TURNPIKE PROPERTY FOR EXTREMALS
OF VARIATIONAL PROBLEMS
WITH VECTOR-VALUED FUNCTIONS

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Abstract. In this paper we study the structure of extremals \( \nu: [0, T] \to \mathbb{R}^n \)
of variational problems with large enough \( T \), fixed end points and an integrand \( f \) from acomplete metric space of functions. We will establish the turnpike property for a generic integrand \( f \). Namely, we will show that for a generic integrand \( f \), any small \( \varepsilon > 0 \) and an extremal \( \nu: [0, T] \to \mathbb{R}^n \) of the variational problem with large enough \( T \), fixed end points and the integrand \( f \), for each \( \tau \in [L_1, T - L_1] \) the set \( \{ \nu(t) : t \in [\tau, \tau + L_2] \} \) is equal to a set \( H(f) \) up to \( \varepsilon \) in the Hausdorff metric. Here \( H(f) \subset \mathbb{R}^n \) is a compact set depending only on the integrand \( f \) and \( L_1 > L_2 > 0 \) are constants which depend only on \( \varepsilon \) and \( |\nu(0)|, |\nu(T)| \).

1. Introduction

In this paper we analyse the structure of optimal solutions of the variational problem

\[
(P) \quad \int_0^T f(z(t), z'(t)) \, dt \to \min, \quad z(0) = x, \ z(T) = y,
\]

where \( T > 0, x, y \in \mathbb{R}^n \) and \( f: \mathbb{R}^{2n} \to \mathbb{R}^1 \) is an integrand.

An optimal solution \( \nu: [0, T] \to \mathbb{R}^n \) of the variational problem \( (P) \) always depends on the integrand \( f \) and on \( x, y, T \). We say that the integrand \( f \) has the turnpike property if for large enough \( T \) the dependence on \( x, y, T \) is not essential. Namely, for any \( \varepsilon > 0 \) there exist constants \( L_1 > L_2 > 0 \) which depend only on \( |x|, |y|, \varepsilon \) such that for each \( \tau \in [L_1, T - L_1] \) the set \( \{ \nu(t) : t \in [\tau, \tau + L_2] \} \)
is equal to a set \( H(f) \) up to \( \varepsilon \) in the Hausdorff metric where \( H(f) \subset \mathbb{R}^n \) is a compact set depending only on the integrand \( f \).

More formally we say that an integrand \( f = f(x, u) \in C(\mathbb{R}^{2n}) \) has the turnpike property if there exists a compact set \( H(f) \subset \mathbb{R}^n \) such that for each bounded set \( K \subset \mathbb{R}^n \) and each \( \varepsilon > 0 \) there exist numbers \( L_1 > L_2 > 0 \) such that for each

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$T \geq 2L_1$, each $x, y \in K$ and an optimal solution $\nu: [0, T] \rightarrow R^n$ for the variational problem (P) the relation
\[
dist(H(f), \{\nu(t): t \in [\tau, \tau + L_2]\}) \leq \varepsilon
\]
holds for each $\tau \in [L_1, T - L_1]$. (Here $dist(., .)$ is the Hausdorff metric.)

The turnpike property is well known in mathematical economics. It was studied by many authors for optimal trajectories of a von Neumann-Gale model determined by a superlinear set-valued mapping (see Makarov and Rubinov [14] and the survey [16]) and for optimal trajectories of convex autonomous systems (see Carlson, Haurie and Leizarowitz [7, Ch. 4.6].) In the control theory the turnpike property was established by Artstein and Leizarowitz [1] for a tracking periodic problem. In all these cases we have an optimal control problem with a convex cost function and a convex set of trajectories. Asymptotic turnpike properties for optimal control problems with infinite time horizon were studied by Brock and Haurie [4], Carlson [5], Carlson, Haurie and Jabrane [6], Leizarowitz [10] and Zaslavski [19].

Our goal is to show that the turnpike property is a general phenomenon which holds for a large class of variational problems with vector-valued functions. We consider the complete metric space of integrands $\mathfrak{M}_k$ ($k$ is a nonnegative integer) described below and establish the existence of a set $\mathcal{F} \subset \mathfrak{M}_k$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}_k$ and such that each integrand $f \in \mathcal{F}$ has the turnpike property.

Moreover we show that the turnpike property holds for approximate solutions of variational problems with a generic integrand $f$ and that the turnpike phenomenon is stable under small perturbations of a generic integrand $f$.

A better understanding of the general nature of the turnpike phenomenon was achieved by our recent study of discrete-time control systems [17, 18] for which we established a weak version of the turnpike property. More recently in Zaslavski [20] employing the reduction to finite rewards by Leizarowitz [11, 12] and the representation formula by Leizarowitz and Mizel [13] an analogous result was established for optimal solutions of the variational problem (P) with $x, y \in R^n$, large enough $T$ and a generic integrand $f$ belonging to the space of functions $\mathfrak{A}$ described below.

In the weak version of the turnpike property established in [20] for an optimal solution of the problem (P) with $x, y \in R^n$, large enough $T$ and a generic integrand $f \in \mathfrak{A}$ the relation
\[
dist(H(f), \{\nu(t): t \in [\tau, \tau + L_2]\}) \leq \varepsilon
\]
with $L_2$ which depends on $\varepsilon$ and $|x|, |y|$ and a compact set $H(f) \subset R^n$ depending only on the integrand $f$, holds for each $\tau \in [0, T] \setminus E$ where $E \subset [0, T]$ is a measurable subset such that the Lebesgue measure of $E$ does not exceed a constant which depends on $\varepsilon$ and $|x|, |y|$.

The turnpike property which will be established in the present work guarantees that we may take $E = [0, L_1] \cup [T - L_1, T]$ where $L_1 > 0$ is a constant which depends on $\varepsilon$ and $|x|, |y|$.

Denote by $|.|$ the Euclidean norm in $R^n$ and denote by $\mathfrak{A}$ the set of continuous functions $f: R^n \times R^n \rightarrow R^1$ which satisfy the following assumptions:

(A)(i) for each $x \in R^n$ the function $f(x, .): R^n \rightarrow R^1$ is convex;

(ii) $f(x, u) \geq \sup \{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x, u) \in R^n \times R^n$ where $a > 0$ is a constant and $\psi: [0, \infty) \rightarrow [0, \infty)$ is an increasing function such that $\psi(t) \rightarrow +\infty$ as $t \rightarrow \infty$ (here $a$ and $\psi$ are independent on $f$);
(iii) for each $M, \varepsilon > 0$ there exist $\Gamma, \delta > 0$ such that
\[
|f(x_1, u_1) - f(x_2, u_2)| \leq \varepsilon \sup \{f(x_1, u_1), f(x_2, u_2)\}
\]
for each $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ which satisfy
\[
|x_i| \leq M, \ |u_i| \geq \Gamma \ (i = 1, 2), \ \sup \{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta.
\]
It is an elementary exercise to show that an integrand $f = f(x, u) \in C^1(\mathbb{R}^2n)$ belongs to $\mathfrak{A}$ if $f$ satisfies assumptions (Ai), (Aii) with a constant $a > 0$ and a function $\psi: [0, \infty) \rightarrow [0, \infty)$ and there exists an increasing function $\psi_0: [0, \infty) \rightarrow [0, \infty)$ such that for each $x, u \in \mathbb{R}^n$
\[
\sup \{ |\partial f / \partial x(x, u)|, |\partial f / \partial u(x, u)| \} \leq \psi_0(|x|)(1 + \psi(|u|)|u|).
\]
For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base
\[
E(N, \varepsilon, \lambda) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A}: |f(x, u) - g(x, u)| \leq \varepsilon \ (u, x \in \mathbb{R}^n, |x|, |u| \leq N), \ (|f(x, u)| + 1)(|g(x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \ (x, u \in \mathbb{R}^n, |x| \leq N)\}
\]
where $N > 0, \ \varepsilon > 0, \ \lambda > 1$ (see Kelley [9]).
It was shown in Zaslavski [20] that the uniform space $\mathfrak{A}$ is metrizable and complete. We consider functionals of the form
\[
I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t)) \, dt
\]
where $f \in \mathfrak{A}, 0 \leq T_1 \leq T_2 < +\infty$ and $x: [T_1, T_2] \rightarrow \mathbb{R}^n$ is an absolutely continuous (a.c.) function.
For $f \in \mathfrak{A}, y, z \in \mathbb{R}^n$ and numbers $T_1, T_2$ satisfying $0 \leq T_1 < T_2$ we set
\[
U^f(T_1, T_2, y, z) = \inf \{ I^f(T_1, T_2, x): x: [T_1, T_2] \rightarrow \mathbb{R}^n \text{ is an a.c function}
\text{satisfying } x(T_1) = y, \ x(T_2) = z \}.
\]
It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < +\infty$ for each $f \in \mathfrak{A}$, each $y, z \in \mathbb{R}^n$ and all numbers $T_1, T_2$ satisfying $0 \leq T_1 < T_2$.
Let $f \in \mathfrak{A}$. For any a.c. function $x: [0, \infty) \rightarrow \mathbb{R}^n$ we set
\[
J(x) = \lim_{T \rightarrow \infty} \inf T^{-1} I^f(0, T, x).
\]
Of special interest is the minimal long-run average cost growth rate
\[
(1.4) \quad \mu(f) = \inf \{ J(x): x: [0, \infty) \rightarrow \mathbb{R}^n \text{ is an a.c function} \}.
\]
Clearly $-\infty < \mu(f) < +\infty$. By a simple modification of the proof of Proposition 4.4 in Leizarowitz and Mizel [13] (see [20, Theorems 8.1, 8.2]) we established the representation formula
\[
(1.5) \quad U^f(0, T, x, y) = T \mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x, y), \quad x, y \in \mathbb{R}^n, \ T \in (0, \infty),
\]
where $\pi^T: \mathbb{R}^n \to \mathbb{R}^1$ is a continuous function and $(T, x, y) \to \theta^T(x, y) \in \mathbb{R}^1$ is a continuous nonnegative function defined for $T > 0$, $x, y \in \mathbb{R}^n$.

\begin{equation}
\pi^T(x) = \inf\left\{ \liminf_{T \to +\infty} |I^T(0, T, \nu) - \mu(f)T| : \nu: [0, \infty) \to \mathbb{R}^n \right\}
\end{equation}

is an a.c. function satisfying $\nu(0) = x$, $x \in \mathbb{R}^n$,

and for every $T > 0$, every $x \in \mathbb{R}^n$ there is $y \in \mathbb{R}^n$ satisfying $\theta^T(x, y) = 0$.

