REDUCIBILITY OF SOME INDUCED REPRESENTATIONS OF \( p \)-ADIC UNITARY GROUPS

FIONA MURNAGHAN AND JOE REPKA

Abstract. In this paper we study reducibility of those representations of quasi-split unitary \( p \)-adic groups which are parabolically induced from supercuspidal representations of general linear groups. For a supercuspidal representation associated via Howe’s construction to an admissible character, we show that in many cases a criterion of Goldberg for reducibility of the induced representation reduces to a simple condition on the admissible character.

1. Introduction

Let \( K \) be a quadratic extension of a \( p \)-adic field \( F \) of characteristic zero and odd residue characteristic. Let \( G' \) and \( G'' \) be the \( F \)-rational points of the quasi-split unitary groups in \( 2n \) and \( 2n+1 \) variables, respectively, defined with respect to the extension \( K/F \). Let \( G = GL_n(K) \). Denote the kernel of the norm map from \( K^\times \) to \( F^\times \) by \( K_1 \). The group \( G' \), resp. \( G'' \), has a maximal parabolic subgroup \( P' \), resp. \( P'' \), with Levi factor isomorphic to \( G \), resp. \( G \times K_1 \). Let \( \pi \) be an irreducible unitary supercuspidal representation of \( G \), and \( \xi \) a character of \( K_1 \). Define a supercuspidal representation of \( \Pi_\xi \) of \( G \times K_1 \) by \( \Pi_\xi(x,\alpha) = \pi(x)\xi(\det_0(x\eta(x))\alpha) \), for \( x \) in \( G \) and \( \alpha \in K_1 \). Here \( \det_0 \) is the determinant on \( G \), and \( \eta \) is the automorphism of \( G \) taking \( x \) to \( \bar{x}^{-1} \), where the bar denotes the usual action of the non-trivial element of \( \text{Gal}(K/F) \) on matrices with entries in \( K \). Set

\[
I(\pi) = \text{Ind}_{P'}^{G'}(\pi \otimes 1)
\]
and

\[
I(\Pi_\xi) = \text{Ind}_{P''}^{G''}(\Pi_\xi \otimes 1).
\]

As it is a necessary condition for reducibility of \( I(\pi) \), and also for \( I(\Pi_\xi) \), we assume that \( \pi \) is equivalent to \( \pi \circ \eta \). In [G2], Goldberg proves that, under this assumption, \( I(\pi) \) is reducible, resp. \( I(\Pi_\xi) \) is irreducible, if and only if the sum of two particular \( \eta \)-twisted orbital integrals vanishes for every choice of matrix coefficient of \( \pi \).

Suppose that \( \pi \) arises via the construction of Howe ([H]) from an admissible character \( \theta \) of the multiplicative group of a tamely ramified degree \( n \) extension \( E \) of \( K \). We show that \( \pi \) is equivalent to \( \pi \circ \sigma \) if and only \( \theta \circ \sigma = \theta^{-1} \) for some involutive automorphism of \( E/F \) which is non-trivial on \( K \). In this paper, we prove that, for many such \( \pi \), Goldberg’s reducibility criterion reduces to a simple condition on \( \theta \). If \( L \) is the fixed field of \( \sigma \), then either \( \theta \mid L^\times \) is trivial or is equal to the quadratic
character of $L^\times$ associated to $E/L$ by class field theory. When $E$ is ramified over $L$ and $\theta | L^\times$ is trivial, we show that the sum of $\eta$-twisted orbital integrals which appears in the reducibility criterion is non-zero for a particular choice of matrix coefficient of $\pi$. When $E$ is unramified over $L$, we get a similar result under some additional assumptions on $\theta$. In an earlier paper ([MR]), using a reducibility criterion of Shahidi ([Sh]), we obtained the same type of results for representations of split classical groups induced from self-contragredient supercuspidal representations of general linear groups. Many of the results of this paper are proved by modifying proofs of analogous results of [MR].

In §2, we derive the relation between the equivalence of $\pi$ and $\pi \circ \eta$ and existence of $\sigma$ as above. In particular, it follows from a result of Adler ([A]) that existence of such an involution $\sigma$ guarantees existence of such supercuspidal representations $\pi$. We also discuss properties of the Howe factorization of $\theta$ relative to $\sigma$.

The $\eta$-twisted orbital integrals in Goldberg’s criterion can be expressed as integrals over certain sets of fixed points in $G$ of an involutive anti-automorphism $\varphi$ of $\mathfrak{gl}_n(K)$. The third section contains a description of the action of $\varphi$ on filtrations of the parahoric subalgebra attached to the extension $E/K$, and on related subgroups of $G$.

The representation $\pi$ is induced from an irreducible representation $\kappa$ of an open compact subgroup $H_0$ of $G$. In §4, we state the reducibility criterion of [G2], and show that for an appropriately chosen finite sum $f_\pi$ of matrix coefficients of $\pi$, each of the two relevant $\eta$-twisted orbital integrals $\Phi_\eta(h_k, f_\pi)$, $k = 1, 2$, reduces to the integral of the character of $\kappa$ over a certain $\varphi$-invariant subset of $H_0$.

In §5, we give some values of the character of $\kappa$, and summarize some results from [MR] relating properties of $\kappa$ and certain extensions of $F$ contained in $E$. We prove that if $\kappa$ is one-dimensional, then $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$.

Up to a character of $H_0$, the inducing representation $\kappa$ is a tensor product of finitely many representations $\kappa_i$ corresponding to the Howe factors $\theta_i$, $i = 1, \ldots, r$, of the admissible character $\theta$. In §6, we show that if a Heisenberg representation is used in the construction of one of these factors, then the character $\chi_i$ of $\kappa_i$ is real-valued on the set of $\varphi$-invariant points in $H_0$. We then compute the value of certain signs appearing in the formula for $\chi_i$.

Next, in §7, we consider the case when the representation $\kappa_r$ is defined in terms of a cuspidal representation of a finite general linear group. Assuming that $\kappa_i$ is one-dimensional for $1 \leq i \leq r - 1$, we outline how to modify the arguments of [MR] to express $\Phi_\eta(h_k, f_\pi)$, $k = 1, 2$, in terms of values of $\theta$ and sums of $\chi_r$ over various subsets of $H_0$. As shown in [MR], these sums of values of $\chi_r$ can be expressed in terms of Deligne-Lusztig characters of non-connected finite reductive groups which were computed in [MR]. This allows us to relate the signs of $\Phi_\eta(h_k, f_\pi)$, $k = 1, 2$, and $\theta | L^\times$.

The main results of the paper are Theorems 8.1 and 8.3. We state conditions on $\theta | L^\times$ which guarantee that $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$, and hence that $I(\pi)$ is irreducible, resp. $I(\Pi_\xi)$ is reducible.

In analogy with the situation in [Sh], the reducibility criterion of [G2] can be interpreted in terms of the conjectural theory of twisted endoscopy ([KS1],[KS2]). For $n = 2$ and 3, this is discussed in [G1] and in §4 of [G2], respectively. Under the conditions on $\theta$ given in §8 of this paper, the representation $\pi$ should be a lift from the unitary group in $n$ variables (see §§4,6 of [G2]).
2. Howe factorizations of admissible characters

Let $F$ be a $p$-adic field of characteristic zero and odd residual characteristic. If $F'$ is a finite extension of $F$, we will use the notation $\mathcal{O}_{F'}, p_{F'},$ and $\varpi_{F'}$ for the ring of integers in $F'$, maximal ideal in the ring of integers, and a uniformizer in $F'$, respectively. The norm and trace maps from $F'$ to $F$ will be denoted by $N_{F'/F}$ and $\text{tr}_{F'/F}$, respectively. Fix a quadratic extension $K$ of $F$. For $n \geq 2$, set $G = GL_n(K)$: we let $x \mapsto \bar{x}$ denote the action of the non-trivial element of the Galois group of $K/F$ on $G$ (apply the automorphism to matrix entries). Set $\eta(x) = f\bar{x}^{-1}$. Let $\pi$ be an irreducible supercuspidal representation of $G$ such that $\pi \circ \eta$ is equivalent to $\pi$ (denoted by $\pi \circ \eta \sim \pi$). Now suppose that $\pi$ arises via Howe’s construction from an admissible character $\theta$ of $E^x$, where $E/K$ is tamely ramified of degree $n$. Note that $E/F$ may not be Galois; we use the notation $\text{Aut}(E/F)$ to refer to the set of automorphisms of $E$ that fix $F$ pointwise, and similarly for $\text{Aut}(E/K)$. Note that $\theta$ is admissible over $K$, but might not be admissible over $F$. Assume that $\pi$ (hence $\theta$) is unitary. The above condition on $\pi$ translates into a condition on $\theta$.

**Lemma 2.1.** $\pi \sim \pi \circ \eta$ if and only if there exists an involution $\sigma \in \text{Aut}(E/F)$ such that $\sigma | K \neq \text{id}$ and $\theta \circ \sigma = \theta^{-1}$.

