

Lemma 3.3. *For some constants c_1, C_1 under control,*

$$(3.9) \quad 0 < c_1 \theta \leq |x^1| \leq C_1 \theta.$$

Proof. We have

$$\underline{u}(x^1) = \underline{u}(0) + \underline{u}_\xi(0)|x^1| + \underline{u}_{\xi\xi}(tx^1)|x^1|^2, \quad \text{for some } 0 < t < 1.$$

But $\underline{u}(0) = \underline{u}(x^1) = 0$ and $\underline{u}_\xi(0) = \underline{u}_n(0) \sin \theta$, since $\underline{u}_i(0) = 0$ for $1 \leq i \leq n - 1$; thus

$$\underline{u}_{\xi\xi}(tx^1)|x^1| = -\underline{u}_n(0) \sin \theta,$$

and (3.9) follows from the local strict convexity of \underline{u} and the comparison principle. □

Lemma 3.3 also implies that $\theta > 0$; that is, S cannot be tangential to $\partial\Omega$ at 0.

Lemma 3.4. *There exist uniform positive constants ε_0 and θ_0 sufficiently small so that if $\theta \leq \theta_0$ then*

$$\rho_{11}(0) \geq \varepsilon_0 > 0.$$

Proof. For any $\mu > 0$, there exists $u^\lambda \in C^{2,1}(\overline{\Omega})$ satisfying (3.1) for some $\lambda > 0$ such that

$$\|u^\lambda - u\|_{C^{0,1}(\overline{\Omega})} \leq \mu.$$

Since $u(0) = u^\lambda(0) = 0$, it follows that

$$0 \leq u(0, t) - u^\lambda(0, t) \leq t\mu, \quad \text{for any } t > 0 \text{ with } (0, t) \in \Omega.$$

But $u(0, t) \geq 0$ for all t sufficiently small; thus

$$(3.10) \quad u_n^\lambda(0) \geq -\mu.$$

As in the proof of Lemma 3.3, we have

$$u_{\xi\xi}^\lambda(tx^1)|x^1| = -u_n^\lambda(0) \sin \theta, \quad \text{for some } 0 < t < 1,$$

since $u^\lambda(x^1) = u^\lambda(0) = 0$ and $u_\xi^\lambda(0) = u_n^\lambda(0) \sin \theta$. Since $u^\lambda \in C^2(\overline{\Omega})$, it follows from Lemma 3.3 that

$$|u_{\xi\xi}^\lambda(tx^1) - u_{\xi\xi}^\lambda(0)| < \mu, \quad \text{when } \theta \text{ is sufficiently small.}$$

By (3.10) and Lemma 3.3 we thus obtain

$$(3.11) \quad u_{\xi\xi}^\lambda(0) \leq C_2\mu, \quad \text{for } \theta \text{ sufficiently small.}$$

Next,

$$u_{\xi\xi}^\lambda(0) = u_{11}^\lambda(0) \cos^2 \theta + u_{nn}^\lambda(0) \sin^2 \theta + 2u_{1n}^\lambda(0) \sin \theta \cos \theta.$$

We have $u_{nn}^\lambda(0) > 0$ and, from the proof of (2.15) in Section 2, $|u_{1n}^\lambda(0)| \leq C$ for some constant C independent of λ . Thus, by (3.11),

$$(3.12) \quad u_{11}^\lambda(0) \leq C_3\mu + C_4\theta, \quad \text{for } \theta \text{ sufficiently small.}$$

Finally, it follows from (see (2.6)) $\underline{u}_{11}(0) - u_{11}^\lambda(0) = (u_n^\lambda(0) - \underline{u}_n(0))\rho_{11}(0)$ and (3.2), (3.12) that

$$\rho_{11}(0) \geq c_0(\underline{u}_{11}(0) - C_3\mu - C_4\theta),$$

where $c_0 > 0$ is a uniform constant. By the strict convexity of \underline{u} we can first fix μ small, then choose θ_0 sufficiently small to complete the proof of Lemma 3.4. \square

Returning to the proof of Theorem 3.2, we first consider the case $\theta \leq \theta_0$, where θ_0 is fixed such that Lemma 3.4 holds for some $\varepsilon_0 > 0$. To set up notation we fix a positive constant r_0 depending only on Ω such that $\Gamma \equiv \{(x', \rho(x')) : |x'| \leq r_0\} \subset \partial\Omega$; by Lemma 3.3 we may assume θ_0 is sufficiently small so that $x^1 \in \Gamma$. As in [7], using Lemma 3.3 and the hypothesis that $\varphi \in C^{3,1}(\partial\Omega)$, one can prove that

$$(3.13) \quad |\varphi_{11}(0)| \leq A\theta^2, \quad |\varphi_{1j}(0)| \leq A\theta, \quad |\varphi_{ij}(0)| \leq A, \quad 1 < i, j \leq n - 1,$$

where A is a constant under control.

