

ERGODIC SEQUENCES IN THE FOURIER-STIELTJES
ALGEBRA AND MEASURE ALGEBRA OF
A LOCALLY COMPACT GROUP

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ABSTRACT. Let G be a locally compact group. Blum and Eisenberg proved that if G is abelian, then a sequence of probability measures on G is strongly ergodic if and only if the sequence converges weakly to the Haar measure on the Bohr compactification of G . In this paper, we shall prove an extension of Blum and Eisenberg's Theorem for ergodic sequences in the Fourier-Stieltjes algebra of G . We shall also give an improvement to Milnes and Paterson's more recent generalization of Blum and Eisenberg's result to general locally compact groups, and we answer a question of theirs on the existence of strongly (or weakly) ergodic sequences of measures on G .

0. INTRODUCTION

Let G be a locally compact group and π be a continuous unitary representation of G on a Hilbert space H . Let H_f denote the fixed point set of π in H , i.e.

$$H_f = \{\xi \in H; \pi(x)\xi = \xi \text{ for all } x \in G\}.$$

A sequence $\{\mu_n\}$ of probability measures on G is called a *strongly* (resp. *weakly*) *ergodic* sequence if for every representation π of G on a Hilbert space H and for every $\xi \in H$, $\{\pi(\mu_n)\xi\}$ converges in norm (resp. weakly) to a member of H_f . When G is abelian or compact, or G is a [Moore]-group (i.e. every irreducible representation of G is finite dimensional), then every weakly ergodic sequence is strongly ergodic. However, this is not true in general (see [8, Proposition 1 and Proposition 5]).

In [1], Blum and Eisenberg proved that if G is a locally compact abelian group, and $\{\mu_n\}$ is a sequence of probability measures on G , then the following are equivalent:

- (i) $\{\mu_n\}$ is strongly ergodic.
- (ii) $\widehat{\mu}_n(\gamma) \rightarrow 0$ for all $\gamma \in \widehat{G} \setminus \{1\}$.
- (iii) $\{\mu_n\}$ converges weakly to the Haar measure on the Bohr compactification of G .

More recently Milnes and Paterson [8] obtained the following generalization of Blum and Eisenberg's result to general locally compact groups:

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Theorem A (Milnes and Paterson [8]). *Let G be a second countable locally compact group. Then the following statements about a sequence $\{\mu_n\}$ of probability measures in $M(G)$ are equivalent:*

- (i) $\{\mu_n\}$ is a weakly ergodic sequence.
- (ii) $\pi(\mu_n) \rightarrow 0$ in the weak operator topology for every $\pi \in \widehat{G} \setminus \{1\}$.
- (iii) $\widehat{\mu}_n$ converges to the unique invariant mean on $B_I(G)$, the closure in $C(G)$ of the linear span of the set of coefficient functions of the irreducible representations of G .

(Here \widehat{G} denotes the set of irreducible continuous representations of G which is the same as the dual group of G when G is abelian.)

Let $P_1(G)$ denote the continuous positive definite functions ϕ on G such that $\phi(e) = 1$ (where e is the identity of G). When G is abelian, $P_1(G)$ corresponds to the set of probability measures on the dual group \widehat{G} of G (by Bochner's Theorem). In this paper, we shall prove an extension of Blum and Eisenberg's Theorem for ergodic sequences in $P_1(G)$ (Theorems 3.1 and 3.3). We shall give an improvement to condition (iii) of Theorem A by replacing " $B_I(G)$ " by the Fourier-Stieltjes algebra " $B(G)$ " for any G (Theorems 4.1 and 4.4) and remove the condition of second countability (= separability in [8]) in Theorem A. We shall also show that (Theorem 4.6) G is σ -compact if and only if it has a strongly (or weakly) ergodic sequence of measures. This completely answers a question in [8, p. 693].

A "strongly ergodic sequence" is called a "general summing sequence" by Blum and Eisenberg in [1]. It was also introduced by Rindler under the name "unitarily distributed sequences" in Def. 4 of [13] for point sequences and their Cesàro averages and by Maxones and Rindler in [9] for sequences of measures.

1. SOME PRELIMINARIES

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure μ . Integration with respect to μ will be given by $\int \cdots dx$. Let $C(G)$ denote the Banach space of bounded continuous functions on G with the supremum norm. Then G is *amenable* if there exists a positive linear functional ϕ on $C(G)$ of norm one such that $\phi(\ell_a f) = \phi(f)$ for each $a \in G$ and $f \in C(G)$ where $(\ell_a f)(x) = f(ax)$, $x \in G$. Amenable groups include all solvable groups and all compact groups. However, the free group on two generators is not amenable (see [11] or [12] for more details).

Let $C^*(G)$ denote the completion of $L^1(G)$ with respect to the norm $\|f\|_c = \sup \{\|T_f\|\}$, where the supremum is taken over all $*$ -representations T of $L^1(G)$ as an algebra of bounded operators on a Hilbert space. Let $P(G)$ denote the subset of $C(G)$ consisting of all continuous positive definite functions on G , and let $B(G)$ be its linear span. Then $B(G)$ (the *Fourier-Stieltjes algebra* of G) can be identified with the dual of $C^*(G)$, and $P(G)$ is precisely the set of positive linear functionals on $C^*(G)$.