**Proof.** $(\Rightarrow)$ Take an embedding $\tau$ of $E$ into the algebraic closure of $F$ having the property that $\tau | K$ is the non-trivial element of $\text{Gal}(K/F)$. Let $E' = \tau(E)$. Then we can set $\theta'(\tau(\alpha)) = \theta(\alpha), \alpha \in E^x$ and observe that $\theta'$ is attached to the representation $x \mapsto \pi(\bar{x})$. But we also know that $x \mapsto \pi'(x^{-1})$ is attached to $\theta^{-1}$. So the condition on $\pi$ forces $\theta'$ and $\theta^{-1}$ to be conjugate (over $K$): there is a field isomorphism $\tau' : E' \rightarrow E$ which fixes $K$ pointwise such that $\theta^{-1}(\tau'(\alpha')) = \theta'(\alpha')$, $\alpha' \in E'^x$. Set $\sigma = \tau' \circ \tau$. Then $\sigma \in \text{Aut}(E/F)$. The automorphism $\sigma$ has the property that $\sigma | K$ is the non-trivial element of $\text{Gal}(K/F)$ and also that $\theta \circ \sigma = \theta^{-1}$.

What remains is to show that $\sigma$ is an involution. Note that $\theta \circ \sigma^2 = \theta$, and $\sigma^2 \in \text{Aut}(E/K)$. Suppose the order of $\sigma^2$ is $k > 1$. Write $E^{\sigma^2}$ for the fixed field of $\sigma^2$. Then $[E : E^{\sigma^2}] \leq k$. But $1, \sigma^2, \sigma^4, \ldots, \sigma^{2(k-1)}$ are $k$ distinct automorphisms of $E$ fixing $E^{\sigma^2}$ pointwise. This shows that $E/E^{\sigma^2}$ is normal, with $[E : E^{\sigma^2}] = k$, and therefore Galois. Since $\theta \circ \sigma^2 = \theta$, we find that for any $t \in E^x$, $\theta(\frac{t}{\sigma^{2(k-1)}(t)}) = 1$. By Hilbert 90, this shows that $\theta$ is trivial on the elements of norm 1, so $\theta$ factors through the norm $N_{E/E^{\sigma^2}}$. This contradicts the admissibility of $\theta$, proving that $\sigma$ is indeed an involution.

$(\Leftarrow)$ If there is an involution $\sigma$ as in the statement of the lemma, then, as above, $x \mapsto \pi(\bar{x})$ is equivalent to $x \mapsto \pi'(x^{-1})$, so $\pi \sim \pi \circ \eta$. □

Note that in contrast to the situation in [MR], $\sigma$ acts non-trivially on the base field $K$ over which the supercuspidal representation is defined.

**Lemma 2.2.** Suppose $E/K$ is a tamely ramified extension of degree $n$. The following are equivalent:

(i) There exists an involution $\sigma \in \text{Aut}(E/F)$ such that $\sigma | K \neq \text{id}$.

(ii) There exist irreducible unitary supercuspidal representations $\pi$ of $G$ associated by the construction of Howe to admissible characters $\theta$ of $E^x$ and satisfying $\pi \sim \pi \circ \eta$.

**Proof.** Part (ii) implies (i) by Lemma 2.1.
(i) ⇒ (ii): The fixed field of $\sigma$ is of index 2 in $E$. The argument given in the proof of Theorem 6.1 of [A] shows that there exists a character $\theta$ of $E^\times$ that is admissible over $F$ and such that $\theta \circ \sigma = \theta^{-1}$. Admissibility over $F$ implies admissibility over $K$, and (ii) follows by Lemma 2.1.

Assume that $\pi$ and $\theta$ are as in Lemma 2.1. The admissible character $\theta$ of $E^\times$ has a Howe factorization (see [H], [M]):

$$\theta = (\Lambda \circ N_{E/K}) \theta_1(\theta_{r-1} \circ N_{E/E_{r-1}}) \cdots (\theta_2 \circ N_{E/E_2})(\theta_1 \circ N_{E/E_1}).$$

Here $\theta$ uniquely determines the tower of fields $K = E_0 \subset E_1 \subset \cdots \subset E_r = E$ and $\Lambda$, $\theta_1, \ldots, \theta_r$ are quasi-characters of $E_0^\times$, $E_1^\times$, $\ldots$, $E_r^\times$, respectively. Comparison of the Howe factorizations of $\theta$ and $\theta \circ \sigma$ shows that $\sigma(E_i) = E_i$ for each $i$, although we shall see that $\sigma$ does not fix $E_i$ pointwise. Each quasi-character $\theta_i$ is generic over $E_{i-1}$ ([H]). The conductoral exponents are unique and satisfy

$$f_E(\theta_1 \circ N_{E/E_1}) > \cdots > f_E(\theta_r) > 0.$$ 

If $f_E(\Lambda \circ N_{E/K}) \leq f_E(\theta_1 \circ N_{E/E_1})$, note that it is possible to absorb $\Lambda \circ N_{E/K}$ into $\theta_1 \circ N_{E/E_1}$ and write $(\Lambda \circ N_{E/K})(\theta_1 \circ N_{E/E_1}) = (\theta'_1 \circ N_{E/E_1})$ for a $\theta'_1$ that is still generic over $E_1$. Because of this, we can choose $\theta_1$ such that either $\Lambda \equiv 1$ or $f_E(\Lambda \circ N_{E/K}) > f_E(\theta_1 \circ N_{E/E_1})$. For each $i = 1, \ldots, r-1$, choose an element $c_i \in E_i$ that "represents" $\theta_i$ in the sense that

$$\theta_i(1+x) = \psi([\text{tr}_{E/K}(c_i)\cdot x]), \quad x \in \mathfrak{p}_{E_i}^{f_E(\theta_i)+1},$$

where $\psi = \psi_0 \circ \text{tr}_{K/F}$ and $\psi_0$ is a character of the additive group $F$ with conductor $\mathfrak{p}_F$; we must have $c_i \in \mathfrak{p}_{E_i}^{-f_E(\theta_i)+1} \setminus \mathfrak{p}_{E_i}^{-f_E(\theta_i)+2}$ (see [H], [M]). Note that the genericity of $\theta_i$ implies that $c_i$ generates $E_i$ over $E_{i-1}$. If $i = r$ and $f_E(\theta_r) > 1$, choose $c_r$ as above.

Let $\sigma$ be as in Lemma 2.1.

**Lemma 2.3.** The characters $\Lambda$ and $\theta_i$, and the elements $c_i$ can be chosen so that

(i) $\Lambda$, $\theta_i$ are unitary,
(ii) $\Lambda \circ N_{E/K} \circ \sigma = (\Lambda \circ N_{E/K})^{-1}$, $\theta_i \circ N_{E/E_i} \circ \sigma = (\theta_i \circ N_{E/E_i})^{-1}$,
(iii) $\sigma(c_i) = -c_i$, if $f_E(\theta_i) > 1$.

**Proof.** The proof is the same as the proof of Lemma 2.5 in [MR], noting that the adjustments made in that proof to the various characters do not affect whether or not $f_E(\Lambda \circ N_{E/K}) > f_E(\theta_1 \circ N_{E/E_1})$ (and hence whether or not $\Lambda \equiv 1$).

From now on we assume that $\Lambda$, $\theta_i$, and $c_i$ are as in Lemma 2.3.

3. Filtrations and the map $\varphi$

Let the notation be as in §2. We will define an antimorphism $\varphi$ of $\mathfrak{gl}_n(K)$ whose action on $E$ is given by $\sigma$, and so that the integrals we will be discussing can be expressed in terms of integrals over certain sets of $\varphi$-invariant points in a subgroup $H_0$. The subgroup $H_0$ is the intersection with $G$ of the subgroup $H$ of $GL_{2n}(F)$ defined in [MR], and the map $\varphi$ is the restriction to $G$ of the map $\varphi$ defined there, so various properties of these maps relative to intermediate extensions, filtrations, and parahoric subgroups will follow immediately from results of [MR].

Let $L$ be the fixed field of $\sigma$ in $E$. We begin by fixing embeddings $L \hookrightarrow \mathfrak{gl}_n(F)$ and $E \hookrightarrow \mathfrak{gl}_2(L) \subset \mathfrak{gl}_{2n}(F)$ and a symmetric matrix $s \in GL_n(F)$ such that $w =
The only difference from the definitions of [MR] is that here such that for 

\[(iii) \ \varphi \in \mathfrak{gl}_n(F) \to \mathfrak{gl}_n(F) \text{ by} \]

\[\varphi(X) = w^{-1} \tau w.\]

By Lemma 3.4 of [MR], there is a symmetric matrix \(S \in \text{GL}_n(F) \subset \text{GL}_n(K)\) such that for \(X \in \mathfrak{gl}_n(K)\), we have

\[
\varphi(X) = S^{-1} \overline{X} S,
\]

where here and from now on \(\overline{\cdot}\) refers to the transpose in \(\mathfrak{gl}_n(K)\) and \(\overline{X}\) refers to the conjugate of \(X\) by \(\sigma\) acting on the entries of \(X\). If \(E/L\) is ramified, take \(a_L\) to be a non-square root of unity in \(L\). Otherwise, let \(a_L = \varpi_L\). Then let

\[
h_1 = S^{-1} \quad \text{and} \quad h_2 = a_L h_1 = a_L S^{-1}.
\]

Note that \(h_1\) and \(h_2\) are hermitian as elements of \(\text{GL}_n(K)\) relative to the action of \(\sigma\) described above. Because of the choice of \(a_L\), \(\det(h_1) = \det(h_2) = N_{E/K}(a_L) \det(h_1)\) both belong to \(F^\times\) and, under the assumptions on \(E\) and \(\sigma\) (see Lemma 2.2(i)) they lie in different cosets of \(N_{K/F}(K^\times)\). This implies that \(h_1\) and \(h_2\) are representatives of the two equivalence classes of hermitian matrices in \(G\).