It follows from Lemma 3.4 that for any point $x \in \Omega$ with $|x - x^0| \leq \delta$ sufficiently small (depending on ε_0 and possibly on x^0), the ray from x^1 to x strikes Γ at a point $\bar{x} = (\bar{x}', \bar{x}_n)$ with $\bar{x}' = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1})$ satisfying

$$(3.14) \quad \theta^2(\bar{x}_1)^2 + (\bar{x}_2)^2 + \dots + (\bar{x}_{n-1})^2 \leq C|x - x^0|^2$$

for a constant C under control (here we also use the fact that $|x^1 - x^0| \geq |x^0|$). Since $u(x^1) = 0$ it follows from the local convexity of u that

$$(3.15) \quad u(x) \leq u(\bar{x}) = \varphi(\bar{x}') = \sum_{i,j=1}^{n-1} \varphi_{ij} \bar{x}_i \bar{x}_j + O(|\bar{x}'|^3).$$

Consequently by (3.13) and (3.14),

$$u(x) \leq C \left(A + \frac{\delta}{\theta^3} \right) |x - x^0|^2,$$

with C under control. Now we may fix $\delta = \delta(x^0) \leq \theta^3$ to obtain (3.7) for $\theta \leq \theta_0$.

The case $\theta > \theta_0$ is simple, and we refer the reader to [7] for the details.

The proof of Theorem 3.2 is complete, and thus so is that of Theorem 1.2. \square

4. SPACELIKE HYPERSURFACES OF PRESCRIBED GAUSS CURVATURE

Recall that Minkowski space $\mathbb{R}^{n,1}$ is the space $\mathbb{R}^n \times \mathbb{R}$ endowed with the Lorentz metric $ds^2 = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2$, where $x = (x_1, \dots, x_n)$ and x_{n+1} are the coordinates in \mathbb{R}^n and \mathbb{R} . A spacelike hypersurface of $\mathbb{R}^{n,1}$ is a codimension-one submanifold whose induced metric is Riemannian. Locally a spacelike hypersurface M is given as the graph of a function $x_{n+1} = u(x)$ satisfying the spacelike condition $|Du| < 1$. (We will also denote the hypersurface by u when it is globally given as the graph of u). The first and second fundamental forms of M are given respectively by

$$g_{ij} = \delta_{ij} - u_i u_j, \quad A_{ij} = \frac{u_{ij}}{(1 - |Du|^2)^{\frac{1}{2}}}.$$

We say M is a *locally strictly convex* hypersurface if its second fundamental form is positive definite everywhere. The Gauss-Kronecker curvature of M has the expression

$$\mathcal{K}[M] = \frac{\det(u_{ij})}{(1 - |Du|^2)^{\frac{n+2}{2}}}.$$

Thus the equation

$$(4.1) \quad \det(u_{ij}(x)) = K(x, u(x))(1 - |Du(x)|^2)^{\frac{n+2}{2}}$$

locally describes spacelike hypersurfaces with prescribed Gauss-Kronecker curvature K . As an immediate consequence of Theorem 1.1, we first state an existence result for spacelike graphs with prescribed boundary value and Gauss-Kronecker curvature. (By a graph in $\mathbb{R}^{n,1}$, we always mean a submanifold, with or without boundary, that can be represented globally as the graph of a function $x_{n+1} = u(x)$ defined in a subset of \mathbb{R}^n .)

Theorem 4.1. *Let Ω be a bounded smooth domain in \mathbb{R}^n . For given $\varphi \in C^\infty(\partial\Omega)$ and $K \in C^\infty(\bar{\Omega} \times \mathbb{R})$, $K > 0$, suppose there exists a spacelike locally strictly convex hypersurface $\underline{u} \in C^2(\bar{\Omega})$ with $\mathcal{K}[\underline{u}](x) \geq K(x, \underline{u}(x))$ for $x \in \bar{\Omega}$ and $\underline{u}|_{\partial\Omega} = \varphi$. Then there exists a spacelike locally strictly convex hypersurface $u \in C^\infty(\bar{\Omega})$ with prescribed boundary value $u|_{\partial\Omega} = \varphi$ and Gauss-Kronecker curvature $\mathcal{K}[u](x) = K(x, u(x))$ for $x \in \bar{\Omega}$, i.e., u satisfies (4.1) in Ω and the gradient bound*