Let $\mathcal{B}(L^2(G))$ be the algebra of bounded linear operators from $L^2(G)$ into $L^2(G)$ and let $VN(G)$ denote the weak operator topology closure of the linear span of $\{\rho(a) : a \in G\}$, where $\rho(a)f(x) = f(a^{-1}x)$, $x \in G$, $f \in L^2(G)$, in $\mathcal{B}(L^2(G))$. Let $A(G)$ denote the subalgebra of $C_0(G)$ (continuous complex-valued functions vanishing at infinity), consisting of all functions of the form $h * \tilde{k}$ where $h, k \in L^2(G)$ and $\tilde{k}(x) = \overline{k(x^{-1})}$, $x \in G$. Then each $\phi = h * \tilde{k}$ in $A(G)$ can be regarded as an

ultraweakly continuous functional on $VN(G)$ defined by

$$\phi(T) = \langle Th, k \rangle \quad \text{for each } T \in VN(G).$$

Furthermore, as shown by Eymard in [3, pp. 210, Theorem 3.10], each ultraweakly continuous functional on $VN(G)$ is of this form. Also $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup \{|\phi(T)|\}$, where the supremum runs through all $T \in VN(G)$ with $\|T\| \leq 1$, is a semisimple commutative Banach algebra with spectrum G ; $A(G)$ is called the *Fourier algebra* of G and it is an ideal of $B(G)$.

There is a natural action of $A(G)$ on $VN(G)$ given by $\langle \phi \cdot T, \gamma \rangle = \langle T, \phi \cdot \gamma \rangle$ for each $\phi, \gamma \in A(G)$ and each $T \in VN(G)$. A linear functional m on $VN(G)$ is called a *topological invariant mean* if

- (i) $T \geq 0$ implies $\langle m, T \rangle \geq 0$,
- (ii) $\langle m, I \rangle = 1$ where $I = \rho(e)$ denotes the identity operator, and
- (iii) $\langle m, \phi \cdot T \rangle = \phi(e)\langle m, T \rangle$ for $\phi \in A(G)$.

As known, $VN(G)$ always has a topological invariant mean. However $VN(G)$ has a unique topological invariant mean if and only if G is discrete (see [14, Theorem 1] and [6, Corollary 4.11]).

Let $C_\delta^*(G)$ denote the norm closure of the linear span of $\{\rho(a); a \in G\}$. Let $B_\delta(G)$ denote the linear span of $P_\delta(G)$, where $P_\delta(G)$ is the pointwise closure of $A(G) \cap P(G)$. Then $B_\delta(G)$ can be identified with $C_\delta^*(G)^*$ by the map $\pi(\phi)(f) = \sum \{\phi(t)f(t), t \in G\}$ for each $f \in \ell_1(G)$ and $\phi \in B_\delta(G)$ (see [3, Proposition 1.21]). Furthermore $B_\delta(G)$ with pointwise multiplication and dual norm is a commutative Banach algebra. If m is topological invariant mean on $VN(G)$, then $m' =$ restriction of m to $C_\delta^*(G)$, is also a topological invariant mean on $C_\delta^*(G)$. Furthermore, if m'' is another topological invariant mean on $C_\delta^*(G)$, then $m' = m''$, by commutativity of $B_\delta(G)$. If G is amenable, then $B(G) \subseteq B_\delta(G)$. In particular, each $\phi \in B(G)$ corresponds to a continuous linear functional on $C_\delta^*(G)$ defined by $\langle \phi, \rho(a) \rangle = \phi(a)$, $a \in G$. Also if G is abelian, then $C_\delta^*(G) \cong AP(\widehat{G})$, the space of continuous almost periodic functions on \widehat{G} (see [5]).

2. SOME LEMMAS

Let G be a locally compact group, and $M^+(G)$ be the positive finite regular Borel measures on G .

Lemma 2.1. *Let $\mu \in M^+(G)$. For each $\phi \in A(G)$, define S_ϕ an operator on $L_2(G, \mu)$ by*

$$S_\phi h = \phi h, \quad h \in L_2(G, \mu).$$

Then the mapping $\phi \rightarrow S_\phi$ is a cyclic $$ -representation of $A(G)$ as bounded operators on $L_2(G, \mu)$.*

Proof. It is easy to see that $\phi \rightarrow S_\phi$ is a $*$ -representation as bounded operators on $L_2(G, \mu)$. Also the element $1 \in L_2(G, \mu)$ is a cyclic vector for S , since we have $\{S_\phi 1; \phi \in A(G)\} = \{\phi; \phi \in A(G)\}$. Let $f \in C_{00}(G)$ (continuous function with compact support); then there exists $\{\phi_n\} \subseteq A(G)$ such that $\|\phi_n - f\|_\infty \rightarrow 0$. In particular, $\phi_n \rightarrow f$ in the L_2 -norm of $L_2(G, \mu)$. The result now follows by density of $C_{00}(G)$ in $L_2(G, \mu)$, and $\mu(G) < \infty$. □

Lemma 2.2. *Let $\{T, H\}$ be a cyclic $*$ -representation of $A(G)$. There exists a measure $\mu \in M^+(G)$ such that T is unitarily equivalent to a representation S defined by μ as in Lemma 2.1.*

Proof. Indeed, for any $\phi \in A(G)$,

$$\|T(\phi)\|_{\text{sp}} \leq \|\phi\|_{\text{sp}}$$

($\|\cdot\|_{\text{sp}}$ denotes the spectral-radius). Since $A(G)$ is commutative, $\|T(\phi)\|_{\text{sp}} =$ operator norm in $\mathcal{B}(H)$ and $\|\phi\|_{\text{sp}} = \sup \{|\phi(x)| : x \in G\}$ (by semi-simplicity of $A(G)$, and the fact that the spectrum of $A(G)$ is G). Hence

$$\|T(\phi)\| \leq \|\phi\|_{\infty}.$$

In particular T extends to a $*$ -representation of the C^* -algebra $C_0(G)$ (by density of $A(G)$ in $C_0(G)$). Let $\eta \in H$ be a cyclic vector of $\{T, H\}$. Then

$$f \rightarrow \langle T(f)\eta, \eta \rangle, \quad f \in C_0(G),$$

defines a positive linear functional on the C^* -algebra $C_0(G)$. Let $\mu \in M^+(G)$ which represents this functional and S be the cyclic representation of $A(G)$ as defined in Lemma 2.1. Then T and S are unitarily equivalent. Indeed, define a map $W : \{T(\phi)\eta; \phi \in A(G)\} \rightarrow \{\phi \cdot 1; \phi \in A(G)\} \subseteq L_2(G, \mu)$ by $W(T(\phi)\eta) = \phi \cdot 1$.