We define various subalgebras and subgroups as in [MR]. The parahoric \(O_F\)-subalgebra \(B \subset \mathfrak{gl}_n(F)\) attached to the embedding \(E \hookrightarrow \mathfrak{gl}_n(F)\) is defined by

\[
B = \{X \in \mathfrak{gl}_n(F) \mid Xp_E^k \subset p_E^k, \text{for all } k\}.
\]

For any integer \(j\), we also define

\[
B_j = \{X \in \mathfrak{gl}_n(F) \mid Xp_E^k \subset p_E^{k+j}, \text{for all } k\}.
\]

The parahoric subgroup \(P \subset GL_2n(F)\) is the units

\[P = B^\times,\]

and we let

\[P_0 = P; \quad P_j = 1 + B_j, \quad \text{for } j \geq 1.\]

We define a function \(\nu\) on \(\mathfrak{gl}_n(F)\) by \(\nu(X) = j\), where \(j\) is the unique integer such that \(X \in B_j \setminus B_{j+1}\). Note that if \(X \in E\), then \(\nu(X) = \text{ord}_E(X)\). We embed \(\mathfrak{gl}_{|E|}(E_i) \subset \mathfrak{gl}_n(E_0) = \mathfrak{gl}_n(K) \subset \mathfrak{gl}_n(F)\) as the set of all elements of \(\mathfrak{gl}_n(K)\) that centralize \(E_i \subset E \subset \mathfrak{gl}_n(K)\). We will refer to this realization of \(\mathfrak{gl}_{|E|}(E_i)\) as \(M_i\). In this situation, for \(i = 0, \ldots, r\), we will define

\[
B_j(i) = \{X \in M_i \mid Xp_{E_i}^k \subset p_{E_i}^{k+j}, \text{for all } k\} = B_j \cap M_i,
\]

\[
P_j(i) = P_j \cap M_i,
\]

and

\[
B(i) = B_0(i), \quad P(i) = P_0(i) = B(i) \cap P.
\]

The only difference from the definitions of [MR] is that here \(B_j(0) = B_j \cap \mathfrak{gl}_n(K) \subset B_j\) (since \(E_0 = K\)), while in the previous paper \(E_0 = F\) and \(B_j(0) = B_j\).

**Lemma 3.1.** ([MR], Corollary 3.5). For \(0 \leq i \leq r\),

(i) \(\varphi(M_i) = M_i\),

(ii) \(\varphi(B_j(i)) = B_j(i), \quad j \in \mathbb{Z}\),

(iii) \(\varphi(P_j(i)) = P_j(i), \quad j \geq 0\).
For $1 \leq i \leq r$, write $\xi_i = \lfloor \frac{f_E(\theta_n N_E/E_i)}{2} \rfloor$. Set

$$H_0 = E^\times P_r(r-1) \cdots P_1(1) P_i(0),$$

$$K_i = P_r(r-1) \cdots P_{i+1}(i), \quad 0 \leq i \leq r-1; \quad K_r = \{1\},$$

$$\mathcal{L}_i = P_r(i-1) \cdots P_1(0), \quad 1 \leq i \leq r.$$ 

If $H$, $K_i$, $L_i$ are the corresponding subgroups defined in §3 of [MR], then we note that $H_0 = H \cap G$, $K_i = K_i \cap G$, $L_i = L_i \cap G$. For any subset $A \subset g_{0n}(K)$, we will write $A^\varphi$ for the $\varphi$-fixed points in $A$.

Lemma 3.2. (i) ([MR], Corollary 3.8). Let $x \in H_0^r$, and $1 \leq i \leq r$. Then there exist $y \in (E^\times K_i)^F$ and $z \in \mathcal{L}_i$ such that $x = yz$. 

(ii) ([MR], Lemma 3.9). Let $0 \leq i \leq r$, $j \geq 1$, and $\tau \in (H_0 \cap M_j)^F$. Then the map $x \mapsto x\tau \varphi(x)$ from $P_j(i)$ to $(\tau P_j(i))^F$ is onto. 

4. Goldberg’s reducibility criterion

Suppose that $\bar{\omega}$ is a character of $K^\times$ of the form $\bar{\omega}(z) = \omega(z/\bar{z})$, $z \in K^\times$, for some character $\omega$ of the kernel $K^1$ of $N_{K/F}$. Let $C(G, \bar{\omega})$ be the space of locally constant complex-valued functions on $G$ which are compactly supported modulo the centre $Z_K$ of $G$, and satisfy $f(zy) = \bar{\omega}^{-1}(z)f(g)$, $z \in Z_K$, $g \in G$. Let $Z_F$ denote the $F$-scalar matrices in $G$. Given $x \in G$, let

$$G_{x,\eta,Z_F} = \{ g \in G \mid g \xi x^{-1} \in Z_F \}.$$ 

If $f \in C(G, \bar{\omega})$ and $x$ is $\eta$-semisimple, that is, $(x, \eta)$ is a semisimple element of $G \times \langle \eta \rangle$, the $\eta$-twisted orbital integral of $f$ at $x$ is defined by ([G2], Def 1.9):

$$\Phi_\eta(x, f) = \int_{G/G_{x,\eta,Z_F}} f(gx\eta x^{-1}) \, dg^\times,$$ 

where $dg^\times$ is the $G$-invariant measure on the quotient coming from Haar measures on $G$ and $G_{x,\eta,Z_F}$.

Let $G'$, resp. $G''$, be the $F$-rational points of the quasi-split unitary group in $2n$, resp. $2n+1$, variables defined with respect to $K/F$. Let $P'$, resp. $P''$, be a maximal parabolic subgroup of $G'$, resp. $G''$, having Levi component isomorphic to $G$, resp. $G \times K^1$ (see [G2], §§2.6). Let $\pi$ be an irreducible supercuspidal representation of $G$. Given a character $\xi$ of $K^1$, define a supercuspidal representation $\Pi_\xi$ of $G \times K^1$ by

$$\Pi_\xi(x, \alpha) = \pi(x) \xi(\text{det}_0(\eta(x))\alpha), \quad x \in G, \alpha \in K^1.$$ 

Here $\text{det}_0$ denotes the determinant on $G$. Extend $\pi$, resp. $\Pi_\xi$, trivially across the unipotent radical to obtain a representation $\pi \otimes 1$, resp. $\Pi_\xi \otimes 1$, of $P'$, resp. $P''$. Set $I(\pi) = \text{Ind}_{P'}/G\pi \otimes 1$ and $I(\Pi_\xi) = \text{Ind}_{P''}/G\Pi_\xi \otimes 1$. When $n = 1$, $I(\pi)$ and $I(\Pi_\xi)$ are principal series representations, and it is known when such representations are reducible ([K1], [K2]). Thus we will assume that $n \geq 2$.

Let $h_1$ and $h_2$ be inequivalent hermitian matrices in $G$. Then $h_1$ is stably $\eta$-conjugate to $h_2$ ([G2], Definition 1.3, Corollary 1.7). This implies ([R]) that $G_{h_1,\eta,Z_F}$ is (the $F$-rational points of) an inner form of $G_{h_2,\eta,Z_F}$. Using an inner twisting, we define compatible measures on $G_{h_1,\eta,Z_F}$ and $G_{h_2,\eta,Z_F}$, and hence on the quotients $G/G_{h_k,\eta,Z_F}$, $k = 1, 2$. 

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Theorem 4.1. ([G2], Theorem 2.9, Theorem 6.3) Let π be an irreducible unitary supercuspidal representation of G such that π ◦ η ∼ π. Then the following are equivalent:
(i) I(π) is reducible.
(ii) I(Πξ) is irreducible (for any ξ).
(iii) Φη(h1, f) + Φη(h2, f) = 0 for every matrix coefficient f of π.

Remark 4.2.
(1) The condition π ◦ η ∼ π is necessary for reducibility of either of I(π) and I(Πξ) ([G2]).
(2) The sum Φη(h1, f) + Φη(h2, f) can be expressed as a κ-twisted orbital integral of f (see §1 of [G2]).
(3) The reducibility of I(Πξ) is independent of the choice of ξ ([G2], §6).

