TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 12, December 1998, Pages 4839–4847
S 0002-9947(98)02359-9

BAIRE AND σ-BOREL CHARACTERIZATIONS OF WEAKLY COMPACT SETS IN $M(T)$

T. V. PANCHAPAGESAN

Abstract. Let T be a locally compact Hausdorff space and let $M(T)$ be the Banach space of all bounded complex Radon measures on T. Let $\mathcal{B}_o(T)$ and $\mathcal{B}_c(T)$ be the σ-rings generated by the compact G_δ subsets and by the compact subsets of T, respectively. The members of $\mathcal{B}_o(T)$ are called Baire sets of T and those of $\mathcal{B}_c(T)$ are called σ-Borel sets of T (since they are precisely the σ-bounded Borel sets of T). Identifying $M(T)$ with the Banach space of all Borel regular complex measures on T, in this note we characterize weakly compact subsets A of $M(T)$ in terms of the Baire and σ-Borel restrictions of the members of A. These characterizations permit us to give a generalization of a theorem of Dieudonné which is stronger and more natural than that given by Grothendieck.

1. Introduction

For a locally compact Hausdorff space T, let $C_o(T)$ be the Banach space of all continuous complex functions vanishing at infinity in T, endowed with the supremum norm. Let $\mathcal{B}(T)$ be the σ-algebra of Borel sets in T. The dual $M(T)$ of $C_o(T)$ is the Banach space of all bounded complex Radon measures μ on T and by Theorem 5.3 of [P2] is isometrically isomorphic to the Banach space of all Borel regular complex measures μ on $\mathcal{B}(T)$, endowed with the norm $||\mu|| = |\mu|(T)$, where $|\mu|$ denotes the variation of μ in $\mathcal{B}(T)$. Hence, we identify $M(T)$ with the Banach space of all Borel regular complex measures on $\mathcal{B}(T)$. Let X be a quasicomplete locally convex Hausdorff space (briefly, a quasicomplete lcHs).

In the present note we characterize weakly compact sets A in $M(T)$ in terms of the Baire and σ-Borel restrictions of the members of A. As a consequence of the Baire characterizations, we obtain a generalization of Proposition 8 of Dieudonné [Die], which is stronger and more natural than that of Grothendieck given on p. 150 of [G].

These characterizations will be powerful enough to replace the use of Theorem 3 and Proposition 11 of [G] in the study of weakly compact operators [P3]. In fact, these results play a key role in [P3] to provide a unified approach to the study of

Received by the editors November 17, 1995.
1991 *Mathematics Subject Classification.* Primary 28A33, 28C05, 28C15; Secondary 46E27.
Key words and phrases. Bounded complex Radon measures, uniform σ-additivity, uniform Baire inner regularity, uniform σ-Borel inner regularity, uniform Borel inner regularity, weakly compact sets.
Supported by the C.D.C.H.T. project C-586 of the Universidad de los Andes, Mérida, and by the international cooperation project between CONICIT-Venezuela and CNR-Italy.

weakly compact operators $u : C_o(T) \to X$ and of regular Borel extension of X-valued σ-additive Baire measures on T. In this context we would like to point out that the study of weakly compact operators was carried out by Grothendieck [G] for complete lcHs-valued operators on $C_o(T)$, and by Bartle, Dunford and Schwartz [BDS] for Banach space valued operators on $C(\Omega)$, Ω compact, while the regular Borel extension problem for quasicomplete lcHs-valued Baire measures was studied by Dinculeanu and Kluvánek in [DK, K] by vector measure methods. As far as we know, such a unification study has not been presented earlier in the literature.

2. Preliminaries

In this section we fix notation and terminology and also give some definitions and results which will be needed in the sequel.

Let T be a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex continuous functions vanishing at infinity in T, endowed with the supremum norm. Let $\mathcal{B}(T)$ be the σ-algebra of Borel sets in T, which is the σ-algebra generated by the class of all open sets in T. Then the dual of $C_o(T)$ is the Banach space $M(T)$ of all bounded complex Radon measures μ on T, and by Theorem 5.3 of [P2] it is isometrically isomorphic to the Banach space of all Borel regular complex measures μ on $\mathcal{B}(T)$ endowed with the norm $||\mu|| = |\mu|(T)$, where $|\mu|$ denotes the variation of μ in $\mathcal{B}(T)$. Consequently, we identify $M(T)$ with the Banach space of all Borel regular complex measures on $\mathcal{B}(T)$. $\mathcal{C}(T)$ (resp. $\mathcal{C}_o(T)$) is the class of all compact subsets (resp. compact G_δ subsets) of T. $\mathcal{B}_c(T)$ (resp. $\mathcal{B}_o(T)$) is the σ-ring generated by $\mathcal{C}(T)$ (resp. by $\mathcal{C}_o(T)$)), and the members of $\mathcal{B}_c(T)$ (resp. of $\mathcal{B}_o(T)$) are called σ-Borel (resp. Baire) sets of T. Since the members of $\mathcal{B}_c(T)$ are precisely the σ-bounded Borel sets of T, we call them σ-Borel.