$$(4.2) \quad |Du| < 1 \quad \text{on } \bar{\Omega}.$$

Proof. Note that since \underline{u} is a locally strictly convex subsolution of (4.1), Theorem 4.1 will follow from Theorem 1.1 once the *a priori* gradient bound (4.2) is established. By the local convexity of u we see that $|Du|$ attains its maximum value on $\partial\Omega$. Next, we want to show that

$$(4.3) \quad \max_{\partial\Omega} |Du| \leq \max_{\partial\Omega} |D\underline{u}|,$$

which implies (4.2) since \underline{u} is a spacelike hypersurface.

Let γ denote the interior normal vector to $\partial\Omega$ and let $\xi \in \mathbb{R}^n$ be a unit vector. Consider an arbitrary point $\bar{x} \in \partial\Omega$. If $\xi \cdot \gamma(\bar{x}) \leq 0$, then

$$(4.4) \quad u_\xi(\bar{x}) \leq \underline{u}_\xi(\bar{x}) \leq |D\underline{u}(\bar{x})|,$$

since $u \geq \underline{u}$ in Ω and $u = \underline{u}$ on $\partial\Omega$. Now suppose $\xi \cdot \gamma(\bar{x}) > 0$, and let $y \in \partial\Omega$ be the first point where the ray $\bar{x} + t\xi$, $t > 0$, touches $\partial\Omega$. Then we have

$$(4.5) \quad u_\xi(\bar{x}) \leq u_\xi(y) \leq \underline{u}_\xi(y) \leq |D\underline{u}(y)|.$$

The first inequality follows from the local convexity of u , the second from (4.4) since $\xi \cdot \gamma(y) \leq 0$. Finally, suppose $|Du(\bar{x})| \neq 0$ and take $\xi = Du(\bar{x})/|Du(\bar{x})|$. From (4.4) and (4.5) it follows that

$$|Du(\bar{x})| = u_\xi(\bar{x}) \leq \max_{\partial\Omega} |D\underline{u}|.$$

This proves (4.3). □

We note that if M is a compact spacelike hypersurface and ∂M is a graph over the boundary of a domain $\Omega \subset \mathbb{R}^n$, then M is necessarily a graph over Ω . Thus Theorem 1.3 follows from Theorem 4.1 and Theorem 1.2 when Γ is a graph. To prove Theorem 1.3 in the general situation, we formulate an extension of Theorem 4.1 as follows: Let U be a compact domain that immerses into \mathbb{R}^n with smooth boundary ∂U , and let $\pi : U \rightarrow \mathbb{R}^n$ denote this immersion. Given a function $u : U \rightarrow \mathbb{R}$, one obtains a hypersurface of $\mathbb{R}^{n,1}$ defined by

$$(4.6) \quad X : U \rightarrow \mathbb{R}^{n,1}, \quad X(q) = (\pi(q), u(q)) \quad \text{for } q \in U.$$

Theorem 4.2. *Let $\varphi \in C^\infty(\partial U)$ and $K \in C^\infty(\mathbb{R}^{n+1})$, $K > 0$. Suppose there exists a spacelike locally strictly convex hypersurface \tilde{M} of $\mathbb{R}^{n,1}$ represented by*

$$(4.7) \quad q \in U \mapsto (\pi(q), \underline{u}(q)) \in \mathbb{R}^{n,1}$$

with $\underline{u} \in C^2(\bar{U})$ and $\underline{u}|_{\partial U} = \varphi$, such that $\mathcal{K}[\tilde{M}](q) \geq K(\pi(q), \underline{u}(q))$ for $q \in U$. Then there exists a spacelike locally strictly convex hypersurface M given by (4.6) with $u \in C^\infty(\bar{U})$ satisfying

$$(4.8) \quad \mathcal{K}[M](q) = K(\pi(q), u(q)) \quad \text{for } q \in U, \quad u|_{\partial U} = \varphi.$$

Proof. We observe that it suffices to prove that, with respect to the metric on U induced by the immersion $\pi : U \rightarrow \mathbb{R}^n$, the Monge-Ampère equation

$$(4.9) \quad \det(u_{i,j}) = K(1 - |Du|^2)^{\frac{n+2}{2}} \quad \text{in } U, \quad u = \varphi \quad \text{on } \partial U$$

has a locally strictly convex solution in $C^\infty(\bar{U})$ that satisfies the spacelike condition

$$(4.10) \quad |Du| < 1 \quad \text{in } U.$$

The existence of such a solution will follow from Theorem 5.1 in Section 5 once (4.10) is derived. To complete the proof, one observes that (4.10) can be derived as in the proof of Theorem 4.1 with some slight modification. □

The case $K \equiv 0$ leads to the degenerate Monge-Ampère equation.