Then $\langle T(\phi)\eta, \eta \rangle = \int \phi d\mu = 0$ whenever $\phi = 0$ μ -a.e. Hence W is well-defined. Also

$$\begin{aligned} \langle T(\phi)\eta, T(\phi)\eta \rangle &= \langle T^*(\phi)T(\phi)\eta, \eta \rangle = \langle T(\bar{\phi}\phi)\eta, \eta \rangle \\ &= \int \bar{\phi}\phi(x) d\mu(x) = \langle \phi, \phi \rangle. \end{aligned}$$

Consequently W extends to a linear isometry from H onto $L^2(G, \mu)$. Finally, if $\psi, \phi \in A(G)$, then

$$S(\phi)W(T(\psi)\eta) = S(\phi)(\psi \cdot 1) = \phi\psi \cdot 1$$

and

$$WT(\phi)(T(\psi)\eta) = WT(\phi\psi)\eta = \phi\psi \cdot 1.$$

Hence $\{S, L_2(G, \mu)\}$ and $\{T, H\}$ are unitarily equivalent. \square

Assume that G is amenable. Then it is well known that $A(G)$ has an approximate identity bounded by 1. Let $\{T, H\}$ be a $*$ -representation of $A(G)$ which is non-degenerate. Next, we will show that for each $\psi \in B(G)$, there is a unique bounded linear operator $\tilde{T}(\psi)$ on H such that

$$(i) \quad \tilde{T}(\psi)T(\phi) = T(\psi\phi) \text{ for all } \phi \in A(G).$$

Uniqueness is clear from the fact that vectors of the form $\{T(\phi)\xi, \phi \in A(G), \xi \in H\}$ span H . For existence, consider first the case when T is cyclic, and let $\xi_0 \in H$ be such that $[T(A(G))\xi_0] = H$. We claim that

$$(ii) \quad \|T(\psi\phi)\xi_0\| \leq \|\psi\| \|T(\phi)\xi_0\| \text{ for each } \phi \in A(G).$$

Indeed, let $\phi \in A(G)$ be fixed. Choose a bounded approximate identity $\{\psi_n\}$ in $A(G)$ such that $\|\psi_n\| \leq 1$. Then

$$\begin{aligned} \|T(\psi\phi)\xi_0\| &= \lim \|T(\psi\psi_n\phi)\xi_0\| \\ &= \lim_n \|T(\psi\psi_n)T(\phi)\xi_0\| \\ &\leq \|\psi\| \|T(\phi)\xi_0\| \\ &\leq \|\psi\| \|T(\phi)\xi_0\| \end{aligned}$$

(since any $*$ -homomorphism from an involutive Banach algebra into a C^* -algebra is norm decreasing) as asserted.

Now it follows that the map $T(\phi)\xi_0 \rightarrow T(\psi\phi)\xi_0$ (where $\phi \in A(G)$) extends uniquely to an operator $\tilde{T}(\psi)$ on $[T(A(G))\xi_0] = H$ having norm at most $\|\psi\|$. The relation $\tilde{T}(\psi)T(\phi) = T(\psi\phi)$ holds on all vectors of the form $T(\theta)\xi_0$, $\theta \in A(G)$, so it holds on H .

For a general non-degenerate $*$ -representation T of $A(G)$, we simply write $T = \sum \bigoplus T_\alpha$, each T_α cyclic, and define $\tilde{T}(\psi) = \sum \bigoplus \tilde{T}_\alpha(\psi)$, $\psi \in B(G)$.

3. ERGODIC SEQUENCES IN $B(G)$

A sequence $\{\phi_n\}$ in $A(G) \cap P_1(G)$ is called *strongly* (respectively *weakly*) *ergodic* if whenever $\{T, H\}$ is a $*$ -representation of $A(G)$, $\xi \in H$, the sequence $T(\phi_n)\xi$ converges in the norm (resp. weak) topology to a member of the fixed point set:

$$H_f = \{\xi \in H; T(\phi)\xi = \xi \text{ for all } \phi \in A(G) \cap P_1(G)\}.$$

Theorem 3.1. *Let G be a locally compact group. The following are equivalent for a sequence $\{\phi_n\}$ in $A(G) \cap P_1(G)$:*

- (i) $\{\phi_n\}$ is strongly ergodic.
- (ii) $\{\phi_n\}$ is weakly ergodic.
- (iii) For each $g \in G$, $g \neq e$, $\phi_n(g) \rightarrow 0$.
- (iv) For each $T \in C_\delta^*(G)$, $\langle \phi_n, T \rangle \rightarrow \langle m, T \rangle$, where m is the unique topological invariant mean on $C_\delta^*(G)$.

Proof. We first observe that $\langle m, \rho(g) \rangle = 0$ for all $g \in G \setminus \{e\}$, and $\langle m, \rho(e) \rangle = 1$. Indeed, if $\phi \in A(G) \cap P_1(G)$, then $\langle m, \rho(g) \rangle = \langle m, \phi \cdot \rho(g) \rangle = \langle m, \phi(g)\rho(g) \rangle = \phi(g)\langle m, \rho(g) \rangle$ (note $\langle \phi \cdot \rho(g), \psi \rangle = \langle \rho(g), \phi\psi \rangle = \phi(g)\psi(g) = \phi(g)\langle \rho(g), \psi \rangle$; hence $\phi \cdot \rho(g) = \phi(g)\rho(g)$). Now if $g \neq e$, then there exists $\phi \in A(G) \cap P_1(G)$ such that $\phi(g) \neq 1$ so $\langle m, \rho(g) \rangle = 0$. Consequently, (iii) and (iv) are equivalent.

