THE STABLE HOMOTOPY TYPES OF STUNTED LENS SPACES MOD 4

HUAIJIAN YANG

Abstract. Let $L^{n+k}_n$ be the mod 4 stunted lens space $L^{n+k}/L^{n-1}$. Let $\nu(m)$ denote the exponent of 2 in $m$, and $\phi(k)$ the number of integers $j$ satisfying $j \equiv 0,1,2,4 \pmod{8}$, and $0 < j \leq k$. In this paper we complete the classification of the stable homotopy types of mod 4 stunted lens spaces. The main result (Theorem 1.3 (i)) is that, under some appropriate conditions, $L^{n+k}_n$ and $L^{m+k}_m$ are stably equivalent iff $\nu(n-m) \geq \phi(k) + \delta$, where $\delta = -1, 0$ or 1.

1. Introduction

Let $L^{n+k}_n = L^{n+k}/L^{n-1}$ and $P^{n+k}_n = P^{n+k}/P^{n-1}$ be respectively the mod 4 stunted lens space and the stunted real projective space. Let $\nu(m)$ denote the exponent of 2 in $m$, and $\phi(k)$ the number of integers $j$ satisfying $j \equiv 0,1,2,4 \pmod{8}$, and $0 < j \leq k$. Two stunted lens spaces $L^{n+k}_n$ and $L^{m+k}_m$ are said to be stably equivalent, denoted by $L^{n+k}_n \sim L^{m+k}_m$, if there exists a homotopy equivalence $\Sigma^N L^{n+k}_n \rightarrow \Sigma^{N+n-m} L^{m+k}_m$ for some $N$.

Classification of the stable homotopy types of stunted real projective spaces was begun by Feder, Gitler, and Mahowald in 1977 ([7]), and finished by Davis and Mahowald in 1986 ([4]). For an odd prime $p$, the classification of mod $p$ stunted lens spaces was recently finished by Gonzalez ([9]). The classification of mod $2^r$ stunted lens spaces, in particular the mod 4 stunted lens spaces, has been a favorite subject of some mathematicians for many years ([8], [11], [13], [14], [20]).

Let $\delta(n,k)$ and $\epsilon(n,k)$ be functions defined by the mod 4 values of $n$ and the mod 8 values of $k$, as given by the following tables:

<table>
<thead>
<tr>
<th>$k \equiv n \pmod{4}$</th>
<th>4 5 6 7</th>
<th>$k \equiv n \pmod{8}$</th>
<th>0 1 2 3 4 5 6 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 3 3 3</td>
<td>0</td>
<td>-1 -1 -1 -1 -1 0 0</td>
</tr>
<tr>
<td>1</td>
<td>3 3 3 4</td>
<td>1</td>
<td>0 0 -1 0 -1 0 0 1</td>
</tr>
<tr>
<td>2</td>
<td>3 3 4 4</td>
<td>2</td>
<td>0 -1 0 0 -1 -1 1 1</td>
</tr>
<tr>
<td>3</td>
<td>3 4 4 4</td>
<td>3</td>
<td>0 0 0 0 -1 1 1 1</td>
</tr>
</tbody>
</table>

Function $\epsilon(n,k)$ Function $\delta(n,k)$

Received by the editors June 6, 1995.
1991 Mathematics Subject Classification. Primary 55T15, 55T25.

©1998 American Mathematical Society

4775
Recall that $L_n^{n+k}$ is $S$-coreducible if there exists a map $f : \Sigma^N L_n^{n+k} \to S^{N+n}$ for some $N \geq 0$ such that the composite $S^{N+n} \to \Sigma^N L_n^{n+k} \to S^{N+n}$ is of degree one, where $i$ is the inclusion to the bottom cell. While $F_n^{n+k}$ is $S$-reducible if there is a map $g : S^{N+n+k} \to \Sigma^N L_n^{n+k}$ for some $N \geq 0$ such that $S^{N+n+k} \to \Sigma^N L_n^{n+k} \to S^{N+n+k}$ is of degree one, where $p$ is the standard projection. It is known that the $S$-dual of $L_n^{n+k}$ is $\Sigma L_n^{n+k}$. Thus it is clear that $L_n^{n+k}$ is $S$-coreducible if its $S$-dual $DL_n^{n+k}$ is $S$-reducible.

In this paper we complete the classification of the stable homotopy types of mod 4 stunted lens spaces by giving proofs to the following Theorems 1.1-1.3. Here Theorem 1.1 follows from the order of $J(\alpha_k-2)$ in $J(L^k)$, the $J$-group of $L^k$. In Theorem 1.3, results for $n \equiv 2 \pmod{4}$ and $k \equiv 2, 3, 6, 7 \pmod{8}$; $n \equiv 3 \pmod{4}$ and $k \equiv 2, 5, 6 \pmod{8}$; can be found in [8, Theorem 1.6] and [13, Theorem 1].

**Theorem 1.1.** (i) Let $0 < k < 4$. Then $L_n^{n+k} \sim L_m^{m+k}$ iff $\nu(n-m) \geq 1$ when $k = 1$, and $2$ when $k = 2, 3$.

(ii) Let $k \geq 4$. Then $L_n^{n+k}$ is $S$-coreducible (resp. $S$-reducible) iff

$$\nu(n) \ (\text{resp. } \nu(n+k+1)) \geq \begin{cases} \phi(k) + 1 & \text{when } k \equiv 0, 1, 4, 5, 6, 7 \pmod{8}, \\ \phi(k) & \text{when } k \equiv 2, 3 \pmod{8}. \end{cases}$$

(iii) Let $k \geq 4$. Suppose either $L_n^{n+k}$ or $L_m^{m+k}$ is $S$-coreducible (or $S$-reducible). Then $L_n^{n+k} \sim L_m^{m+k}$ iff

$$\nu(n-m) \geq \begin{cases} \phi(k) + 1 & \text{when } k \equiv 0, 1, 4, 5, 6, 7 \pmod{8}, \\ \phi(k) & \text{when } k \equiv 2, 3 \pmod{8}. \end{cases}$$

**Theorem 1.2.** Let $4 < k < 8$. Suppose neither of $L_n^{n+k}$, $L_m^{m+k}$ is $S$-coreducible nor $S$-reducible. Then $L_n^{n+k} \sim L_m^{m+k}$ iff $\nu(n-m) \geq \epsilon(n,k)$.

**Theorem 1.3.** Let $k \geq 8$. Suppose neither of $L_n^{n+k}$, $L_m^{m+k}$ is $S$-coreducible nor $S$-reducible.

(i) If none of $\{\nu(n), \nu(m), \nu(n+k+1), \nu(m+k+1)\}$ is

$$\begin{cases} 4b+2 & \text{when } k = 8b+4, 5; \\ 4b & \text{when } k = 8b+1; \end{cases}$$

then $L_n^{n+k} \sim L_m^{m+k}$ iff $\nu(n-m) \geq \phi(k) + \delta(n,k)$.

(ii) If one of $\{\nu(n), \nu(m), \nu(n+k+1), \nu(m+k+1)\}$ is $4b+2$ when $k = 8b+4, 5$, or is $4b$ when $k = 8b+1$, then $L_n^{n+k} \sim L_m^{m+k}$ iff $\nu(n-m) \geq 4b+3$ when $k = 8b+4, 5$, or $\nu(n-m) \geq 4b+1$ when $k = 8b+1$.

The paper is organized as follows. In section 2, we study the Adams operation $\Psi^3$ on $KO$-homology. Theorems 1.1-1.3 are proved in section 3 assuming sections 4-6. Results in section 4 on $J$-homology and coextension will be used in section 6. In section 6, we study the triviality of the composite

$$\beta_t : S^{2n+1+t} \xrightarrow{\Omega_{b,c}} S^{2n+1} \vee L_{2n-b}^2 \to L_{2n-b}^{2n+t+c},$$

where $a, c, t$ are positive integers, and $b \geq 0$ is an appropriate integer such that $L_{2n-b}^{2n+t+c}$ is $S$-reducible, while $\beta_t$ is the generator of $ImJ$ on a $t$-stem. In section 5, we study the Adams filtration of $\Omega_{b,-1}$. As in [4], the determination of whether the element $\beta_t$ is null-homotopic will be crucial. $J$-homology, Adams spectral sequence (ASS), and Mahowald’s table 8.1 in [16], will be used actively.
The paper is substantially the author’s Ph.D thesis under Donald M Davis. The author is very grateful to Professor Davis for his constant encouragement and enthusiastic guidance. The author thanks the referee for many valuable suggestions, and Dr. Kono for informing him of his result [13, Theorem 2 (ii)], which leads to our Theorem 1.3 (ii) here.

2. $KO$-HOMOLOGY

**Lemma 2.1.** Let $f : L^{n+k} \to \Sigma^{-m} L^{m+k}$ be a stable equivalence and $\nu(n - m) \geq 3$.

(i) Let $\omega \in KO^{-j}((\Sigma^{-m} L^{m+k})$ be of order $2^e$. Suppose $\Psi^3(\omega) = 3^{2j+n-m)/2} \omega$ and $\Psi^3(f^* (\omega)) = 3^{2j} f^* (\omega)$. Then $\nu(n - m) \geq e - 1$.

(ii) Let $\omega \in KO^{-1}((\Sigma^{-m} L^{m+k})$ be of order $2^e$. Suppose $\Psi^3(\omega) = 3^{-2j} \omega$ and $\Psi^3(f^* (\omega)) = 3^{-2j} \omega / 2 f^* (\omega)$. Then $\nu(n - m) \geq e - 1$.

**Proof.** Notice that (ii) is dual to (i). For (i), we have $(3^{-(n-m)/2} - 1) \omega = 0$ since $\Psi^3(f^* (\omega)) = f^* (\Psi^3(\omega))$. Thus $\nu(n - m) \geq e - 1$ by [1, Lemma 8.1].

Let $L^\infty$ be the mod 4 infinite lens space with a CW-structure as indicated in [21, p. 91]. It is known that $H^*(L^\infty; Z_2) \cong Z_2[u, v]/(u^2)$, where deg $u = 1$ and deg $v = 2$, satisfying

$$ Sq^{2i}(u^j) = \left( \begin{array}{c} j \\ i \end{array} \right) u^{i+j}, \quad Sq^{2i+1}(uv^j) = \left( \begin{array}{c} j \\ i \end{array} \right) u^{i+j} v^j, \quad Sq^{2i+1}(-) = 0. $$

Let $\rho : P^{\infty} \to L^\infty$ be the covering map. Then $\rho$ induces a stable map $P^{n+k} \to L^{n+k}$ denoted also by $\rho$. The following lemma is immediate.

**Lemma 2.2.** (i) The map $\rho : P^{m+k} \to L^{m+k}$ is of odd degree on even dimensional cells, and even degree on odd dimensional cells.

(ii) If $m \leq 2n + 1 \leq m + k$, then $\rho_* : H_{2n+1}(P^{m+k}; Z) \to H_{2n+1}(L^{m+k}; Z)$ is a monomorphism $Z_2 \to Z_4$, and $\rho_* : H_{2n+1}(P^{m+k}; Z) \to H_{2n+1}(L^{m+k}; Z)$ is trivial, while $\rho_* : H_{2n}(P^{m+k}; Z) \to H_{2n}(L^{m+k}; Z) / Z_2$ is an isomorphism.

(iii) If $m \leq 2n \leq m + k$, then $\rho_* : H_{2n}(L^{m+k}; Z) \to H_{2n}(P^{m+k}; Z)$ is an epimorphism $Z_4 \to Z_2$, and $\rho_* : H_{2n}(L^{m+k}; Z) \to H_{2n}(P^{m+k}; Z)$ is an isomorphism, while $\rho_* : H_{2n+1}(L^{m+k}; Z) \to H_{2n+1}(P^{m+k}; Z)$ is trivial.

**Lemma 2.3** ([20, 2.5]). Let $(E^2_{p,q}, d_1)$ be the Atiyah-Hirzebruch spectral sequence (AHS) for $KO^*(L^\infty)$ with $E^2_{p,q} = H^p(L^\infty; KO^q(pt))$. Then differentials $d_2^{8k} : E_2^{i,8k} \to E_2^{i+2,8k-1}$, $d_2^{8k-1} : E_2^{i,8k-1} \to E_2^{i+2,8k-2}$ and $d_2^{8k-2} : E_3^{i,8k-2} \to E_3^{i+3,8k-4}$ are given by $Sq^2 \rho_2$, $Sq^2$ and $\beta_2 Sq^2$, respectively, where $\rho_2$ is the mod 2 reduction, and $\beta_2$ is the Bockstein operation associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$.

Let $\tilde{\eta} \in KO(-)$ (or $\tilde{K}(-)$) denote the reduction of a vector bundle of $\eta$. Let $\lambda_k$ be the realification of the complex Hopf bundle $\xi_k$ over $L^k$. The next lemma is from [14, Prop. 3.3 (2)] and [8, Theorem 2.5].

**Lemma 2.4.** (i) The order of $\tilde{\xi}_k$ in $\tilde{K}(L^k)$ is $2^{k/2} + 1$.

(ii) If $k \geq 4$, then the order of $\lambda_k^i$ in $KO(L^k)$ is

$$
\begin{cases}
2^{k/2} - 2i + 1 & \text{if } k \equiv 1 \pmod{4} \text{ or } k \equiv 4 \pmod{8}, \\
2^{k/2} - 2i & \text{otherwise}.
\end{cases}
$$
For a given $k$, let $F^n(X)$ be the subgroup of $KO_k(X)$ (or $KO^k(X)$) of elements of CW-filtrations $\geq n$ in AHSS.