We need the following lemma before we give some definitions.

Lemma 1. *For $\mu \in M(T)$, let $|\mu|(\cdot) = var(\mu, (\cdot))$ in $\mathcal{B}(T)$. Then*

$$|\mu||_{\mathcal{B}_o(T)}(\cdot) = var(\mu|_{\mathcal{B}_o(T)}, (\cdot)) \text{ and } |\mu||_{\mathcal{B}_c(T)}(\cdot) = var(\mu|_{\mathcal{B}_c(T)}, (\cdot)).$$

Proof. Let $\mathcal{D}(\mathcal{C}_o(T))$ be the δ-ring generated by $\mathcal{C}_o(T)$. Then by Lemma 3.2 of [P2]

$$|\mu||_{\mathcal{D}(\mathcal{C}_o(T))}(\cdot) = var(\mu|_{\mathcal{D}(\mathcal{C}_o(T))}, (\cdot)).$$

Given $E \in \mathcal{B}_o(T)$, there exists a disjoint sequence $(E_n)_{n=1}^\infty \subset \mathcal{D}(\mathcal{C}_o(T))$ such that $E = \bigcup_{n=1}^\infty E_n$. Then

$$\begin{aligned} |\mu|(E) &= \sum_{n=1}^\infty |\mu|(E_n) = \sum_{n=1}^\infty var(\mu|_{\mathcal{D}(\mathcal{C}_o(T))}, E_n) \\ &= \sum_{n=1}^\infty var(\mu|_{\mathcal{B}_o(T)}, E_n) = var(\mu|_{\mathcal{B}_o(T)}, E). \end{aligned}$$

Now, let $E \in \mathcal{B}_c(T)$. Since it is easy to check that each $F \subset E$ with $F \in \mathcal{B}(T)$ is σ-Borel, it follows that $|\mu||_{\mathcal{B}_c(T)}(E) = var(\mu|_{\mathcal{B}_c(T)}, E)$. □

Notation 1. For $\mu \in M(T)$, let $|\mu|(\cdot) = var(\mu, (\cdot))$ on $\mathcal{B}(T)$.

In the light of the above lemma, the variations used in the following definition are unambiguously defined. The first part is an adaptation of Definition 3.2 of [P1].

Definition 1. Let $\mathcal{S}$ be a σ-ring of sets in T such that $\mathcal{C}(T) \subset \mathcal{S}$ or $\mathcal{C}_o(T) \subset \mathcal{S}$. A complex measure μ on $\mathcal{S}$ is said to be *$\mathcal{S}$-regular* if, given $E \in \mathcal{S}$ and $\varepsilon > 0$, there exist a compact set $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ such that $|\mu(B)| < \varepsilon$ for every $B \in \mathcal{S}$ with $B \subset U \backslash K$. When $\mathcal{S} = \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_o(T)$), we use the terminology *Borel (*resp. *σ-Borel, Baire) regularity* in place of $\mathcal{S}$-regularity. Let A be a subset of $M(T)$. We say that A is *uniformly Baire inner regular (*resp. *Baire regular) in a set* $E \in \mathcal{B}_o(T)$ if, given $\varepsilon > 0$, there exists a compact $K \in \mathcal{B}_o(T)$ with $K \subset E$ (resp. and an open Baire set O in T with $K \subset E \subset O$) such that $\sup_{\mu \in A} |\mu|(E \backslash K) < \varepsilon$ (resp. such that $\sup_{\mu \in A} |\mu|(O \backslash K) < \varepsilon$). If A is uniformly Baire inner regular (resp. Baire regular) in each Baire set, then A is said to be *uniformly Baire inner regular (*resp. *Baire regular).* Similarly, the *uniform Borel (*resp. *σ-Borel) regularity* and *inner regularity* of A and those of A in a Borel (resp. σ-Borel) set E are defined.

By virtue of Theorem 51.D of [H], we note that a compact $K \in \mathcal{B}_o(T)$ is necessarily a G_δ.

It is well known that every complex Baire measure μ_o is Baire regular and that it has a unique extension μ to $\mathcal{B}(T)$(resp. μ_c to $\mathcal{B}_c(T)$) such that μ is a Borel (resp. μ_c is a σ-Borel) regular complex measure. Moreover, $\mu|_{\mathcal{B}_c(T)} = \mu_c$. (See, for example, Theorem 2.4 of [P2].)

Definition 2. A family $\mathcal{F}$ of complex measures defined on a σ-ring Σ of sets is said to be *uniformly σ-additive*, if for each decreasing sequence (E_n) of members of Σ with $E_n \searrow \emptyset$, $\lim_n \mu(E_n) = 0$ uniformly in $\mu \in \mathcal{F}$.