(ii) \implies (iii). Consider, for $g \in G$ (fixed), the representation $\{T, H\}$, where $H = \mathbb{C}$, $T(\phi)\lambda = \phi(g)\lambda$. If $g \neq e$, then $H_f = \{\lambda; T(\phi)\lambda = \lambda, \phi \in A(G) \cap P_1(G)\} = \{\lambda; \phi(g)\lambda = \lambda, \phi \in A(G) \cap P_1(G)\} = \{0\}$. Hence if $g \neq e$, then

$$\phi_n(g) = T(\phi_n) = T(\phi_n)1 = \phi_n(g) \cdot 1 \rightarrow 0$$

by ergodicity of the sequence $\{\phi_n\}$.

(iii) \implies (i). We first assume that $\{T, H\}$ is a cyclic $*$ -representation of $A(G)$. By Lemma 2.2 there exists a measure $\mu \in M^+(G)$ such that T is unitarily equivalent to a representation S on $L_2(G, \mu)$ as in Lemma 2.1. Hence we may assume that $T = S$, and $H = L_2(G, \mu)$.

Let $h \in L_2(G, \mu)$. Then for each n, m ,

$$\|T(\phi_n)h - T(\phi_m)h\|^2 = \int (\phi_n - \phi_m)(x)h(x) \overline{(\phi_n - \phi_m)(x)h(x)} d\mu(x).$$

The integrand converges pointwise to “0” as $n, m \rightarrow \infty$, and it is dominated by the integrable function $4|h|^2$. Hence by the dominated convergence theorem

$$\|T(\phi_n)h - T(\phi_m)h\|^2 \rightarrow 0 \quad \text{as } n, m \rightarrow \infty,$$

i.e. $\{T(\phi_n)h\}$ is Cauchy. Let f be the limit of $T(\phi_n)h$ in $L_2(G, \mu)$. Now if $\phi \in A(G) \cap P_1(G)$, $h \in L_2(G, \mu)$, then

$$\|T(\phi)(T(\phi_n)h) - T(\phi_n)h\|^2 = \int (\phi \cdot \phi_n - \phi_n)h \cdot \overline{(\phi \cdot \phi_n - \phi_n) \cdot h} d\mu$$

which again converges to zero as $n \rightarrow \infty$ by the dominated convergence theorem. So $T(\phi)f = f$, i.e. f is a fixed point of $\{T, H\}$.

If $\{T, H\}$ is any $*$ -representation of $A(G)$, then $T = \{T_0, H_0\} \oplus \sum_{\alpha \in \Gamma} \{T_\alpha, H_\alpha\}$ where $\{T_0, H_0\}$ is the degenerate part of $\{T, H\}$ and $\{T_\alpha, H_\alpha\}$ is cyclic. The result follows by applying the cyclic case to each $\{T_\alpha, H_\alpha\}$ to obtain a fixed point $f_\alpha \in H_\alpha$ of $\{T_\alpha, H_\alpha\}$. Then $f = (f_\alpha)$ is the limit of the sequence $\{T(\phi_n)h\}$ in H , and $T(\phi)f = f$ for all $\phi \in A(G) \cap P(G)$. \square

Corollary 3.2. *A locally compact group G is first countable if and only if $A(G)$ contains an ergodic sequence.*

Proof. Let $\{U_n\}$ be a sequence of compact symmetric neighborhoods of the identity of G , such that

- (i) $U_n \downarrow \{e\}$,
- (ii) $U_n \cdot U_n \subseteq U_{n-1}$.

For each n , let $\phi_n = \frac{1}{\lambda(U_n)} (1_{U_n} * 1_{U_n})$. Then $\phi_n \in A(G) \cap P_1(G)$, and $\phi_n(g) \rightarrow 0$ for each $g \in G$ ($g \neq e$). Hence $\{\phi_n\}$ is ergodic by Theorem 3.1. Conversely if $\{\phi_n\}$ is an ergodic sequence on $A(G)$, then the topology on G defined by the sequence of pseudometrics $\{d_n\}$, where $d_n(x, y) = |\phi_n(x) - \phi_n(y)|$ is Hausdorff (by Theorem 3.1(iii)) and hence must agree on any compact neighbourhood of x , $x \in G$. Consequently G is first countable. \square

For G amenable, a sequence $\{\phi_n\}$ in $P_1(G)$ is called *strongly* (resp. *weakly*) *ergodic* if whenever $\{T, H\}$ is a non-degenerate $*$ -representation of $A(G)$, the sequence $\tilde{T}(\phi_n)\xi$ converges in norm (resp. weakly) to a member of the fixed point set:

$$H_f = \{\xi \in H; T(\phi)\xi = \xi \quad \text{for all } \phi \in A(G) \cap P_1(G)\}$$

where \tilde{T} is the unique extension of T to $B(G)$ as defined earlier in Section 2.

Theorem 3.3. *Let G be an amenable locally compact group. The following are equivalent for a sequence $\{\phi_n\}$ in $P_1(G)$:*

- (i) $\{\phi_n\}$ is strongly ergodic.
- (ii) $\{\phi_n\}$ is weakly ergodic.
- (iii) For each $g \in G$, $g \neq e$, $\phi_n(g) \rightarrow 0$.
- (iv) For each $T \in C_\delta^*(G)$, $\langle \phi_n, T \rangle \rightarrow \langle m, T \rangle$, where m is the unique topological invariant mean on $C_\delta^*(G)$.