**Lemma 2.5.** $\widetilde{KO}(L^2) \approx \widetilde{KO}(L^3) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

**Proof.** By AHSS, we see that $\widetilde{KO}(L^2) \approx \widetilde{KO}(L^3)$, and $\widetilde{KO}(L^2)$ is of order 4. Thus it suffices to show $2x = 0$ for $x \in \widetilde{KO}(L^2)$. The left square

\[
\begin{array}{ccc}
S^1 & \xrightarrow{\delta} & S^1 \\
\downarrow & & \downarrow \\
S^3 & \xrightarrow{\delta} & S^3
\end{array}
\]

commutes. Thus there exists a stable map $f : L^2 \to P^2$ whose degrees on the top and bottom cells are respectively 2 and 1. It is known that $\widetilde{KO}(P^2) \approx \mathbb{Z}_4$. If $\widetilde{KO}(L^2) \approx \mathbb{Z}_4$, then any element of $\widetilde{KO}(L^2)$ from $F^1(L^2)$ would be a generator, and $f^* : \widetilde{KO}(P^2) \to \widetilde{KO}(L^2)$ would be an isomorphism because $f$ is of degree 1 on the bottom cell. But it is not since $f$ is of degree 2 on the top cell. \(\square\)

We assume in the following that $L^n_m$ is the base point if $m > n$. Suppose $p$ is either the composite $L_k \to L_{k+1}^b = S^{8b} \cup L_{8b+1}^k \to S^{8b}$ when $8b \leq k \leq 8b + 3$, or $L_k \to L_k_{8b+4} = S^{8b+4} \cup L_{8b+5}^k \to S^{8b+4}$ when $8b + 4 \leq k \leq 8b + 7$.

**Theorem 2.6.** Let $g_1, g_2$ generate $\widetilde{KO}(S^{8b})$, $\widetilde{KO}(S^{8b+4})$ respectively.

(i) If $k = 8b, 8b+1$, then there exists an odd integer $a_1$ and a stable vector bundle $\eta_1$ over $S^{8b}$ such that $\eta_1 = a_1g_1$, and $2^{4b-1}\lambda_k = p^*(2\eta_1)$. Moreover, $2^{4b-1}\lambda_k \in \widetilde{KO}(L^k)$ is of order 2 (resp. 4) if $k = 8b$ (resp. $8b + 1$).

(ii) If $k = 8b+2, 8b+3$, then there exists an odd integer $a_2$ and a stable vector bundle $\eta_2$ over $S^{8b}$ such that $\eta_2 = a_2g_1$ in $\widetilde{KO}(S^{8b})$, and $2^{4b}\lambda_k = p^*(4\eta_2)$. Moreover, $2^{4b}\lambda_k \in \widetilde{KO}(L^k)$ is of order 2.

(iii) If $8b + 4 \leq k \leq 8b + 7$, then there exists an odd integer $a_3$ and a stable vector bundle $\eta_3$ over $S^{8b+4}$ such that $\eta_3 = a_3g_2$, and $2^{4b+1}\lambda_k = p^*(\eta_3)$. Moreover, $2^{4b+1}\lambda_k \in \widetilde{KO}(L^k)$ is of order 4.

**Proof.** The orders of the claimed elements follows from Lemma 2.4.

Let $\{E_2^{p,q}(i), d_2^{i,j}\}$, $i = 1, 2$, be respectively the AHSS for $K^*(L^k)$ and $KO^*(L^k)$. By [20, p. 240, (1)], elements of $E_2^{p,q}(i)$ survive to $E_\infty$ if $p + q \equiv 0 \pmod{4}$.

Consider (i). By Lemma 2.4, $2^{4b-1}\lambda_k$ is stably trivial over $L^{8b-1}$. Thus there is a stable vector bundle $\omega_1$ over $L_{8b}^k$ such that $2^{4b-1}\lambda_k = p_1^*(\omega_1)$, where $p_1 : L^k \to L_{8b}^k$ is the standard projection. In this case $L_{8b}^k = S^{8b} \cup L_{8b+1}^k$ and $\widetilde{KO}(L_{8b+1}^k)$ is either trivial or $\mathbb{Z}_2$. Since $2^{4b}\lambda_k$ is trivial over $L^{8b}$ and is of order 2 over $L^{8b+1}$, we see that $\widetilde{KO}(S^{8b+1}) \to \widetilde{KO}(L^{8b+1})$ maps the generator to $2^{4b}\lambda_k$. Thus we can ignore the part $L_{8b+1}^k$ and let $\omega_2$ be the restriction of $\omega_1$ on $S^{8b}$. Then $\omega_2$ is not divisible by 4, otherwise $p^*(\omega_2)$ is trivial over $L^{8b}$ and $p^*(\omega_1) \in \widetilde{KO}(L^{8b+1})$ is of order $\leq 2$. This is contrary to Lemma 2.4. Next we claim that $\omega_2 \in \widetilde{KO}(S^{8b})$ is divisible by 2, which will imply (i). Note that $2^{4b-1}\lambda_k \in F^{8b}$. If $\omega_2$ is not divisible by 2, then $2^{4b-1}\lambda_k$ generates $F^{8b}/F^{8b+1} \approx \mathbb{Z}_4$. This means $2^{4b}\lambda_k$ is of order 2 in $F^{8b}/F^{8b+1}$. However, we will show that $2^{4b}\lambda_k$ is trivial in $F^{8b}/F^{8b+1}$.

Recall that $2^{4b}\lambda_k$ is the realisation of $2^{4b}\xi_k$, where $\xi_k$ is the complex Hopf bundle over $L^k$, and $2^{4b}\xi_k$ is stably trivial over $L^{8b-1}$ by Lemma 2.4. Since $2^{4b+1}\xi_{8b+1}$ is stably trivial over $L^{8b+1}$, $2^{4b}\xi_k$ corresponds to an element of $E_2^{8b-8b}(1)$ of order 2.
By [19, p. 304] the realification \(\pi_{8j}(K) \approx \mathbb{Z} \rightarrow \pi_{8j}(KO) \approx \mathbb{Z}\) is a multiplication by 2, so is the morphism \(E_{2}^{s b, -s b}(1) \approx \mathbb{Z}_{4} \rightarrow E_{2}^{s b, -s b}(2) \approx \mathbb{Z}_{4}\). Therefore \(2^{s b} \lambda_{k}\) is trivial in \(F^{s b}/F^{s b+1}\).

Consider (ii). As in (i), we have an element \(\omega_{3} \in KO(L_{s b}^{k} = S^{s b} \lor L_{s b+1}^{k}\) such that \(p^{*}(\omega_{3}) = 2^{s b-1} \tilde{\lambda}_{k}\) and the restriction of \(\omega_{3}\) on \(S^{s b}\) is \(2\alpha_{2} g_{1} \in KO(S^{s b})\) for some odd integer \(\alpha_{2}\). By Lemma 2.5, \(2\omega_{3}\) is trivial over \(L_{s b+1}^{k}\), so we can choose the desired \(\eta_{2}\) and \(\alpha_{2}\).

For (iii), notice that by Lemma 2.4, the bundle \(2^{s b+1} \lambda_{k}\) is stably trivial over \(L^{sb+3}\), and is of order 4 in \(KO(L^{sb+4})\). It follows by a similar argument and the fact that \(L_{s b+4}^{k} = S^{s b+4} \lor L_{s b+5}^{k}\).

**Theorem 2.7.** Suppose \(n \not\equiv 0 \pmod{4}\), and \(m \geq n + 4\).

(i) The Adams operation \(\Psi^{3}\) satisfies \(\Psi^{3}(x) = 32^{j} x\) for \(x \in KO^{-4j}(L_{n}^{m})\).

(ii) \(KO^{-4j}(L_{n}^{m}) \approx \mathbb{Z}/2^{j(m+4j,n+4j-1)} \lor \mathbb{Z}/2^{(m+4j,n+4j-1)}\)

where \(h(u) = [u/4] + [(u + 4)/8] + [(u + 7)/8]\), while \(a(u, v)\) and \(b(u, v)\) are defined by

\[
a(u, v) = h(u) - [(v + 1)/4] - [(v + 6)/8] - [(v + 1)/8],
\]

\[
b(u, v) = [u/8] + [(u + 6)/8] - [(v + 7)/8] - [(v + 5)/8],
\]

satisfying \(a(u, v) \geq b(u, v)\).

(iii) There is an element in \(KO^{4k}(L_{4k+1}^{4k+1})\) of order \(2^{c(t)}\) (the maximum), where

\[
c(t) = \begin{cases} 
4l + 2 & \text{if } t = 8l; \\
4l + 3 & \text{if } t = 8l + 1, 8l + 2, 8l + 3; \\
4l + 5 & \text{if } t = 8l + 4, 8l + 5, 8l + 6, 8l + 7.
\end{cases}
\]

**Proof.** Consider (i). Let \(\{E_{r}^{p, q}, d_{r}\}\) be the AHSS for \(KO^{*}(L_{m+n}^{m+k})\). Since each element of \(E_{2}^{q}\) survives to \(E_{\infty}\) when \(p + q \equiv 0 \pmod{4}\), the morphism

\[
KO^{-4j}(L_{n}^{m}) \rightarrow KO^{-4j}(L^{m})
\]

is injective when \(n \not\equiv 0 \pmod{4}\) and \(m > n\). Thus (i) follows from [14, Lemma 4.2].

Part (ii) is from [14, Theorem 2 (1)]. Here note that \(a(m + 8, n + 8) = a(m, n), b(m + 4j, n + 4j - 1) \geq 0\) when \(m \geq n + 4\). The formula is even true for integer \(j < 0\).

Consider (iii). First we have

\[
h(u) = \begin{cases} 
u/2 & \text{if } u \equiv 0, 6 \pmod{8}, \\
(u + 1)/2 & \text{if } u \equiv 1, 5 \pmod{8}, \\
(u - 1)/2 & \text{if } u \equiv 3, 7 \pmod{8}, \\
u/2 + 1 & \text{if } u \equiv 4 \pmod{8}.
\end{cases}
\]

\[
= \begin{cases} \phi(u) & \text{if } u \not\equiv 2, 3 \pmod{8}, \\
\phi(u) - 1 & \text{otherwise.}
\end{cases}
\]

By (ii), there is an element of order \(2^{a(t, -2)}\) in \(KO^{4k}(L_{4k+1}^{4k+1})\). Let \(c(t) = a(t, -2)\). Then

\[
c(t) = h(t) + 2 = \begin{cases} \phi(t) + 2 & \text{if } t \not\equiv 2, 3 \pmod{8}, \\
\phi(t) + 1 & \text{otherwise.}
\end{cases}
\]
Lemma 2.8. Let $4j + 1 \geq 4n + 2k$.

(i) The Adams operation on $KO_{4j+1}(L^{4n+2k}_{4n+1})$ satisfies $(\Psi^3 - 1)(g) = 4g$. If $2k - 1 < 8$, then $4g = 0$ for $g \in KO_{4j+1}(L^{4n+2k}_{4n+1})$. If $2k - 1 \geq 8$, then

$$KO_{4j+1}(L^{4n+2k}_{4n+1}) \approx \mathbb{Z}/2[(2k-3)/4] \oplus A \oplus B,$$

where $A \approx \mathbb{Z}_2$ if $j - n$ even, and 0 otherwise; and

$$B = \begin{cases} \mathbb{Z}_2 \text{ if } 4(j - n) - 2k \equiv 0, 2 \pmod{8}, \\ 0 \text{ otherwise.} \end{cases}$$

(ii) Suppose $k \geq 5$. Then each element $x \in KO_{4j+1}(L^{4n+2k}_{4n+1})$ is divisible by 4 if $x$ is in $F^{4n+1}$ and is in the image of the projection

$$p_* : KO_{4j+1}(L^{4n+2k}_{4n+1}) \rightarrow KO_{4j+1}(L^{4n+2k}_{4n+1}).$$

Proof. Part (i) is from [14, Theorem 2 (2) and (5.10) (1)], where the summand $A$ is from the bottom cell, and $B$ is from the cells near the top. For (ii), consider the composite

$$L^{4n+2k}_{4n+1} \rightarrow L^{4n+2k}_{4n+1} \rightarrow L^{4n+2k}_{4n+1}, \ n' < n.$$  

With an appropriate $n'$ and using (i), we can assume that the image of

$$KO_{4j+1}(L^{4n+2k}_{4n+1}) \rightarrow KO_{4j+1}(L^{4n+2k}_{4n+1})$$

is in $\mathbb{Z}/2[(2k-3)/4] \oplus B$. Since $k \geq 5$, $[(2k-3)/4] \geq 3$. Then (ii) follows from the fact that $x$ is of CW-filtration $4n + 1$.

3. PROOFS OF THEOREMS

It is known that if $J((A - B)\lambda_k) = 0$ over $L^k$, then $L^2_{-A+k}$ and $L^2_{-B+k}$ are Thom spaces of $J$-equivalent vector bundles, therefore a stable equivalence exists. We call a stable equivalence obtained in this way an equivalence (5), to distinguish from those four equivalences in Theorem 3.2.

Proof of Theorem 1.1. When $k > 0$, $L^{n+k}_m$ is not $S$-coreducible if $n$ is odd, and $L^{n+k}_m$ is not $S$-reducible if $n + k$ is even. So (i) is immediate when $k = 1$. By Steenrod operations, and noting that by Lemma 2.5 the order of $\lambda_k$ is 2 when $k = 2, 3$, we see that $L^{n+k}_m \sim L^{m+k}_m$ if $\nu(n - m) \geq 2$ when $k = 2, 3$.

Consider (ii) and (iii). Assume that $L^{n+k}_m$ is $S$-coreducible (if $S$-reducible use $S$-duality). By [3, Proposition 2.8], $L^{n+k}_m$ is $S$-coreducible iff $J(\frac{1}{2}\lambda_k) = 0$, where $J$ is the standard homomorphism $KO(-) \rightarrow J(-)$. By [8, Theorem 2.1], the order of $J(\lambda_k)$ is

$$\begin{align*}
4l & \quad \text{if } k = 8l; \\
4l + 1 & \quad \text{if } k = 8l + 1, 8l + 2, 8l + 3; \\
4l + 3 & \quad \text{if } k = 8l + 4, 8l + 5, 8l + 6, 8l + 7.
\end{align*}$$

Hence we have (ii). For (iii), if $L^{n+k}_m \sim L^{m+k}_m$, then $L^{n+k}_m$ is $S$-coreducible iff $L^{m+k}_m$ is $S$-coreducible, so the inequalities follow from (ii). Conversely, if the inequalities in (iii) are satisfied, then an equivalence (5) exists.

By Adams operation $\Psi^5$ on $K^*(-)$, we have

Lemma 3.1 ([11, Theorem 1.1 and p. 292]). If $L^{n+k}_m \sim L^{m+k}_m$, then

$$\nu(n - m) \geq \lfloor (n + k)/2 \rfloor - \lfloor n/2 \rfloor.$$  

Proof of the necessity in Theorems 1.2 and 1.3. The necessity in Theorem 1.3 (ii) follows from Lemma 5.11 (ii) (or [13, Theorem 2 (2), p. 699]). By Steenrod op-
erations, we have \( \nu(n - m) \geq 3 \) when \( 4 \leq k \leq 7 \). The proof of the necessity in Theorem 1.3 (i) follows from the following observations.