Theorem 4.3. *Let $\varphi \in C^{3,1}(\partial U)$ and suppose there exists a spacelike locally strictly convex hypersurface \tilde{M} of $\mathbb{R}^{n,1}$ given by (4.7) with $\underline{u} \in C^2(\bar{U})$ and $\underline{u}|_{\partial U} = \varphi$. Then there exists a locally convex spacelike hypersurface M given by (4.6) with $u \in C^{1,1}(\bar{U})$ and $u|_{\partial U} = \varphi$, whose Gauss-Kronecker curvature vanishes everywhere.*

Proof. The existence of M of the form (4.6) with $u \in C^{0,1}(\bar{U})$ follows from Theorem 4.2 by approximation. In order to obtain the desired $C^{1,1}$ regularity, we observe that for an arbitrary point $q \in U$, since it is locally convex, M has a supporting hyperplane, T , at $Q \equiv (\pi(q), u(q)) \in M$. By Lemma 3.1, there is a simplex $S \subset M \cap T$, containing Q , of dimension $k \leq n$ with vertices on ∂M . In a neighborhood of S , M lies above T and is given as a graph $x_{n+1} = \tilde{u}(x)$ which solves $\det(\tilde{u}_{i,j}) = 0$ weakly. Now we can repeat the proof of Theorem 3.2 to show that \tilde{u} satisfies an inequality of the form (3.4), which implies that $u \in C^{1,1}(\bar{U})$ since, clearly, $\tilde{u} = u \circ \pi$ locally. \square

Finally, in order to see that Theorem 1.3 is a consequence of Theorems 4.2 and 4.3, we take $U = \tilde{M}$ and let $\pi : U \rightarrow \mathbb{R}^n$ be the orthogonal projection from $\tilde{M} \subset \mathbb{R}^{n,1}$ to \mathbb{R}^n . By the spacelike condition, we see π is an immersion and \tilde{M} can be represented in the form (4.7) with $\underline{u} \in C^2(\bar{U})$. Theorem 1.3 thus follows.

Entire spacelike hypersurfaces with constant or prescribed mean curvature have also been studied in Minkowski space and in more general Lorentzian manifolds as well; for references please see, for example, [10]. In [9], Cheng and Yau proved a Bernstein type theorem for entire maximal spacelike hypersurfaces. It seems of interest to study entire spacelike hypersurfaces of constant Gauss-Kronecker curvature.

5. MONGE-AMPÈRE EQUATIONS ON MANIFOLDS

In this section we extend Theorem 1.1 to Monge-Ampère equations on Riemannian manifolds. Let M^n be a smooth Riemannian manifold of dimension $n \geq 2$ and $\Omega \subset M^n$ a smooth domain with compact closure $\bar{\Omega}$. We consider the Dirichlet problem

$$(5.1) \quad g^{-1} \det(\nabla_{i,j} u) = \psi(x, u, \nabla u) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where $g = \det(g_{i,j}) > 0$, $g_{i,j}$ denotes the metric of M^n , ∇ is the Levi-Civita connection, and $\nabla_{i,j} u$ denotes the Hessian of u (with respect to the metric $g_{i,j}$). We assume $\varphi \in C^\infty(\partial\Omega)$ and $\psi > 0$ is C^∞ with respect to $(x, z, p) \in \bar{\Omega} \times \mathbb{R} \times T_x M$; here $T_x M$ denotes the tangent space at $x \in M$. The main result of this section is the following analogue of Theorem 1.1, which extends some of the results in [14].

Theorem 5.1. *There exists a locally strictly convex solution of (5.1) in $C^\infty(\bar{\Omega})$, provided that there exists a locally strictly convex subsolution $\underline{u} \in C^2(\bar{\Omega})$ to (5.1). Furthermore, the solution is unique if $\psi_z \geq 0$.*

Proof. Let \mathcal{A} be as in Section 2. We shall prove the existence of a solution of (5.1) in \mathcal{A} . It is clear that the proof of Theorem 1.1 in Section 2 still works in this general case once C^2 a priori estimates are established for such solutions. According to [14], we need only estimate the second derivatives at the boundary.