Here we follow Leizarowitz [11] in defining “good functions” for the infinite horizon variational problem with the integrand $f$.

An a.c. function $x: [0, \infty) \to \mathbb{R}^n$ is called an $(f)$-good function if the function $\Phi^T_f: T \to I^T(0, T, x) - \mu(f)T$, $T \in (0, \infty)$ is bounded. In [20] we showed that for each $f \in \mathcal{A}$ and each $z \in \mathbb{R}^n$ there exists an $(f)$-good function $\nu: [0, \infty) \to \mathbb{R}^n$ satisfying $\nu(0) = z$.

Propositions 1.1 and 3.2 in Zaslavski [20] imply the following result.

**Proposition 1.1.** For any a.c. function $x: [0, \infty) \to \mathbb{R}^n$ either

\[ I^T(0, T, x) - T\mu(f) \to +\infty \text{ as } T \to \infty \]

or

\[ \sup\{|I^T(0, T, x) - T\mu(f)| : T \in (0, \infty)\} < \infty. \]

Moreover any $(f)$-good function $x: [0, \infty) \to \mathbb{R}^n$ is bounded.

We denote $d(x, B) = \inf\{|x - y| : y \in B\}$ for $x \in \mathbb{R}^n$, $B \subset \mathbb{R}^n$. Denote by $\text{dist}(A, B)$ the Hausdorff metric for two sets $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$. For every bounded a.c. function $x: [0, \infty) \to \mathbb{R}^n$ define

\begin{equation}
\Omega(x) = \{y \in \mathbb{R}^n : \text{there exists a sequence } \{t_i\}_{i=0}^\infty \subset (0, \infty) \text{ for which} \]
\end{equation}

\[ t_i \to \infty, x(t_i) \to y \quad \text{as} \quad i \to \infty. \]

We say that an integrand $f \in \mathcal{A}$ has Property B if $\Omega(\nu_2) = \Omega(\nu_1)$ for all $(f)$-good functions $\nu_i: [0, \infty) \to \mathbb{R}^n$, $i = 1, 2$.

In Zaslavski [20, Theorem 2.1] we establish the following result which describes the limit behaviour of $(f)$-good functions for a generic $f \in \mathcal{A}$.

**Theorem 1.1.** There exists a set $F \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ and such that each $f \in F$ has Property B.

By Proposition 1.1 for each integrand $f \in \mathcal{A}$ which has Property B there exists a compact set $H(f) \subset \mathbb{R}^n$ such that $\Omega(\nu) = H(f)$ for each $(f)$-good function $\nu: [0, \infty) \to \mathbb{R}^n$.

Denote by $\mathfrak{M}$ the set of all functions $f \in C^1(\mathbb{R}^{2n})$ satisfying the following assumptions which ensure that each solution of (P) belongs to $C^2([0, T]; \mathbb{R}^n)$:

\[ \partial f/\partial u_i \in C^1(\mathbb{R}^{2n}) \quad \text{for } i = 1, \ldots, n; \]

the matrix $(\partial^2 f/\partial u_i \partial u_j)(x, u)$, $i, j = 1, \ldots, n$, is positive definite for all $(x, u) \in \mathbb{R}^{2n}$;

\[ f(x, u) \geq \sup\{|\psi(x)|, |\psi(u)|u|\} - a \quad \text{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^n; \]
there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i \colon [0, \infty) \to [0, \infty)$, $i = 0, 1, 2$, such that

$$
\phi_0(t)^{-1} \to +\infty \text{ as } t \to +\infty, \quad f(x, u) \geq \phi_0(c_0|u|) - \phi_1(|x|), \quad x, u \in \mathbb{R}^n;
$$

$$
\sup\{|\partial f/\partial x_i(x, u)|, |\partial f/\partial u_i(x, u)|\} \leq \phi_2(|x|)(1 + \phi_0(|u|)),
$$

$x, u \in \mathbb{R}^n$, $i = 1, \ldots, n$.

It is easy to see that $\mathfrak{M} \subset \mathfrak{A}$. We will establish the following result.

**Theorem 1.2.** Assume that an integrand $f \in \mathfrak{M}$ has Property B and $\varepsilon, K > 0$. Then there exists a neighborhood $U$ of $f$ in $\mathfrak{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in U$, each $T \geq 2l_0$ and each a.c. function $\nu \colon [0, T] \to \mathbb{R}^n$ which satisfies

$$
|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta
$$

the relation $|\nu(t)| \leq M$ holds for all $t \in [0, T]$ and

$$
(1.8) \quad \text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) \leq \varepsilon
$$

for each $\tau \in [l_0, T - l_0]$. Moreover if $d(\nu(0), H(f)) \leq \delta$, then (1.8) holds for each $\tau \in [0, T - l_0]$ and if $d(\nu(T), H(f)) \leq \delta$, then (1.8) holds for each $\tau \in [l_0, T - l]$.

Let $k \geq 1$ be an integer. Denote by $\mathfrak{A}_k$ the set of all integrands $f \in \mathfrak{A} \cap C^k(\mathbb{R}^{2n})$. For $p = (p_1, \ldots, p_{2n}) \in \{0, \ldots, k\}^{2n}$ and $f \in C^k(\mathbb{R}^{2n})$ we set

$$
|p| = \sum_{i=1}^{2n} p_i, \quad D^p f = \partial^{p_1} f/\partial y_1^{p_1} \cdots \partial^{p_{2n}} y_{2n}^{p_{2n}}.
$$

For the set $\mathfrak{A}_k$ we consider the uniformity which is determined by the following base.

$$
E(N, \varepsilon, \lambda) = \{(f, g) \in \mathfrak{A}_k \times \mathfrak{A}_k : |D^p f(x, u) - D^p g(x, u)| \leq \varepsilon
$$

$$
(u, x \in \mathbb{R}^n, |x|, |u| \leq N, p \in \{0, \ldots, k\}^{2n}, |p| \leq k),
$$

$$
|f(x, u) - g(x, u)| \leq \varepsilon (u, x \in \mathbb{R}^n, |x|, |u| \leq N),
$$

$$
((f(x, u)) + 1)(g(x, u)) + 1)^{-1} \in [\lambda^{-1}, \lambda](x, u \in \mathbb{R}^n, |x| \leq N)\}
$$

where $N > 0$, $\varepsilon > 0$, $\lambda > 1$ (see Kelley [9]). It is easy to verify that the uniform space $\mathfrak{A}_k$ is metrizable and complete (see [20], Section 2).

For each integer $k \geq 1$ we define $\mathcal{M}_k = \mathfrak{M} \cap \mathfrak{A}_k$. Set

$$
\mathfrak{A}_0 = \mathfrak{A}, \quad \mathcal{M}_0 = \mathfrak{M}.
$$

Let $k \geq 0$ be an integer. Denote by $\overline{\mathcal{M}}_k$ the closure of $\mathcal{M}_k$ in $\mathfrak{A}_k$ and consider the topological subspace $\overline{\mathcal{M}}_k \subset \mathfrak{A}_k$ with the relative topology. We will establish the following result.

**Theorem 1.3.** Let $q \geq 0$ be an integer. Then there exists a set $\mathcal{F}_q \subset \overline{\mathcal{M}}_q$ which is a countable intersection of open everywhere dense subsets of $\overline{\mathcal{M}}_q$ and such that each $f \in \mathcal{F}_q$ has Property B and the following property:

For each $\varepsilon, K > 0$ there exist a neighborhood $U$ of $f$ in $\mathfrak{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in U$, each $T \geq 2l_0$ and each a.c. function $\nu \colon [0, T] \to \mathbb{R}^n$ which satisfies

$$
|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta
$$
 Proposition 2.4 holds.

Proposition 2.5 holds.

Theorems 1.2 and 1.3 are extensions to the class of variational problems with vector-valued functions of the main result in Zaslavski [21] established for a class of one-dimensional variational problems arising in continuum mechanics which was discussed in Leizarowitz and Mizel [13] and Coleman, Marcus and Mizel [8]. In the approach used in [21] the following property of this class of one-dimensional variational problems established in Leizarowitz and Mizel [13] played a crucial role.

Property C. In the space of integrands there exists an everywhere dense subset $E$ such that for each $f \in E$ there exists an $(f)$-good periodic trajectory.

It is not clear whether Property C holds in general. In Zaslavski [20] and in the present paper we develop a more general approach based on the idea that the validity of Property B implies the weak version of the turnpike property for an integrand $f \in \mathfrak{A}$ and implies the turnpike property for an integrand $f \in \mathfrak{M}$.

2. Auxiliary results

In [20] we established the following results.

**Proposition 2.1** ([20, Proposition 3.1]). For each $f \in \mathfrak{A}$ there exists a neighborhood $U$ of $f$ in $\mathfrak{A}$ and a number $M > 0$ such that for each $g \in U$ and each $(g)$-good function $x : [0, \infty) \to \mathbb{R}^n$

$$\limsup_{t \to \infty} |x(t)| < M.$$ 

**Proposition 2.2** ([20, Proposition 3.2]). Let $f \in \mathfrak{A}$ and $M_1, M_2, c > 0$. Then there exist a neighborhood $U$ of $f$ in $\mathfrak{A}$ and $S > 0$ such that for each $g \in U$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following property holds.