As in §2, let E be a tamely ramified degree n extension of K, and take θ to be a unitary character of E× which is admissible over K and satisfies θ−1 = θ ◦ σ for some σ ∈ Aut(E/F) such that σ | K is non-trivial. Let π be the irreducible supercuspidal representation of G associated to θ via Howe’s construction. Let H0 be the open compact-mod-centre subgroup of G defined in §3. Then π = IndH0G κ for some irreducible representation κ of H0. Let χκ denote the character of κ. Set

\[ χκ(x) = \begin{cases} \chiκ(x), & \text{if } x \in H0, \\ 0, & \text{otherwise}. \end{cases} \]

Then the function fπ defined by fπ(x) = χκ(xh1−1) is a finite sum of matrix coefficients of π.

Let h_k, k = 1, 2, and φ be as in §3. Recall that h2 = aLh1 and aL ∈ L×, where L = Eσ. Then, noting that φ(g) = h1η(g−1)h1−1, we have

\[ Φη(h1, fπ) = \int_{G/G_{h1}η, ZF} χκ(gφ(g)) dg×, \]

\[ Φη(h2, fπ) = \int_{G/G_{h2}η, ZF} χκ(gaLφ(g)) dg×. \]

Our aim is to show that under certain conditions on θ, both of the integrals Φη(h_k, fπ), k = 1, 2, are positive and hence, by Theorem 4.1, that I(π) is irreducible, and I(Πξ) is reducible.

5. Preliminary results

Let the subgroups H0, K_i, L_i, etc. be defined as in §3. For 0 ≤ i ≤ r, let det_i : M_i → E_i denote the determinant on M_i ∼ gl(E_i). The notation tr will be used for the trace map on gl_n(K). Recall ([H], [M]) that π = IndH0G κ, where the inducing representation κ is a tensor product:

\[ κ = (Λ ◦ det0) ⊗ κ1 ⊗ ⋯ ⊗ κ_r, \]

and κ_i is defined using the character θ_i of E_i× which appears in the Howe factorization of θ. We continue to assume that Λ and θ_i, 1 ≤ i ≤ r, are chosen as in Lemma 2.3. When fE(θ_i ◦ N_E/E_i) > 1, the representation κ_i is first defined on E×K_i−1 and then extended across L_i−1 by ψ(tr(c_i(−1))) to get a representation of H0 = E×K_i−1L_i−1. Here, c_i ∈ E_i is an element representing θ_i as in Lemma 2.3.
If \( f_E(\theta_r) = 1 \) then \( \kappa_r \) is defined in terms of the cuspidal representation of the finite general linear group \( P(r - 1)/P_1(r - 1) \) parametrized by \( \theta_r \mid \mathcal{O}_E^0 \). This case will be discussed in §7.

Recall that \( m_i = \left[ \frac{f_E(\theta_r \circ N_{E/E_i}) + 1}{2} \right] \) and \( \ell_i = \left[ \frac{f_E(\theta_r \circ N_{E/E_i})}{2} \right], 1 \leq i \leq r \). If \( i \leq r - 1 \) or if \( i = r \) and \( f_E(\theta_r) \neq 1 \), define a character \( \omega_i \) of \( E^xK_i \) \( P_m(i - 1)L_{i-1} \subset H_0 \) by

\[
\omega_i | E^xK_i = \theta_i \circ \det_i \quad \text{and} \quad \omega_i | P_m(i - 1)L_{i-1} = \psi(\text{tr}(c_i \cdot (-1))).
\]

The condition \( 2m_i \geq f_E(\theta_i \circ N_{E/E_i}) \) guarantees that the two definitions coincide on the intersection \( E^xK_i \cap P_m(i - 1)L_{i-1} \) ([H]).

If \( x \in H_0^\circ \), then, by Lemma 3.2(i), \( x \in L^x P_i(0) \). For \( x \in H_0^\circ \), define

\[
\mu(x) = \begin{cases} 
1, & \text{if } x \in N_{E/L}(E^x)P_1(0), \\
\alpha_L, & \text{otherwise}.
\end{cases}
\]

Lemma 5.1. ([MR], Lemma 5.1) If \( E/L \) is ramified, then \( f_E(\theta_r) > 1 \).

Lemma 5.2. ([MR], Lemma 5.2)

(i) Suppose that \( x \in E^xK_i \) \( P_m(i - 1)L_{i-1} \) and \( \varphi(x) = x \). If \( f_E(\theta_r) = 1 \), make the additional assumption that \( x \in E^xP_1(0) \). Then \( \omega_i(x) = \theta_i(N_{E/E}, \mu(x)) \).

(ii) If \( x \in H_0^\circ \), then \( \Lambda(\det_0(x)) = \Lambda(N_{E/K}(\mu(x))) \).

The conductoral exponent \( f_E(\theta_i \circ N_{E/E_i}) \) is even if and only if \( m_i = \ell_i \). In this case, \( E^xK_i \) \( P_m(i - 1) \) \( E^xK_{i-1} \), so \( \omega_i \) is defined on all of \( H_0 \), and \( \kappa_i = \omega_i \).

In particular, if \( m_i = \ell_i \), then \( \dim \kappa_i = 1 \). If \( i = r \) and \( f_E(\theta_r) = 1 \), since the construction of \( \kappa_r \) involves a cuspidal representation of a finite general group, we have \( \dim \kappa_r > 1 \). Otherwise, \( m_i = \ell_i + 1 \geq 2 \) and a Heisenberg construction is used to define \( \kappa_i \) on \( E^xK_i \), and \( \dim \kappa_i > 1 \).

Proposition 5.3. If \( \dim \kappa = 1 \) and \( \theta \mid L^x \equiv 1 \), then \( \Phi_{\eta}(h_k, f_\pi) > 0 \), \( k = 1, 2 \).

Proof. By the above remarks, \( m_i = \ell_i + 1 \), \( 1 \leq i \leq r \), and \( \kappa_i = \omega_i \). If \( x = g\varphi(g) \in H_0 \), then \( \varphi(x) = x \), so Lemma 5.2 applies and

\[
\kappa(g\varphi(g)) = \Lambda(\det_0(g\varphi(g))) \prod_{i=1}^r \kappa_i(g\varphi(g))
\]

\[
= \Lambda(N_{E/K}(\mu(g\varphi(g)))) \prod_{i=1}^r \theta_i(N_{E/E}, \mu(g\varphi(g))) = \theta(\mu(g\varphi(g))), \quad \text{if } g\varphi(g) \in H_0.
\]

Similarly, if \( ga_L\varphi(g) \in H_0 \), by Lemma 5.2,

\[
\kappa(ga_L\varphi(g)) = \Lambda(N_{E/K}(\mu(ga_L\varphi(g)))) \prod_{i=1}^r \kappa_i(ga_L\varphi(g)) = \theta(\mu(ga_L\varphi(g))).
\]

Since \( \mu(x) \in L^x \) for \( x \in \mathcal{H}_{0}^\circ \) and \( \theta \mid L^x \equiv 1 \), it follows from (4.2) that \( \Phi_{\eta}(h_k, f_\pi) = \Phi_{\eta}(h_k, 1_{H_{0h_1}}), k = 1, 2 \), where \( 1_{H_{0h_1}} \) denotes the characteristic function of \( H_0h_1 \).

Since \( P_{1}(0) \) and \( a_L P_{1}(0) \) are contained in \( H_0 \) for sufficiently large \( j \), it is a simple matter to show, using Lemma 3.2(ii), that \( \Phi_{\eta}(h_k, 1_{H_{0h_1}}) > 0, k = 1, 2 \).

We collect some results of [MR] which will be used later in this paper.

Lemma 5.4. ([MR], Lemmas 5.4–5.7)

(i) Suppose that \( K \subset N_1 \subset N_2 \subset E \), \( \sigma(N_j) = N_j \), \( j = 1, 2 \), and \( N_2/(N_2 \cap L) \) is ramified. Then \( N_1/(N_1 \cap L) \) is ramified and \( e(N_2/N_1) \) is odd.

(ii) If \( E/L \) is ramified, then \( \dim \kappa = 1 \).
(iii) If a Heisenberg construction is required for one of the $\kappa_i$’s, then $E/L$ is unramified.
(iv) If $r > 1$, $f_E(\theta_r) = 1$, and $e(E_{r-1}/(E_{r-1} \cap L)) = 2$, then $\dim \kappa_i = 1$ for $1 \leq i \leq r - 1$.