Suppose \( n \equiv 0 \pmod{4} \). By Lemma 3.1, \( \nu(n - m) \geq 4l + 3 \) when \( k - 8l = 7, 6; \nu(n - m) \geq 4l + 2 \) when \( k - 8l = 5, 4; \nu(n - m) \geq 4l + 1 \) when \( k - 8l = 3, 2; \nu(n - m) \geq 4l \) when \( k - 8l = 1, 0 \).

Suppose \( n \equiv 1 \pmod{4} \). By Lemma 3.1, \( \nu(n - m) \geq 4l + 4 \) when \( k - 8l = 7; \nu(n - m) \geq 4l + 3 \) when \( k - 8l = 6, 5; \nu(n - m) \geq 4l + 2 \) when \( k - 8l = 4, 3; \nu(n - m) \geq 4l + 1 \) when \( k - 8l = 2, 1; \nu(n - m) \geq 4l \) when \( k = 8l \).

Suppose \( n \equiv 2 \pmod{4} \). By Lemma 2.1, Theorem 2.7 (i), (iii), and using S-duality, we have \( \nu(n - m) \geq 4l + 4 \) when \( k - 8l = 6, 5 \); \( \nu(n - m) \geq 4l + 2 \) when \( k - 8l = 4, 3 \); \( \nu(n - m) \geq 4l + 1 \) when \( k - 8l = 2, 1 \). By Lemma 3.1, \( \nu(n - m) \geq 4l \) when \( k - 8l = 1, 0 \).

Suppose \( n \equiv 3 \pmod{4} \). By Lemma 2.1, Theorem 2.7 (i), (iii), we have \( \nu(n - m) \geq 4l + 4 \) when \( k - 8l = 6, 5 \); \( \nu(n - m) \geq 4l + 2 \) when \( k - 8l = 4, 3 \); \( \nu(n - m) \geq 4l + 1 \) when \( k - 8l = 2, 1 \). By Lemma 3.1, \( \nu(n - m) \geq 4l \) when \( k - 8l = 1, 0 \).

**Theorem 3.2.** Suppose \( b \geq 0 \) in (1), and \( b \geq 1 \) in (2)-(4). Assume none of the stunted lens spaces in question is \( S \)-coreducible or \( S \)-reducible.

\[(1) \quad L_{4n+8b+7}^{4n+8b+7} \sim L_{4n}^{4n+8b+7} \text{ if } \nu(4n - 4m) \geq 4b + 3;\]
\[(2) \quad L_{4n}^{4n+8b+5} \sim L_{4n}^{4n+8b+5} \text{ if } \nu(4n - 4m) \geq 4b + 2 \text{ or } 4b + 3 \text{ when Theorem 1.3 (ii) is satisfied};\]
\[(3) \quad L_{4n}^{4n+8b+3} \sim L_{4n}^{4n+8b+3} \text{ if } \nu(4n - 4m) \geq 4b + 1;\]
\[(4) \quad L_{4n+8b+1}^{4n+8b+1} \sim L_{4n+8b+1}^{4n+8b+1} \text{ if } \nu(4n - 4m) \geq 4b \text{ or } 4b + 1 \text{ when Theorem 1.3 (ii) is satisfied}.\]

We call a stable equivalence given by Theorem 3.2 (j), an equivalence (j). Assuming the above theorem, we can complete the sufficiency in Theorems 1.2, 1.3.

**Proof of the sufficiency in Theorem 1.2.** (a) \( n \equiv 0 \pmod{4} \). If \( k = 7 \), use an equivalence (1). By removing from both sides of an equivalence \( L_{4A}^{4A+7} \sim L_{4B}^{4B+7} \), the top cell, the top two cells, top three cells respectively, we have the desired equivalences for \( k = 6, 5, 4 \).

(b) \( n \equiv 1 \pmod{4} \). If \( k = 7 \), use an equivalence (4) to get an equivalence \( L_{4A}^{4A+9} \sim L_{4B}^{4B+9} \), then remove the top cell and the bottom cell from both sides of that equivalence. If \( k = 6, \) remove the bottom cell from both sides of an equivalence \( L_{4A}^{4A+7} \sim L_{4B}^{4B+7} \) given by an equivalence (1); if \( k = 5, 4 \), remove respectively the top cell, the top two cells from both sides of an equivalence \( L_{4A+1}^{4A+7} \sim L_{4B+1}^{4B+7} \).

(c) \( n \equiv 2 \pmod{4} \). If \( k = 7, 6 \), use an equivalence (5); if \( k = 5 \), remove the bottom two cells from both sides of an equivalence \( L_{4A}^{4A+7} \sim L_{4B}^{4B+7} \); if \( k = 4 \), remove the top cell from both sides of an equivalence \( L_{4A+2}^{4A+7} \sim L_{4B+2}^{4B+7} \).

(d) \( n \equiv 3 \pmod{4} \). If \( k = 7 \), use an equivalence (4) and S-duality to get an equivalence \( L_{4A+11}^{4A+11} \sim L_{4B+11}^{4B+11} \), then remove the bottom cell and the top cell; if \( k = 6, \) remove respectively the top cell, the top two cells from both sides of an equivalence \( L_{4A+10}^{4A+10} \sim L_{4B+10}^{4B+10} \); if \( k = 4 \), remove the bottom three cells from both sides of an equivalence \( L_{4A+7}^{4A+7} \sim L_{4B+7}^{4B+7} \) given by an equivalence (1).

**Proof of the sufficiency in Theorem 1.3.** (a) \( n \equiv 0 \pmod{4} \). If \( k = 7, 6 \pmod{8} \), use equivalences (1); if \( k = 5, 4 \pmod{8} \), use equivalences (2); if \( k = 3, 2 \pmod{8} \), use equivalences (3); if \( k \equiv 1, 0 \pmod{8} \), use equivalences (4).
Proof of Theorem 3.2. Maps will be stable here.

(1) \( L_{4n}^{4n+8b+7} \sim L_{3m}^{3m+8b+7} \) when \( \nu(4n-4m) \geq 4b+3 \).

Let \( \lambda = \lambda_{8b+7} \). Note that \( L_{4n}^{4n+8b+7} = T(2n\lambda) \), \( L_{4n}^{4n-4m+8b+7} = T(2(n-m)\lambda) \) and \( L_{4m}^{4m+8b+7} = T(2m\lambda) \). Since \( \Delta^*(2(n-m)\lambda \times 2m\lambda) = 2n\lambda \), where \( \Delta : L_{8b+7} \to L_{8b+7}^{\times} \) is the diagonal map, we have a map

\[
T(2n\lambda) \to T(2(n-m)\lambda \times 2m\lambda) = T(2(n-m)\lambda) \wedge T(2m\lambda),
\]

namely a map \( f_1 : L_{4n}^{4n+8b+7} \to L_{4n}^{4n-4m+8b+7} \wedge L_{4m}^{4m+8b+7} \). We may assume \( 4(n-m) \geq 0 \). Since \( \nu(4n-4m) \geq 4b+3 \), by Theorem 2.6 (iii), there is a stable vector bundle \( \eta \) over \( S_{8b+4}^{\times} \) such that \( p^*(\eta) = 2(n-m)\lambda \), where \( p \) is the projection \( L_{8b+7}^{\times} \to L_{8b+4}^{\times} \vee L_{8b+7}^{\times} \to S_{8b+4}^{\times} \). Thus we have a map

\[
f_2 : L_{4n}^{4n-4m+8b+7} = T(2(n-m)\lambda) \to T(2\eta) = S^{4n-4m} \cup_{2\beta} e^{4n-4m+8b+4},
\]

where \( \beta \) is the image of \( \bar{\eta} \) under the \( J \)-homomorphism \( \pi_{8b+4}(BO) \to \pi^a_{8b+3} \), thus \( \beta = a\beta_{8b+3} \) for some integer \( a \). Let \( f_3 \) be the composite

\[
L_{4n}^{4n+8b+7} \overset{f_1}{\longrightarrow} L_{4n}^{4n-4m+8b+7} \wedge L_{4m}^{4m+8b+7} \overset{f_2 \wedge 1}{\longrightarrow} (S^{4n-4m} \cup_{2\beta} e^{4n-4m+8b+4}) \wedge L_{4m}^{4m+8b+7}.
\]

Taking a CW-approximation of \( f_3 \), we get a map

\[
f_4 : L_{4n}^{4n+8b+7} \to \Sigma^4(n-m) \left( L_{4m}^{4m+8b+7} \cup_{2\beta \vee 2\beta} (e^{4m+8b+4} \vee CM \vee e^{4m+8b+7}) \right)
\]

where \( CM \) is the cone on the mod 4 Moore space \( M = S^{4m+8b+4} \cup_{4} e^{4m+8b+5} \). Since \( L_{4m}^{4m+8b+7} \) is not \( S \)-reducible, by Theorem 1.1, we have \( \nu(4m + 8b + 8) \leq 4b + 3 \).

Thus by Lemma 6.2, the top part of the 3-part wedge splits off. By Lemma 5.7, the next to top part \( CM \) also splits off. Consequently we have a projection

\[
r : \Sigma^4(n-m) \left( L_{4m}^{4m+8b+7} \cup_{2\beta \vee 2\beta} (e^{4m+8b+4} \vee CM \vee e^{4m+8b+7}) \right) \to \Sigma^4(n-m) \left( L_{4m}^{4m+8b+7} \cup_{2\beta} e^{4m+8b+4} \right).
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
which is the identity on the part $L_{4m}^{4m+8b+7}$. Let

$$f_5 = r f_4 : L_{4n}^{4n+8b+7} \to \Sigma^4(n-m) (L_{4m}^{4m+8b+7} \cup \beta e^{4m+8b+4}).$$

Taking the $S$-dual of $f_5$, we have a map

$$g : X = (S^{4m-8b-5} \cup L_{-4m-1}^{4m+8b-8}) \cup (\beta \cup \alpha) e^{4m-1} \to \Sigma^4(n-m) L_{-4n-8b-8}^{4n-1}.$$

Since $L_{4n}^{4n+8b+7}$ is not $S$-coreducible, $\nu(4n) \leq 4b + 3$. By Lemma 6.2, the composite

$$S^{-4m-2} \beta \nu \to \Sigma^4(n-m) L_{-4n-8b-8}^{4n-1}$$

is null, so $g : L_{-4m-8b-8}^{4n-8b} \to \Sigma^4(n-m) L_{-4n-8b}^{4n-1}$ extends to

$$L_{-4m-8b-8}^{4n-8b} \to \Sigma^4(n-m) L_{-4n-8b-8}^{4n-1},$$

which is an equivalence below the top cell. We claim it is an equivalence.

If $\nu(4n) = \nu(4m)$, then it follows from Lemma 5.11 (ii). Since $\nu(4n-4m) \geq 4b+3$, the only possible case that $\nu(4n) \neq \nu(4m)$ was when

$$\nu(4n) > \nu(4m) \geq 4b + 3 \text{ or } \nu(4m) > \nu(4n) \geq 4b + 3,$$

which would imply one of the original stunted lens spaces is $S$-coreducible.

(2) $L_{4n}^{4n+8b+5} \sim L_{4m}^{4m+8b+5}$ when $\nu(4n-4m) \geq 4b + 2$.

If both $\nu(4n)$ and $\nu(4m) \geq 4b + 3$, then an equivalence follows from case (1). Suppose $\nu(4n) \leq 4b + 2$. By the diagonal map and then a CW-approximation, we have a map

$$f_1 : L_{4n}^{4n+8b+5} \to \Sigma^4(n-m) (L_{4m}^{4m+8b+5} \cup \beta \cup \alpha e^{4m+8b+4} \cup e^{4m+8b+5})$$

where $\beta = a\beta_{8b+3}$ for some integer $a$. By Lemma 6.1, the top part of the 2-part wedge splits off, and we have a map $f_2 : L_{4n}^{4n+8b+5} \to \Sigma^4(n-m) (L_{4m}^{4m+8b+5} \cup \beta \cup e^{4m+8b+4})$.

Let

$$g : (S^{4m-8b-5} \cup L_{-4m-2}^{4m+8b-6} \cup \beta \cup \alpha) e^{4m-1} \to \Sigma^4(n-m) L_{-4n-8b-6}^{4n-1}$$

be the dual of $f_2$. By Lemmas 6.4 (iii), 6.5 (ii), and with an argument as in the previous case, we see that $g : L_{-4m-8b-6}^{4m-1} \to \Sigma^4(n-m) L_{-4n-8b-6}^{4n-1}$ extends to a map $g_5 : L_{-4m-8b-6}^{4m-1} \to \Sigma^4(n-m) L_{-4n-8b-6}^{4n-1}$ which is an equivalence below the top.

If $\nu(4n) = \nu(4m)$, then an equivalence follows from Lemma 5.11 (ii). Since $\nu(4n-4m) \geq 4b + 2$, the only possible case that $\nu(4n) \neq \nu(4m)$ is when

(3) $\nu(4n) \geq 4b + 2$ or $\nu(4m) \geq 4b + 2$.

If $\nu(4n)$ and $\nu(4m)$ both $> 4b + 2$, then the one of the original stunted lens spaces was $S$-coreducible.

If $\nu(4n)$ (or $\nu(4m)$) is $4b+2$, then the condition of (2) requires $\nu(4n-4m) \geq 4b + 3$. So (3.3) is ruled out.

(4) $L_{4n}^{4n+8b+1} \sim L_{4m}^{4m+8b+1}$ when $\nu(4n-4m) \geq 4b$.