For each $x, y \in \mathbb{R}^n$ satisfying $|x|, |y| \leq M_1$ and each a.c. function $\nu : [T_1, T_2] \to \mathbb{R}^n$ satisfying $\nu(T_1) = x$, $\nu(T_2) = y$, $I^g(T_1, T_2, \nu) \leq U^g(T_1, T_2, x, y) + M_2$ the following relation holds: $|\nu(t)| \leq S$ $(t \in [T_1, T_2])$.

**Proposition 2.3** ([20, Proposition 3.8]). Let $f \in \mathfrak{A}$, $0 < c_1 < c_2 < \infty$, $D, \varepsilon > 0$. Then there is a neighborhood $V$ of $f$ in $\mathfrak{A}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each a.c. function $x : [T_1, T_2] \to \mathbb{R}^n$ satisfying $\inf \{I^f(T_1, T_2, x), I^g(T_1, T_2, x)\} \leq D$ the relation $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq \varepsilon$ holds.

**Proposition 2.4** ([20, Proposition 3.9]). Let $f \in \mathfrak{A}$, $0 < c_1 < c_2 < \infty$, $c_3, \varepsilon > 0$. Then there exists a neighborhood $V$ of $f$ in $\mathfrak{A}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $z \in \mathbb{R}^n$ satisfying $|g|, |z| \leq c_3$ the relation $|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \leq \varepsilon$ holds.

**Proposition 2.5** ([20, Theorem 5.1]). Assume that $f \in \mathfrak{A}$ and there exists a compact set $H(f) \subset \mathbb{R}^n$ such that $\Omega(v) = H(f)$ for each $(f)$-good function $v : [0, \infty) \to \mathbb{R}^n$. Let $\varepsilon$ be a positive number. Then there exists an integer $L \geq 1$ such that for each $(f)$-good function $v : [0, \infty) \to \mathbb{R}^n$

$$\text{dist}(H(f), \{v(t) : t \in [T, T + L]\}) \leq \varepsilon$$

for all large $T$. 

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Proposition 2.6 ([20, Theorem 6.1]). Assume that \( f \in \mathfrak{A} \). Then the mapping \( (T_1, T_2, x, y) \to U^f(T_1, T_2, x, y) \) is continuous for \( T_1 \in [0, \infty) \), \( T_2 \in (T_1, \infty) \), \( x, y \in \mathbb{R}^n \).

Proposition 2.7 ([20, Theorem 2.3]). Assume that \( f \in \mathfrak{A} \) and there exists a compact set \( H(f) \subset \mathbb{R}^n \) such that \( \Omega(\nu) = H(f) \) for each \( (f) \)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \). Let \( \varepsilon > 0 \) be a positive number. Then there exist an integer \( L \geq 1 \) and a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathfrak{A} \) such that for each \( g \in \mathcal{U} \) and each \((g)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \)

\[
\text{dist}(H(f), \{ \nu(t) : t \in [T, T + L] \}) \leq \varepsilon \quad \text{for all large} \ T.
\]

Proposition 2.8 ([20, Lemma 10.2]). Assume that \( f \in \mathfrak{A} \) and \( H(f) \subset \mathbb{R}^n \) is a compact set such that \( \Omega(\nu) = H(f) \) for each \( (f) \)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \). Let \( \varepsilon_0 \in (0, 1) \), \( K_0, M_0 > 0 \) and let \( l \) be a positive integer such that for each \( (f) \)-good function \( x : [0, \infty) \to \mathbb{R}^n \)

\[
\text{dist}(H(f), \{ x(t) : t \in [T, T + l] \}) \leq 8^{-1} \varepsilon_0
\]

for all large \( T \) (the existence of \( l \) follows from Proposition 2.5). Then there exist an integer \( N \geq 10 \) and a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathfrak{A} \) such that for each \( g \in \mathcal{U} \), each \( S \in [0, \infty) \) and each a.c. function \( x : [S, S + NL] \to \mathbb{R}^n \) satisfying

\[
|x(S)|, |x(S + NL)| \leq K_0,
\]

\[
I^g(S, S + NL, x) \leq U^g(S, S + NL, x(S), x(S + NL)) + M_0
\]

there exists an integer \( i_0 \in [0, N - 8] \) such that for all \( T \in [S + i_0l, S + (i_0 + 7)l] \)

\[
\text{dist}(H(f), \{ x(t) : t \in [T, T + l] \}) \leq \varepsilon_0.
\]

Proposition 2.9 ([20, Lemma 10.3]). Assume that \( f \in \mathfrak{A} \) and \( H(f) \subset \mathbb{R}^n \) is a compact set such that \( \Omega(\nu) = H(f) \) for each \( (f) \)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \). Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that for each \( x_1, x_2 \in \mathbb{R}^n \) which satisfy

\[
d(x_1, H(f)) \leq \delta, \quad i = 1, 2 \text{ there exists an a.c. function } \nu : [0, T] \to \mathbb{R}^n \text{ for which}
\]

\[
T \geq 1, \nu(0) = x_1, \nu(T) = x_2, \quad I^f(0, T, \nu) - \pi^f(x_1) + \pi^f(x_2) - T\mu(f) \leq \varepsilon.
\]

Proposition 2.10 ([20, Lemma 10.4]). Assume that \( f \in \mathfrak{A} \) and \( H(f) \subset \mathbb{R}^n \) is a compact set such that \( \Omega(\nu) = H(f) \) for each \( (f) \)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \). Let \( \varepsilon \in (0, 1) \) and let \( L \) be a positive integer such that for each \( (f) \)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \)

\[
\text{dist}(H(f), \{ \nu(t) : t \in [S, S + L] \}) \leq \varepsilon
\]

for all large \( S \) (the existence of \( L \) follows from Proposition 2.5).

Then there exists \( \delta > 0 \) such that for each \( T \in [L, \infty) \) and each a.c. function \( \nu : [0, T] \to \mathbb{R}^n \) which satisfies

\[
d(\nu(0), H(f)) \leq \delta, \quad d(\nu(T), H(f)) \leq \delta,
\]

\[
I^f(0, T, \nu) - T\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(T)) \leq \delta
\]

relation (2.1) holds for every \( S \in [0, T - L] \).

Proposition 2.11 ([20, Lemma 9.1]). Assume that \( f \in \mathfrak{A} \). Then there exists a compact set \( H^* \subset \mathbb{R}^n \) which has the following properties:

there exists an \((f)\)-good function \( u : [0, \infty) \to \mathbb{R}^n \) such that \( \Omega(u) = H^* \);

for each \((f)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \) either \( \Omega(\nu) = H^* \) or \( \Omega(\nu) \setminus H^* \neq \emptyset \).
Proposition 2.12 ([20, Lemmas 9.3, 9.4]). Let \( f \in \mathcal{A} \) and let \( H^* \) be as guaranteed in Proposition 2.11. Assume that \( \phi: \mathbb{R}^n \to [0, \infty) \) is a continuous bounded function such that \( H^* = \{ x \in \mathbb{R}^n : \phi(x) = 0 \} \). For \( r \in (0, 1] \) we set

\[
f_r(x, u) = f(x, u) + r \phi(x), \quad x, u \in \mathbb{R}^n.
\]

Then \( f_r \in \mathcal{A} \), \( r \in (0, 1] \) and for each \( r \in (0, 1] \) and each \( (f_r) \)-good function \( \nu: [0, \infty) \to \mathbb{R}^n \)

\[
\Omega(\nu) = H^*.
\]

Moreover for any neighborhood \( U \) of \( f \) in \( \mathcal{A} \) there exists \( r_0 \in (0, 1) \) such that \( f_r \in \mathcal{A} \) for every \( r \in (0, r_0) \).

Proposition 2.13 ([20, Proposition 3.4]). Assume that \( f \in \mathcal{A} \), \( M_1 > 0 \), \( 0 \leq T_1 < T_2 \), \( x_i: [T_1, T_2] \to \mathbb{R}^n \), \( i = 1, 2, \ldots \) is a sequence of a.c. functions such that \( I^f(T_1, T_2, x_i) \leq M_1 \), \( i = 1, 2, \ldots \). Then there exist a subsequence \( \{ x_{i_k} \}_{k=1}^\infty \) and an a.c. function \( x: [T_1, T_2] \to \mathbb{R}^n \) such that \( I^f(T_1, T_2, x) \leq M_1 \), \( x_{i_k}(t) \to x(t) \) as \( k \to \infty \) uniformly in \( [T_1, T_2] \) and \( x_{i_k}' \to x' \) as \( k \to \infty \) weakly in \( L^1(\mathbb{R}^n; (T_1, T_2)) \).

Theorem 8.1, (8.2) and Proposition 7.3 in [20] imply the following result.

Proposition 2.14. Let \( f \in \mathcal{A} \). Then \( \pi^f(x) \to +\infty \) as \( |x| \to \infty \).

The following result was established in Aubin and Ekeland [2, Ch. 2, Sec. 3].

Proposition 2.15. Let \( \Omega \) be a closed subset of \( \mathbb{R}^n \). Then there exists a bounded nonnegative function \( \phi \in C^\infty(\mathbb{R}^n) \) such that \( \Omega = \{ x \in \mathbb{R}^n : \phi(x) = 0 \} \) and for each sequence of nonnegative integers \( p_1, \ldots, p_q \) the function \( \partial^{p_1} \phi / \partial x_1^{p_1}, \ldots, \partial x_q^{p_q} : \mathbb{R}^q \to \mathbb{R}^1 \) is bounded where \( |p| = \sum_{i=1}^q p_i \).

We can establish the following proposition which is a higher dimensional version of a well known result (Ball and Mizio [3], Morrey [15]).