6. THE HEISENBERG CONSTRUCTION

Fix $i$, $1 \leq i \leq r$. Suppose that $f_E(\theta_i \circ N_{E/E_i})$ is odd, that is, $m_i = \ell_i + 1$. If $i = r$, assume in addition that $\ell_r \geq 1$. Recall that in this case (Lemma 5.4(iii)) $E/L$ must be unramified. Set

\[ H_i = K^x(1 + p_E)(K_i P_{\ell_i}(i - 1) \cap P_1(0)), \]
\[ H'_i = K^x(1 + p_E)(K_i P_{m_i}(i - 1) \cap P_1(0)). \]

Let $\omega_i$ be the character of $E^xK_iP_{\ell_i}(i - 1)L_{i-1}$ defined in §5. Let $\chi_i$ denote the character of $\kappa_i$. A Heisenberg construction is used to define $\kappa_i|E^xK_iP_{\ell_i}(i - 1)$ in such a way that the restriction of $\chi_i$ to $H'_i$ is a multiple of $\omega_i|H'_i$. Then, if $i \geq 2$, $\kappa_i$ is extended by $\psi(\text{tr}(c_i(-1)))$ on $L_{i-1}$ to produce a representation of $H_0$. In this section, we see that, for $x \in (E^x H_i)^{\mathbb{F}}$, $\chi_i(x)$ is a real scalar multiple of $\theta_i(N_{E/E_i}(\mu(x)))$. When the scalar multiple is non-zero, we compute its sign (Corollary 6.5).

If $F \subset N \subset E$, let $\zeta_N$ denote the set of roots of unity in $N$ of order prime to $p$. We assume that a uniformizer $\varpi_N \in N$ is chosen so that $\varpi_N^{(N/F)} = \varpi_{F}$, where $\varpi$ is a uniformizer in $F$. Let $C_N$ be the subgroup of $N^x$ generated by $\varpi_N$ and $\zeta_N$.

Lemma 6.1. Let $x \in L^x(H_i \cap P_1(0))$.
(i) There exists a unique $c_L(x) \in C_L$ such that $x \in c_{L}(x)(H_i \cap P_1(0))$.

(ii) Suppose that $y^{-1}xy \in E^x H'_i$ for some $y \in E^x H_i$. Then, given any subfield $N$ of $E$ containing $E_0 = K$,

\[ y^{-1}xy \in N^x H'_i \iff c_L(x) \in N^x. \]

Remark 6.2. In [MR], an analogue of the above lemma was proved for points which were $\varphi$-invariant, but the proof only required $x \in L^x(H_i \cap P_1(0))$.

Define

\[ S_i = \{ N \mid E_{i-1} \subset N \subset E, \ N \not\ni E_i \}. \]

To each $N \in S_i$, there are attached a sign $\text{sgn}(N) \in \{ \pm 1 \}$, and a positive integer $D(N)$ as defined in (3.6.47) of [M]. Set

\[ \text{sgn}(x) = \prod_{\{N \in S_i, |c_L(x)\notin N^x\}} \text{sgn}(N), \quad x \in L^x(H_i \cap P_1(0)). \]

Let $q_{E_{i-1}}$ denote the cardinality of the residue class field of $E_{i-1}$.

Lemma 6.3. Let $x \in (E^x H_i)^{\mathbb{F}}$. If $x$ is conjugate to an element of $E^x H'_i$, then

\[ \chi_i(x) = \sum_{N \in S_i, |c_L(x)\in N^x|} D(N) \text{sgn}(x) \theta_i(N_{E/E_i}(\mu(x))). \]

Otherwise $\chi_i(x) = 0$. Here, $\mu$ is as defined in §5.

Proof. The second statement of the lemma follows from [M], §3.6. Thus, without loss of generality, we assume that there exists $y \in E^x H_i$ such that $y^{-1}xy \in E^x H'_i$. 


Let $\omega_i$ and $\mu$ be defined as in §5. It follows from results of [M] (see Lemma 6.1 of [MR]) and Lemma 6.1, that
\[
\chi_i(x) = \sum_{E_i \subseteq E} q_{E_i/N}^{(E_i \times E_i)} D(N) \ s\ s\ g\ s\ n\ (y^{-1}xy).
\]
To complete the proof, arguing as for Lemma 6.4 of [MR] results in:
\[
\omega_i(y^{-1}xy) = \theta_i(N_E/E, (\mu(x))).
\]

\[\square\]

**Lemma 6.4.** Let $L' = L_{un}(\mathbb{Q}_{L, \sqrt{\varepsilon}})$, where $L_{un}$ is the maximal unramified extension of $F$ contained in $L$ and $\varepsilon$ is a non-square in $\mathbb{Q}_{L, \sqrt{\varepsilon}}$. Suppose that $N \in S_i$ and $\sigma(N) = N$.

(i) If $K/F$ is unramified, then $\sgn(N) = 1$.

(ii) If $K/F$ is ramified and $e(E/K)$ is even, then $\sgn(N) = 1$.

(iii) If $K/F$ is ramified, $e(E/K)$ is odd, and $e(E_i/(E_i \cap L)) = e(E_i/(E_i \cap L))$, then $\sgn(N) = 1$.

(iv) If $K/F$ is ramified, $e(E/K)$ is odd, $e(E_i/(E_i \cap L)) = 1$ and $e(E_i/(E_i \cap L)) = 2$, then

\[
\sgn(N) = \begin{cases} -1, & \text{if } N = L', \\ 1, & \text{otherwise}. \end{cases}
\]

**Proof.** As shown in Proposition 3.6.55 of [M], $\sgn(N) = 1$ whenever $f(E/N) > 2$. By arguing as in the second part of the proof of Lemma 7.4 of [MR], we see that $\sgn(N) = 1$ whenever $f(E/N) = 1$. Thus we need only consider the case $f(E/N) = 2$.

Suppose that $K/F$ is unramified. As $E/L$ is also unramified, and $K \not\subseteq L$, we have $f(E/K) = f(L/F)$ odd. In particular, as $K \subset N$, $f(E/N)$ must be odd, and so (i) follows.

Suppose that $K/F$ is ramified. Assume that $N \in S_i$, $f(E/N) = 2$ and $\sigma(N) = N$. Then, by Proposition 7.6 of [MR], $\sgn(N) = 1$ if $[E : N] > 2$, and $\sgn(N) = -1$ if $[E : N] = 2$. By Lemma 7.5(i) of [MR], $L$ and $L'$ are the only two extensions $N'$ of $F$ in $E$ satisfying $\sigma(N') = N'$ and $[E : N'] = f(E/N') = 2$. By Lemma 7.5(ii) of [MR], $K \subset E_i \subset L'$ is equivalent to $e(E/K)$ odd and $e(E_i/(E_i \cap L)) = 2$. Also, if $e(E/K)$ is odd, then $L' \not\subset E_i$ is equivalent to $e(E_i/(E_i \cap L)) = 1$. Thus, by definition of $S_i$, $L' \in S_i$ is equivalent to the three conditions $e(E/K)$ odd, $e(E_i/(E_i \cap L)) = 2$ and $e(E_i/(E_i \cap L)) = 1$. Parts (ii)–(iv) now follow. \[\square\]

**Corollary 6.5.** Let $x \in L^\times (H_i \cap P_0(0))$. Then, if $\nu$ is as defined in §3,

\[
\sgn(x) = \begin{cases} (-1)^{\nu(x)}, & \text{if } e(E/K) \text{ is odd, } e(E_{i-1}/(E_{i-1} \cap L)) = 2, \\
1, & \text{otherwise}. \end{cases}
\]

**Proof.** First note that if $\sigma(N) \neq N$, then $\sigma(E_{i-1}) = E_{i-1}$ implies that $E_{i-1} \subset \sigma(N)$. Also, $E_i \not\subset N$ and $\sigma(E_i) = E_i$ implies that $E_i \not\subset \sigma(N)$. Thus if $\sigma(N) \neq N$, we have $N \in S_i$ if and only if $\sigma(N) \in S_i$. It follows easily from the definitions in [M] that $\sgn(N) = \sgn(\sigma(N))$. Therefore, when computing $\sgn(x)$, we need only consider those $N \in S_i$ such that $c_i(x) \notin N^\times$ and $\sigma(N) = N$.

It now follows from Lemma 6.4 that we need only consider the case where $e(E/K)$ is odd, $e(E_{i-1}/(E_{i-1} \cap L)) = 2$, and $e(E_i/(E_i \cap L)) = 1$. (Note that in this case
\( K/F \) is ramified, by Lemma 5.4(i). By Lemma 6.4(iv),
\[
\text{sgn}(x) = \begin{cases} 
-1, & \text{if } c_L(x) \not\in L' \cap \nu, \\
1, & \text{if } c_L(x) \in L' \cap \nu.
\end{cases}
\]
By definition, \( L' \) is a quadratic extension of \( L_{an}(w_L^2) \) containing \( \zeta_L \) and not containing \( w_L \). It is immediate that \( c_L(x) \in L' \cap \nu \) if and only if \( c_L(x) \in w_L^{2k} \zeta_L \) for some integer \( k \); that is, if and only if \( \nu(c_L(x)) = \nu(x) \) is even. 

We can predict precisely when there will be a Heisenberg construction with \( \text{sgn}(x) = -1 \) for some \( x \), as follows:

**Lemma 6.6.** Assume that \( E/L \) is unramified.

(i) Suppose that \( K/F \) is ramified and \( e(E/K) \) is odd. Then there exists a unique \( j, 1 \leq j \leq r \), having the property that \( m_j = \ell_j + 1 \) and \( \text{sgn}(x) = (-1)^{r(x)} \), \( x \in L^\times(H_j \cap P_1(0)) \). In particular, for all \( i \neq j, 1 \leq i \leq r \), such that \( m_i = \ell_i + 1 \), we have \( \text{sgn}(x) = 1 \), for every \( x \in L^\times(H_i \cap P_1(0)) \).