First assume $b > 1$. By taking the diagonal map and then a CW-approximation, we have a map

$$f_1 : L_{4n}^{4n+8b+1} \to \Sigma^4(n-m) (L_{4m}^{4m+8b+1} \cup \beta \cup \beta' (e_{2}^{4m+8b} \cup e_{1}^{4m+8b+1} \cup e_{2}^{4m+8b+1})).$$

where $\beta, \beta'$ are respectively multiples of $\beta_{8b-1}, \beta_{8b}$. By Lemma 6.1, the top part $e_{1}^{4m+8b+1}$ splits off, thus we have a map

$$f_2 : L_{4n}^{4n+8b+1} \to \Sigma^4(n-m) (L_{4m}^{4m+8b+1} \cup \beta \cup \beta' (e_{1}^{4m+8b} \cup e_{4m+8b+1})).$$
Let $\xi$ be the real Hopf bundle over $P^{8b}$. Since $\nu(4n-4m) \geq 4b$, both $(n-m)\xi$ and $2(n-m)\lambda_{8b-1}$ are trivial. Thus we have an equivalence $f : P_{4n+8b}^{4} \to \Sigma^{4}(n-m) P_{4m+8b}^{4m+8b}$ by the $S$-coreducibility of $P_{4}^{4(n-m)+8b}$ and the map

$$P_{4n+8b}^{4} \to P_{4(n-m)}^{4n+8b} \wedge P_{4m}^{4m+8b} \to \Sigma^{4n-4m} P_{4m+8b}^{4m+8b}.$$ 

Consider the diagram

$$\begin{array}{ccc}
S^{4n+8b-1} & \xrightarrow{\alpha'} & P_{4n}^{4+8b-1} \\
\downarrow 1 & & \downarrow 1 \\
S^{4n+8b-1} & \xrightarrow{\alpha''} & \Sigma^{4}(n-m) P_{4m}^{4m+8b-1} \\
& & \downarrow f' \\
& & \Sigma^{4}(n-m) P_{4m}^{4m+8b} \\
& & \xrightarrow{\rho} \Sigma^{4}(n-m) L_{4m+8b}
\end{array}$$

where $\alpha'$, $\alpha''$ are the attaching maps for the top cells of $P_{4n+8b}^{4}$ and $\Sigma^{4}(n-m) P_{4m}^{4m+8b}$, and $f'$, $f''$ are restrictions of $f$. Let $\alpha'_{1}$ be the attaching map for the top cell of $\Sigma^{4}(n-m) L_{4m+8b}$. Then both $\alpha_{1}$ and $\alpha'_{1}$ are the attaching maps for the even dimensional cells, by Lemma 2.2 (i) or the CW-structure given by [21, p. 91], we have $\alpha_{1} = \rho \alpha'_{1}$ and $\alpha'_{1} = \rho \alpha''_{1}$. Thus the composite of the bottom row is homotopic to

$$S^{4n+8b-1} \xrightarrow{\alpha''} P_{4n+8b-1} \xrightarrow{f''} \Sigma^{4}(n-m) L_{4m+8b},$$

which is null-homotopic because $f''$ extends to $f$. Here the commutativity of the middle and the right squares is immediate, while the left follows from the existence of $f$. Hence $f_{3}$ extends to a map $f_{4} : L_{4n+8b}^{4} \to \Sigma^{4n-4m} L_{4m+8b+1}^{4m+8b+1}$, which is an equivalence on $(4n + 8b)$-skeletons, by $S_{2}$ and Lemma 5.11 (i).

On the other hand, it is well known that the first morphism in the exact sequence

$$\pi_{4n+8b}(S^{4n+8b-1}) \xrightarrow{2\beta} \pi_{4n+8b}(\Sigma^{4}(n-m)(L_{4m+8b+1}^{4m+8b+1} \cup \rho \epsilon^{4m+8b+1}))$$

$$\xrightarrow{1} \pi_{4n+8b}(\Sigma^{4}(n-m)(L_{4m}^{4m+8b+1} \cup 2\beta \epsilon \rho \epsilon^{4m+8b+1}))$$

is 0 since $\beta$ is a multiple of $\beta_{8b-1}$.

Now that $i_{*}(f_{4} \alpha_{2}) = f_{2} \alpha_{2} = 0$, we conclude that $f_{4} \alpha_{2}$ is null-homotopic. Thus $f_{4}$ extends to a map

$$g : L_{4n+8b+1}^{4} \to \Sigma^{4}(n-m)(L_{4m+8b+1}^{4m+8b+1} \cup \rho \epsilon^{4m+8b+1})$$

such that $g^{*}$ on $H^{*}(\cdot; Z_{2})$ is an isomorphism on the part $\Sigma^{4}(n-m) L_{4m+8b+1}^{4m+8b+1}$.

Taking the $S$-dual of $g$ as in the final stage of (1) or (2), using Lemmas 5.10, 6.3 5.11 (ii), we have a desired equivalence.

Assume $b = 1$. By $S$-duality, it suffices to show $L_{4n+11}^{4} \sim L_{4m+2}^{4m+3}$. If both $\nu(4m+12)$ and $\nu(4n+12) \geq 5$, then an equivalence (5) follows because neither of the spaces is $S$-reducible implies $\nu(4m+12) = \nu(4n+12) = 5$. So we may assume $2 \leq \nu(4m+12) \leq 4$.

Case (a). Suppose both $n$ and $m$ are odd. Taking the diagonal and then a CW-approximation, we have a map

$$f_{1} : L_{4n+2}^{4n+11} \to \Sigma^{4}(n-m)(L_{4m+2}^{4m+11} \cup \rho \epsilon \rho \epsilon^{4m+11})$$

such that $g^{*}$ on $H^{*}(\cdot; Z_{2})$ is an isomorphism on the part $\Sigma^{4}(n-m) L_{4m+2}^{4m+11}$.
Since $3 \leq \nu(4m + 12) \leq 4$, by Lemmas 6.3 (ii), 5.10 the top parts $e_1^{4n+11} \cup e_2^{4n+11}$ split off, and we have a map

$$f_2 : L_{4n+2}^{4n+11} \to \Sigma^{4(n-m)} (L_{4m+2}^{4m+11} \cup 2\mathbf{e}_r e^{4m+10}).$$

Just as in (1), taking the $S$-dual and using Lemma 6.1, we have a desired equivalence.

**Case (b).** Suppose both $n$ and $m$ are even. By $S$-duality, the statement that $L_{4n+1}^{4n+11} \sim L_{4m+2}^{4m+11}$ when $\nu(4(n - m)) \geq 4$ is equivalent to that $L_{8A+4}^{8A+13} \sim L_{8B+4}^{8B+13}$ when $\nu(8(A - B)) \geq 4$. By Case (a) and $S$-duality, we have an equivalence $g_0 : L^{8(A-B)+9} \to \Sigma^{8(A-B)-2} L^{2^2+9}$. Since $2^3 \lambda_9$ is trivial over $L^7$ and $L^{2^2+9} = T(2^4 \lambda_9)$, we have a map

$$g_1 : L^{2^2+9} \to S^{26} \cup 4\mathbf{e}_r e^{2^5+8}$$

of Thom spaces. Let $g_2$ be the composite

$$L_{8A+4}^{8A+13} \to L_{8B+4}^{8B+13} \to \Sigma^{8(A-B)-2} L^{2^2+9} \cup L_{8B+4}^{8B+13}$$

$$\nu \to \Sigma(\Sigma^{8(A-B)} \cup 4\mathbf{e}_r e^{8(A-B)+8}) \cup L_{8B+4}^{8B+13}.$$ 

Taking a $CW$-approximation, we have a map

$$g_3 : L_{8A+4}^{8A+13} \to \Sigma^{8(A-B)} (L_{8B+4}^{8B+13} \cup 4\mathbf{e}_r e^{8B+12} \cup e^{8B+13}).$$

By Lemma 5.2, the top part of the 2-part wedge splits off, so we have map

$$g_4 : L_{8A+4}^{8A+13} \to \Sigma^{8(A-B)} (L_{8B+4}^{8B+13} \cup 4\mathbf{e}_r e^{8B+12})$$

inducing an isomorphism on $H^*(\cdot; \mathbb{Z}_2)$ when restricted on the part $\Sigma^{8(A-B)} L_{8B+4}^{8B+13}$. Let

$$g_5 : (S^{8B-13} \cup L_{-8B-14}^{8B-6}) \cup 4\mathbf{e}_r e^{8B-5} \to \Sigma^{8(A-B)} L_{-8A-14}^{8A-5}$$

be the $S$-dual of $g_4$. Since $g_5 | S^{8B-13}$ is the degree one map into the $(-8B - 13)$-cell of $\Sigma^{8(A-B)} L_{-8A-14}^{8A-5}$, we have $(g_5 | S^{8B-13})_* (4\mathbf{e}_r) = 0$ by Lemma 5.10. Thus $g_5 | L_{-8B-14}^{8B-14}$ extends to a map $L_{-8B-14}^{8B-6} \to \Sigma^{8(A-B)} L_{-8A-14}^{8A-14}$, which is an equivalence by $Sq^8$ and Lemma 5.11 (ii).

**Case (3).** $L_{4n}^{4n+8b+3} \sim L_{4m+8b+3}$ when $\nu(4m - 4m) \geq 4b + 1$.

Suppose $b \geq 2$. By $S$-duality, we can assume that both $n$ and $m$ are even. As in (1), the diagonal map gives a map

$$f_1 : L_{4n}^{4n+8b+3} \to L_{4n}^{4(n-m)+8b+3} \cup L_{4m}^{4m+8b+3}.$$ 

Since $\nu(4m - 4m) \geq 4b + 1$, by Theorem 2.6 (ii), there is a stable vector bundle $\eta$ over $S^{8b}$ such that $p^* (4\eta) = 2(n-m) \lambda_{8b+3}$, where $p$ is the projection $L_{8b+3} \to L_{8b+3}^{8B+3} \to S^{8b}$. Thus we have a map

$$f_2 : L_{4n}^{4(n-m)+8b+3} \to T(4\eta) = S^{4(n-m)} \cup 4\beta e^{4(n-m)+8b},$$

where $\beta = a\beta_{8b-1}$ for some integer $a$. Let

$$f_3 = (f_2 \land 1) f_1 : L_{4n}^{4n+8b+3} \to (S^{4(n-m)} \cup 4\beta e^{4(n-m)+8b}) \cup L_{4m}^{4m+8b+3}.$$ 

Taking a $CW$-approximation of $f_3$, we have a map

$$f_4 : L_{4n}^{4n+8b+3} \to \Sigma^{4n-4m} (L_{4m}^{4m+8b+3} \cup 4\beta \cup 4\beta \cup 4\beta (e^{4m+8b} \cup C M \cup e^{4m+8b+3})$$
Lemma 6.3 (i), the top part of the 3-part wedge splits off. By Lemma 5.2, the next to top part also splits off. Thus we have a map
\[
 f_5 : L_4^{n+8b+3} \to \Sigma_4^{(n-m)}(L_4^{m+8b+3} \cup_4 e^{4m+8b+1}).
\]
View \(L_4^{n+8b+3} = L_4^{m+8b+1} \cup \alpha_1 \cup \alpha_3\) \((e^{4n+8b} \lor CN \lor e^{4n+8b+3})\), where \(CN\) is the cone on \(N = S^{4n+8b} \cup_4 e^{4n+8b+1}\). Let \(f_6 : L_4^{m+8b+1} \to \Sigma_4^{(n-m)}L_4^{m+8b+3}\) be the map obtained by restricting \(f_5\) on the \((4n+8b-1)\)-skeleton.

As in (4), \(f_6\alpha_1 = 0\) in homotopy. Since \(\eta \beta_{b-1} = \beta_{b}\) by [16], and \(4\beta_3 = \eta^3\), we have \(4\beta \beta_{b-1} = \eta^3 \beta_{b-1} = 0\). So the first morphism in the exact sequence
\[
[X, S^{4n+8b-1}] \xrightarrow{\beta} [X, \Sigma_4^{(n-m)}L_4^{m+8b+3}] \xrightarrow{\theta} [X, \Sigma_4^{(n-m)}(L_4^{m+8b+3} \cup_4 e^{4m+8b})]
\]
is null for \(X = S^{4n+8b+2}\). It is also null for \(X = N\) by Lemma 5.2. Thus both \(f_6\alpha_2\) and \(f_6\alpha_3\) are null-homotopic, and \(f_6\) extends to a map over \(L_4^{n+8b+3}\) which can be required to be an equivalence by Lemma 5.11.

Suppose \(b = 1\). By \(s\)-duality, we can assume that both \(n\) and \(m\) are odd. Thus both \(\nu(4n+12)\) and \(\nu(4m+12) \geq 3\). Using Lemma 6.3 (ii), and as for the case \(b \geq 2\), we can find an equivalence \(f_6 : L_4^{n+11} \to \Sigma_4^{(n-m)}L_4^{m+11}\).

4. \(J\)-HOMOLOGY AND COEXTENSION

According to [5], the stable Adams operation \(\psi^3 : bo \to bo\) is defined and \(\psi^3 - 1 : bo \to bo\) lifts to \(\Sigma^4 bsp\). Let \(\theta : bo \to \Sigma^4 bsp\) be the lift and \(J\) its fibre. There is a long exact sequence
\[
\text{bsp}_{n+1}(\Sigma^4 X) \xrightarrow{\theta} \text{J}_n(X) \xrightarrow{\theta} \text{bsp}_{n}(\Sigma^4 X) \to \cdot
\]
Use the fact that \(\pi_t(k) = \pi_t(KO) \otimes \mathbb{Z}_2\) if \(t \geq 0\), and is 0 otherwise; while \(\pi_t(\Sigma^4 bsp) = \pi_t(KO) \otimes \mathbb{Z}_2\) if \(t \geq 0\), and is 0 otherwise, we can compute \(bo_n(X)\) and \(bsp_n(\Sigma^4 X)\) and \(\theta : bo_n(X) \to \text{bsp}_n(\Sigma^4 X)\) in some cases. For example, if \(X = P^{2n}\), then both \(bo_{4j-1}(X)\) and \(bsp_{4j-1}(\Sigma^4 X)\) are cyclic and \(\theta\) sends a generator to \(2^{(j)}g\), where \(g \in \text{bsp}_{4j-1}(\Sigma^4 X)\) is a generator, this is because \((\psi^3 - 1)(x) = 2^{(j)} + 2^j x (mod 2^{(j)+1})\) in \(bo_{4j-1}(X)\) and the projection \(\text{bsp}_{4j-1}(\Sigma^4 X) \to bo_{4j-1}(X)\) is injective with image divisible by 4.

Let \(J : \pi_k(SO) \to \pi_k^e\) be the standard \(J\)-homomorphism. By [18, Theorem 1.1.13], \(IMJ\) is cyclic with 2-component \(\mathbb{Z}_2 / (2(k+1))\) when \(k \equiv 3 \mod 8\), and \(\mathbb{Z}_2\) when \(k \equiv 0 \mod 8\), and the Hurewicz map \(\pi_k^e \to \pi_k(J)\) is injective on \(1mJ\).