Proposition 2.16. Suppose that \( f \in \mathcal{M} \), \( x, y \in \mathbb{R}^n \), \( T_1 \in [0, \infty) \), \( T_2 > T_1 \) and \( w: [T_1, T_2] \to \mathbb{R}^n \) is an a.c. function such that

\[
w(T_1) = x, \quad w(T_2) = y, \quad I^f(T_1, T_2, w) = U^f(T_1, T_2, x, y).
\]

Then \( w \in C^2([T_1, T_2]; \mathbb{R}^n) \).

3. Structure of the proof of Theorem 1.2

Assume that \( f \in \mathcal{M} \) and \( H(f) \subset \mathbb{R}^n \) is a compact set such that \( \Omega(\nu) = H(f) \) for each \( (f) \)-good function \( \nu: [0, \infty) \to \mathbb{R}^n \).

We will describe briefly the proof of Lemma 4.4 which is established in Section 4 and which plays a crucial role in our discussion of Theorem 1.2.

For each a.c. function \( u: [\tau_1, \tau_2] \to \mathbb{R}^n \) where \( \tau_1 \geq 0 \), \( \tau_2 > \tau_1 \) and each \( r_1, r_2 \in [\tau_1, \tau_2] \) satisfying \( r_1 < r_2 \) we set

\[
\sigma(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1) \mu(f).
\]

Let \( \varepsilon > 0 \). To prove Lemma 4.4 we need to show that there is a number \( q \geq 8 \) such that for each \( h_1, h_2 \in H(f) \) there exists an a.c. function \( \nu: [0, q] \to \mathbb{R}^n \) which satisfies

\[
\nu(0) = h_1, \quad \nu(q) = h_2, \quad \sigma(0, q, \nu) \leq \varepsilon.
\]
By Proposition 2.6 there exists a sequence of positive numbers \( \{\delta_i\}_{i=0}^{\infty} \) such that

\[
\delta_0 \in (0, 8^{-1} \varepsilon), \quad \delta_{i+1} < \delta_i, \quad i = 0, 1, \ldots.
\]

and for each integer \( i \geq 0 \), each \( x_1, x_2, y_1, y_2 \in H(f) \) which satisfy \( |x_j - y_j| \leq \delta_i \), \( j = 1, 2 \) the following relations hold:

\[
|U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \leq 2^{-1-i} \varepsilon,
\]

\[
|\pi^f(x_j) - \pi^f(y_j)| \leq 2^{-1-i} \varepsilon, \quad j = 1, 2.
\]

We will show that there exists an \( (f) \)-good function \( \nu_* : [0, \infty) \to H(f) \) such that \( \sigma(T_1, T_2, \nu_*) = 0 \) for each \( T_1 \geq 0, T_2 > T_1 \). Then we will define a function \( \phi : [0, \infty) \to R^3 \) as

\[
\phi(\tau) = I^f(0, 1, \nu_* + P_\tau) - \mu(f) - \pi^f(\nu_*(0)) + \pi^f(\nu_*(\tau)), \quad \tau \in [0, \infty),
\]

where

\[
P_\tau(t) = t(\nu_*(\tau) - \nu_*(1)), \quad t \in R^1, \quad \tau \in [0, \infty),
\]

and verify that

\[
\phi \in C^1([0, \infty); R^3), \quad \phi(1) = 0, \quad \phi(t) \geq 0, \quad t \in [0, \infty).
\]

We can find a number \( L \geq 10 \) and a sequence of numbers \( \{T_p\}_{p=1}^{\infty} \) such that

\[
\text{dist}(H(f), \{\nu_*(t) : t \in [T, T + L]\}) \leq 4^{-1} \delta_0 \quad \text{for all} \quad T \in [0, \infty),
\]

\[
T_p \geq 2L + 8, \quad |\nu_*(0) - \nu_*(T_p)| \leq 2^{-8} \delta_p, \quad p = 1, 2, \ldots.
\]

Fix a positive number \( \varepsilon_0 < 2^{-8}L^{-1} \varepsilon \). By (3.6) there exists a positive number \( \Delta \) such that

\[
\Delta < 2^{-8}, \quad |\phi'(t)| \leq 2^{-1} \varepsilon_0, \quad t \in [1 - \Delta, 1 + \Delta].
\]

Choose an integer \( N > 64(L + 1) \Delta^{-1} \) and set \( q = \sum_{i=1}^{N} T_i + 8L + 8 \).

Let \( h_1, h_1 \in H(f) \). We will construct an a.c. function \( \nu : [0, q] \to R^n \) satisfying (3.1). By the definition of \( L \) there exists numbers \( t_1, t_2 \) such that

\[
t_1 \in [0, L], \quad t_2 \in [8, L + 8], \quad |h_2 - \nu_*(t_2)| \leq 4^{-1} \delta_0, \quad j = 1, 2.
\]

We set \( \Delta_0 = (N - 1)^{-1}(8L + 8 - (t_2 - t_1)) \) and verify that \( \Delta_0 \in (0, \Delta) \). By using (3.8), (3.10) and the definition of \( \{\delta_i\}_{i=0}^{\infty} \) we can construct functions \( w_0 : [0, T_1 - t_1] \to R^n \) and \( u_0 : [0, t_2] \to R^n \) such that

\[
w_0(0) = h_1, \quad w_0(T_1 - t_1) = \nu_*(0), \quad \sigma(0, T_1 - t_1, u_0) \leq 2^{-6} \varepsilon,
\]

\[
w_0(0) = \nu_*(0), \quad u_0(t_2) = h_2, \quad \sigma(0, t_2, w_0) \leq 2^{-7} \varepsilon.
\]

For each integer \( k \geq 1 \) there exists an a.c. function \( w_k : [0, \Delta_0 + T_{k+1}] \to R^n \) such that

\[
w_k(t) = \nu_*(t) + P_{1-\Delta_0}(t), \quad t \in [0, 1], \quad w_k(t) = \nu_*(t - \Delta_0), \quad t \in [1, \Delta_0 + T_{k+1} - 1],
\]

\[
w_k(\Delta_0 + T_{k+1}) = \nu_*(0),
\]

\[
I^f(\Delta_0 + T_{k+1} - 1, \Delta_0 + T_{k+1}, w_k) = U^f(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})).
\]

By using (3.4)–(3.6), (3.9), (3.8) and the definition of \( \{\delta_i\}_{i=0}^{\infty} \) we show that

\[
\sigma(0, T_{k+1} + \Delta_0, w_k) \leq 2^{-1} \varepsilon_0 \Delta_0 + 2^{-k-8} \varepsilon, \quad k = 1, 2, \ldots.
\]

We will finally define \( \nu : [0, q] \to R^n \) as a concatenation of the functions \( w_k, k = 0, \ldots, N - 1, u_0 \) and show that (3.1) holds.
The structure of the proof of Theorem 1.2. For simplicity we will only sketch the proof of the turnpike property for the integrand $f$ and will not discuss the stability of the turnpike phenomenon under small perturbations of $f$. We choose a small enough number $\delta > 0$ and large enough numbers $l_0 > l > 0$ depending on $\varepsilon, K$.

Assume that $T \geq 2l_0$ and an a.c. function $\nu: [0, T] \rightarrow R^n$ satisfies

$$|\nu(0)|, |\nu(T)| \leq K, \quad I^f(0, T, \nu) \leq U^f(0, T, \nu(0), \nu(T)) + \delta.$$  \hfill (3.12)

We will show that for each $\tau \in [l_0, T - l_0]

$$\text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) \leq \varepsilon.$$  \hfill (3.13)

Assume the contrary. Then there is a number $\tau \in [l_0, T - l_0]$ for which

$$\text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) > \varepsilon.$$  \hfill (3.14)

By Proposition 2.8 there are numbers $S_1, S_2 \in [0, T]$ such that

$$d(\nu(S_i), H(f)) \leq \delta, \quad i = 1, 2, \quad S_2 - \tau, \quad \tau - S_1 \in [c_1, c_2],$$  \hfill (3.15)

where $c_1, c_2$ are some positive constants depending on $\varepsilon, K$.

It follows from (3.14), (3.15) and Proposition 2.10 that

$$\sigma(S_1, S_2, \nu) > \delta_0$$  \hfill (3.16)

where $\delta_0 > 8\delta$ is some constant depending on $\varepsilon, K$. By using (3.15) and Lemma 4.4 we show that there exists an a.c. function $u: [0, T] \rightarrow R^n$ such that

$$u(t) = \nu(t), \quad t \in [0, S_1] \cup [S_2, T], \quad \sigma(S_1, S_2, u) < \delta_0 - \delta.$$  \hfill (3.17)

It follows from (3.12), (3.16), (3.17) that

$$\delta \geq I^f(0, T, \nu) - I^f(0, T, u) = \sigma(S_1, S_2, \nu) - \sigma(S_1, S_2, u) > \delta.$$  \hfill (4.1)

The obtained contradiction proves that (3.13) holds for each $\tau \in [l_0, T - l_0]$.

4. Proof of Theorem 1.2

Assume that $f \in \mathfrak{M}$ and $H(f) \subset R^n$ is a compact set such that $\Omega(\nu) = H(f)$ for each $(f)$-good function $\nu: [0, \infty) \rightarrow R^n$.

Lemma 4.1. Let $h \in H(f)$. Then there exists an $(f)$-good function $\nu: [0, \infty) \rightarrow H(f)$ such that $\nu(0) = h$ and

$$I^f(T_1, T_2, \nu) = \mu(f)(T_2 - T_1) + \pi^f(\nu(T_1)) - \pi^f(\nu(T_2))$$  \hfill (4.1)

for each $T_1 \geq 0$, $T_2 > T_1$.