Proof. Note that if m is a topological invariant mean on $C_\delta^*(G)$ (i.e. $\langle m, \phi \cdot T \rangle = \langle m, T \rangle$ for any $\phi \in P_1(G) \cap A(G)$, $T \in C_\delta^*(G)$), then $\langle m, \psi \cdot T \rangle = \langle m, T \rangle$, for $\psi \in P_1(G)$, $T \in C_\delta^*(G)$, where $\langle \psi \cdot T, \phi \rangle = \langle T, \psi \phi \rangle$, for $\phi \in A(G)$: indeed, let $\psi_n \subseteq P(G) \cap A(G)$ be a bounded approximate identity for $A(G)$. Then $\|\psi_n \cdot T - T\| \rightarrow 0$ for all $T \in UC(\widehat{G}) = A(G) \cdot VN(G) \supseteq C_\delta^*(G)$. Hence $\langle m, \psi \cdot T \rangle = \lim_n \langle m, \psi \cdot \psi_n \cdot T \rangle = \langle m, T \rangle$. So (iii) \iff (iv) as in the proof of Theorem 3.1.

(i) \iff (ii) \iff (iii): same as Theorem 3.1 (*Note:* the representation $T(\phi)\lambda = \phi(g)\lambda$, where $\phi \in A(G)$ has a unique extension \widetilde{T} to $B(G)$, $\widetilde{T}(\phi)\lambda = \phi(g)\lambda$, for $\phi \in B(G)$; similarly, the unique extension of S from $A(G)$ to $B(G)$ is $S_\phi h = \phi h$, $h \in L^2(G, \mu)$.) \square

4. ERGODIC SEQUENCES OF MEASURES

Let $M(G)$ denote the space of finite regular Borel measures on G . We put $\langle \mu, f \rangle = \int_G f(t) d\mu(t)$, for $\mu \in M(G)$, $f \in C(G)$ (in [8] this is denoted by $\widehat{\mu}(f)$). If π is a continuous unitary representation of G , let P_f denote the orthogonal projection from H^π onto the closed subspace H_f^π of fixed points.

Theorem 4.1. *Let G be a locally compact group. Then the following statements about a sequence $\{\mu_n\}$ of probability measures on G are equivalent:*

- (i) $\{\mu_n\}$ is a weakly ergodic sequence.
- (ii) $\pi(\mu_n) \rightarrow 0$ in the weak operator topology for every $\pi \in \widehat{G} \setminus \{1\}$.
- (iii) $\mu_n \rightarrow m$ in the weak*-topology ($\sigma(B(G)^*, B(G))$), where m is the unique translation-invariant mean on $B(G)$.

Proof. (i) \implies (ii). Let $\pi \in \widehat{G}$. Then $\pi(\mu_n) \rightarrow P_f$. But $P_f = 0$ or I by irreducibility of π . Hence if $\pi \neq I$, $\pi(\mu_n) \rightarrow 0$ in the weak operator topology.

(ii) \implies (iii). Let $\pi \in \widehat{G}$, $\xi, \eta \in H^\pi$ and $\phi_{\xi, \eta}^\pi(x) = \langle \pi(x)\xi, \eta \rangle$, $x \in G$. Then

$$\begin{aligned} \langle \mu_n, \phi_{\xi, \eta}^\pi \rangle &= \int \phi_{\xi, \eta}^\pi(x) d\mu_n(x) \\ &= \int \langle \pi(x)\xi, \eta \rangle d\mu_n(x) \\ &= \langle \pi(\mu_n)\xi, \eta \rangle \rightarrow 0 \quad \text{if } \pi \neq I. \end{aligned}$$

Let $\mathcal{E}(G)$ denote the extreme points of $P_1(G)$. The above implies that $\langle \mu_n, \ell_y \phi \rangle \rightarrow 0$ for any $y \in G$, $\phi \in \mathcal{E}(G)$, $\phi \neq 1$ where 1 denotes the constant one function on G . We will show that $\langle \mu_n, \phi \rangle \rightarrow \langle m, \phi \rangle$ for all $\phi \in P_1(G)$.

Note that if E is a locally convex space, and C a compact subset of E , and f_n a sequence of continuous linear functionals on E which are uniformly bounded on C and converge to 0 on C , then convergence to 0 holds on the closed convex hull of C (see [15] or [10] for an elementary proof). This applies easily if G is discrete. In the general case, slight complications arise: the set $P_1(G)$ is not weak*-compact, and measures are in general not weak*-continuous on $B(G)$. Nevertheless the method of proof generalizes to this case:

If G is second countable, then the weak*-topology on the unit ball of $B(G)$ is metrizable. Then $P_0(G)$ (= intersection with the cone of positive definite functions) is compact and convex; the extreme points of $P_0(G)$ are 0 and the extreme points of $P_1(G)$. Let $\phi \in P_1(G)$. By Choquet's theorem, there is a probability measure Φ concentrated on $\text{ext}(P_0(G))$ representing ϕ , i.e., for $T \in C^*(G)$, we

have

$$\langle T, \phi \rangle = \int_{P_0(G)} \langle T, \gamma \rangle d\Phi(\gamma) \quad \text{for all } x \in G.$$

Using a bounded approximate unit (v_n) in $L^1(G) \subseteq C^*(G)$, it follows that the map $(x, \gamma) \rightarrow \gamma(x) = \lim \langle v_n, \ell_x \gamma \rangle$ is Borel measurable on $G \times P_0(G)$ and by dominated convergence that

$$\phi(x) = \int_{P_0(G)} \gamma(x) d\Phi(\gamma) \quad \text{for all } x \in G,$$

in particular that 0 has weight zero (take $x = e$). Thus, Φ is concentrated on $\mathcal{E}(G)$. Hence if $\mu \in M(G)$, one gets by Fubini's theorem

$$\langle \phi, \mu \rangle = \int_{\mathcal{E}(G)} \langle \gamma, \mu \rangle d\Phi(\gamma).$$

Hence if $\{\mu_n\}$ is a sequence of probability measures on G , satisfying (ii), it follows from the Lebesgue dominated convergence theorem that $\langle \phi, \mu_n \rangle \rightarrow \Phi(\{1\})$, and similarly $\langle \phi, \ell_y^* \mu_n \rangle \rightarrow \Phi(\{1\})$ for $\phi \in P_1(G)$, $y \in G$. Consequently, μ_n and $\ell_y^* \mu_n$ have the same limit on $P_1(G)$; hence μ_n converges to the unique invariant mean m on $B(G)$.