Putting \(\beta_{b-1}, \beta_{b+3}, \beta_{b+1}\), generators of \(IMJ\), into the bottom cell, we get classes in \(\pi_*(L_{n+k}^m)\).

Lemma 4.2. Let \(b \geq 2\) (i), and \(b \geq 1\) in (ii)-(iv).

(i) \(2\beta_{b-1}\) is null in \(J_4A + s_b(\Sigma_4^{L_{4+1}})\) but not null in \(J_4A + s_b(\Sigma_4^{L_{4+1}})\).

(ii) For odd \(A\), \(\beta_{b-1}\) is null in \(J_4A + s_b-2(\Sigma_4^{L_{4+1}})\) but not null in \(J_4A + s_b-2(\Sigma_4^{L_{4+1}})\).

(iii) \(\beta_{b+3}\) is null in \(J_4A + s_b+s_b+4(\Sigma_4^{L_{4+1}})\) but not null in \(J_4A + s_b+s_b+4(\Sigma_4^{L_{4+1}})\).

(iv) If \(A\) is even, then \(\beta_{b+1}\) is null in \(J_4A + s_b+2(\Sigma_4^{L_{4+1}})\) but not null in \(J_4A + s_b+2(\Sigma_4^{L_{4+1}})\).

Proof. We just show (i) and (ii). The proofs for (iii) and (iv) are similar.

Consider (i). Since \(b \geq 2\), there is a natural isomorphism \(f_4 : \text{bsp}_{4A + s_b+1}(\Sigma_4 L_{4+1}) \approx \text{bsp}_{4A + s_b+1}(\Sigma_4 L_{4+1})\) of Lemma 2.8 (i), the morphism
\[
\theta : \text{bsp}_{4A + s_b+1}(\Sigma_4 L_{4+1}) \to \text{bsp}_{4A + s_b+1}(\Sigma_4 L_{4+1})
\]
in (4.1) satisfies \( \theta_\ast (x) = 4f_\ast (x) \mod (8) \). Note that \( 2\beta \in J_{4A + 8b}(L_{4A + 1}^{4A + 10}) \) is from an element \( z \in bsp_{4A + 8b + 1}(\Sigma^4 L_{4A + 1}^{4A + 10}) \), that is from the bottom cell corresponding to the order 2 element \( z' \in E^2_{4A + 5, 8b - 4} \) in the AHSS for \( bsp_\ast (\Sigma^4 L_{4A + 1}^{4A + 10}) \). Let \( p \) be the projection as in Lemma 2.8 (ii). Since \( z' \) survives in the AHSS for \( bsp_\ast (\Sigma^4 L_{4A + 1}^{4A + 10}) \), \( z \) is in the image of \( p_\ast \). By Lemma 2.8 (ii), \( z \) is divisible by 4 and hence is hit by \( \theta_\ast \). So \( 2\beta \) is null in \( J_{4A + 8b}(L_{4A + 1}^{4A + 10}) \). However \( 2\beta \) is not null in \( J_{4A + 8b}(L_{4A + 1}^{4A + 8}) \) because all elements of \( bsp_{4A + 8b + 1}(\Sigma^4 L_{4A + 1}^{4A + 8}) \) are of order \( \leq 4 \) by Lemma 2.8 (i), thus are not hit by \( \theta_\ast \). Finally the \( d_2 \)-differential on the unique nontrivial element \( y \) of \( E^2_{4A + 9, 8(b - 1)} = H_{4A + 9}(L_{4A + 1}^{4A + 9}; J_{8(b - 1)}) \approx \mathbb{Z}_2 \) in the AHSS for \( J_\ast (L_{4A + 3}) \) is not zero. Thus \( J_{4A + 8b + 1}(L_{4A + 1}^{4A + 3}) \rightarrow J_{4A + 8b + 1}(L_{4A + 1}^{4A + 9}) \) is surjective by exactness. This together with the fact that \( 2\beta \) is nontrivial in \( J_{4A + 8b}(L_{4A + 1}^{4A + 8}) \) imply that the boundary \( J_{4A + 8b + 1}(L_{4A + 1}^{4A + 9}) \rightarrow J_{4A + 8b}(L_{4A + 1}^{4A + 2}) \) does not hit \( 2\beta \in J_{4A + 8b}(L_{4A + 1}^{4A + 2}) \), and (i) follows.

Consider (ii). Note that the composite

\[
bo_{4A + 8b - 1}(P_{4A - 2}^{4A + 5})^\ast \rightarrow bsp_{4A + 8b - 1}(\Sigma^4 P_{4A - 2}^{4A + 5}) \rightarrow bo_{4A + 8b - 1}(P_{4A - 2}^{4A + 5})
\]

sends \( x \) to \( 8x \). Thus \( \beta_{8b - 1} \) is not null in \( J_{4A + 8b - 2}(P_{4A - 2}^{4A + 5}) \) because

\[
bo_{4A + 8b - 1}(P_{4A - 2}^{4A + 5}) \approx KO_{4A + 8b - 1}(P_{4A - 2}^{4A + 5}) \approx \mathbb{Z}_8,
\]

which implies that \( \beta_{8b - 1} \) is not hit by \( \theta_\ast \). However, \( bo_{4A + 8b - 1}(P_{4A - 2}^{4A + 6}) \approx \mathbb{Z}_{16} \), hence \( \beta_{8b - 1} \) is hit by \( \theta_\ast \).

Let \( E_\ast (X) \) be the \( E_\ast \) term in \( \text{ASS} \) for \( \pi_\ast (X) \). Given \( x \in \pi_\ast (X) \), let \( A(x) \) be the Adams filtration of \( x \). Let \( A_k(\pi_\ast (X)) \) be the subgroup of \( \pi_\ast (X) \) of classes of Adams filtrations \( \geq k \).

**Lemma 4.3.** Let \( n \geq a > b \). Let \( I(n, a, b) \) be the image of the projection \( J_\ast (L_n^a) \rightarrow J_\ast (L_n^b) \).

(i) Suppose \( x : S^m \rightarrow L_n^a \) satisfies \( A(x) \geq N \) and \( x \in I(n, a, b) \) when put into \( J_\ast (-) \). If the projection \( X_{\ast n + 1}(\pi_m(S_{a - 1}^n)) \rightarrow J_{m - 1}(S_{a - 1}^n) \) is injective, and each nontrivial element of \( E_{2s}^s + s - 1(S_{a - 1}^n) \) survives to a homotopy class when \( s \geq N + 1 \), then \( x \) extends to a map \( z : S^m \rightarrow L_n^b \) with \( A(z) \geq N \).

(ii) If moreover the projection \( X^{\ast n}(\pi_m(S_{a - 1}^n)) \rightarrow J_m(S_{a - 1}^n) \) is surjective, then the \( z \) in (i) can be chosen to be in \( I(n, a - 1, b) \) when put into \( J_\ast (-) \).

**Proof.** Consider (i). Here the idea is similar to [4, Lemma 6]. First the condition implies that the boundary \( \partial : \pi_m(L_n^a) \rightarrow \pi_{m - 1}(S_{a - 1}^n) \) is null on \( x \). By \( S \)-duality, it suffices to show \( x_0 : L_{a - 1}^a \rightarrow S_{m - 1} \), the dual of \( x_0 \), extends to a map \( x' : L_{a - 1}^a \rightarrow S_{m - 1} \), with \( A(x') \geq N \). Let \( \alpha \) be the attaching map for the top cell of \( L_{a - 1}^a \). Since \( x_0 \) lifts to \( E_N \), the \( N \)-stage of the stable Adams resolution of \( S_{m - 1} \), and \( A(x_0 \alpha) \geq N + 1 \), the only way that the composite \( x_0 \alpha : S_{a - 1} \rightarrow E_N \) fails to be null-homotopic is that in the \( \text{ASS} \) for \( \pi_\ast (S_0) \) there was a nontrivial differential \( E_r^{s + m - a - 1} \rightarrow E_r^{s + r + m - a} \) with \( s < N < s + r \). But this is ruled out by the condition.

Consider (ii). There exists \( y \in J_m(L_n^a) \) which is a coextension of the composite \( S^m \rightarrow L_n^a \rightarrow J \rightarrow L_n^a \). So \( p(y) - z \) pulls back to \( J_m(S_{a - 1}^n) \), where \( p \) is the projection \( L_n^a \rightarrow L_n^a \). Pick up an element \( y' \in \pi_m(S_{a - 1}^n) \) with \( A(y') \geq N \), and \( p(y) - z = y' \) in \( J_m(S_{a - 1}^n) \). Then \( z + y' \) is the desired coextension. \( \square \)
By [17, Theorem 8.2], if $t \geq 11$, then $\beta_1$ has both the same Adams filtration in $\pi_t(S^0)$ and $J_t(S^0)$ except when $t = 8c - 1$. As indicated in [4, p. 344], the composite $S^{2N+8c-2} \to S^{2N-1} \to P_{2N-1}^N$ has Adams filtration $4c - 2$ for $c \geq 2$.

**Lemma 5.1.** Suppose $\nu(n+1) = i$, $4c - 1 \leq i \leq 4c + 2$ and $c \geq 1$. Let $a = i - 4c + 1$. Suppose $\nu_0 : S^n \to P_{n-8c+2}^n$ is the degree one map given by the $S$-reducibility. Let $i$ generate $\pi_n(S^n)$, and $\partial : \pi_n(P_{n-8c+2}^n) \to \pi_{n-1}(P_{n-8c+2}^n)$ the boundary. Then $\partial(e_0) = x$, where

$$x : S^{n-1} \to S^{n-8c} \to P_{n-8c}^{n-8c} \to P_{n-8c+1}^{n-8c+1}.$$

Here the third map is the inclusion, the second is given by the $S$-reducibility.

**Proof.** Choose $m$ such that $\nu(m) = i$. Then $P_m^{n+8c+7} = T(m\xi)$, where $\xi$ is the real Hopf bundle over $P^{8c+7}$. Then $m\xi$ is stably trivial over $P^{8c-1}$. As in Theorem 2.6 (i), there is a stable vector bundle $\eta'$ over $S^{8c}$ corresponding to a generator of $K\tilde{O}(S^{8c}) \otimes \mathbb{Z}_2$, such that $p^* (2^n \eta') = m\xi$, where $p$ is the composite $P^{8c+7} \to P^{8c+7} \to S^{8c}$ in which the last map is given by the $S$-coreducibility. So we have a map $g : P_m^{n+8c+7} \to S^m \cup_{2^n \beta_{8c-1}} e^{n+8c}$ of Thom spaces such that $g^* \eta'$ is injective on $H^*(-; \mathbb{Z}_2)$. By $S$-duality we have a map $g : S^{n-8c} \cup_{2^n \beta_{8c-1}} e^n \to P_{n-8c+7}$ such that $g_*$ is injective on $H_*(-; \mathbb{Z}_2)$. By the diagram,

$$\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{g^*} & \pi_{n-1}(S^{n-8c}) \\
\xrightarrow{\partial} & & \xrightarrow{\partial} \\
\pi_n(P_{n-8c+2}^n) & \xrightarrow{g_*} & \pi_{n-1}(P_{n-8c+1}^{n-8c+1})
\end{array}$$

we have $\partial(e_0) = x$ with $e_0 = g_*(i)$. This completes the proof. \qed

**Lemma 5.2.** Let $M$ be the mod 4 Moore space with the bottom cell of dimension 0. Then $A_2([M, M]) = 0$, and the identity map $1 : M \to M$ is of order 4.

**Proof.** Consider the exact sequence

$$[S^1, M] \xrightarrow{4} [S^1, M] \to [M, M] \to [S^0, M] \xrightarrow{4} [S^0, M]$$

derived from the cofibre sequence $S^0 \xrightarrow{4} S^0 \to M \xrightarrow{4} S^1$.

Then in the ASS for $\pi_*(M)$, we have $E_3^{s,s}(M) = 0$ by [18, Theorem 2.3.4, p. 63] when $s \geq 2$. So the only possible nontrivial elements of $A_2([M, M])$ are from $[S^1, M]$. By the exact sequence

$$[S^1, S^0] \xrightarrow{4} [S^1, S^0] \to [S^1, M] \to [S^1, S^1] \xrightarrow{4} [S^1, S^1],$$

we see that each nontrivial element of $E_2^{s,s+1}(M)$ supports a nontrivial $d_2$-differential when $s \geq 2$. This implies $A_2([M, M]) = 0$, and the order of $1 : M \to M$ is of order $\leq 4$. Therefore $1 : M \to M$ must be of order 4 because $H_1(M; \mathbb{Z}) \approx \mathbb{Z}_4$. \qed

The filtration 4 maps between real stunted projective spaces are studied in detail in [6], [15] and [17]. In the next lemma, $c$, $i$ and $\iota$ are the collapse map, the inclusion and the identity.

**Lemma 5.3.** (i) ([6, Prop. 2.1], [15, Theorem 3.3]) Let $m$ be odd, $n$ even, $m < n$. There is a stable map $g : P_m^{n+8} \to P_m^n$, such that $A(g) = 4$, the composite $P_m^{n+8} \xrightarrow{g} P_m^n \xrightarrow{1} P_m^{n+8}$ is $16\iota$, and $g_* : KO_{8k-1}(P_m^{n+8}) \to KO_{8k-1}(P_m^n)$ is an isomorphism for all $k$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 5.4. There exists an unstable map $f : F_{8n}^{8(n+1)+7} \to S^{8n} \cup_{n,\beta} e^{8(n+1)}$ whose restriction $S^{8n} \to S^{8n} \cup_{n,\beta} e^{8(n+1)}$ induces a surjection on $KO(-)$.

Proof. Since $P^1_{15} = T(8\xi_7)$ and $8\xi_7$ is trivial over $P^7$, we see $P^1_{15} = S^8 \vee P_9^1$ in the unstable case. The bundle $8\xi_{15}$ over $P^1_{15}$ is trivial on $P^7$, so we have an 8-dimensional bundle $\eta$ over $P^1_{15}$. Let $\eta'$ be the restriction of $\eta$ on $S^8$. The composite $P^1_{15} \to P^1_{15} = S^8 \vee P^1_{15} \to S^8$ induces a map

$$
T(8\xi_{15}) = P^8_{8n+15} \to T(\eta') = e^{8n} \cup_{n,\beta} e^{8(n+1)}
$$

of Thom spaces, which is of degree one on the bottom cell.