Proof. Consider any $(f)$-good function $w: [0, \infty) \rightarrow R^n$. Then

$$\Omega(w) = H(f).$$

By Proposition 2.1 the function $w$ is bounded. It is easy to see that the following property holds:

(a) for each $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for each $T_1 \geq T(\varepsilon), T_2 > T_1$

$$I^f(T_1, T_2, w) - \mu(f)(T_2 - T_1) - \pi^f(w(T_1)) + \pi^f(w(T_2)) \leq \varepsilon.$$  \hfill (4.2)

There exists a sequence of numbers $\{T_p\}_{p=0}^\infty \subset [0, \infty)$ such that

$$T_{p+1} \geq T_p + 1, \quad p = 0, 1, \ldots, \quad w(T_p) \rightarrow h \quad \text{as} \quad p \rightarrow \infty.$$  \hfill (4.2)

For every integer $p \geq 1$ we set

$$\nu_p(t) = w(t + T_p), \quad t \in [0, \infty).$$  \hfill (4.3)
Define a function
\[ (4.10) \]
\[
\inf_{h}\phi(h) \rightarrow \nu(t) \quad \text{as} \quad j \rightarrow \infty \quad \text{uniformly in} \quad [0, N],
\]
(4.2)–(4.4) imply that \( \nu(0) = h \) and \( \nu(t) \in H(f), \ t \in [0, \infty) \). It follows from property (a), (4.3), (4.4) that (4.1) holds for each \( T_1 \geq 0, T_2 > T_1 \). The lemma is proved.

By Lemma 4.1 there exists an \( (f) \)-good function \( \nu_* : [0, \infty) \rightarrow H(f) \) such that
\[ (4.5) \]
\[
I^f(T_1, T_2, \nu_*) = \mu(f)(T_2 - T_1) + \pi^f(\nu_*(T_1)) - \pi^f(\nu_*(T_2))
\]
for each \( T_1 \geq 0, T_2 > T_1 \).

It follows from Proposition 2.16 that
\[ (4.6) \]
\[
\nu_* \in C^2([0, \infty); \mathbb{R}^n).
\]

**Lemma 4.2.** The function \( \pi^f \cdot \nu_* \in C^1([0, \infty); \mathbb{R}^1) \).

**Proof.** By (4.5) for each \( T \geq 0 \)
\[ (4.7) \]
\[
\pi^f(\nu_*(T)) = -I^f(0, T, \nu_*) + \mu(f)T + \pi^f(\nu_*(0)).
\]
Together with (4.6) this implies the assertion of the lemma.

For each \( \tau \in [0, \infty) \) we define
\[ (4.8) \]
\[
P_\tau(t) = t(\nu_*(\tau) - \nu_*(1)), \quad t \in \mathbb{R}^1, \quad \psi(\tau) = I^f(0, 1, \nu_* + P_\tau).
\]

**Lemma 4.3.** \( \psi \in C^1([0, \infty); \mathbb{R}^1) \).

**Proof.** For \( \lambda, \tau \in [0, \infty) \) we set
\[ (4.9) \]
\[
B(\lambda, t) = (\nu_*(t) + P_\lambda(t), \nu'_*(t) + P'_\lambda(t)).
\]

Let \( \tau, h \in [0, \infty), \tau \neq h \) and \( t \in [0, 1] \). By (4.7), (4.8) there exists \( \lambda_h(t) \in [\inf\{h, \tau\}, \sup\{h, \tau\}] \) such that
\[ (4.10) \]
\[
(\chi - \tau)^{-1}f(B(h, t)) = \partial f/\partial x(B(\lambda_h(t), t))tu'_*(\lambda_h(t))
\]
\[
+ \partial f/\partial u(B(\lambda_h(t), t))u'_*(\lambda_h(t)) \rightarrow \partial f/\partial x(B(\tau, t))u'_*(\tau) + \partial f/\partial u(B(\tau, t))u'_*(\tau)
\]
as \( h \rightarrow \tau \) uniformly for all \( t \in [0, 1] \). This implies that \( \psi \in C^1([0, \infty); \mathbb{R}^1) \). The lemma is proved.

**Lemma 4.4.** Let \( \varepsilon > 0 \). Then there exists a number \( q \geq 8 \) such that for each \( h_1, h_2 \in H(f) \) there exists an a.c. function \( \nu : [0, q] \rightarrow \mathbb{R}^n \) which satisfies
\[ (4.11) \]
\[
I^f(0, q, \nu) \leq q\mu(f) + \pi^f(\nu(0)) - \pi^f(\nu(q)) + \varepsilon.
\]

**Proof.** Define a function \( \phi : [0, \infty) \rightarrow \mathbb{R}^1 \) as follows:
\[ (4.12) \]
\[
\phi(t) = \psi(t) - \mu(f) - \pi^f(\nu_*(0)) + \pi^f(\nu_*(t)), \quad t \in [0, \infty).
\]
It follows from (4.11), (4.7), Lemmas 4.2, 4.3, (4.5) and the representation formula (see (1.5), (1.6)) that
\[ (4.13) \]
\[
\phi \in C^1([0, \infty); \mathbb{R}^1), \quad \phi(1) = 0, \quad \phi(t) \geq 0, \quad t \in [0, \infty).
\]
By Proposition 2.6 there exists a sequence of positive numbers \( \{\delta_i\}_{i=0}^{\infty} \) such that

\[
(4.13) \quad \delta_0 \in (0, 8^{-1}\varepsilon), \quad \delta_{i+1} < \delta_i, \quad i = 0, 1, \ldots
\]

and for each integer \( i \geq 0 \), each \( x_1, x_2, y_1, y_2 \in H(f) \) which satisfy \( |x_j - y_j| \leq \delta_i \), \( j = 1, 2 \) the following relations hold:

\[
(4.14) \quad |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \leq 2^{-i-8}\varepsilon,
\]

\[
|\pi^f(x_j) - \pi^f(y_j)| \leq 2^{-i-8}\varepsilon, \quad j = 1, 2.
\]

By the definition of \( \nu \), Proposition 2.5 there exists an integer \( L \geq 10 \) such that

\[
(4.15) \quad \text{dist}(H(f), \{\nu_s(t) : t \in [T, T + L]\}) \leq 4^{-1}\delta_0
\]

for all \( T \in [0, \infty) \).

Since \( \Omega(\nu_s) = H(f) \) and \( \nu_s(0) \in H(f) \) there exists a sequence of numbers \( \{T_p\}_{p=1}^{\infty} \) such that

\[
(4.16) \quad T_p \geq 2L + 8, \quad |\nu_s(0) - \nu_s(T_p)| \leq 2^{-8}\delta_p, \quad p = 1, 2, \ldots.
\]

Fix a positive number \( \varepsilon_0 \) for which

\[
(4.17) \quad \varepsilon_0 < 2^{-8}L^{-1}\varepsilon.
\]

It follows from (4.12) that there exists a positive number \( \Delta \) such that

\[
(4.18) \quad \Delta < 2^{-8}, \quad |\phi^f(t)| \leq 2^{-1}\varepsilon_0, \quad t \in [1 - \Delta, 1 + \Delta].
\]

Choose an integer

\[
(4.19) \quad N > 64(L + 1)\Delta^{-1}
\]

and set

\[
(4.20) \quad q = \sum_{i=1}^{N} T_i + 8L + 8.
\]

Let \( h_1, h_1 \in H(f) \). We will construct an a.c. function \( \nu : [0, q] \to R^n \) satisfying (4.9), (4.10). It follows from (4.15) which holds for each \( T \in [0, \infty) \) that there exists numbers \( t_1, t_2 \) such that

\[
(4.21) \quad t_1 \in [0, L], \quad t_2 \in [8, L + 8], \quad |h_j - \nu_s(t_j)| \leq 4^{-1}\delta_0, \quad j = 1, 2.
\]

Set

\[
(4.22) \quad \Delta_0 = (N - 1)^{-1}(8L + 8 - (t_2 - t_1)).
\]

(4.22), (4.21), (4.19), (4.18) imply that

\[
(4.23) \quad \Delta_0 \in (0, \Delta).
\]

For each a.c. function \( u : [\tau_1, \tau_2] \to R^n \) where \( \tau_1 \geq 0, \tau_2 > \tau_1 \) and each \( r_1, r_2 \in [\tau_1, \tau_2] \) satisfying \( r_1 < r_2 \) we set

\[
(4.24) \quad \sigma(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f).
\]

It follows from (4.16), (4.21) and Proposition 2.13 that there exists an a.c. function \( w_0 : [0, T_1 - t_1] \to R^n \) such that

\[
(4.25) \quad w_0(0) = h_1, \quad w_0(t) = \nu_s(t_1 + t), \quad t \in [1, T_1 - t_1 - 1], \quad w_0(T_1 - t_1) = \nu_s(0),
\]

\[
I^f(\tau, \tau + 1, w_0) = U^f(0, 1, w_0(\tau), w_0(\tau + 1)), \quad \tau = 0, T_1 - t_1 - 1.
\]
By (4.24), the definition of $\nu_*$, (4.5), (4.25), (4.21), (4.16) and the definition of $\{\delta_j\}_{j=0}^{\infty}$
\[
\begin{align*}
\sigma(0, T_1 - t_1, w_0) &= \sigma(0, 1, w_0) + \sigma(T_1 - t_1 - 1, T_1 - t_1, w_0) \\
&= U^f(0, 1, h_1, \nu_*(t_1 + 1)) - \pi^f(h_1) + \pi^f(\nu_*(t_1 + 1) - \mu(f)) \\
&+ U^f(0, 1, \nu_*(T_1 - 1), \nu_*(0)) - \pi^f(\nu_*(T_1 - 1) + \pi^f(\nu_*(0)) - \mu(f) \\
&\leq 4 \cdot 2^{-8\varepsilon} + U^f(0, 1, \nu_*(t_1), \nu_*(t_1 + 1)) - \pi^f(\nu_*(t_1)) \\
&+ \pi^f(\nu_*(t_1 + 1)) - \mu(f) + U^f(0, 1, \nu_*(T_1 - 1), \nu_*(T_1)) \\
&- \pi^f(\nu_*(T_1 - 1)) + \pi^f(\nu_*(T_1)) - \mu(f) \leq 2^{-6\varepsilon}.
\end{align*}
\]