About a point $x_0 \in \partial\Omega$, let e_1, \dots, e_n be a local orthonormal frame on M^n obtained by parallel translation of a local orthonormal frame on $\partial\Omega$ and the interior

unit normal vector field to $\partial\Omega$ along the geodesics perpendicular to $\partial\Omega$ on M^n . We assume e_n is the parallel translation of the unit normal field on $\partial\Omega$.

Let $u \in \mathcal{A}$ be a solution of (5.1). Since $u - \underline{u} = 0$ on $\partial\Omega$, it is straightforward to bound the pure tangential second derivatives

$$(5.2) \quad |\nabla_{\alpha\beta}u| \leq C \quad \text{on } \partial\Omega \text{ for } \alpha, \beta < n.$$

Next, we note that Lemma 2.2 readily extends to the present general case. For completeness we restate the lemma. As in (2.9), set

$$\mathcal{L} = u^{ij}\nabla_{ij} - f_{p_i}(x, u, \nabla u)\nabla_i,$$

where $\{u^{ij}\}$ is the inverse matrix of the Hessian $\{\nabla_{ij}u\}$, $f = \log \psi$. Let v be the function as defined in (2.12).

Lemma 5.2. *For N sufficiently large and t, δ sufficiently small,*

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \quad \text{in } \Omega_\delta, \quad v \geq 0 \quad \text{on } \partial\Omega_\delta,$$

where $\Omega_\delta = \Omega \cap B_\delta(x_0)$; here $B_\delta(x_0)$ denotes the geodesic ball of radius δ about x_0 .

Proof. It is the same as that of Lemma 2.2 except that (2.14) takes a simpler form:

$$u^{ij}\nabla_i d \nabla_j d \geq u^{nn}(\nabla_n d)^2,$$

since $\nabla_\beta d = 0$ for all $\beta < n$. □

Using Lemma 5.2 one can estimate $\nabla_{n\alpha}u$ on $\partial\Omega$ for $\alpha \leq n$ as in [14]. For any fixed $\alpha \leq n$, differentiate equation (5.1) and use the formula for commuting the covariant derivatives

$$\nabla_{ijk}w - \nabla_{jik}w = R^l_{kji}\nabla_l w,$$

to find

$$(5.3) \quad |\mathcal{L}\nabla_\alpha(u - \underline{u})| \leq C(1 + \sum u^{ii}).$$

The mixed normal tangential derivatives $\nabla_{\alpha n}u(x_0)$, $\alpha < n$, can be estimated the same way as in Section 2. Namely, by (5.3), we may choose $A \gg B \gg 1$ such that

$$\mathcal{L}(Av + B|x|^2 \pm \nabla_\alpha(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta,$$

where $|x|$ denotes the (geodesic) distance between x and x_0 , and

$$Av + B|x|^2 \pm \nabla_\alpha(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

since $\nabla_\alpha(u - \underline{u}) = 0$ on $\partial\Omega \cap B_\delta(x_0)$, and $|\nabla_\alpha(u - \underline{u})| \leq C$ in Ω . It follows from the maximum principle that $Av + B|x|^2 \geq |\nabla_\alpha(u - \underline{u})|$ in Ω_δ . Consequently,

$$(5.4) \quad |\nabla_{n\alpha}u(x_0)| \leq A\nabla_n v(x_0) + |\nabla_{n\alpha}\underline{u}(x_0)| \leq C, \quad \alpha < n.$$

For the double normal derivative $\nabla_{nn}u$, since u is locally convex, it suffices to derive an upper bound

$$(5.5) \quad \nabla_{nn}u \leq C \quad \text{on } \partial\Omega.$$

We use an idea of Trudinger [25]. For $x \in \partial\Omega$ let

$$\lambda(x) = \min_{|\xi|=1, \xi \in T_x(\partial\Omega)} \nabla_{\xi\xi}u(x),$$

and assume that $\lambda(x)$ is minimized at $x_0 \in \partial\Omega$ with $\xi = e_1(x_0)$, that is, $\nabla_{11}u(x_0) \leq \nabla_{\xi\xi}u(x)$ for all $x \in \partial\Omega$ and any unit vector $\xi \in T_x(\partial\Omega)$. As in [6], (5.5) will follow from

$$(5.6) \quad \nabla_{11}u(x_0) \geq c_0 > 0.$$

To show (5.6), we may assume $\nabla_{11}u(x_0) < \frac{1}{2}\nabla_{11}\underline{u}(x_0)$, since otherwise we are done as $\nabla_{11}\underline{u}(x_0) \geq c_1 > 0$ for some uniform $c_1 > 0$. We have