Then the surjection follows from the fact that $KO^1(S^{8(n+1)}) = 0$ in the exact sequence below

$$
KO^0(S^{8n} \cup_{n,\beta} e^{8(n+1)}) \to KO^0(S^{8n}) \to KO^1(S^{8(n+1)}).$

□

Let $X$ be a CW-complex, $X^n$ its $n$-skeleton. Let $X^{n+k} = X^{n+k}/X^n$.

Lemma 5.5. Let $b$ be an integer. Suppose both $f$ and $g$ make the diagram

$$
\begin{array}{ccc}
X^n & \overset{i}{\to} & X^{n+k} \\
\downarrow h & & \downarrow f \circ g \\
X^n & \overset{p}{\to} & X^{n+k}
\end{array}
$$

commute, where $i$ and $p$ are respectively the inclusion and the projection. Then $h = f - g : X^{n+k} \to X^{n+k}$ factors as $X^{n+k} \overset{p}{\to} X^{n+k} \overset{i}{\to} X^n$ and $X^{n+k} \overset{\rho}{\to} X^{n+k}$.

Proof. The first factoring follows from the fact that the restriction of $f - g$ on $X^n$ is null, while the second factoring is implied by $p(f - g) = 0$.

□

Lemma 5.6. There exists a map $h : L^{2k+2}_{2k+1} \to L^{2k+2}_{2k+1}$ factoring as $L^{2k+2}_{2k+1} \to S^{2k+1} \to L^{2k+2}_{2k+1}$ such that $2 - h : L^{2k+2}_{2k+1} \to L^{2k+2}_{2k+1}$ factors as $L^{2k+2}_{2k+1} \overset{\rho}{\to} L^{2k+2}_{2k+1}$.

Proof. There is a map $\epsilon$ such that the diagram

$$
\begin{array}{ccc}
S^{2k+1} & \overset{4}{\to} & S^{2k+1} \\
\downarrow 2 & & \downarrow 1 \\
S^{2k+1} & \overset{2}{\to} & S^{2k+1} \\
\downarrow 1 & & \downarrow 2 \\
S^{2k+1} & \overset{4}{\to} & S^{2k+1} \\
\downarrow \rho & & \downarrow \epsilon
\end{array}
$$

commutes. By Lemma 5.5, the map $h = 2 - \epsilon \rho$ is the desired.

□

Lemma 5.7. Let $\beta$ generate $ImJ$ on the $(8b + 3)$-stem. Let $N = e^{4m+1} \cup_4 e^{4m+2}$. If $\nu(4m + 8b + 8) \leq 4b + 2$, then the composite

$$
\begin{array}{c}
x : \Sigma^{8b+3} N^{\beta/2} \to N \to L^{4m+8b+7}_{4m}
\end{array}
$$

is null-homotopic, where the second map is the inclusion to $L^{4m+2}_{4m} = S^{4m} \vee N$.
Proof. By Lemma 5.6, the map \( N \to N \to L_{4m+86+7} \) factors as \( N \to P_{4m+86+7} \to L_{4m+86+7} \) up to a map \( N \to S^{4m+1} \to L_{4m+86+7} \), where the second map is the degree one map to the \((4m+1)\)-cell. Thus by [4, Prop. 4] and Lemma 6.1, it is null. \( \square \)

Lemma 5.8. Let \( i = \nu(n+1) \).

(i) If \( 3 \leq i \leq 4b+1 \), then in ASS for \( \pi_*(P_{n-88-6}^n) \) elements in \( E_{2}^{s,n-1}(P_{n-88-6}^n) \) of Adams filtrations \( \geq 4b \) are hit by Adams differentials.

(ii) If \( 3 \leq i \leq 4b \), then in ASS for \( \pi_*(P_{n-88-6}^{n-88-6}) \) elements in \( E_{2}^{s,n-1}(P_{n-88-6}^{n-88-6}) \) of Adams filtrations \( \geq 4b-1 \) are hit by Adams differentials.

Proof. We just show (i). The proof for (ii) is similar. In ASS for \( \pi_*(P_{n-88-6}^{n-88-6}) \) the chart for \( E_{2}^{s,n-1}(P_{n-88-6}^{n-88-6}) \) is as follows.

\[
E_{2}(P_{n-88-6}^{n-88-6})
\]

where the bottom class corresponds to \( s = 4b - 1 \). The chart for \( E_{2}(P_{n-88-6}^{n-88-6}) \) can be obtained by using the pre-spectral sequence (PSS, [16, p. 26]) converging to \( E_{2}(P_{n-88-6}^{n-88-6}) \) with

\[
E_{1}^{s,t} = \Sigma_{k=-1}^{6} \text{Ext}^{s,t}_{A}(H^{*}(S^{8n-88-6}), \mathbb{Z}_{2}),
\]

and repeatedly using [16, table 8.1], the Adams periodicity, [2, Lemma 2.6.1], and [18, Theorem 2.3.4, p. 63].

Case \( b = 1 \). Let \( \partial \) be the boundary \( \pi_{n}(S^{n}) \to \pi_{n-1}(P_{n-14}^{n-14}) \). Then by Lemma 5.1, \( \partial(\nu) \) is the class \( x : S^{n-1} \to S^{n-8} \to P_{n-14}^{n-14} \to P_{n-14}^{n-14} \) when put into \( P_{n-14}^{n-14} \), where \( a = i - 3 \). Let \( y = 2^{2-a}x \). Then \( y \) is of Adams filtration \( \geq 3 \). Since \( y \) is of Adams filtration 4 when put into \( J_{n-1}(P_{n-14}^{n-14}) \), by the chart for \( E_{2}^{s,n-1}(P_{n-14}^{n-14}) \), \( y \) is of Adams filtration 4. This shows the case \( b = 1 \).

Case \( b \geq 2 \). Suppose (i) holds for \( b \). We wish to show (i) holds for \( b + 1 \).

If \( 3 \leq \nu(n+1) \leq 4b+1 \), then it follows from the filtration 4 map \( g : P_{n-88-6}^{n-88-6} \to P_{n-88-6}^{n-88-6} \) and the fact that the filtration 4b element of \( E_{2}^{4b,4b+n-1}(P_{n-88-6}^{n-88-6}) \) is hit by an Adams differential. So we may assume \( 4b+2 \leq \nu(n+1) \leq 4(b+1) + 1 \).

Part A. Assume (i) for \( b = 2 \). Suppose \( i = 4b + 2 \). Let \( \Sigma P_{m}^{m+8(b+1)+6} \) be the S-dual of \( P_{m}^{m+8(b+1)+6} \). By Lemma 5.4, there is an unstable map \( f : P_{88}^{8(b+1)+6} \to S^{88} \cup_{\alpha} e^{8(b+1)} \) and a stable vector bundle \( \eta \) over \( S^{88} \cup_{\alpha} e^{8(b+1)} \) such that \( f^{*}(\eta) \) is a 4-multiple of a generator in \( K\Omega(S^{88}) \) when restricted on the bottom cell \( S^{88} \). So \( (fp)^{*}(\eta) = (m/2)\xi \), where \( p \) is the projection \( P_{88}^{8(b+1)+6} \to P_{88}^{8(b+1)+6} \). Thus there is a map of Thom spaces \( f_{0} : P_{m/2}^{m/2+8(b+1)+6} \to T(\eta) \). Taking the diagonal and then a CW-approximation, we have a composite

\[
\begin{align*}
P_{m}^{m+8(b+1)+6} & \xrightarrow{f_{1}} (P_{m/2}^{m/2+8(b+1)+6} \land P_{m/2}^{m/2+8(b+1)+6})(m+8(b+1)+6) \\
& \xrightarrow{f_{2}} (T(\eta) \land T(\eta))(m+8(b+1)+6)
\end{align*}
\]
where the restriction of \( f_2 \) on \( P_{m/2}^{m/2+8(b+1)+6} \land S^{m/2} \) or \( S^{m/2} \land P_{m/2}^{m/2+8(b+1)+6} \) is exactly \( f_0 \) given above, and

\[
(T(\eta) \land T(\eta))^{(m+8(b+1)+6)} / S^m = \Sigma^{m/2}(T(\eta)/S^{m/2} \lor T(\eta)/S^{m/2})
\]

since \( b \geq 2 \). This observation is important to (5.9).

Let

\[
\partial_0 : \pi_n(S^n) \rightarrow \pi_{n-1}(\Sigma^{n+1}/2 P_{(n+1)/2}^{(n+1)/2-2} \land (n+1)/2-8(b+1)-7)
\]

and

\[
\partial_1 : \pi_n(S^n) \rightarrow \pi_{n-1}(P_{n-8(b+1)-6})^{n-1}
\]

be boundaries.

Let \( g_1 \) and \( g_0 \) be the \( S \)-duals of \( f_1 \) and \( f_0 \), \( \alpha_1 = \partial_0(i) \). By \( S \)-duality, and as in Lemma 5.1, we have

(5.9)

\[
\partial_1(i) = 2g_1(\alpha_1).
\]

In general, let \( a = i - 4b - 1 \), and consider the composite

\[
P_m^{m-8(b+1)+6} \rightarrow P_{m/2}^{m/2+8(b+1)+6} \land \ldots \land P_{m/2}^{m/2+8(b+1)+6} \rightarrow T(\eta) \land \ldots \land T(\eta),
\]

where \( P_{m/2}^{m/2+8(b+1)+6} \land \ldots \land P_{m/2}^{m/2+8(b+1)+6} \) is the smash product of \( 2^a \) copies of \( P_{m/2}^{m/2+8(b+1)+6} \). Taking a \( CW \)-approximation and repeating the preceding argument, we have

\[
\partial_1(i) = 2^a g_1(\alpha_1),
\]

where \( \alpha_1 = \partial_0(i) \), and \( \partial_0 : \pi_n(S^n) \rightarrow \pi_{n-1}(\Sigma^{n+1}/2 P_{(n+1)/2}^{(n+1)/2-2} \land (n+1)/2-8(b+1)-7) \).

Let \( x = 2^a g_1(\alpha_1) \). Then \( 2^{a+1} g_1(\alpha_1) \). By induction, \( \alpha_1 \) is of Adams filtration \( 4b \) when projected to \( P_{n-8(b+1)-6}^{n-1} \). Thus by the filtration 4 map again, we see that filtration \( \geq 4(b+1) \) elements of \( E_2^{∗,n−1} (P_{n-8(b+1)-6}) \) are hit by Adams differentials. Moreover \( \partial_1(i) \) is of Adams filtration \( 4(b+1) \) when \( i = 4(b+1) \).

**Part B.** Consider (i) for \( b = 2 \). First assume \( \nu(m) = 5 \). As before we have an unstable map \( f : P_{m/2}^{m/2+22} \rightarrow S^{m/2} \cup_{β} e^{16} \) and a stable vector bundle \( \eta \) over \( S^{m/2} \cup_{β} e^{16} \), such that \( (fp)^∗(\eta) = (m/2)ξ \). Let \( f_0 : P_{m/2}^{m/2+22} \rightarrow T(\eta) \) be a map of the Thom spaces. Taking the diagonal and then a \( CW \)-approximation, we have

\[
P_m^{m+22} f_1 \rightarrow (P_{m/2}^{m/2+22} \land P_{m/2}^{m/2+22})^{m+22} \rightarrow (T(\eta) \land T(\eta))^{m+22}.
\]

Since \( π_7(S^0) \) is generated by \( β_7 \), \( ν(m/2) \geq 4 \) and \( (f_0)^∗ \) is injective on \( H^∗(−;Z_2) \), we see that \( α \) is odd in the following

\[
T(\eta)/S^{m/2} = S^{m/2+8} \cup_{α} e^{m/2+16}
\]

by \( Sq^8 \). This implies that the attaching map \( α \) indicated below

\[
(T(\eta) \land T(\eta))^{m+22} / S^m = \Sigma^{m/2}((T(\eta)/S^{m/2} \lor T(\eta)/S^{m/2}) \cup_α e^{m/2+16}),
\]

is null. Thus

\[
(T(\eta) \land T(\eta))^{m+22} / S^m = \Sigma^{m/2}(T(\eta)/S^{m/2} \lor T(\eta)/S^{m/2} \lor e^{m/2+16}).
\]

So in this case time (5.9) becomes

\[
\partial_1(i) = 2g_1(\alpha_1) + x_0,
\]
where \(x_0\) is in the image of \(\pi_{n - 1}(S^{n - 16}) \to \pi_{n - 1}(P^{n - 1}_{n - 22})\) induced by the degree one map from \(S^{n - 16}\) to the \((n - 16)\)-cell of \(P^{n - 1}_{n - 22}\) with \(A(x_0) \geq 4\). Here \(\alpha_1 = \partial_0(i), \partial_0 : \pi_n(S^n) \to \pi_{n - 1}(\Sigma^{(n + 1)/2}P^{(n + 1)/2 - 2}_{(n + 1)/2 - 23})\) and \(\partial_1 : \pi_n(S^n) \to \pi_{n - 1}(P^{n - 1}_{n - 22})\) are boundaries.

For \(6 \leq i \leq 9\), we have

\[
\partial_1(t) = 2^a(2\gamma_4(\alpha_1) + x_0)
\]

where \(a = i - 5\) and \(\alpha_1 = \partial_0(i) : \pi_n(S^n) \to \pi_{n - 1}(\Sigma^{n + 1 - (n + 1)/2^{n + 1} - 2}_{(n + 1)/2^{n + 1} - 23})\). Thus

\[
\partial_1(2^{4-a}l) = 2^5\gamma_4(\alpha_1) + 2^4x_0.
\]

Since each element in \(\pi_{n - 1}(P^{n - 15}_{n - 22})\) is of order \(\leq 2^4\), we see \(2^4x_0 = 0\). So as before, the proof follows from the filtration 4 map. Moreover if \(i = 9\), then \(\partial_1(i)\) is of Adams filtration 8.

**Lemma 5.10.** If \(3 \leq \nu(4n) \leq 4b\), then both composites

\[
\begin{align*}
S^{4n-2} & \quad \xrightarrow{4\beta_{8b-1}} \quad S^{4n-8b-1} \to \quad L^{4n-1}_{4n-8b-2}, \\
S^{4n-2} & \quad \xrightarrow{\beta_{8b}} \quad S^{4n-8b-2} \to \quad L^{4n-1}_{4n-8b-2}
\end{align*}
\]

are null-homotopic.