Notation 2. Given a σ-ring Σ of sets, $ca(\Sigma)$ denotes the Banach space of all complex measures μ on Σ with $||\mu|| = \sup_{E \in \Sigma} var(\mu, E)$.

The following result is well known when Σ is a σ-algebra(see, for example, Theorem IV.9.1 of [DS]).

Proposition 1. *Let Σ be a σ-ring of subsets of a nonempty set Ω. A subset A of $ca(\Sigma)$ is relatively weakly compact if and only if A is bounded and uniformly σ-additive.*

Proof. By the Eberlein-Šmulian theorem and by the fact that for each sequence $(\mu_n) \subset ca(\Sigma)$ there exists $E \in \Sigma$ such that $var(\mu_n, F) = 0$ for each $F \in \Sigma$ with $F \cap E = \emptyset$ and for each n, we can replace the space $ca(S, \Sigma, \lambda)$ in the proof of Theorem IV.9.1 of [DS] by the space $ca(\Omega \cap E, \Sigma \cap E, \lambda)$ of all λ-continuous set functions in $ca(\Omega \cap E, \Sigma \cap E)$. Since $\Sigma \cap E$ is a σ-algebra, the rest of the argument in the proof of Theorem IV.9.1 of [DS] holds here to show that the conditions are necessary and sufficient. □

3. Main results

In this section we obtain characterizations of (bounded) relatively weakly compact subsets of $M(T)$ in terms of the Baire and σ-Borel restrictions of the members of the set in question. These characterizations are similar to those obtained by Grothendieck in Theorem 2 of [G] and those of Lemma VI.2.13 of Diestel and Uhl [DU]. As mentioned in the introduction, these results are powerful enough to replace the use of Theorem 3 and Proposition 11 of [G] in our forthcoming paper [P3] where we characterize quasicomplete lcHs-valued weakly compact operators

on $C_o(T)$. Moreover, the isolated results of Dinculeanu and Kluvánek [DK, K] on vector valued σ-additive Baire and Borel measures are deduced in [P3] as corollaries of some of these characterizations. Finally, Theorem 1 below combined with the study of Grothendieck on p.150 of [G] provides a generalization of Proposition 8 of Dieudonné [Die], which is stronger and more natural than that of Grothendieck [G]. See Corollary 1 below.

Theorem 1. *Let A be a bounded set in $M(T)$. Then the following statements are equivalent:*

(i) *A is relatively weakly compact.*
(ii) *For each disjoint sequence (O_i) of open Baire sets in T, $\lim_i \mu(O_i) = 0$ uniformly in $\mu \in A$.*
(iii) *For each disjoint sequence (O_i) of open Baire sets in T, $\lim_i |\mu|(O_i) = 0$ uniformly in $\mu \in A$.*
(iv) a) *A is uniformly Baire inner regular in each open Baire set O in T.*
b) *For each $\varepsilon > 0$, there exists a $K \in \mathcal{C}_0(T)$ such that*

$$\sup_{\mu \in A} |\mu|(T \backslash K) < \varepsilon.$$

(v) *A is uniformly Baire inner regular.*
(vi) *$A|_{\mathcal{B}_o(T)}$ is uniformly σ-additive on $\mathcal{B}_o(T)$.*
(vii) *A is uniformly Baire regular.*

Proof. By Theorem 2 of Grothendieck [G] (which is the same as Theorem 4.22.1 of Edwards [E]), (i) implies (ii).

(ii) $\Rightarrow$ (iii). Since each $\mu|_{\mathcal{B}_o(T)}$ is Baire regular for $\mu \in A$, the argument in the proof of (a) $\Rightarrow$ (b) of Lemma VI.2.13 of Diestel and Uhl [DU] can suitably be modified to show that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). Let O be an open Baire set in T or let $O = T$. Let $\varepsilon > 0$. If there exists no compact G_δ $K \subset O$ such that $\sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon$, then there is a $\mu_1 \in A$ such that $|\mu_1|(O) > \varepsilon$, for otherwise $K = \emptyset$ will provide a contradiction. If $O \in \mathcal{B}_o(T)$, then by the Baire regularity of $|\mu_1||_{\mathcal{B}_o(T)}$ there exists a compact G_δ $K_1 \subset O$ such that $|\mu_1|(K_1) > \varepsilon$. If $O = T$, then by the Borel regularity of $|\mu_1|$ there exists a compact K such that $|\mu_1|(K) > \varepsilon$. Then by Theorem 50.D of Halmos [H] there exists a compact G_δ K_1 such that $K \subset K_1$ and hence $|\mu_1|(K_1) > \varepsilon$. Since K_1 is a subset of O, again by Theorem 50.D of [H] there exist an open Baire set O_1 and a compact G_δ F_1 such that