Let $k \geq 1$ be an integer. By (4.16), (4.7), (4.23), (4.18) and Proposition 2.13 there exists an a.c. function $w_k: [0, \Delta_0 + T_{k+1}] \to \mathbb{R}^n$ such that
\[
\begin{align*}
w_k(t) &= \nu_*(t) + P_{1-\Delta_0}(t), \quad t \in [0, 1], \quad w_k(t) = \nu_*(t - \Delta_0), \quad t \in [1, \Delta_0 + T_{k+1} - 1], \\
w_k(\Delta_0 + T_{k+1}) &= \nu_*(0), \\
I^f(\Delta_0 + T_{k+1} - 1, \Delta_0 + T_{k+1}, w_k) &= U^f(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})).
\end{align*}
\]

By (4.27), (4.7)
\[
w_k(0) = \nu_*(0).
\]

We will estimate $\sigma(0, T_{k+1} + \Delta_0, w_k)$. It follows from (4.27), (4.24), (4.5) that
\[
\begin{align*}
\sigma(0, T_{k+1} + \Delta_0, w_k) &= \sigma(0, 1, w_k) + \sigma(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k).
\end{align*}
\]

(4.27), (4.24), (4.7), (4.11) imply that
\[
\begin{align*}
\sigma(0, 1, w_k) &= \phi(1 - \Delta_0).
\end{align*}
\]

It follows from (4.30), (4.23), (4.18), (4.12) that
\[
\begin{align*}
\sigma(0, 1, w_k) &\leq 2^{-1} \Delta_0 \varepsilon_0.
\end{align*}
\]

By (4.27), (4.24), (4.16), the definition of $\nu_*$, (4.5), the definition of $\{\delta_j\}_{j=0}^{\infty}$ (see (4.13), (4.14))
\[
\begin{align*}
\sigma(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k) &= U^f(0, 1, \nu_*(T_{k+1} - 1), \nu_*(0)) - \pi^f(\nu_*(T_{k+1} - 1)) \\
&+ \pi^f(\nu_*(0)) - \mu(f) \\
&\leq U^f(0, 1, \nu_*(T_{k+1} - 1), \nu_*(T_{k+1})) - \pi^f(\nu_*(T_{k+1} - 1)) \\
&+ \pi^f(\nu_*(T_{k+1})) - \mu(f) + 2 \cdot 2^{-k-9\varepsilon} = 2^{-k-8\varepsilon}.
\end{align*}
\]

Combining (4.29), (4.31), (4.32) we obtain that
\[
\begin{align*}
\sigma(0, T_{k+1} + \Delta_0, w_k) &\leq 2^{-1} \varepsilon_0 \Delta_0 + 2^{-k-8\varepsilon}.
\end{align*}
\]

By Proposition 2.13 there exists an a.c. function $u_0: [0, t_2] \to \mathbb{R}^n$ such that
\[
\begin{align*}
u_*(t) &= \nu_*(t), \quad t \in [0, t_2 - 1], \quad u_0(t_2) = h_2, \\
I^f(t_2 - 1, t_2, u_0) &= U^f(0, 1, u_0(t_2 - 1), u_0(t_2)).
\end{align*}
\]
It follows from (4.34), (4.24), (4.21), the definition of \( \{\delta_i\}_{i=0}^{\infty} \) (see (4.13), (4.14)), the definition of \( \nu \), (4.5) that

\[
\sigma(0, t_2, u_0) = \sigma(t_2 - 1, t_2, u_0)
\]

\[
= U^f(0, 1, \nu(t_2 - 1), h_2) - \pi^f(\nu(t_2 - 1)) + \pi^f(h_2) - \mu(f)
\]

\[
\leq U^f(0, 1, \nu(t_2 - 1), \nu(t_2)) - \pi^f(\nu(t_2)) + \pi^f(h_2) - \mu(f) + 2^{-7}\varepsilon \leq 2^{-7}\varepsilon.
\]

(4.35) implies that

\[
T_1 - t_1 + \sum_{k=1}^{N-1} (\Delta_0 + T_{k+1}) + t_2 = q.
\]

By (4.36), (4.25), (4.27), (4.28), (4.34) there exists an a.c. function \( \nu: [0, q] \to R^n \) such that

\[
\nu(t) = w_0(t), \ t \in [0, T_1 - t_1], \ \nu(t) = w_k \left( t - \left( \sum_{i=1}^{k} T_i + (k-1)\Delta_0 - t_1 \right) \right),
\]

(4.37) implies that

\[
\nu(0) = h_1, \ \nu(q) = h_2.
\]

It follows from (4.24), (4.37), (4.26), (4.33), (4.35), (4.22), (4.21), (4.17) that

\[
I^f(0, q, \nu) - \pi^f(\nu(0)) + \pi^f(\nu(q)) - q\mu(f)
\]

\[
= \sigma(0, T_1 - t_1, u_0) + \sum_{k=1}^{N-1} \sigma(0, T_{k+1} + \Delta_0, w_k) + \sigma(0, t_2, u_0)
\]

\[
\leq 2^{-6}\varepsilon + \sum_{k=1}^{N-1} (2^{-1}\varepsilon_0 + 2^{-k-8}\varepsilon) + 2^{-7}\varepsilon \leq 2^{-5}\varepsilon + 2^{-1}(N-1)\varepsilon_0 \Delta_0
\]

\[
\leq 2^{-5}\varepsilon + 2^{-1}(9L + 16)\varepsilon_0 \leq 2^{-1}\varepsilon.
\]

This completes the proof of the lemma.

Proof of Theorem 1.2. Let \( \varepsilon, K > 0 \). We may assume that

\[
\varepsilon < 1, \ K > \sup\{|h|: h \in H(f)\} + 4.
\]

By Proposition 2.2 there exist a neighborhood \( U_1 \) of \( f \) in \( \mathfrak{Y} \) and a number \( M > K \) such that for each \( g \in U_1 \), each \( T_1 \geq 0 \), \( T_2 \geq T_1 + 1 \) and each a.c. function \( \nu: [T_1, T_2] \to R^n \) which satisfies

\[
|\nu(T_i)| \leq 2K + 4, \ i = 1, 2, \ I^g(T_1, T_2, \nu) \leq U^g(T_1, T_2, \nu(T_1), \nu(T_2)) + 2
\]

(4.38) holds:

\[
|\nu(t)| \leq M (t \in [T_1, T_2]).
\]

(4.39)
By Proposition 2.6 there exists \\
(4.40) \[ \delta_1 \in (0, 8^{-1}\varepsilon) \]
such that for each \( x_1, x_2, y_1, y_2 \in \mathbb{R}^n \) which satisfy \\
(4.41) \[ |x_i|, |y_i| \leq 2M + 4 + 2 \sup\{|h|: h \in H(f)\}, \quad |x_i - y_i| \leq 4\delta_1, \quad i = 1, 2 \]
the following relations hold: \\
(4.42) \[ |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \leq 2^{-8}\varepsilon, \quad |\pi^f(x_i) - \pi^f(y_i)| \leq 2^{-8}\varepsilon, \quad i = 1, 2. \]

By Proposition 2.5 there exists an integer \( l \geq 1 \) such that for each \((f)-\text{good}\) function \( \nu: [0, \infty) \to \mathbb{R}^n \) 
(4.43) \[ \text{dist}(H(f), \{\nu(t): t \in [T, T + l]\}) \leq \varepsilon \]
for all large \( T \). By Proposition 2.10 there exists \\
(4.44) \[ \delta_0 \in (0, 2^{-1}\delta_1) \]
such that for each \( T \in [l, \infty) \) and each a.c. function \( \nu: [0, T] \to \mathbb{R}^n \) which satisfies \\
(4.45) \[ d(\nu(0), H(f)) \leq \delta_0, \quad d(\nu(T), H(f)) \leq \delta_0, \]
(4.46) \[ I^f(0, T, \nu) - T\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(T)) \leq \delta_0 \]
the relation \\
(4.47) \[ \text{dist}(H(f), \{\nu(t): t \in [S, S + l]\}) \leq \varepsilon \]
holds for every \( S \in [0, T - l] \). By Proposition 2.6 there exists \\
(4.48) \[ \delta \in (0, 32^{-1}\delta_0) \]
such that for each \( x_1, x_2, y_1, y_2 \in \mathbb{R}^n \) which satisfy \\
(4.49) \[ |x_i|, |y_i| \leq 2M + 4 + 2 \sup\{|h|: h \in H(f)\}, \quad |x_i - y_i| \leq 4\delta, \quad i = 1, 2, \]
the following relations hold: \\
(4.50) \[ |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \leq 2^{-8}\delta_0, \quad |\pi^f(x_i) - \pi^f(y_i)| \leq 2^{-8}\delta_0, \quad i = 1, 2. \]

By Proposition 2.5 there exists an integer \( L \geq 1 \) such that for each \((f)-\text{good}\) function \( \nu: [0, \infty) \to \mathbb{R}^n \) 
(4.51) \[ \text{dist}(H(f), \{\nu(t): t \in [T, T + L]\}) \leq 8^{-1}\delta \]
for all large \( T \).