For general G , if there is a weakly ergodic sequence of measures in $M(G)$ (resp. (ii) holds), then G has to be σ -compact (see Theorem 4.6 and Remark 4.3).

If G is σ -compact, and π is a cyclic representation of G on a Hilbert space H , then H is separable, and hence the strong operator topology on $\mathcal{B}(H)$ is metrizable on bounded sets. Consequently, the quotient group $G/\text{Ker } \pi$ is second countable, and the above argument applies.

(iii) \implies (i). Let π be a continuous unitary representation of G . Then, by (iii), $\{\langle \pi(\mu_n)\xi, \eta \rangle\}$ converges for all $\xi, \eta \in H^\pi$, and hence $\pi(\mu_n) \rightarrow T$ in the weak operator topology for some $T \in \mathcal{B}(H^\pi)$. Clearly, $\langle T\xi, \eta \rangle = \langle m, \phi_{\xi, \eta}^\pi \rangle$, and since m is translation-invariant, we have $\pi(y)T = T = T\pi(y)$ for all $y \in G$. So, $T = P_f$ i.e. $\pi(\mu_n) \rightarrow P_f$ in the weak operator topology for all π . Hence (iii) holds. \square

Lemma 4.2. *If H is an open subgroup of G with G/H infinite, (μ_n) a weakly ergodic sequence of measures, then $\mu_n(H) \rightarrow 0$.*

Proof. Let π be the regular representation on $\ell^2(G/H)$, $\xi = 1_H$. Then $\langle \pi(\mu)\xi, \xi \rangle = \mu(H)$. If G/H is infinite, it follows easily that $\ell^2(G/H)_f = (0)$; hence $\mu_n(H) \rightarrow 0$. \square

Remark 4.3. By a similar argument one shows that if the sequence (μ_n) satisfies (ii) of Theorem 4.1, then the measures μ_n cannot be concentrated on a subgroup H as above: We have $1_H \in P_1(G)$ and the set of $\phi \in P_1(G)$ for which $\phi(x) = 1$ for $x \in H$ is easily seen to be weak*-compact in $B(G)$. Hence it has an extreme point $\phi \neq 1$ and this is also an extreme point of $P_1(G)$. Thus we get $\pi \in \widehat{G} \setminus \{1\}$, $\xi \in H^\pi \setminus \{0\}$ with $\pi(x)\xi = \xi$ for $x \in H$. If all μ_n would be concentrated on H , we would get $\pi(\mu_n)\xi = \xi$ for all n , contradicting (ii).

Theorem 4.4. *Let G be a locally compact group. Then the following statements about a sequence (μ_n) of probability measures on G are equivalent:*

- (i) (μ_n) is a strongly ergodic sequence.
- (ii) $\pi(\mu_n) \rightarrow 0$ in the strong operator topology for every $\pi \in \widehat{G} \setminus \{1\}$.

Proof. (i) \implies (ii) follows as in Theorem 4.1.

(ii) \implies (i): Let (π, H) be a (continuous, unitary) representation of G , P_f denotes the orthogonal projection onto H_f . As in the proof of Theorem 4.1, (ii) \implies (iii), we may assume that H is separable, G second countable. Then $C^*(G)$ is separable. By [16, Theorem IV.8.32] there exists a disintegration $(\pi, H) = \int_{\Gamma}^{\oplus} (\pi_{\gamma}, H(\gamma)) d\nu(\gamma)$ of the representation (π, H) of $C^*(G)$ such that π_{γ} is an irreducible representation of $C^*(G)$ for almost all γ . Each π_{γ} defines a representation of G and, putting $\Gamma_f = \{\gamma : \pi_{\gamma} = 1\}$, we have clearly $H_f = \int_{\Gamma_f}^{\oplus} H(\gamma) d\nu(\gamma)$. For $\xi = \int_{\Gamma}^{\oplus} \xi(\gamma) d\nu(\gamma) \in H$, we get $P_f \xi = \int_{\Gamma_f}^{\oplus} \xi(\gamma) d\nu(\gamma)$. If μ is a bounded measure on G , it follows as in [2, 18.7.4] that $\pi(\mu) = \int_{\Gamma}^{\oplus} \pi_{\gamma}(\mu) d\nu(\gamma)$; hence $\pi(\mu)\xi = \int_{\Gamma}^{\oplus} \pi_{\gamma}(\mu)\xi(\gamma) d\nu(\gamma)$. Since $\pi_{\gamma}(\mu_n)\xi(\gamma) \rightarrow 0$ for almost all $\gamma \notin \Gamma_f$, it follows as in the proof of Theorem 3.1, (iii) \implies (i), from Lebesgue's dominated convergence theorem that $\pi(\mu_n)\xi \rightarrow P_f \xi$. \square

Remark 4.5. The question of the existence of weakly ergodic sequences of measures was stated as a problem in [8]. In fact, the case of separable groups G had already been settled before in [7], Theorem 3: for the sequences (x_n) constructed there, $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$ has the property that $\pi(\mu_n)$ converges to P_f in the strong operator topology for any continuous representation of G on a Banach space B for which all orbits $\{\pi(x)b : x \in G\}$ are relatively weakly compact. In particular, (μ_n) is even a strongly ergodic sequence. More generally, the following result holds:

Theorem 4.6. *The following statements about a locally compact group G are equivalent:*

- (i) *There exists a strongly ergodic sequence of measures.*
- (ii) *There exists a weakly ergodic sequence of measures.*
- (iii) *G is σ -compact.*

Proof. (i) \implies (ii) is trivial.