$$O \supset F_1 \supset O_1 \supset K_1.$$

Moreover, $|\mu_1|(O_1) \geq |\mu_1|(K_1) > \varepsilon$. Since F_1 is a compact G_δ subset of O, by our assumption there exists $\mu_2 \in A$ such that $|\mu_2|(O \backslash F_1) > \varepsilon$. If $O \neq T$, then using the Baire regularity of $|\mu_2|$ in $O \backslash F_1$, and if $O = T$, then using the Borel regularity of $|\mu_2|$ in $O \backslash F_1$ and then applying Theorem 50.D of [H], we can choose a compact G_δ $C_1 \subset O \backslash F_1$ such that $|\mu_2|(C_1) > \varepsilon$. Let $K_2 = F_1 \cup C_1$. Then K_2 is a compact G_δ, $O \supset K_2 \supset F_1$ and $|\mu_2|(K_2 \backslash F_1) = |\mu_2|(C_1) > \varepsilon$. Again by Theorem 50.D of [H] there exist an open Baire set O_2 and a compact G_δ F_2 such that

$$O \supset F_2 \supset O_2 \supset K_2 \supset F_1 \supset O_1 \supset K_1.$$

Accordingly, $|\mu_2|(O_2 \backslash F_1) \geq |\mu_2|(K_2 \backslash F_1) > \varepsilon$. Next, by our assumption there exists $\mu_3 \in A$ such that $|\mu_3|(O \backslash F_2) > \varepsilon$. If $O \neq T$, then using the Baire regularity of $|\mu_3|$

in $O\backslash F_2$, and if $O = T$, then using the Borel regularity of $|\mu_3|$ in $O\backslash F_2$ and then applying Theorem 50.D of [H], we can choose a compact G_δ $C_2 \subset O\backslash F_2$ such that $|\mu_3|(C_2) > \varepsilon$. Let $K_3 = F_2 \cup C_2$. Then K_3 is a compact G_δ, $O \supset K_3 \supset F_2$ and $|\mu_3|(K_3\backslash F_2) = |\mu_3|(C_2) > \varepsilon$. Again by Theorem 50.D of [H] there exist an open Baire set O_3 and a compact G_δ F_3 such that

$$O \supset F_3 \supset O_3 \supset K_3 \supset F_2 \supset O_2$$

and hence $|\mu_3|(O_3\backslash F_2) \geq |\mu_3|(K_3\backslash F_2) > \varepsilon$.

Proceeding as in the proof of (b) $\Rightarrow$ (c) of Lemma VI.2.13 of [DU], applying Theorem 50.D of [H] in each step and using the Baire regularity of each $|\mu||_{\mathcal{B}_o(T)}$ for $\mu \in A$ or using the Borel regularuty of $\mu \in A$ and then applying Theorem 50.D of [H], as the case may be, we can produce an increasing sequence (O_n) of open Baire sets in T, another two increasing sequences (K_n) and (F_n) of compact G_δs in T and a sequence (μ_n) in A such that

$$O \supset \cdots \supset F_{n+1} \supset O_{n+1} \supset K_{n+1} \supset F_n \supset O_n \supset \cdots \supset K_2 \supset F_1 \supset O_1 \supset K_1$$

and

$$|\mu_{n+1}|(O_{n+1}\backslash F_n) > \varepsilon$$

for all $n \geq 1$. Let $G_{n+1} = O_{n+1}\backslash F_n,\ n \geq 1$. Then (G_{n+1}) is a disjoint sequence of open Baire sets in T and satisfies $|\mu_{n+1}|(G_{n+1}) > \varepsilon$ for $n \geq 1$. This contradicts (iii), and hence (iv) holds.

(iv) $\Rightarrow$ (v). Let $\varepsilon > 0$. By (iv)(b) there exists a compact G_δ Ω in T such that

$$\sup_{\mu \in A} |\mu|(T\backslash\Omega) < \frac{\varepsilon}{2}. \tag{3.1}$$

We shall now modify the proof of (c) $\Rightarrow$ (d) of Lemma VI.2.13 of [DU] to show that (v) holds. Let $\mathcal{C}_o(\Omega) = \{K \subset \Omega : K$ compact G_δ in Ω with respect to the relative topology of $\Omega\}$. It is easy to check that

$$\mathcal{C}_o(\Omega) = \mathcal{C}_o(T) \cap \Omega = \{K \subset \Omega : K \in \mathcal{C}_o(T)\}. \tag{3.2}$$

Let

$$\begin{aligned} \mathcal{S} \quad = \quad & \{E \in \mathcal{B}_o(\Omega) : \text{for each } \varepsilon' > 0, \text{ there exists } K \in \mathcal{C}_o(\Omega) \\ & \text{such that } E \cap K \text{ is compact and } \sup_{\mu \in A} |\mu|(\Omega\backslash K) \leq \varepsilon'\} \ . \end{aligned}$$