(ii) If the conditions of (i) are not satisfied, then for all \( i, 1 \leq i \leq r \), such that \( m_i = \ell_i + 1 \), we have \( \text{sgn}(x) = 1 \), for every \( x \in L^\times(H_i \cap P_1(0)) \).

**Proof.** First suppose that \( K/F \) is ramified and \( e(E/K) \) is odd. Then, by Lemma 5.4(i), there exists a unique \( j, 1 \leq j \leq r \), such that \( e(E_j/(E_j \cap L)) = 1 \) and \( e(E_{j-1}/(E_{j-1} \cap L)) = 2 \). As \( e(E_j/E_j) \) is odd, it follows from
\[
f_{E_j}(\theta_j \circ N_{E_j/E_j}) = e(E_j/E_j)(f_{E_j}(\theta_j) - 1) + 1
\]
that \( m_j = \ell_j + 1 \) if and only if \( f_{E_j}(\theta_j) \) is odd. To show that \( f_{E_j}(\theta_j) \) is odd, argue as in the proof of Corollary 7.11 of [MR]. All statements concerning \( \text{sgn}(x) \) are now immediate consequences of Corollary 6.5.

\[ \square \]

7. The case \( f_{E_j}(\theta_r) = 1 \).

Throughout this section, we assume that \( f_{E_j}(\theta_r) = 1 \) and that if \( r > 1 \), then \( \kappa_j \) is one-dimensional for \( 1 \leq j \leq r - 1 \). Using a modification of the arguments of §10 of [MR], we express each \( \Phi_{\sigma}(h_k, x) \), \( k = 1, 2 \), in terms of sums of the character \( \chi_{r-1} \) of \( \kappa_r \) over subsets of \( \overline{D} \). Certain conditions on \( \theta \) imply that \( \Phi_{\sigma}(h_k, x) > 0 \). Omitting some of the details, we indicate how to adapt the results of §10 of [MR] to this setting.

We now define prime elements in \( E, L, E_{r-1} \) and \( E_{r-1} \cap L \) as in [MR]. Recall that \( f_{E_j}(\theta_r) = 1 \) implies that \( E \) is unramified over \( L \) (Lemma 5.1) and over \( E_{r-1} \).

Set \( e_0 = e(E_{r-1}/E_{r-1} \cap L) \) and \( f_0 = f(E_{r-1}/E_{r-1} \cap L) \). Fix a prime element \( \varpi_0 \) in \( E_{r-1} \cap L \) and a non-square root of unity \( \varepsilon \) in \( L \). If \( e_0 = 1 \), then \( E/(E_{r-1} \cap L) \) is unramified and we choose prime elements in \( E \) and \( L \) as follows: \( \varpi_E = \varpi_L = \varpi_0 \). If \( e_0 = 2 \), then \( \varpi_E = \sqrt{\varpi_0} \) is a prime element in \( E \) which generates \( E_{r-1} \) over \( E_{r-1} \cap L \) and satisfies \( \sigma(\varpi_E) = -\varpi_E \). Furthermore, the element \( \varpi_L = \sqrt{\varepsilon \varpi_0} = \sqrt{\varepsilon \varpi_E} \) is a prime element in \( L \).

Let \( \overline{M} \) denote the residue class field of a \( p \)-adic field \( M \). Set
\[
\overline{P}_0 = (H_0 \cap P(0))/(H_0 \cap P_1(0)).
\]

It follows from the definition of \( H_0 \) that
\[
\overline{P}_0 \simeq P(r-1)/P_1(r-1) \simeq GL_{[E:E_{r-1}]}(E_{r-1}).
\]
If \( r > 1 \), then \( \overline{P}_0 = \overline{P} \), where \( \overline{P} = (H \cap P)/(H \cap P_1) \) is as in §9 of [MR]. If \( r = 1 \), then since \( E_0 = K \) here and the \( E_0 \) of [MR] was \( F \), we have \( \overline{P}_0 = \overline{P} \cap GL_n(K) \).
We can now apply the results of §9 of [MR], remembering to replace $\overline{H}$ by $\overline{H}_0$ in the case $r = 1$.

As $f_E(\theta_r) = 1$ and $\theta_r$ is generic over $E_{r-1}$, the character $\theta_r | \mathcal{O}_E^\times$ determines a character of $\overline{H}^\times$ which corresponds to an irreducible cuspidal representation $\overline{\pi}_r$ of $\overline{\mathcal{H}}_0$. The restriction of $\kappa_r$ to $H_0 \cap P(0)$ is trivial on $H_0 \cap P_1(0)$ and induces $\overline{\kappa}_r$ on $\overline{H}_0$. As the prime element $\varpi_E$ above is a prime element in $E_{r-1}$, setting $\kappa_r(\varpi_E) = \theta_r(\varpi_E) \kappa_r(1)$ extends $\kappa_r$ to $H_0$.

Let $\mathcal{C}_L$, resp. $\mathcal{C}_L^\prime$, be the set of elements in $\overline{H}_0$ whose semisimple part is conjugate to an element of $E$, resp. $\mathcal{T}$. Next, define $S_{E-L}$, resp. $S_L$, to be the set of $x \in H_0 \cap P(0)$ such that the image of $x$ in $\overline{H}_0$ belongs to $\mathcal{C}_L \cap \mathcal{C}_L^\prime$, resp. $\mathcal{C}_L^\prime$. It follows from properties of the cuspidal representation $\overline{\pi}_r$ of $\overline{\mathcal{H}}_0$ that if $x \in H_0 \cap P(0)$ does not belong to $S_L \cup S_{E-L}$, then $\chi_r(x) = 0$. As we will see in Lemma 7.2, we need only consider values of $\kappa_r$ for $x \in (\varpi_E^k(H_0 \cap P(0)))^\times$, $k = 1, 2$. The following lemma gives information on properties of such $x$, when $\varpi_E^{-k} x \in S_L \cup S_{E-L}$, $k = 1, 2$.

**Lemma 7.1.**
(i) Suppose that $x \in P(r-1)\varpi$. Then there exists $g \in P(r-1)$ such that $x = g\varphi(g)$.
(ii) Suppose that $x \in (\varpi_E P(r-1))\varphi$. If $e_0 = 2$ and $\varpi_E^{-1} x \in S_{E-L}$, or if $e_0 = 1$, then there exists $g \in P(r-1)$ such that $x = g\varphi(g)$.
(iii) Suppose that $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is even. Fix $\delta \in P(r-1) \cap S_L$ such that $\varphi(\varpi_E \delta) = \varphi(\varphi(\varpi_E \delta))$. If $x \in (\varpi_E P(r-1))\varphi$ and $\varpi_E^{-1} x \in S_L$, then there exists $g \in P(r-1)$ such that $x = g\varphi(\varphi(g))$. Furthermore, $x = g_1\varphi(g_1)$ for some $g_1 \in G$.

**Proof.** Statements (i), (ii), and the first part of (iii) are proved as in Lemma 10.3 of [MR].

Recall that $h_2 = \varpi_L h_1$ (§3). Given $y \in G$, $y \in G^\varphi$ if and only if $y h_1$ is hermitian. Recall (§3) that $h_1$ and $h_2$ belong to distinct equivalence classes of hermitian matrices. It follows that $G^\varphi$ is the disjoint union of the sets $\{g \varpi_L \varphi(g) \mid g \in G\}$, $\ell = 0, 1$, the elements of the first set, resp. second set, having determinants in $N_{K/F}(K^\times)$, resp. in $N_{E/K}(\varpi_E) N_{K/F}(K^\times) = F^\times \backslash N_{K/F}(K^\times)$. Assume that $\delta$ is as in (iii). As $\varpi_E \delta \in G^\varphi$ by assumption, to show that $\varpi_E \delta = y \varphi(y)$ for some $y \in G$, it suffices to show that $\det_0(\varpi_E \delta) \in N_{K/F}(K^\times)$.

Because $\delta \in S_L$,

$$\det_0(\delta) \in \det_0(\mathcal{O}_E^\times) = \det_0(N_{E/L}(\mathcal{O}_E^\times)) \subset N_{K/F}(\mathcal{O}_K^\times).$$

Also, by choice of the prime element $\varpi_E \in E_{r-1}$, since

$$[E : E_{r-1}] = 2f(L/(E_{r-1} \cap L))$$

is divisible by 4, $N_{E/E_{r-1}}(\varpi_E) = \det_{r-1}(\varpi_E) = \varpi_E^{[E : E_{r-1}]} \in (E_{r-1} \cap L)^\times$. Thus $\det_0(\varpi_E) = N_{E/K}(\varpi_E) \in (F^\times)^2$. We conclude that $\det_0(\varpi_E \delta) \in N_{K/F}(K^\times)$. Thus $\varpi_E \delta = y \varphi(y)$, for some $y \in G$. Taking $x$ as in (iii), there exists $g \in P(r-1)$ such that $x = g\varphi(\varphi(g)) = g y \varphi(\varphi(g))$. Set $g_1 = g y$.