(ii) \implies (iii): See Lemma 4.2 (any sequence of finite measures is supported by a countable union of compact sets, hence by an open σ -compact subgroup).

(iii) \implies (i): By the Kakutani-Kodaira theorem, G has a compact normal subgroup N such that G/N is metrizable. In particular, G/N is separable. Let λ be the normalized Haar measure on N and let M be a closed separable subgroup of G such that $G = M \cdot N$. Let (x_n) be a sequence in M satisfying the properties of [7], Theorem 3, mentioned above. Put $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j} * \lambda$. We claim that (μ_n) is strongly ergodic. Let (π, H) be a unitary representation of G . Put $H_{f,N} = \{\xi \in H : \pi(x)\xi = \xi \text{ for all } x \in N\}$, similarly for $H_{f,M}$, and denote the orthogonal projections on these spaces by $P_{f,N}$ resp. $P_{f,M}$. Clearly, $P_{f,N} = \pi(\lambda)$. Since N is normal, $H_{f,N}$ is a π -invariant subspace; hence the same is true for $H_{f,N}^{\perp}$. This entails that $P_{f,N}$ and $P_{f,M}$ commute; hence $P_f = P_{f,M} \circ P_{f,N}$. By assumption, $(\frac{1}{n} \sum_{j=1}^n \pi(x_n))$ converges strongly to $P_{f,M}$; hence $(\pi(\mu_n))$ converges to $P_{f,M} \circ P_{f,N} = P_f$. \square

Examples. a) Let H be the Heisenberg group. If (μ_n) is a sequence of probability measures, we claim that the following statements are equivalent:

- (i) (μ_n) is strongly ergodic.
- (ii) (μ_n) is weakly ergodic.
- (iii) $\widehat{\mu}_n(\gamma) \rightarrow 0$ for all $\gamma \in \widehat{H} \setminus \{1\}$.

Here \widehat{H} denotes the set of *abelian* continuous characters of H , i.e. in this example strong (or weak) ergodicity is uniquely determined by the projections of μ_n to H/Z , where $Z = [H, H]$ is the center of H .

Proof. We use the notation of [8], Proposition 6. Condition (iii) is clearly necessary, since \widehat{G} describes the one-dimensional representations of G . Hence it is sufficient to show that (iii) implies (i). We write $H = \mathbb{R}^3$ (as a set). Then the infinite dimensional irreducible representations of H act on $H^\pi = L^2(\mathbb{R})$ by

$$(\pi(x)f)(t) = e^{2\pi i(x_1-x_2t)a} f(t-x_3)$$

($x = (x_1, x_2, x_3)$, $a \in \mathbb{R} \setminus \{0\}$ is a fixed parameter). It is clearly enough to show that $\pi(\mu_n)f \rightarrow 0$ for f with bounded support, i.e. $\text{supp } f \subseteq [-K, K]$ for some $K > 0$. Then $\langle \pi(x)f, f \rangle = 0$ if $|x_3| > 2K$.

Put $A = \{(x, y) \in H \times H : |x_3 - y_3| \leq 2K\}$. Then it follows that $\|\pi(\mu)f\|^2 \leq \|f\|^2 \mu \otimes \mu(A)$. Hence it is sufficient to show that

$$\mu_n \otimes \mu_n(A) \rightarrow 0$$

for every sequence (μ_n) satisfying (iii).

Put $A_j = \mathbb{R}^2 \times]2K(j-1), 2K(j+1)[$, $\alpha_{nj} = \mu_n(A_j)$. Then $A \subseteq \bigcup_{j \in \mathbb{Z}} A_j \times A_j$; hence

$$\mu_n \otimes \mu_n(A) \leq \sum_j \alpha_{nj}^2.$$

Furthermore, $\sum_j \alpha_{nj} \leq 2$ (observe that $A_j \cap A_k = \emptyset$ for $|j - k| \geq 2$).

Put $\bar{\mu}_n(M) = \mu_n(\mathbb{R}^2 \times M)$. Then $(\bar{\mu}_n)$ is a sequence of probability measures on \mathbb{R} . By assumption (iii), the sequence $(\bar{\mu}_n)$ converges to the Bohr-von Neumann mean m on $AP(\mathbb{R})$ (we have $Z = \{(x_1, 0, 0)\}$).

Fix $t \in \mathbb{N}$ with $t \geq 6$. Let f be a continuous function on \mathbb{R} with period tK , satisfying $0 \leq f \leq 1$ and

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq 2K, \\ 0 & \text{for } 3K \leq x \leq (t-3)K. \end{cases}$$

Then

$$m(f) = \frac{1}{tK} \int_0^{tK} f(x)dx < \frac{6}{t}.$$

Hence there exists n_0 such that $\langle f, \bar{\mu}_n \rangle < 6/t$ for $n \geq n_0$. Then it follows that $\alpha_{nj} < 6/t$ for $n \geq n_0$, $j = 0, \pm t, \pm 2t, \dots$. Considering appropriate translates of f , we get the same estimate for the other residue classes mod t , i.e.

$$\alpha_{nj} < \frac{6}{t} \quad \text{for } n \geq n_1, \quad j \in \mathbb{Z}.$$

This implies $\sum_j \alpha_{nj}^2 < 2 \cdot \frac{6}{t}$ for $n \geq n_1$, and for $t \rightarrow \infty$ our claim follows.