If $E \in \mathcal{B}_o(\Omega)$ and $K \in \mathcal{C}_o(\Omega)$ are such that $E \cap K$ is compact, then by Theorem 51.D of Halmos [H], $E \cap K \in \mathcal{C}_o(\Omega)$. Clearly, $\mathcal{C}_o(\Omega) \subset \mathcal{S}$, since for $C \in \mathcal{C}_o(\Omega)$ we know that $C \cap \Omega = C$ is compact and $|\mu|(\Omega\backslash\Omega) = 0$ for $\mu \in A$.

Claim 1. For each open Baire set O in T, $O \cap \Omega \in \mathcal{S}$.

In fact, $O\cap\Omega \in \mathcal{B}_o(T)\cap\Omega = \mathcal{S}(\mathcal{C}_o(T))\cap\Omega = \mathcal{S}(\mathcal{C}_o(T)\cap\Omega) = \mathcal{S}(\mathcal{C}_o(\Omega)) = \mathcal{B}_o(\Omega)$ by (3.2) and by Theorem 5.E of Halmos [H], where $\mathcal{S}(\mathcal{E})$ denotes the σ-ring generated by the class $\mathcal{E}$. Given $\varepsilon' > 0$, by (iv)(a) there exists $K \in \mathcal{C}_o(T)$ with $K \subset O$ such that

$$\sup_{\mu \in A} |\mu|(O\backslash K) \leq \varepsilon'. \tag{3.3}$$

Let $K_o = K \cap \Omega$. Then $K_o \in \mathcal{C}_o(\Omega)$ by (3.2), and moreover, $O \cap \Omega \cap K_o = K_o$ is compact. Further, as $(O \cap \Omega)\backslash K_o \subset O\backslash K$, by (3.3) we have

$$\sup_{\mu \in A} |\mu|((O \cap \Omega)\backslash K_o) \leq \varepsilon'. \tag{3.4}$$

Let $K_1 = K_o \cup (\Omega\backslash O)$. Then by Theorem 51.D of Halmos [H] and by (3.2), $\Omega\backslash O \in \mathcal{C}_o(\Omega)$ and hence $K_1 \in \mathcal{C}_o(\Omega)$. Moreover, $O \cap \Omega \cap K_1 = K_o$ is compact, and by (3.4)

$$\sup_{\mu \in A} |\mu|(\Omega\backslash K_1) = \sup_{\mu \in A} |\mu|((O \cap \Omega)\backslash K_o) \leq \varepsilon'.$$

Thus $O \cap \Omega \in \mathcal{S}$.

Claim 2. For $K \in \mathcal{C}_o(\Omega)$, $\Omega\backslash K \in \mathcal{S}$.

In fact, by (3.2) K is of the form $K = \bigcap_1^\infty V_n$, where the V_n are open Baire sets in T (see Proposition 14, §14, Chapter III of [Din]). Then $\Omega\backslash K = \bigcup_1^\infty (\Omega\backslash V_n)$. Now, by Theorem 50.D of Halmos [H] there exists an open Baire set W_n in T such that $\Omega\backslash V_n \subset W_n$ for $n \geq 1$. Let $W = \bigcup_1^\infty W_n$. Then W is an open Baire set in T and $\Omega\backslash K = (\Omega\backslash K) \cap W = \Omega \cap (W\backslash K)$. Since $W\backslash K$ is an open Baire set in T, by Claim 1 we conclude that $\Omega\backslash K \in \mathcal{S}$.

To show that $\mathcal{S}$ is closed under countable intersections, let (E_n) be a sequence in $\mathcal{S}$ and let $\varepsilon' > 0$. Then, proceeding as on p.158 of [DU], we get a sequence (K_n) in $\mathcal{C}_o(\Omega)$ such that $E_n \cap K_n$ is compact and $\sup_{\mu \in A} |\mu|(\Omega\backslash K_n) \leq \frac{\varepsilon'}{2^n}$ for each $n \geq 1$. Then the set

$$(\bigcap_{n=1}^\infty E_n) \cap (\bigcap_{n=1}^\infty K_n) = \bigcap_{n=1}^\infty (E_n \cap K_n)$$

is compact and

$$\sup_{\mu \in A} |\mu|(\Omega\backslash \bigcap_1^\infty K_n) \leq \sum_{n=1}^\infty \sup_{\mu \in A} |\mu|(\Omega\backslash K_n) \leq \varepsilon'.$$

Thus $\bigcap_1^\infty E_n \in \mathcal{S}$.