Let $\mathcal{F}_k = f_\pi \cdot 1_{(H_0 \cap P(0))h_k}$, $k = 1, 2$, where we write $1_S$ for the characteristic function of a subset $S$ of $G$.

**Lemma 7.2.** Set $e = e(E/F)$. Let $(\cdot, \cdot)$ denote gcd.
(i) $\Phi_0(h_1, f_\pi) = \frac{1}{(2\pi)} (\Phi_0(h_1, \mathcal{F}_1) + \Phi_0(h_1, \mathcal{F}_2)).$
(ii) $\Phi_0(h_2, f_\pi) = \frac{e}{(2\pi)} \Phi_0(h_2, \mathcal{F}_2).$
Proof. By arguing as in the proof of Lemma 10.1 of [MR],
\[ \chi_\kappa(\varpi_E^1 x \varphi(\varpi_E^j)) = \chi_\kappa(x), \quad x \in H_0. \]

Let \( C_k = \{ g \varpi_E^{k-1} \varphi(g) \mid g \in G \}, \ k = 1, 2. \) Recall (see above) that \( G^r \) is the disjoint union of \( C_1 \) and \( C_2. \) Given \( j \in \mathbb{Z} \) and \( \alpha \in \mathcal{O}_E^k, \) define a map \( \lambda_{\alpha,j} \) from \( G \) to \( G \) by \( \lambda_{\alpha,j}(x) = \varpi_E^j \alpha x \varphi(\varpi_E^j \alpha). \) For \( 1 \leq k, \ell \leq 2, \) the map \( \lambda_{\alpha,j} \) restricts to a measure-preserving bijection between
\[ C_k \cap \varpi_E^{\ell-1}(H_0 \cap P(0)) \quad \text{and} \quad C_k \cap \varpi_E^{\ell-1+2j}(H_0 \cap P(0)), \]
where the measure is the one on \( G/G_{h_k n_i Z_F}. \) Thus, using the map \( \lambda_{\alpha,j} \) and the fact that \( \chi_\kappa \circ \lambda_{\alpha,j} = \chi_\kappa \circ \lambda_{\alpha,0} \) (see above),
\[ \int_{G/G_{h_k n_i Z_F}} (\hat{\chi}_\kappa 1_{\varpi_E^{\ell-1}(H_0 \cap P(0))})(g \varpi_E^{k-1} \varphi(g)) \, dg \]
\[ = \int_{G/G_{h_k n_i Z_F}} (\hat{\chi}_\kappa (\alpha g \varpi_E^{k-1} \varphi(\alpha g))) 1_{\varpi_E^{\ell-1+2j}(H_0 \cap P(0))}(g \varpi_E^{k-1} \varphi(g)) \, dg \]
\[ = \int_{G/G_{h_k n_i Z_F}} (\hat{\chi}_\kappa 1_{\varpi_E^{\ell-1+2j}(H_0 \cap P(0))})(g \varpi_E^{k-1} \varphi(g)) \, dg. \]

To obtain the second equality, we have used the fact that \( \lambda_{\alpha,0} \) fixes the set \( \varpi_E^{\ell-1+2j}(H_0 \cap P(0)). \)

The smallest positive integer \( j \) such that \( N_{E/L}(\varpi_E^j \mathcal{O}_E^k) \cap F^r \neq \emptyset, \) that is, such that \( \varpi_E^j \mathcal{O}_E^k \cap G_{h_k n_i Z_F} \neq \emptyset, \) is \( j = e/(2, e). \) Therefore, applying (7.1) (which is independent of the choice of \( \alpha \in \mathcal{O}_E^k \)), we conclude from (4.1) and \( H_0 = \bigcup_{j \in \mathbb{Z}} \varpi_E^j(H_0 \cap P(0)), \) that
\[ \Phi_\eta(h_k, f_\pi) = \frac{e}{(2, e)} \sum_{1 \leq \ell \leq 2} \int_{G/G_{h_k n_i Z_F}} (\hat{\chi}_\kappa 1_{\varpi_E^{\ell-1}(H_0 \cap P(0))})(g \varpi_E^{k-1} \varphi(g)) \, dg, \]
\[ k = 1, 2. \]

As \( h_2 = \varpi_L h_1, \varpi_L \) normalizes \( H_0 \cap P(0), \) and \( \varpi_L \in \varpi_L \mathcal{O}_E^\times \subset \varpi_L(H_0 \cap P(0)), \) it follows that \( (H_0 \cap P(0))h_2 = \varpi_E(H_0 \cap P(0))h_1. \) Therefore (see comments preceding (4.1))
\[ \Phi_\eta(h_k, F_\pi) = \int_{G/G_{h_k n_i Z_F}} (\hat{\chi}_\kappa 1_{\varpi_E^{\ell-1}(H_0 \cap P(0))})(g \varpi_E^{k-1} \varphi(g)) \, dg, \quad k, \ell = 1, 2. \]

Comparing this with the above expression for \( \Phi_\eta(h_k, f_\pi), \) we see that it remains to show that \( \Phi_\eta(h_2, F_1) = 0. \)

Let \( x \in (H_0 \cap P(0))^r. \) By Lemma 3.2(i), there exists \( y \in (E^r K_{r-1})^r = (E^r P(r-1))^r \) and \( z \in L_{r-1} \) such that \( x = yz. \) As \( x \in P(0) \) and \( z \in P_1(0), \) it follows that \( y \in P(r-1)^r. \) By Lemma 7.1(i), there exists \( y_1 \in P(r-1) \) such that \( y = y_1 \varphi(y_1). \) Since \( z \in P_1(0), \) and \( x = yz \) and \( y \) are \( \varphi \)-invariant, it follows that \( \det_0(z) \in 1 + \mathfrak{p}_F. \) Thus \( \det_0(x) \in N_{K/F}(\det_0(y_1))(1 + \mathfrak{p}_F) \subset N_{K/F}(K^\times). \) Since \( x \in C_1 \cup C_2, \) \( \det_0(C_1) \subset N_{K/F}(K^\times), \) and \( \det_0(C_2) \subset F^\times \setminus N_{K/F}(K^\times), \) we must have \( x \in C_1. \) It follows from \( (H_0 \cap P(0))^r \subset C_1, C_1 \cap C_2 = \emptyset, \) and (7.2) that \( \Phi_\eta(h_2, F_1) = 0. \) \( \square \)
As the $\kappa_i$'s, $1 \leq i \leq r - 1$, are one-dimensional, their values on the relevant $\varphi$-invariant elements in $\omega^i_E(H_0 \cap P(0))$, $j = 1, 2$, are easily computed in terms of the characters $\theta_i$. In [MR], this was done in Lemma 10.2. Here, the result still holds, and it is proved the same way (with $\mathcal{H}_0$ replacing $\mathcal{H}$). The computation for $\Lambda$ is handled in exactly the same way. Combining this with the definition of $\kappa_r$ we get, for $x \in (H_0 \cap P(0))^\varphi \cup (\omega_E(H_0 \cap P(0))^\varphi$,

\[
\chi(x) = \begin{cases} 
\theta(\omega^i_E(x)) \chi(\omega^i_E(x)), & \text{if } \omega^i_E(x) \in S_L, \\
\theta(\varphi_L) \theta_r(\sqrt{z})^{-1} \chi_r(\omega^{-1}_E(x)), & \text{if } \nu(x) = 1 \text{ and } \omega^i_E(x) \in S_{E-L}, \\
0, & \text{if } \omega^i_E(x) \notin S_L \cup S_{E-L}.
\end{cases}
\]

(7.3)

Let $c = (-1)^{f_0}$. For $x \in (H_0 \cap P(0))^\varphi \cup (\omega_E(H_0 \cap P(0))^\varphi$, observe that the image of $\omega^{-\nu(x)}_E(x)$ in $\mathcal{H}_0$ belongs to $\mathcal{H}_0^\varphi$ if $\nu(x) = 0$, and to $\mathcal{H}^\varphi_0$ if $\nu(x) = 1$ (see the proof of Proposition 10.5 of [MR]). Here, we are using the same notation for $\varphi$ and the map which $\varphi$ induces on $\mathcal{H}_0$. The next step is to express the integrals $\Phi_k(h_k, F_k)$, $1 \leq k, \ell \leq 2$, in terms of sums of $\chi_r$ over certain $\varphi$ or $c\varphi$-invariant subsets of $\mathcal{H}_0$. This is the analogue of Proposition 10.5 of [MR]. In order to do this, we use Lemma 7.1 to write elements of $(\omega^{-1}_E(H_0 \cap P(r - 1))^\varphi$ in the form $g^T \varphi(g)$, where $g \in P(r - 1)$, $T = 1$ if $j = 1$, and $T \in \{\omega_L, \omega_E \delta\}$ if $j = 2$ (with $\delta$ as in Lemma 7.1). Using these results together with (7.2), (7.3) and Lemma 3.2, and following the proof of Proposition 10.5 of [MR], except with $I(F_0)$, $I(F_1)$, $H \cap P_1$, and $\mathcal{H}$ of [MR] replaced by $\Phi_1(h_1, F_1)$, $\Phi_1(h_1, F_2)$, $\Phi_1(h_2, F_2)$, $H_0 \cap P(0)$, and $\mathcal{H}_0$, respectively, results in

**Proposition 7.3.** Suppose that $f_{E}(\theta_r) = 1$. If $f_0 = 2$, assume that dim $\kappa_i = 1$ for $1 \leq i \leq r - 1$.