Further results of this type (in the setting of uniform distribution) have been shown in [17].

b) A similar description holds for the ‘ $ax+b$ ’-group (compare [8], Proposition 7). In particular, a) and b) provide examples of non-Moore groups for which strong and weak ergodicity are equivalent.

c) For $G = \mathbb{C} \times_{\sigma} \mathbb{T}$, the *euclidean motion* group of the plane, the situation is different. For measures μ_n on G , let as before $\bar{\mu}_n$ be the projections to \mathbb{T} , m denotes normalized Haar measure on \mathbb{T} . Then we have

- (i) (μ_n) is weakly ergodic if and only if $\bar{\mu}_n \rightarrow m (w^*)$ and $\mu_n \rightarrow 0$ (with respect to $C_0(G)$).
- (ii) (μ_n) is strongly ergodic if and only if $\bar{\mu}_n \rightarrow m (w^*)$ and $\delta_{x_n} * \mu_n \rightarrow 0$ (with respect to $C_0(G)$) for arbitrary sequences $(x_n) \subseteq G$, i.e. the convergence $\mu_n(xK) \rightarrow 0$ holds uniformly for the translates of a given compact set K .

(δ_x denotes the Dirac measure concentrated at x .) For example, $\mu_n = \delta_{x_n} * m$, where x_n is a sequence in G tending to infinity, establishes a sequence of measures that is weakly but not strongly ergodic. □

Proof. (i) follows immediately from [8], Proposition 8.

(ii) results from the following lemma. (Necessity of the condition is obvious since (μ_n) strongly ergodic implies $(\delta_{x_n} * \mu_n)$ strongly ergodic.) □

Lemma 4.7. *Let G be a locally compact group, π a unitary representation of G whose coefficients $\phi_{\xi,\eta}^{\pi}$ belong to $C_0(G)$ and let (μ_n) be a sequence of probability measures on G such that $\mu_n(xK) \rightarrow 0$ uniformly for $x \in G$ (where K is a fixed compact subset of G with non-empty interior). Then $\pi(\mu_n) \rightarrow 0$ in the strong operator topology.*

Proof. Assume $\|\xi\| \leq 1$. We have

$$\|\pi(\mu_n)\xi\|^2 = \int_G \int_G \phi_{\xi,\xi}^{\pi}(y^{-1}x) d\mu_n(x) d\mu_n(y).$$

For $\varepsilon > 0$ choose K such that $|\phi_{\xi,\xi}^{\pi}(z)| < \varepsilon$ for $z \notin K$ (the condition for (μ_n) does not depend on the choice of K). Then $\mu_n(yK) < \varepsilon$ for $n \geq n_0$, $y \in G$. Since $y^{-1}x \in K$ is equivalent to $x \in yK$, and $|\phi_{\xi,\xi}^{\pi}(z)| \leq 1$ for all z , this gives combined

$$\left| \int_G \phi_{\xi,\xi}^{\pi}(y^{-1}x) d\mu_n(x) \right| < 2\varepsilon \quad \text{for } n \geq n_0, y \in G.$$

Hence $\|\pi(\mu_n)\xi\|^2 < 2\varepsilon$ for $n \geq n_0$. □

d) A similar description (as in c) but taking into account that there are no non-trivial finite dimensional unitary representations) holds for the case of non-compact, connected, simple Lie groups with finite center (compare [8], Proposition 5).

REFERENCES

- [1] J. Blum and B. Eisenberg, *Generalized summing sequences and the mean ergodic theorem*, Proc. Amer. Math. Soc. **42** (1974), 423-429. MR **48**:8749
- [2] J. Dixmier, *C*-Algebras*, North-Holland, Amsterdam - New York - Oxford, 1977. MR **56**:16388
- [3] P. Eymard, *L'algèbre de Fourier d'une groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236. MR **37**:4208
- [4] E. Granirer and M. Leinert, *On some topologies which coincide on the unit sphere of the Fourier-Stieltjes algebra $B(G)$ and of the measure algebra $M(G)$* , Rocky Mountain J. Math. **11** (1981), 459-472. MR **85f**:43009
- [5] A.T. Lau, *The second conjugate algebra of the Fourier algebra of a locally compact group*, Trans. Amer. Math. Soc. **267** (1981), 53-63. MR **83e**:43009
- [6] A.T. Lau and V. Losert, *The C*-algebra generated by operators with compact support on a locally compact group*, Journal of Functional Analysis **112** (1993), 1-30. MR **94d**:22005

- [7] V. Losert and H. Rindler, *Uniform distribution and the mean ergodic theorem*, *Inventiones Math.* **50** (1978), 65-74. MR **80f**:22001
- [8] P. Milnes and A. Paterson, *Ergodic sequences and a subspace of $B(G)$* , *Rocky Mountain Journal of Mathematics* **18** (1988), 681-694. MR **90a**:43002
- [9] W. Maxones and H. Rindler, *Einige Resultate ueber unitär gleichverteilte Massfolgen*, *Anz. Österreich Akad. Wiss., Math.-Natur. Kl.* (1977/2), 11-13. MR **58**:6063
- [10] I. Namioka, *A substitute for Lebesgue's bounded convergence theorem*, *Proc. Amer. Math. Soc.* **12** (1961), 713-716. MR **23**:A2729
- [11] A.T. Paterson, *Amenability*, *Mathematical Surveys Monographs*, Vol. **29**, Amer. Math. Soc. Providence, R.I., 1988. MR **90e**:43001
- [12] J.P. Pier, *Amenable Locally Compact Groups*, Wiley, New York, 1984. MR **86a**:43001
- [13] H. Rindler, *Gleichverteilte Folgen in lokalkompakten Gruppen*, *Monatsh. Math.* **82** (1976), 207-235. MR **55**:567
- [14] P.F. Renaud, *Invariant means on a class of von Neumann algebras*, *Trans. Amer. Math. Soc.* **170** (1972), 285-291. MR **46**:3688
- [15] H.H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York-Heidelberg-Berlin, 1971. MR **49**:7722
- [16] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York-Heidelberg-Berlin, 1979. MR **81e**:46038
- [17] K. Gröchenig, V. Losert, H. Rindler, *Uniform distribution in solvable groups*, *Probability Measures on Groups VIII, Proceedings, Oberwolfach, Lecture Notes in Mathematics 1210*, Springer, Berlin-Heidelberg-New York, 1986, pp. 97-107. MR **88d**:22010

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