$$(5.7) \quad \nabla_{11}u = \nabla_{11}\underline{u} - B_{11}\nabla_n(u - \underline{u}) \quad \text{on } \partial\Omega,$$

where $B_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$, $1 \leq \alpha, \beta \leq n - 1$. It follows that

$$B_{11}(x_0)\nabla_n(u - \underline{u})(x_0) \geq \frac{1}{2}\nabla_{11}\underline{u}(x_0) \geq \frac{c_1}{2},$$

and for $x \in \partial\Omega$ near x_0 , since $\nabla_{11}u|_{\partial\Omega}$ is minimized at x_0 ,

$$B_{11}(x)\nabla_n(u - \underline{u})(x) \leq \nabla_{11}\underline{u}(x) - \nabla_{11}\underline{u}(x_0) + B_{11}(x_0)\nabla_n(u - \underline{u})(x_0).$$

Because B_{11} is smooth near $\partial\Omega$ and $0 < \nabla_n(u - \underline{u}) \leq C$, we must have $B_{11} \geq c_2 > 0$ on Ω_δ for some uniform $c_2 > 0$, if δ is chosen sufficiently small. Therefore,

$$(5.8) \quad \nabla_n(u - \underline{u})(x) \leq \Psi(x) \quad \text{for } x \in \Omega_\delta \cap \partial\Omega \quad \text{and } \nabla_n(u - \underline{u})(x_0) = \Psi(x_0)$$

where $\Psi(x) = B_{11}^{-1}(x)[\nabla_{11}\underline{u}(x) - \nabla_{11}\underline{u}(x_0) + B_{11}(x_0)\nabla_n(u - \underline{u})(x_0)]$.

We observe that since Ψ is smooth in Ω_δ , by (5.3), (5.8) and Lemma 5.2 we may choose $A \gg B \gg 1$ such that

$$Av + B|x|^2 + \Psi - \nabla_n(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

$$\mathcal{L}(Av + B|x|^2 + \Psi - \nabla_n(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta.$$

By the maximum principle, $v + \Psi - \nabla_n(u - \underline{u}) \geq 0$ in Ω_δ , and therefore

$$\nabla_{nn}u(x_0) \leq C.$$

This shows that the eigenvalues of $\{\nabla_{ij}u(x_0)\}$ are all bounded (and all positive). On the other hand, equation (5.1) says the product of these eigenvalues is bounded below away from zero by a uniform positive constant (ψ_0 as in (2.3)). Thus each of them must be bounded below away from zero. In particular, we obtain the estimate (5.6), which in turn implies (5.5). \square

We conclude this section by a remark on the following equation of Monge-Ampère type on \mathbb{S}^n :

$$(5.9) \quad g^{-1} \det(ug_{ij} + \nabla_{ij}u) = \psi(x, u, \nabla u).$$

This equation arises in various geometric problems related to Gauss curvature such as the Minkowski problem (see for example Cheng and Yau [8] and the references therein). The Dirichlet problem was studied, in connection with the boundary value problem of finding hypersurfaces in \mathbb{R}^{n+1} of prescribed Gauss-Kronecker curvature, by J. Spruck and the author in [15] and [13] under the hypothesis that $\psi^{1/n}$ is a convex function with respect to the gradient ∇u . By employing a better barrier similar to that in Lemma 5.2, we may refine the argument in [13] to prove

Theorem 5.3. *Let $\Omega \subset \mathbb{S}^n$ be a smooth domain that does not contain any hemisphere. Let $\varphi \in C^\infty(\partial\Omega)$, and let $\psi > 0$ be a smooth function. Then (5.9) has a solution $u \in C^\infty(\bar{\Omega})$ satisfying $\{ug_{ij} + \nabla_{ij}u\} > 0$ in $\bar{\Omega}$ and $u = \varphi$ on $\partial\Omega$, provided that there exists a subsolution $\underline{u} \in C^2(\bar{\Omega})$ with $\{\underline{u}g_{ij} + \nabla_{ij}\underline{u}\} > 0$ in $\bar{\Omega}$ and $\underline{u} = \varphi$ on $\partial\Omega$.*

ACKNOWLEDGEMENT

The author wishes to thank Joel Spruck, Yan-Yan Li and Rob Kusner for helpful conversations. The author would also like to thank Rick Schoen for his interest and encouragement.

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