By Proposition 2.8 there exists an integer \( N \geq 10 \) and a neighborhood \( U_2 \) of \( f \) in \( \mathcal{A} \) such that for each \( g \in U_2 \), each \( S \in [0, \infty) \) and each a.c. function \( x: [S, S + NL] \to \mathbb{R}^n \) which satisfies \\
(4.52) \[ |x(S)|, |x(S + NL)| \leq 2M + 2, \quad I^g(S, S + NL, x) \leq U^g(S, S + NL, x(S), x(S + NL)) + 4 \]
there exists an integer \( i_0 \in [0, N - 8] \) such that for all \( T \in [S + i_0L, S + (i_0 + 7)L] \) 
(4.53) \[ \text{dist}(H(f), \{x(t): t \in [T, T + L]\}) \leq \delta. \]
By Lemma 4.4 there exists a number \( q \geq 8 \) such that for each \( h_1, h_2 \in H(f) \) there exists an a.c. function \( \nu : [0, q] \to R^n \) which satisfies
\begin{equation}
(4.54)
\nu(0) = h_1, \quad \nu(q) = h_2, \quad I^f(0, q, \nu) - q\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(q)) \leq 8^{-1}\delta.
\end{equation}
By Proposition 2.3 there exists a neighborhood \( U_3 \) of \( f \) in \( A \) such that for each \( g \in U_3 \), each \( T_1 \geq 0, T_2 \in [T_1 + 8^{-1}, T_1 + 6N(q + l + L)] \) and each a.c. function \( x : [T_1, T_2] \to R^n \) which satisfies
\begin{equation}
(4.55)
\inf\{I^f(T_1, T_2, x), I^g(T_1, T_2, x)\} \leq 4 + 2\sup\{|\pi^f(h) : h \in R^n, |h| \leq \sup\{|z| : z \in H(f)\} + 4\} + 4\mu(f)N(q + l + L)
\end{equation}
the following relation holds
\begin{equation}
(4.56)
|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq 4^{-1}\delta.
\end{equation}
By Proposition 2.4 there exists a neighborhood \( U_4 \) of \( f \) in \( A \) such that for each \( g \in U_4 \), each \( x_1, x_2 \in R^n \) which satisfy
\begin{equation}
(4.57)
|x_1|, |x_2| \leq 2M + 4 + 2\sup\{|z| : z \in H(f)\}
\end{equation}
the following relation holds:
\begin{equation}
(4.58)
|U^f(0, 1, x_1, x_2) - U^g(0, 1, x_1, x_2)| \leq 2^{-8}\delta.
\end{equation}
Set
\begin{equation}
(4.59)
l_0 = 2l + 2q + 2NL + 6,
\end{equation}
\begin{equation}
(4.60)
U = \bigcap_{i=1}^4 U_i.
\end{equation}
Assume that \( g \in U \), \( T \geq 2l_0 \) and an a.c. function \( \nu : [0, T] \to R^n \) satisfies
\begin{equation}
(4.61)
|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta.
\end{equation}
It follows from the definition of \( U_4 \) (see (4.38), (4.39)) and (4.60) that
\begin{equation}
(4.62)
|\nu(t)| \leq M, \quad t \in [0, T].
\end{equation}
Assume that there exist numbers \( S_1, S_2 \in [0, T] \) such that
\begin{equation}
(4.63)
d(\nu(S_i), H(f)) \leq \delta, \quad i = 1, 2, \quad S_2 - S_1 \in [1 + l + q, 5N(L + l + q)].
\end{equation}
We will show that for each \( \tau \in [S_1, S_2 - l] \)
\begin{equation}
(4.64)
\text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) \leq \varepsilon.
\end{equation}
Let us assume the converse. Then there exists a number \( \tau \) such that
\begin{equation}
(4.65)
\text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) > \varepsilon.
\end{equation}
It follows from (4.62), (4.64), (4.48) and the definition of \( \delta_0 \) (see (4.44)–(4.47)) that
\begin{equation}
(4.66)
I^f(S_1, S_2, \nu) - (S_2 - S_1)\mu(f) - \pi^f(\nu(S_1)) + \pi^f(\nu(S_2)) > \delta_0.
\end{equation}
We will show that
\begin{equation}
(4.67)
I^g(S_1, S_2, \nu) \leq 2\sup\{|\pi^f(z) : z \in R^n, d(z, H(f)) \leq 1\} + |\mu(f)|(S_2 - S_1) + 1.
\end{equation}
It follows from this relation, (4.62) and the definition of $U_3$ (see (4.55), (4.56)) that

$$|I^f(S_1, S_2, \nu) - I^g(S_1, S_2, \nu)| \leq 4^{-1} \delta.$$ 

Together with (4.65) this implies (4.66). The obtained contradiction proves that (4.66) holds.

By (4.62) there exist $h_1, h_2 \in H(f)$ such that

(4.67) $$|\nu(S_i) - h_i| \leq \delta, \quad i = 1, 2.$$ 

By Lemma 4.1 there exists an $a.c.$ function $\tilde{w}_0 : [0, \infty) \rightarrow H(f)$ such that

(4.68) $$w_0 : [0, \infty) \rightarrow H(f) \quad \text{such that} \quad w_0(0) = h_1,$$

(4.69) $$I^f(t_1, t_2, w_0) - (t_2 - t_1)\mu(f) - \pi^f(w_0(t_1)) + \pi^f(w_0(t_2)) = 0$$

for each $t_1 \geq 0$, $t_2 > t_1$.

It follows from the definition of $q$ (see (4.54), (4.62), (4.68)) that there exists an a.c. function $w_1 : [0, q] \rightarrow \mathbb{R}^n$ such that

(4.70) $$w_1(0) = w_0(S_2 - S_1 - q), \quad w_1(q) = h_2,$$

$$I^f(0, q, w_1) - q\mu(f) - \pi^f(w_1(0)) + \pi^f(w_1(q)) \leq 8^{-1}\delta.$$ 

By (4.62), Proposition 2.13, (4.68), (4.70) there exists an a.c. function $u : [0, T] \rightarrow \mathbb{R}^n$ such that

(4.71) $$u(t) = \nu(t), \quad t \in [0, S_1] \cup [S_2, T],$$

$$u(t) = w_0(t - S_1), \quad t \in [S_1 + 1, S_2 - q],$$

$$u(t) = w_1(t - (S_2 - q)), \quad t \in [S_2 - q, S_2 - 1],$$

$$I^g(r, r + 1, u) = U^g(0, 1, u(r), u(r + 1)), \quad r = S_1, S_2 - 1.$$ 

For each a.c. function $y : [a, b] \rightarrow \mathbb{R}^n$ where $a \geq 0$, $b > a$ and each $r_1, r_2 \in [a, b]$ satisfying $r_1 \leq r_2$ we set

(4.72) $$\sigma(r_1, r_2, y) = I^g(r_1, r_2, y) - \pi^f(y(r_1)) + \pi^f(y(r_2)) - (r_2 - r_1)\mu(f).$$ 

It follows from (4.60), (4.71), (4.72) that

(4.73) $$\delta \geq I^g(0, T, \nu) - I^g(0, T, u)$$

$$= \sigma(0, T, \nu) - \sigma(0, T, u) = \sigma(S_1, S_2, \nu) - \sigma(S_1, S_2, u).$$ 

By (4.70), (4.68) and the definition of $M$ (see (4.38), (4.39))

(4.74) $$|w_1(t)| \leq M, \quad t \in [0, q].$$ 

By (4.70) there exists an a.c. function $\tilde{w} : [S_1, S_2] \rightarrow \mathbb{R}^n$ such that

(4.75) $$\tilde{w}(t) = w_0(t - S_1), \quad t \in [S_1, S_2 - q],$$

$$\tilde{w}(t) = w_1(t - (S_2 - q)), \quad t \in [S_2 - q, S_2].$$

(4.73), (4.72), (4.66), (4.71), (4.75) imply that

(4.76) $$\delta \geq 2^{-1}\delta_0 - \sigma(S_1, S_2, u) = 2^{-1}\delta_0 - \sigma(S_1, S_2, \tilde{w})$$

$$+ |\sigma(S_1, S_1 + 1, \tilde{w}) - \sigma(S_1, S_1 + 1, u)|$$

$$+ |\sigma(S_2 - 1, S_2, \tilde{w}) - \sigma(S_2 - 1, S_2, u)|.$$ 

We will estimate $\sigma(S_1, S_2, \tilde{w})$ and $\sigma(h, h + 1, \tilde{w}) - \sigma(h, h + 1, u)$, $h = S_1, S_2 - 1$. 

Let $h \in \{S_1, S_2 - 1\}$. (4.70), (4.75), (4.71), (4.74), (4.68), (4.61), (4.62), (4.67) imply that

$$|\tilde{w}(h)|, |\tilde{w}(h + 1)|, |u(h)|, |u(h + 1)| \leq M + \sup\{z : z \in H(f)\},$$

$$|\tilde{w}(h) - u(h)|, |\tilde{w}(h + 1) - u(h + 1)| \leq \delta.$$

It follows from these relations, (4.71), (4.72), the definition of $U$ (see (4.57)) and $\delta$ (see (4.49), (4.50), (4.48)) that

$$\sigma(h, h + 1, \tilde{w}) - \sigma(h, h + 1, u)$$

$$\geq U^g(0, 1, \tilde{w}(h), \tilde{w}(h + 1)) - \pi^f(\tilde{w}(h)) + \pi^f(\tilde{w}(h + 1))$$

$$- [U^g(0, 1, u(h), u(h + 1)) - \pi^f(u(h)) + \pi^f(u(h + 1))]$$

$$\geq U^f(0, 1, \tilde{w}(h), \tilde{w}(h + 1)) - \pi^f(\tilde{w}(h)) + \pi^f(\tilde{w}(h + 1))$$

$$- [U^f(0, 1, u(h), u(h + 1)) - \pi^f(u(h)) + \pi^f(u(h + 1))] - 2^{-7}\delta$$

$$\geq -2^{-6}\delta_0, \quad h \in \{S_1, S_2 - 1\}.$$

We will estimate $\sigma(S_1, S_2, \tilde{w})$. It follows from (4.69), (4.70), (4.75) that

$$I^f(S_1, S_2, \tilde{w}) - \pi^f(\tilde{w}(S_1)) + \pi^f(\tilde{w}(S_2)) - (S_2 - S_1)\mu(f) \leq 8^{-1}\delta.$$ 

By this relation, (4.62), (4.75), (4.68), (4.70) and the definition of $U_3$ (see (4.55), (4.56))

$$|I^f(S_1, S_2, \tilde{w}) - I^g(S_1, S_2, \tilde{w})| \leq 4^{-1}\delta.$$

Together with (4.78), (4.72) this implies that

$$\sigma(S_1, S_2, \tilde{w}) \leq 3 \cdot 8^{-1}\delta.$$

It follows from this relation, (4.76), (4.77) that

$$\delta \geq 2^{-1}\delta_0 - 3 \cdot 8^{-1}\delta - 2^{-5}\delta_0.$$

This is contradictory to (4.48). The obtained contradiction proves that (4.63) holds for each $\tau \in [S_1, S_2 - l]$. Therefore we have shown that the following property holds:

**Property D.** For each $S_1, S_2 \in [0, T]$ which satisfy (4.62) relation (4.63) holds for each $\tau \in [S_1, S_2 - l]$.