**Proof.** Notice that the boundary \(\pi_{4n-1}(S^{4n-8b}) \to \pi_{4n-2}(L^{4n-8b-1}_{4n-8b-2})\) maps \(\beta_{8b-1}\) to

\[
S^{4n-2} \quad \xrightarrow{\beta_{8b} \vee 4\beta_{8b-1}} \quad S^{4n-8b-2} \vee S^{4n-8b-1} \to \quad L^{4n-8b-1}_{4n-8b-2}.
\]

So in \(\pi_{4n-2}(L^{4n-1}_{4n-8b-2})\) both the composites in question are equal. The lemma follows from Lemma 5.8. \(\square\)

We have repeatedly used the next lemma in section 3.

**Lemma 5.11.** Suppose neither the top cell of \(L^{n+k}_m\) nor of \(L^{m+k}_m\) splits off. Let \(g : L^{n+k}_m \to \Sigma^{n-m}L^{m+k}_m\) be an equivalence below the top cell.

(i) If \(g\) is of odd degree on the top cell, then \(g\) can be adjusted to an equivalence.

(ii) Suppose \(\nu(m + k + 1) \equiv \nu(n + k + 1) \equiv 0 \pmod{4}\). Then \(g\) is of odd degree on the top cell (hence an equivalence) \(\iff\ \nu(m + k + 1) = \nu(n + k + 1)\).

**Proof.** Consider (i). The case when \(n + k\) is even is immediate. Assume \(n + k\) is odd. Let \(\alpha_1\) and \(\alpha_2\) be the attaching maps for the top cells of \(L^{n+k}_m\) and \(\Sigma^{n-m}L^{m+k}_m\).

Then \(t\alpha_2 = g_*(\alpha_1)\) for an odd \(t\). Note that \(\alpha_2\) is the image of an element \(\alpha'_2 \in \pi_{n+k-1}(\Sigma^{n-m}L^{m+k-1})\) under the projection \(\pi_*(\Sigma^{n-m}L^{m+k-1}) \to \pi_*(\Sigma^{n-m}L^{m+k-1})\). Consider the diagram

\[
\begin{array}{ccc}
S^{n+k-1} & \xrightarrow{\alpha'_2} & \Sigma^{n-m}L^{m+k-1} \\
\downarrow 1 & & \downarrow l \\
S^{n+k-1} & \xrightarrow{t\alpha'_2} & \Sigma^{n-m}L^{m+k-1}
\end{array}
\]

Since \(t : \Sigma^{n-m}L^{m+k} \to \Sigma^{n-m}L^{m+k}\) is an equivalence, we can view \(t\alpha'_2\) (thus \(t\alpha_2\)) as the attaching map for the top cell of \(\Sigma^{n-m}L^{m+k}\). Write \(t\alpha_2\) as \(\alpha_2\). Then \(\alpha_2 = g_*(\alpha_1)\), and (i) follows.
Consider (ii). Let 0 ≤ d < k be an integer satisfying
\[ k - d = \begin{cases} 8b & \text{if } \nu(n + k + 1) = 4b, \\ 8b - 4 & \text{if } \nu(n + k + 1) = 4b - 1, 4b - 2, \\ 8b - 7 & \text{if } \nu(n + k + 1) = 4b - 3. \end{cases} \]

Then by Theorem 1.1, \( L_{n+d}^{n+k} \) is \( S \)-reducible but \( L_{n+d}^{n+k+1} \) is not. Consider the map
\[ g': L_{n+d}^{n+k} \to \Sigma^{n-m} L_{m+d}^{m+k} \]
induced by \( g \), and the diagram
\[ \begin{array}{ccc} \pi_{n+k}(S^{n+k}) & \xrightarrow{g'_*} & \pi_{n+k}(S^{n+k}) \\ \downarrow \partial_1 & & \downarrow \partial_2 \\ \pi_{n+k-1}(L_{n+k-1}^{n+k-1}) & \xrightarrow{g'_*} & \pi_{n+k-1}(\Sigma^{n-m} L_{m+k-1}^{m+k-1}). \end{array} \]

Suppose \( \nu(m + k + 1) = \nu(n + k + 1) \). As in Lemma 5.1, then \( \partial_i(e_0), i = 1, 2 \)
and \( g'_*(\partial_i(e_0)) \) are the classes corresponding to \( \epsilon \beta_{k-1} \) for the same \( \epsilon \). \( L_{m+k}^{m+k} \) is not \( S \)-reducible implies that \( \epsilon \beta_{k-1} \) is not null in \( \pi_{n+k-1}(\Sigma^{n-m} L_{m+k-1}^{m+k-1}) \).
Therefore \( g' \) (and hence \( g \)) must be of odd degree on the top cell.

If \( g \) is an equivalence, then so is \( g' \). We have \( g'_*(\partial_1(e_0)) = \partial_2(e_0) \). This implies \( \nu(m + k + 1) = \nu(n + k + 1) \).

6. Triviality of \( \beta_i \) in (1.4)

In this section, we study under some appropriate conditions the triviality of \( \beta_i \) in (1.4) for \( \beta_i = \beta_{8b-1} \) or \( \beta_{8b+3} \).

**Lemma 6.1.** Let \( b \geq 2 \) in (i), and \( b \geq 1 \) in (ii). Let \( \partial_1, \partial_2 \) be respectively the boundaries \( \pi_*(L_{4A+3}^{4A+10}) \to \pi_{*-1}(L_{4A+2}^{4A+2}) \), \( \pi_*(L_{4A+2}^{4A+10}) \to \pi_{*-1}(L_{4A}^{4A+1}) \).

(i) Let \( 2^i+1 \) be the order of \( \beta_{8(b-1)-1} \). There is an integer \( i \) such that \( e - 1 \leq i \leq e \) and \( 2^i \beta_{8(b-1)-1} \) coextends to a map \( x_1: S^{4A+1+b} \to L_{4A+4}^{4A+10} \) satisfying \( A(x_1) \geq 4(b - 1) - 1 \), and \( \partial_1(x_1) = z_1 \), where \( z_1 \) is the composite \( S^{4A+8b} \to S^{4A+1} \to L_{4A}^{4A+4} \).

(ii) There is an integer \( i \) with \( 1 \leq i \leq 2 \) such that \( 2^i \beta_{8b-5} \) coextends to a map \( x_2: S^{4A+8b+5} \to L_{4A+4}^{4A+10} \) satisfying \( A(x_2) \geq 4b - 2 \), and \( \partial_2(x_2) = z_2 \), where \( z_2 \) is the composite
\[ S^{4A+8b+5} \to L_{4A+4}^{4A+10} \to L_{4A}^{4A+4}. \]

(iii) Both composites \( S^{4A+8b} \to S^{4A+1} \to L_{4A}^{4A+4} \) and \( S^{4A+8b+4} \to L_{4A}^{4A+4} \) are null-homotopic.

**Proof.** Part (iii) is just a corollary of (i) and (ii).

Consider (i). Let \( \beta = \beta_{8b-1} \). By Lemma 4.2 (i), \( 2\beta \) is null in \( J_{4A+8b}(L_{4A+4}^{4A+10}) \) but not null in \( J_{4A+8b+1}(L_{4A+4}^{4A+10}) \). So the only possible element in \( J_{4A+8b+1}(L_{4A+4}^{4A+10}) \) that might hit \( 2\beta \) is \( J_{*}(L_{4A+4}^{4A+10}) \) under the boundary
\[ g': J_{4A+8b+1}(L_{4A+4}^{4A+10}) \to J_{4A+8b}(L_{4A+4}^{4A+10}). \]

is from the top cell of \( L_{4A+4}^{4A+10} \), that is, there is an element \( x \in J_{4A+8b+1}(L_{4A+4}^{4A+10}) \) that is from a nontrivial element
\[ u_0 \in E_4^{2}(L_{4A+4}^{4A+10} ; J(8(b-1)-1)) \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
in AHSS satisfying $\partial'(x) = 2\beta$ in $J_{4A+8b}(L_{4A+1}^{4A+2})$. Let $2^i\beta_{8(b-1)-1} : S^{4A+8b+1} \rightarrow S^{4A+10}$ correspond to $u_0$, then $e-1 \leq i \leq e$. Denote $2^i\beta_{8(b-1)-1}$ also by $u_0$.

We want to coextend $u_0$ to a map $x_1 : S^{4A+8b+1} \rightarrow L_{4A+1}^{4A+10}$ with $A(x_1) \geq 4(b-1) - 1$. By [16, table 8.1], we see that when $-1 \leq t \leq 5$,

$$A_{4(b-1)}(\pi_{8(b-1)+t}(S^0)) \rightarrow J_{8(b-1)+t}(S^0)$$

is injective, and each nontrivial element of $E_2^{s,t}(S^0)$ survives to a homotopy class in the image of $J$ if $s \geq 4(b-1)$; moreover the projection

$$A_{4(b-1)-1}(\pi_{8(b-1)+t}(S^0)) \rightarrow J_{8(b-1)+t}(S^0)$$

is surjective when $0 \leq t \leq 6$. By repeatedly applying Lemma 4.3 (i) and (ii), we have the desired coextension $x_1$.

Next we show $\partial_1(x_1) = z_1$ in $\pi_{4A+8b}(L_{4A+1}^{4A+2})$. Consider the Adams spectral sequences for $\pi_*(L_{4A+1}^{4A+2})$ and $\pi_*(L_{4A})^{4A+2})$ and notice that

$$E_2^{s,t}(L_{4A+1}^{4A+2}) = \sum_{k=0}^{1} \text{Ext}^{s,t}(S^{4A+1+k})$$

and

$$E_2^{s,t}(L_{4A}^{4A+2}) = \sum_{0 \leq k \leq 2} \text{Ext}^{s,t}(S^{4A+1+k}).$$

By [16, table 8.1] and [18, Theorem 2.3.4, p.63], we get the charts for $E_3^{s,t}(L_{4A+1}^{4A+2})$ and $E_3^{s,t}(L_{4A}^{4A+2})$, where $s \geq 4(b-1)$ and $4A+8b \leq t - s \leq 4A+8b+1$, as follows.

$$E_3(L_{4A+1}^{4A+2})$$

where the bottom elements $u_1, u_2, v_1$ and $v_2$ are in the same position $(t-s,s) = (4A+8b, 4(b-1)+1)$, and $u_1$ corresponds to $\beta_{8b-1}$ into $\pi_{4A+8b}(L_{4A+1}^{4A+2})$, $u_2$ supports a nontrivial $d_1$-differential to $S^{4A-1}$ in the ASS for $\pi_*(L_{4A+2})^{4A+2})$. Also $v_2$, $v_3$ and $v_4$ support nontrivial $d_1$-differentials to $S^{4A-2}$ in the ASS for $\pi_*(L_{4A+2})^{4A+2})$, so those homotopy classes which $v_2$, $v_3$ and $v_4$ converge to, can not be in the image of $\partial_1$. Since $\partial'(x_1) = 2\beta$ in $J_{4A+8b+4}(L_{4A+1}^{4A+2})$ and $A(\partial_1(x_1)) \geq 4(b-1)$, we see that $\partial_1(x_1)$ corresponds to $2v_1$, and hence $\partial_1(x_1) = z_1$ in $\pi_{4A+8b}(L_{4A+2}^{4A+2})$ because $\eta(x_1) = 0$, while $\eta$ is injective on the subgroup generated by classes corresponding to $v_2$, $v_3$ and $v_4$. Here $\eta = 2\beta$.

Consider (ii). Let $\beta = \beta_{8b+3}$. By Lemma 4.2 (iii), $\beta$ is null in $J_{4A+8b+4}(L_{4A+1}^{4A+10})$ but not null in $J_{4A+8b+4}(L_{4A+9}^{4A+10})$. So the only possible element in $J_{4A+8b+4}(L_{4A+10})^{4A+10}$ that might hit $\beta$ is $J_{4A+1}^{4A+11}$ under the boundary

$$\partial' : J_{4A+8b+5}(L_{4A+2}^{4A+10}) \rightarrow J_{4A+8b+4}(S^{4A+1})$$

is from the top cell of $L_{4A+10}^{4A+2}$, and there is an element $x \in J_{4A+8b+5}(L_{4A+10}^{4A+10})$ that is from a nontrivial element $u_0 \in E_2^{4A+10,8(b-1)+3} = H_{4A+10}(L_{4A+2}^{4A+10}, J_8(b-1)+3)$ in
Applying Lemma 4.3 (i) and (ii) repeatedly, we get the desired coextension to a generator of \(\tilde{\nu}\) injective on \(\partial\).

Let \(u_0\) be the composite \(\beta_{8b-1}\) under \(\pi_{4A+8b+5}(S^{4A+6}) \mapsto \pi_{4A+8b+5}(L^{4A+12}_{4A+6})\), where \(2^t\) is the order of \(\beta_{8b-1}\). If \(u_1 \notin I(4A + 12, 4A + 6, 4A + 2)\) when put into \(J_\ast\), then \(u_1 = v_1 + \epsilon v_2\) for some \(\epsilon\). Here the homotopy class to which \(v_1\) survives is also denoted by \(v_1\). Choose a new \(u_1\) to be \(u_1 - \epsilon v_2\). Then \(u_1\) is a coextension of \(u_0\) with \(A(u_1) = 4b - 2\) and \(u_1 \in I(4A + 12, 4A + 6, 4A + 2)\) when put into \(J_\ast\).

Applying Lemma 4.3 (i) and (ii) repeatedly, we get the desired coextension \(x_2\).

Finally in the ASS the charts for \(E_{2}^{s,t}(L^{4A+12}_{4A+6})\) and \(E_{2}^{s,t}(S^{4A+1})\), where \(s \geq 4b - 1\) and \(t - s = 4A + 8b + 4\), coincide as the right chart above, where \(u\) is in the position \((t - s, s) = (4A + 8b + 4, 4b - 1)\) surviving to \(\beta_{8b+3} \in \pi_{4A+8b+4}(S^{4A+1})\) or \(z_2 \in \pi_{4A+8b+4}(L^{4A+12}_{4A+6})\). Since \(d'(x_2) = \beta\) in \(J_{4A+8b+4}(S^{4A+1})\) and \(A(d_2(x_2)) \geq 4b - 1\), there must hold \(d_2(x_2) = z_2\).