To verify that $\mathcal{S}$ is also closed under complements in Ω, let $E \in \mathcal{S}$ and let $\varepsilon' > 0$. Then there exists $K_1 \in \mathcal{C}_o(\Omega)$ such that $E \cap K_1$ is compact and $\sup_{\mu \in A} |\mu|(\Omega\backslash K_1) \leq \frac{\varepsilon'}{2}$. Now, by Claim 2 and by Theorem 51.D of [H], we have $\Omega\backslash(E \cap K_1) \in \mathcal{S}$. Therefore there exists $K_2 \in \mathcal{C}_o(\Omega)$ such that $(\Omega\backslash(E \cap K_1)) \cap K_2$ is compact and $\sup_{\mu \in A} |\mu|(\Omega\backslash K_2) \leq \frac{\varepsilon'}{2}$. Then $(K_1 \cap K_2) \cap (\Omega\backslash E) = K_1 \cap K_2 \cap (\Omega\backslash(E \cap K_1))$ is compact and

$$\sup_{\mu \in A} |\mu|(\Omega\backslash(K_1 \cap K_2)) \leq \sup_{\mu \in A} |\mu|(\Omega\backslash K_1) + \sup_{\mu \in A} |\mu|(\Omega\backslash K_2) < \varepsilon'.$$

Thus $\Omega\backslash E \in \mathcal{S}$. Consequently, $\mathcal{S}$ is a σ-algebra in Ω.

Since $\mathcal{C}_o(\Omega) \subset \mathcal{S} \subset \mathcal{B}_o(\Omega)$, it follows that $\mathcal{S} = \mathcal{B}_o(\Omega)$. Thus, for each $E \in \mathcal{B}_o(\Omega)$ and $\varepsilon' > 0$, there exists $K \in \mathcal{C}_o(\Omega)$ such that $E \cap K$ is compact and $\sup_{\mu \in A} |\mu|(\Omega\backslash K) \leq \varepsilon'$. Then

$$\sup_{\mu \in A} |\mu|(E\backslash(E \cap K)) \leq \sup_{\mu \in A} |\mu|(\Omega\backslash K) \leq \varepsilon'. \tag{3.5}$$

Now let $E \in \mathcal{B}_o(T)$ and $\varepsilon' = \frac{\varepsilon}{2}$. Then $E \cap \Omega \in \mathcal{B}_o(\Omega)$ by (3.2) and by Theorem 5.E of [H]. Consequently, using $E \cap \Omega \in \mathcal{B}_o(\Omega)$ in place of E above, as in (3.5) there

exists $K \in \mathcal{C}_o(\Omega)$ such that $(E \cap \Omega) \cap K = K_o$ (say) is compact and

$$\sup_{\mu \in A} |\mu|((E \cap \Omega) \backslash K_o) \leq \frac{\varepsilon}{2}. \tag{3.6}$$

Thus $K_o \in \mathcal{C}_o(T)$, $K_o \subset E$ and

$$\sup_{\mu \in A} |\mu|(E \backslash K_o) \leq \sup_{\mu \in A} |\mu|((E \cap \Omega) \backslash K_o) + \sup_{\mu \in A} |\mu|(T \backslash \Omega) < \varepsilon$$

by (3.6) and (3.1). Thus (v) holds.

Replacing in the proof of (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) of Lemma VI.2.13 of [DU] compact sets, Borel sets and open sets in Ω respectively by compact G_δ sets in T, Baire sets in T and open Baire sets in T, one can easily show that (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (ii).

Finally, to show that (vi) $\Rightarrow$ (i), let $\Phi(\mu) = \mu|_{\mathcal{B}_o(T)}$ for $\mu \in M(T)$ and let $\mathcal{M}_o(T) = \{\mu : \mathcal{B}_o(T) \to \mathbb{C}, \mu\ \sigma\text{-additive}\}$ with $||\mu||_o = \sup_{E \in \mathcal{B}_o(T)} |\mu|(E)$ for $\mu \in \mathcal{M}_o(T)$. Then by Lemma 1 and by Theorem 5.3 of [P2], Φ is an isometric isomorphism of $M(T)$ onto $\mathcal{M}_o(T)$. By Proposition 1, and by (vi), $\Phi(A)$ is relatively weakly compact in $\mathcal{M}_o(T)$. Consequently, A is relatively weakly compact in $M(T)$. □

Corollary 1 (Generalization of Proposition 8 of Dieudonné [Die])**.** *A bounded sequence (μ_i) in $M(T)$ is weakly convergent if and only if, for each open Baire set O in T, $\lim_i \mu_i(O)$ exists in $\mathbb{C}$.*