It follows from (4.60), (4.61) and the definition of $U_2, N$ (see (4.52), (4.53)) that for each $r_0 \in [0, T - (1 + l + q + L(N + 2))]$ there exists a number $r_1$ such that

$$r_1 - r_0 \in [1 + l + q + 2L, 1 + l + q + L(N + 2)], \quad d(\nu(r_1), H(f)) \leq \delta.$$

This implies that there exists a finite sequence of numbers $\{S_i\}_{i=0}^Q \subset [0, T]$ such that

$$S_0 = 0, S_{i+1} - S_i \in [1 + l + q + 2L, 1 + l + q + L(N + 2)], \quad i = 0, \ldots, Q - 1,$$

$$T - S_Q \leq 1 + l + q + L(N + 2), \quad d(\nu(S_i), H(f)) \leq \delta, \quad i = 1, \ldots, Q.$$

The assertion of the theorem follows from these relations and Property D.
5. Proof of Theorem 1.3

Lemma 5.1. Assume that an integrand \( f \in \mathfrak{M} \) has Property B and \( \varepsilon > 0 \). Then there exists a neighborhood \( U \) of \( f \) in \( \mathfrak{A} \) such that for each \( g \in U \) and each \((g)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \)

\[
\text{dist}(\Omega(\nu), H(f)) \leq \varepsilon.
\]

Proof. By Proposition 2.1 there exist a neighborhood \( U_1 \) of \( f \) in \( \mathfrak{A} \) and a number \( K > 0 \) such that for each \( g \in U_1 \) and each \((g)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \)

\[
\limsup_{t \to \infty} |\nu(t)| < K.
\]

By Theorem 1.2 there exist a neighborhood \( U \) of \( f \) in \( \mathfrak{A} \) which satisfies \( U \subseteq U_1 \) and numbers \( l_0 > l > 0 \), \( \delta > 0 \) such that for each \( g \in U \), each \( T \geq 2l_0 \) and each a.c. function \( \nu : [0, T] \to \mathbb{R}^n \) which satisfies

\[
|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta
\]

the relation \( \text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) \leq \varepsilon \) holds for each \( \tau \in [l_0, T - l_0] \).

Assume that \( g \in U \) and \( \nu : [0, \infty) \to \mathbb{R}^n \) is a \((g)\)-good function. It follows from the definition of \( U, U_1, K \) and Proposition 1.1 that there exists a number \( T_0 \geq 0 \) such that

\[
|\nu(t)| \leq K, \quad t \in [T_0, \infty),
\]

\[
I^g(t_1, t_2, \nu) \leq U^g(t_1, t_2, \nu(t_1), \nu(t_2)) + \delta \quad \text{for each } t_1 \geq T_0, \ t_2 > t_1.
\]

It follows from these relations and the definition of \( U, l_0, \delta \) that

\[
\text{dist}(H(f), \Omega(\nu)) \leq \varepsilon.
\]

The lemma is proved.

Construction of the set \( \mathcal{F}_q \). Suppose that \( q \) is a nonnegative integer. By Propositions 2.11, 2.12, 2.15 there exists a set \( E_q \subset \mathfrak{M}_q \) which is an everywhere dense subset of \( \mathfrak{M}_q \) and such that each integrand \( f \in E_q \) has Property B. Therefore for each \( f \in E_q \) there exists a compact set \( H(f) \subset \mathbb{R}^n \) such that \( \Omega(\nu) = H(f) \) for each \((f)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \).

By Theorem 1.2 and Lemma 5.1 for each \( f \in E_q \) and each integer \( p \geq 1 \) there exist an open neighborhood \( U(f, p) \) of \( f \) in \( \mathfrak{A} \) and numbers \( M(f, p) > p, l_0(f, p) > l(f, p) > 0, \delta(f, p) \in (0, p^{-1}) \) such that for each \( g \in U(f, p) \) and each \((g)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \)

\[
\text{dist}(H(f), \Omega(\nu)) \leq 4^{-1} \delta(f, p);
\]

for each \( g \in U(f, p) \), each \( T \geq 2l_0(f, p) \) and each a.c. function \( \nu : [0, T] \to \mathbb{R}^n \) which satisfies

\[
|\nu(0)|, |\nu(T)| \leq p, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta(f, p)
\]

the relation \( |\nu(t)| \leq M(f, p) \) holds for all \( t \in [0, T] \) and the following properties hold:

(i) for each \( \tau \in [l_0(f, p), T - l_0(f, p)] \)

\[
\text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l(f, p)]\}) \leq p^{-1};
\]

(ii) if \( d(\nu(0), H(f)) \leq \delta(f, p) \), then \( (5.2) \) holds for each \( \tau \in [0, T - l_0(f, p)] \);

(iii) if \( d(\nu(T), H(f)) \leq \delta(f, p) \), then \( (5.2) \) holds for each \( \tau \in [l_0(f, p), T - l(f, p)] \).
We define

\[ \mathcal{F}_q = \left[ \bigcup_{p=1}^{\infty} \mathcal{U}(f, p) : f \in E_q \right] \cap \mathcal{M}_q. \]

Clearly \( \mathcal{F}_q \) is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_q \).

Assume that \( f \in \mathcal{F}_q, \varepsilon, K > 0 \). Fix a natural number \( p \) such that

\[ p > 2K + 4 + 8\varepsilon^{-1}. \]

There exists \( G \in E_q \) such that

\[ f \in \mathcal{U}(G, p). \]

It follows from (5.4), (5.5) and the definition of \( \mathcal{U}(G, p), \delta(G, p) \) that for each \((f)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \)

\[ \text{dist}(H(G), \Omega(\nu)) \leq 4^{-1}\delta(G, p) < (4p)^{-1} < 8^{-1}\varepsilon. \]

This implies that for each \((f)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n, i = 1, 2, \)

\[ \text{dist}(\Omega(\nu_1), \Omega(\nu_2)) \leq \varepsilon. \]

Since \( \varepsilon \) is any positive number we conclude that \( f \) has Property B and there exists a compact set \( H(f) \subset \mathbb{R}^n \) such that \( \Omega(w) = H(f) \) for each \((f)\)-good function \( w : [0, \infty) \to \mathbb{R}^n \). It follows from (5.6) that

\[ \text{dist}(H(G), H(f)) \leq 4^{-1}\delta(G, p). \]

Set

\[ \mathcal{U} = \mathcal{U}(G, p), \quad M = M(G, p), \]

\[ l_0 = l_0(G, p), \quad l = l(G, p), \quad \delta = 8^{-1}\delta(G, p). \]

Assume that \( g \in \mathcal{U}, T \geq 2l_0 \) and an a.c. function \( \nu : [0, T] \to \mathbb{R}^n \) satisfies

\[ |\nu(0)|, |\nu(T)| \leq K, \quad F^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta. \]

It follows from (5.9), (5.8), (5.5), (5.4) and the definition of \( \mathcal{U}(G, p), M(G, p), l_0(G, p), l(G, p), \delta(G, p) \) that

\[ |\nu(t)| \leq M, \quad t \in [0, T], \]

and properties (i)–(iii) hold with \( f = G \). Together with (5.7), (5.8), (5.4) this implies that

\[ \text{dist}(H(f), \{\nu(t) : t \in [\tau, \tau + l]\}) \leq \varepsilon \]

for each \( \tau \in [l_0, T - l_0] \); if \( d(\nu(0), H(f)) \leq \delta \), then (5.11) holds for each \( \tau \in [0, T - l_0] \).

If \( d(\nu(T), H(f)) \leq \delta \), then (5.11) holds for each \( \tau \in [l_0, T - l] \). This completes the proof of the theorem.

6. Examples

Fix a constant \( a > 0 \) and set \( \psi(t) = t, t \in [0, \infty) \). Consider the complete metric space \( \mathfrak{M} \) of integrands \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) defined in Section 1.

**Example 1.** Consider an integrand \( f(x, u) = |x|^2 + |u|^2 \), \( x, u \in \mathbb{R}^n \). It is easy to see that \( f \in \mathfrak{M}_q \) for each integer \( q \geq 0 \) if the constant \( a \) is large enough. We can show (see [20, Section 14]) that \( \Omega(\nu) = \{0\} \) for every \((f)\)-good function \( \nu : [0, \infty) \to \mathbb{R}^n \). Therefore Theorem 1.2 holds with the integrand \( f \).
Example 2. Fix a number $q > 0$ and consider an integrand

$$g(x, u) = q|x|^2|x - e|^2 + |u|^2, \quad x, u \in \mathbb{R}^n,$$

where $e = (1, 1, \ldots, 1)$. It is easy to see that $g \in \mathfrak{M}$ if the constant $a$ is large enough. Clearly $f$ does not have the turnpike property.

References

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