**Lemma 6.2.** (i) Let \(x_3\) be the composite \(S^{4A+8b+2} \xrightarrow{\beta_{8b+3}} S^{4A-1} \xrightarrow{P_{4A-4}}\). If \(\nu(4A + 8b + 4) \leq 4b + 2\), then \(x_3\) is null-homotopic in \(\pi_{4A+8b+2}(P_{4A-4})\).

(ii) Let \(x_4\) be the composite \(S^{4A+8b+2} \xrightarrow{2\beta_{8b+3}} S^{4A-1} \xrightarrow{L_{4A-4}}\). If \(\nu_2(4A + 8b + 4) \leq 4b + 3\), then \(x_4\) is null-homotopic in \(\pi_{4A+8b+2}(L_{4A+8b+4})\).

**Proof.** Let \(\beta = \beta_{8b+3}\). Consider (i). The case \(\nu(4A + 8b + 4) < 4b + 2\) is from [4, Proposition 5]. Assume \(\nu(4A + 8b + 4) = 4b + 2\). Let \(DP_{4A-4} = \Sigma P_{4A-4}\), where \(\nu(n) = 4b + 2\). Since \(P_{n+8b+7} = T(n\xi)\), where \(\xi\) is the Hopf real line bundle over \(P^{8b+7}\), and \(\nu(n) = 4b + 2\), there is a stable vector bundle \(\eta\) over \(S^{8b+4}\) corresponding to a generator of \(KO(S^{8b+4}) \otimes \mathbb{Z}_2\) and \(n\xi = P'(\eta)\), where \(P\) is the projection \(P^{8b+7} \xrightarrow{\text{det}} P^{8b+7} \xrightarrow{S^{8b+4}} \vee P_{8b+5}^{8b+7} \xrightarrow{S^{8b+4}}\). Note that \(T(\eta) = S^n \cup \mathbb{Z}_2\). We have a map \(f : P_{n+8b+7} \rightarrow T(\eta)\) such that \(f'\) is injective on \(H^*(\&; \mathbb{Z}_2)\), and the induced map \(P_{n+8b+7}^{8b+7} \rightarrow T(\eta)/S^n\) is precisely the projection \(S^{n+8b+4} \vee P_{n+8b+5}^{8b+7} \rightarrow S^{n+8b+4}\). Thus by duality we have a map \(g : S^{4A-1} \cup \mathbb{Z}_2 \xrightarrow{4A+8b+3} P^{4A-4}\) such that \(g_*\) is injective on \(H^*(\&; \mathbb{Z}_2)\) and the restriction of \(g\) on \(S^{4A-1}\) is the map \(S^{4A-1} \xrightarrow{4A+8b+3} P^{4A-4}\).
\[ P_{4A-4}^{4A-2} = P_{4A-4}^{4A-1}. \] There is a diagram

\[
\begin{array}{ccc}
\pi_{4A+8b+3}(S^{4A+8b+3}) & \xrightarrow{\partial_1} & \pi_{4A+8b+2}(S^{4A-1}) \\
\downarrow g_* & & \downarrow g_* \\
\pi_{4A+8b+3}(P_{4A}^{4A+8b+3}) & \xrightarrow{\partial_2} & \pi_{4A+8b+2}(S^{4A-1} \cup P_{4A-4}^{4A-2}) \\
\end{array}
\]

whose rows are exact sequences. Then \(\partial_1(1) = \beta, \partial_2(g_*(1)) = x_3.\) This means

\[ i_*x_3 = 0 \text{ in } \pi_*(P_{4A-4}^{4A+8b+3}) \text{ and (i) follows.} \]

Consider (ii). If \(\nu(4A + 8b + 4) \leq 4b + 2,\) then (ii) follows from (i) by the map

\[
\rho : P_{4A-4}^{4A+8b+3} \to L_{4A-4}^{4A+8b+3}. \]

Suppose \(\nu(4A + 8b + 4) = 4b + 3.\) Let \(\lambda = \lambda_{8b+7}\) and

\[
DL_{4A-4}^{4A+8b+7} = \Sigma L_{4A-4}^{n+8b+7}. \]

Then \(\nu(n) = 4b + 3.\) Thus \(\frac{n}{4} \lambda\) is stably trivial over \(L^{8b+3},\)

and \(\frac{n}{2} \lambda = p^*(2\eta)\), where \(\eta\) is a stable vector bundle over \(S^{8b+4}\)

corresponding to a generator of \(\bar{KO}(S^{8b+4}) \otimes \mathbb{Z}(2).\) Then the proof follows just as (i).

**Lemma 6.3.** (i) Let \(\nu(4A + 8b) = 2\) and \(b \geq 2.\) The composite

\[
x_5 : S^{4A+8b-2} \to S^{4A-1} \to L_{4A-4}^{4A-1}
\]

is null-homotopic in \(L_{4A-4}^{4A+6}.\)

(ii) If \(3 \leq \nu(4A + 8) \leq 4,\) then the composite \(x_5' : S^{4A+4} \to S^{4A-1} \to L_{4A-4}^{4A-1}\) is

null-homotopic in \(L_{4A-4}^{4A+7}.\)

**Proof.** Consider (i). Let \(\partial, \partial'\) be respectively the boundaries from \(\pi_*(P_{4A+6}^{4A+6})\), \(\pi_*(L_{4A+3}^{4A+6})\) to \(\pi_{-1}(P_{4A+2}^{4A+2}), \pi_{-1}(L_{4A+4}^{4A+2}).\)

Let \(i\) be a suitable inclusion. Let \(x'_5\) be the composite \(S^{4A+8b-2} \xrightarrow{\beta_{4b+1}} S^{4A-1} \to P_{4A-4}^{4A-1}.\)

Then \(x_5 = 2\rho(x'_5)\). Lemma 4.2 (ii) implies \(x'_5\) is null in \(J_{4A+8b-2}(P_{4A-4}^{4A+6})\) but not null in \(J_{4A+8b-2}(P_{4A-4}^{4A+5})\).

So each element \(x \in J_{4A+8b-1}(P_{4A+6}^{4A+6})\) that is from the nontrivial element

\[
w_1 \in E_2 = H_{4A+6}(P_{4A+3}^{4A+6}, J_{8(b-1)+1}^{4A+4}) \approx \mathbb{Z}_2
\]

in AHSS will satisfy \(\partial(x) = i_*(x'_5)\) when put into \(J_*(P_{4A-4}^{4A+2}).\)

By Lemma 4.3, it is easy to see that \(w_1 : S^{4A+8b-1} \xrightarrow{\beta_{4b+1} + 1} S^{4A+6}\) extends to a map \(w_2 : S^{4A+8b-1} \to P_{4A+6}^{4A+6}\) of Adams filtration \(4(b - 1).\)

Thus \(A(\partial(w_2)) \geq 4b - 3,\) so \(\partial'(2\rho_* (w_2))\) is of Adams filtration \(\geq 4b - 2,\) and \(\partial'(2\rho_* (w_2)) = i_*(x'_5)\) when put into \(J_*(L_{4A-4}^{4A+2})\) because \(\partial(w_2) = i_*(x'_5)\) when put into \(J_*(P_{4A-4}^{4A+2})\) and because \(x'_5 = 2\rho(x'_5)\). In ASS the chart for \(E_2^{t, v}(L_{4A-4}^{4A+2})\) with \(s \geq 4b - 2\) and \(4A + 8b - 2 \leq t - s \leq 4A + 8b - 1\) is given in the below
where $v_1$ is in the position $(t - s, s) = (4A + 8b - 2, 4b - 2)$ surviving to an element in $J_*(L^{4A+2}_{4A-1})$ of Adams filtration $4b - 2$. Since $A(i_*(x_5)) = 4b - 1, \partial'(2p_*(w_2)) = i_*(x_5)$ because $\eta(x_5) = 0$, while $\eta$ is injective on classes to which $v_2$ and $v_3$ converge.

Consider (ii). As in Lemma 6.2, if $\nu(4A + 8) = 3$ the composite

$$S^{4A+6} \xrightarrow{\beta_0} S^{4A-1} \xrightarrow{P_{4A-4}}$$

is null-homotopic and (ii) follows by the map $\rho$. The case $\nu(4A + 8) = 4$ can be proved in the same way by noting that $((4A + 8)/2)\lambda_{11}$ is stably trivial over $L^7$. □

**Lemma 6.4.** (i) There is a map $i_0 : S^{4A-1} \cup \eta e^{4A+1} \xrightarrow{P_{4A-2}}$ such that $g_*$ on $H_*(-; \mathbb{Z}_2)$ is injective.

(ii) $S^{4A+8b+2} \xrightarrow{\beta_{4b+3}} S^{4A+1}$ coextends to a map $w : S^{4A+8b+2} \xrightarrow{S^{4A-1} \cup \eta \times e^{4A+1}}$ $S^{4A+1}$ such that $A(w) \geq 4b$, and $(\rho i_0)_* (w) = z$ in $\pi_{*}(L^{4A+1}_{4A-2})$, where $z$ is the composite

$$S^{4A+8b+2} \xrightarrow{\beta_{4b+3}} S^{4A-1} \xrightarrow{\bigvee S^{4A-1} \cup S^{4A-2} = L^{4A-1}_{4A-2} \xrightarrow{L^{4A+1}_{4A-2}}}$

(iii) If $\nu (4A + 8b + 4) = 2$, then $i_0w$ is null-homotopic in $\pi_{*}(P^{4A+8}_{4A-2})$ (hence $z$ is null-homotopic in $\pi_{*}(L^{4A+1}_{4A-2})$).

**Proof.** Consider (i). Let $DP^{4A+1}_{4A-2} = \Sigma P^{2B+3}_{2B} \xrightarrow{\Sigma T(2B \xi)}$, where $\xi$ is the real Hopf bundle over $P^3$. There is a stable vector bundle $\eta'$ over $S^2$ corresponding to the generator of $KO(S^2)$ with $p^*(\eta') = 2B \xi$. So we have a map $f_1 : P^{2B+3} \xrightarrow{2B} S^{2B} \xrightarrow{\eta e^{2B+2}}$ inducing a monomorphism on $H^*(-; \mathbb{Z}_2)$, where $\eta = \beta_1$. Thus (i) follows by duality.

The first half of (ii) is from [4, Lemma 6] by applying the map $i_0$. Consider the second half. In the ASS, the chart for $E_3^*_u ((L^{4A+1}_{4A-2})$ with $s \geq 4b$ and $t - s = 4A + 8b + 2$, is as follows.

$$\begin{array}{ccc}
E_3^*(L^{4A+1}_{4A-2}) & \xrightarrow{i_0} & E_2^*(P^{4A+4}_{4A-2})
\end{array}$$

where $u$ is in the position $(t - s, s) = (4A + 8b + 2, 4b)$ and $2u$ corresponds to $z$ in homotopy. Since $(\rho i_0)^*$ induces zero morphism on $H^*(-; \mathbb{Z}_2)$, we have $A((\rho i_0)(w)) \geq 4b + 1$. Putting $i_0 w$ into $J_*(-; \mathbb{Z}_2)$, we see $A((\rho i_0)(w)) = 4b + 1$. Thus $(\rho i_0)_* (w)$ must be $z$.

Consider (iii). Lemma 4.2 (iv) implies that $i_0 w$ is null in $J_{4A+8b+2}^*(P^{4A+8}_{4A-2})$ but not null in $J_{4A+8b+2}^*(P^{4A+1}_{4A-2})$. As before, each coextension $x : S^{4A+8b+3} \xrightarrow{\beta_{4b+3}} S^{4A+8}$ will satisfy $\partial(x) = i_0 w$ in $J_* (P^{4A+4}_{4A-2})$. By Lemma 4.3, there exists a coextension $x : S^{4A+8b+3} \xrightarrow{\beta_{4b+3}} S^{4A+8}$ of $4\beta_{(b-1)+3} : S^{4A+8b+3} \xrightarrow{4\beta_{(b-1)+3}} S^{4A+8}$ will satisfy $\partial(x) = i_0 w$ in $J_* (P^{4A+4}_{4A-2})$. By Lemma 4.3, there exists a coextension $x : S^{4A+8b+3} \xrightarrow{\beta_{4b+3}} S^{4A+8}$ of $4\beta_{(b-1)+3} : S^{4A+8b+3} \xrightarrow{4\beta_{(b-1)+3}} S^{4A+8}$ with $A(x) \geq 4(b - 1) + 3$, hence $A(\partial(x)) \geq 4b$. Then $\partial(x) = i_0 w$ follows from the chart $E_2^*(P^{4A+4}_{4A-2})$ with $s \geq 4b$ and $t - s = 4A + 8b + 2$ as given above. Here $v$ is in the position $(t - s, s) = (4A + 8b + 2, 4b)$, and corresponds to $i_0 w$ when put into $P^{4A+4}_{4A-2}$, because $A((i_0 w) = 4b$.

The case $\nu(4A + 8b + 4) \geq 3$ in Lemma 6.4 is handled in the next lemma.
Lemma 6.5. Let \( i_0 \) and \( w, z \) be as in Lemma 6.4.

(i) If \( 3 \leq \nu(4A+8b+4) \leq 4b+1 \), then the composite
\[
x_6 : S^{4A+8b+2} \xrightarrow{\text{inv}} P^{4A+1} \xrightarrow{\text{inv}} P^{4A+8b+3}
\]
is null-homotopic.

(ii) If \( 3 \leq \nu(4A+8b+4) \leq 4b+2 \), then \( z \) is null in \( \pi_* (L^{4A+8b+3}_{4A-2}) \).

Proof. Part (i) is implied by Lemma 6.4 (ii) and Lemma 5.8. Consider (ii). Note that \( z = \rho x_6 \). If \( 3 \leq \nu(4A+8b+4) \leq 4b+1 \), then \( z \) is null-homotopic in \( \pi_* (L^{4A+8b+3}_{4A-2}) \) by (i). If \( \nu(4A+8b+4) = 4b+2 \), let \( \lambda = \lambda_{8b+5} \), then \( \frac{3}{2} \lambda \) is stably trivial over \( L^{8b+3} \), and (ii) follows from an argument as in the proof of Lemma 6.2.

\[ \square \]

References


Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015