Proof. We only have to show that the condition is sufficient. By regularity, each complex Baire measure μ in T is determined by its restriction on the lattice of all open Baire sets. Moreover, each complex Baire measure has a unique regular Borel extension. These facts and the Eberlein-Šmulian theorem ensure that it suffices to show that (μ_i) is relatively weakly compact in $M(T)$. Arguing as in the proof of Corollary 4.22.2 of Edwards [E], one can show that $\lim_i \mu_n(O_i) = 0$ uniformly in $n \in \mathbb{N}$ for each disjoint sequence (O_i) of open Baire sets in T. Then by the equivalence of (i) and (ii) of Theorem 1 it follows that (μ_i) is relatively weakly compact in $M(T)$. □

Remark 1. When T is compact, the proof of Proposition 9 in [Die] holds verbatim to show that the hypothesis that $\lim_i \mu_i(U)$ exists in $\mathbb{C}$ for each open set U in T implies that (μ_i) is bounded. When T is locally compact, one can argue with its one-point compactification as on p.177 of Thomas [T] to show that the above hypothesis also ensures the boundedness of (μ_i) in $M(T)$. Again, when T is compact, using Theorem 50.D of [H] and the Baire regularity of the $\mu_i|_{\mathcal{B}_o(T)}$ we can modify the proof of Proposition 9 in [Die] to show that (μ_i) is bounded when $\lim_i \mu_i(O)$ exists in $\mathbb{C}$ for each open Baire set O in T. However, when T is locally compact and not compact, we do not know whether the boundedness condition can be dispensed with in the above corollary. When T is metrizable and compact $\mathcal{B}(T) = \mathcal{B}_o(T)$, and hence the above corollary reduces to Proposition 8 of Dieudonné [Die]. Thus the present generalization is more natural, and also stronger, than that of Grothendieck on p.150 of [G].

Theorem 2. *Let A be a bounded set in $M(T)$. Then the following statements are equivalent:*

(i) *A is relatively weakly compact.*

(ii) *For each disjoint sequence* (O_i) *of* σ*-Borel open sets (resp. (ii)*$'$ *open sets) in* T*,*

$$\lim_i \mu(O_i) = 0$$

uniformly in $\mu \in A$*.*

(iii) *For each disjoint sequence* (O_i) *of* σ*-Borel open sets (resp. (iii)*$'$ *open sets) in* T*,*

$$\lim_i |\mu|(O_i) = 0$$

uniformly in $\mu \in A$*.*

(iv) (a) A *is uniformly* σ*-Borel inner regular in each* σ*-Borel open set* O *in* T*.*
(b) *For each* $\varepsilon > 0$*, there exists a compact* K *in* T *such that*

$$\sup_{\mu \in A} |\mu|(T \backslash K) \leq \varepsilon.$$

(resp. (iv)$'$ A *is uniformly Borel inner regular in each open set* O *in* T*).*

(v) A *(resp. (v)*$'$ A*) is uniformly* σ*-Borel (resp. Borel) inner regular.*

(vi) $A|_{\mathcal{B}_c(T)}$ *(resp. (vi)*$'$ A*) is uniformly* σ*-additive on* $\mathcal{B}_c(T)$ *(resp. on* $\mathcal{B}(T)$*).*

(vii) A *(resp. (vii)*$'$ A*) is uniformly* σ*-Borel (resp. Borel) regular.*

Proof. Let

$$\mathcal{M}_c(T) = \{\mu : \mathcal{B}_c(T) \to \mathbb{C}, \mu \ \sigma\text{-additive and } \sigma\text{-Borel regular}\}$$

with $||\mu||_c = \sup_{E \in \mathcal{B}_c(T)} |\mu|(E)$, and let $\Psi : M(T) \to \mathcal{M}_c(T)$ be given by $\Psi(\mu) = \mu|_{\mathcal{B}_c(T)}$. Then by Lemma 1 and by Theorem 5.3 of [P2], Ψ is an isometric isomorphism of $M(T)$ onto $\mathcal{M}_c(T)$. This fact and an argument similar to that in the proof of Theorem 1 can be used to show that (i) $\Rightarrow$ (ii) (resp. (ii)$'$) $\Rightarrow$ (iii) (resp. (iii)$'$) $\Rightarrow$ (iv) (resp.(iv)$'$); (v) (resp. (v)$'$) $\Rightarrow$ (vi) (resp. (vi)$'$) $\Rightarrow$ (vii) (resp. (vii)$'$) $\Rightarrow$ (ii) (resp. (ii)$'$) and (vi) (resp.(vi)$'$) $\Rightarrow$ (i).

Now we shall prove (iv) (resp. (iv)$'$) $\Rightarrow$ (v) (resp. (v)$'$). Given $\varepsilon > 0$, by (iv)(b) (resp. by (iv)$'$) there exists a compact set Ω in T such that

$$\sup_{\mu \in A} |\mu|(T \backslash \Omega) < \frac{\varepsilon}{2}. \tag{3.7}$$

Let

$$\begin{aligned} \Sigma \quad = \quad & \{E \in \mathcal{B}(\Omega) : \text{for each } \varepsilon' > 0, \text{ there exists a compact } K \subset \Omega \\ & \text{such that } E \cap K \text{ is compact and } \sup_{\mu \in A} |\mu|(\Omega \backslash K) \leq \varepsilon'\}. \end{aligned}$$

Clearly, $\mathcal{C}(\Omega) = \{K \subset \Omega : K \quad \text{compact}\} \subset \Sigma$.

We claim that $O \cap \Omega \in \Sigma$ for each σ-Borel open set (resp. open set) O in T. In fact, given $\varepsilon' > 0$, by (iv)(a) (resp. (iv)$'$) there exists a compact K in T with $K \subset O$ such that $\sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon'$. Let $K_o = K \cap \Omega$. Then $O \cap \Omega \cap K_o = K_o$ is compact, and clearly $O \cap \Omega \in \mathcal{B}(\Omega)$. Moreover,

$$\sup_{\mu \in A} |\mu|((O \cap \Omega) \backslash K_o) \leq \sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon'.$$

Setting $K_1 = K_o \cup (\Omega \backslash O)$, we note that K_1 is compact , $K_1 \subset \Omega$, $(\Omega \cap O) \cap K_1 = K_o \in \mathcal{C}(T)$ and $\sup_{\mu \in A} |\mu|(\Omega \backslash K_1) \leq \varepsilon'$. Thus $O \cap \Omega \in \Sigma$.

We also claim that $\Omega \backslash K \in \Sigma$ for each compact $K \subset \Omega$. In fact, by Theorem 50.D of [H] there exists a relatively compact open set U in T such that $\Omega \subset U$. Clearly

U is a σ-Borel open set in T, and $\Omega\backslash K = (\Omega\backslash K) \cap U = \Omega \cap (U\backslash K)$ with $U\backslash K$ a σ-Borel open set in T. Then by the foregoing claim it follows that $\Omega\backslash K \in \Sigma$.

Proceeding as on p.158 of [DU], one can show that Σ is closed under countable intersections. The argument used in the proof of (iv) $\Rightarrow$ (v) of Theorem 1 to show that $\mathcal{S}$ is closed under complements can be modified here to prove that Σ is also closed under complements in Ω. Thus Σ is a σ-algebra in Ω. As $\mathcal{C}(\Omega) \subset \Sigma \subset \mathcal{B}(\Omega)$, it follows that $\Sigma = \mathcal{B}(\Omega)$. Then arguing as in the last part of (iv) $\Rightarrow$ (v) of Theorem 1 but using (3.7) in place of (3.1), we conclude that (v) (resp. (v)$'$) holds. $\square$

References

[BDS] R.G. Bartle, N. Dunford, and J.T. Schwartz, *Weak compactness and vector measures*, Canad. J.Math. **7**, (1955), 289–305. MR **16:**1123c

[DU] J. Diestel, and J.J. Uhl, *Vector Measures*, Survey No.15, Amer. Math. Soc. Providence, R.I., (1977). MR **56:**12216

[Die] J. Dieudonné, *Sur la convergence des suites de mesures de Radon*, Anais Acad. Bras. Ciencias **23**, (1951), 21–38. MR **13:**121a

[Din] N. Dinculeanu, *Vector Measures*, Pergamon Press, New York, (1967). MR **34:**6011

[DK] N. Dinculeanu, and I. Kluvánek, *On vector measures*, Proc. London Math. Soc. **17**, (1967), 505–512. MR **35:**5571

[DS] N. Dunford, and J.T. Schwartz, *Linear Operators, Part I, General Theory*, Interscience, New York, (1958). MR **22:**8302

[E] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, New York, (1965). MR **36:**4308

[G] A. Grothendieck, *Sur les applications linéares faiblement compactes d'espaces du type C(K)*, Canad. J. Math. **5**, (1953), 129–173. MR **15:**438b

[H] P. R. Halmos, *Measure Theory*, Van Nostrand, New York, (1950). MR **11:**504d

[K] I. Kluvánek, *Characterization of Fourier-Stieltjes transforms of vector and operator valued measures*, Czech. Math. J. **17**, (1967), 261–277. MR **37:**6430

[P1] T.V. Panchapagesan, *On complex Radon measures I*, Czech. Math. J. **42**, (1992), 599–612. MR **94c:**28014

[P2] ———, *On complex Radon measures II*, Czech. Math. J. **43**, (1993), 65–82. MR **94i:**28011

[P3] ———, *Characterizations of weakly compact operators on $C_o(T)$*, Trans. Amer. Math. Soc. **350** (1998), 4849–4867.

[T] E. Thomas, *L'intégration par rapport a une mesure de Radon vectorielle*, Ann. Inst. Fourier (Grenoble) **20**, (1970), 55–191. MR **57:**3348

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela

E-mail address: panchapa@ciens.ula.ve