(i) $\Phi_1(h_1, F_1) = \Phi_1(h_1, 1_{(H_0 \cap P(0) \cap \kappa_k)}) \left( \sum_{x \in \Pi_0^\varphi} \chi_r(x) \right)$.

(ii) If $e_0 = 1$, then $\Phi_1(h_1, F_2) = 0$ and

$$
\Phi_1(h_2, F_2) = \theta(\omega_L) \Phi_1(h_2, 1_{(H_0 \cap P(0) \cap \kappa_k)}) \left( \sum_{x \in \Pi_0^\varphi} \chi_r(x) \right).
$$

(iii) If $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is odd, then $\Phi_1(h_1, F_2) = 0$ and

$$
\Phi_1(h_2, F_2) = \Phi_1(h_2, 1_{(H_0 \cap P(0) \cap \kappa_k)}) \theta(\omega_L) \theta_r(\sqrt{z})^{-1} \left( \sum_{x \in \Pi_0^\varphi} \chi_r(x) \right).
$$

(iv) If $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is even, let $\delta$ be as in Lemma 7.1(iii). Then

$$
\Phi_1(h_1, F_2) = \Phi_1(h_1, 1_{(H_0 \cap P(0) \cap \delta \sqrt{z}^{-1})}) \theta(\omega_L) \chi_r(x) \chi_r(x) \left( \sum_{x \in \Pi_0^\varphi} \chi_r(x) \right).
$$

$$
\Phi_1(h_2, F_2) = \Phi_1(h_2, 1_{(H_0 \cap P(0) \cap \delta \sqrt{z}^{-1})}) \theta(\omega_L) \chi_r(x) \chi_r(x) \left( \sum_{x \in \Pi_0^\varphi} \chi_r(x) \right).
$$

**Remark 7.4.** We have used the facts that $(H_0 \cap P(1))h_2 = \omega_L(H_0 \cap P(1))h_1$ and that, when $e_0 = 1$, $(H_0 \cap P(1))h_2 = \omega_E \delta(H_0 \cap P(1))h_1$. By arguing along the same lines as in the last part of the proof of Lemma 7.2, we can use Lemma 7.1 to show that if $x \in (\omega_E(H_0 \cap P(0)))^\varphi$ satisfies $\omega^{-1}_E(x) \in S_{E-L} \cup S_L$, then $x = g \varphi(g)$
for some $g \in G$ if and only if $\varpi^{-1} x \in S_L$, and that can happen only when $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is even. This leads to the conditions on $\Phi_\eta(h_1, F_2)$ in parts (ii)-(iv) (see (7.2)).

The signs of the sums appearing in Proposition 7.3 are evaluated as in [MR], using results of §9 of [MR], yielding

$$\Phi_\eta(h_1, F_1) > 0 \quad \text{and} \quad \Phi_\eta(h_1, F_2) = 0,$$

$$(−1)^e \theta(\varpi_L)^{-1} \Phi_\eta(h_2, F_2) > 0.$$ 

Combining this with Lemma 7.2 results in:

**Theorem 7.5.** Suppose that $f_E(\theta_r) = 1$. If $f_0 = 2$, assume that $m_i = \ell_i$ for $1 \leq i \leq r - 1$.

(i) If $e_0 = 1$ and $\theta | L^\times \equiv 1$, then $\Phi_\eta(h_k, f_{\pi}) > 0$, $k = 1, 2$.

(ii) If $e_0 = 2$ and $\theta | L^\times \neq 1$, then $\Phi_\eta(h_k, f_{\pi}) > 0$, $k = 1, 2$.

8. Main results

Recall that $E$ is a tamely ramified degree $2n$ extension of $F$, $n \geq 2$, and $\theta$ is a unitary character of $E^\times$, admissible over the quadratic extension $K$ of $F$, having the property that $\theta \circ \sigma = \theta^{-1}$ for some involution $\sigma$ in $Aut(E/F)$ whose restriction to $K$ is non-trivial. As discussed in §2 (Lemma 2.1), the supercuspidal representation $\pi$ of $G = GL_n(K)$ associated to $\theta$ via Howe’s construction ([H]) has the property that $\pi \circ \eta \sim \pi$. The fixed field of $\sigma$ is denoted by $L$. Our main results are stated in terms of the values of $\theta$ on $L^\times$ and certain ramification degrees. We continue to assume that the residue characteristic $p$ of $F$ is odd.

**Theorem 8.1.** Let $f_{\pi}$ be the finite sum of matrix coefficients of $\pi$ defined in §4. If $\theta$ satisfies one of the following conditions, then $\Phi_\eta(h_k, f_{\pi}) > 0$, $k = 1, 2$.

(i) $E$ is ramified over $L$ and $\theta | L^\times \equiv 1$.

(ii) $E$ is unramified over $L$ and

$$\theta | L^\times = (−1)^{ord_{E}(\cdot)}(e(K/F)−1)e(E/K),$$

with the additional assumption that if $r > 1$ and $f_E(\theta_r) = 1$, then $m_i = \ell_i$, $1 \leq i \leq r - 1$.

**Remark 8.2.** The purpose of the additional assumption in (ii) is to exclude the case where a Heisenberg construction and a representation of a finite general linear group both occur in the inducing data for $\pi$. As remarked in [MR], we expect that the result still holds in that case.

**Proof of Theorem 8.1.** If (i) holds, the result follows from Proposition 5.3 and Lemma 5.4(ii).

Assume that (ii) holds. If $f_E(\theta_r) = 1$ and $f_0 = 1$, then $e(K/F) = 2$ and $e(E_{r-1}/K)$ is odd, by Lemma 5.4(i). Therefore $e(E/K) = e(E_{r-1}/K)$ is odd. Note that in this case $m_i = \ell_i$ is guaranteed by Lemma 5.4(iv). If $f_E(\theta_r) = 1$, $f_0 = 2$, and $e(K/F) = 2$, then, by Lemma 6.6(i), the assumption $m_i = \ell_i$, $1 \leq i \leq r - 1$, implies that $e(E/K)$ is even. We conclude that in the case $f_E(\theta_r) = 1$, Theorem 7.5 coincides with this theorem.
For the remainder of the proof, suppose that (ii) holds and \( f_E(\theta_i) > 1 \). Let \( \mu \) be as defined in §5. It follows from Lemma 5.2 that, given \( 1 \leq i \leq r \) and \( x \in H_0^i \),
\begin{align}
(8.1) \quad & \text{If } m_i = \ell_i, \text{ then } \chi_i(x) = \kappa_i(x) = \theta_i(N_{E/E_i}(\mu(x))), \\
& \quad \Lambda(\det_0(x)) = \Lambda(N_{E/K}(\mu(x))).
\end{align}

Next, suppose that \( m_i \neq \ell_i \) for some \( i \). Let \( H_i \) and \( H_i' \) be as in §6. By Lemma 3.2(i), given \( x \in H_0^i \), there exist \( y \in (E^x H_i)^{\overline{\rho}} = (E^x K_{i-1})^{\overline{\rho}} \) and \( z \in \mathcal{L}_{i-1} \) such that \( x = yz \). By definition of \( \kappa_i \) (see the beginning of §5),
\[ \chi_i(x) = \chi_i(y) \psi(\text{tr}(c_i(z - 1))). \]

Note that \( y'(z - 1)y'^{-1} \in (z - 1) + B_{\ell_i+x_i}(0) \subset (z - 1) + B_{f_E(\theta_i)N_{E/E_i}}(0) \), if \( y' \in P_{\ell_i}(0) \).

Now \( y \in E^x K_{i-1} = E^x K_i P_{\ell_i}(i-1) \), and \( c_i \) commutes with \( E^x K_i \), so
\[ \text{tr}(c_i(y(z - 1)y^{-1})) = \text{tr}(c_i(z - 1)). \]

By definition of \( \varphi \), \( \text{tr}(\varphi(X)) = \text{tr}(X) \), \( X \in g \). As \( x \) and \( y \) are \( \varphi \)-invariant, it follows that \( \varphi(z) = yz y^{-1} \). Thus, using Lemma 2.3(iii), we find
\[ \sigma(\text{tr}(c_i(z - 1))) = -\text{tr}(c_i(y(z - 1)y^{-1})) = -\text{tr}(c_i(z - 1)). \]

Combining this with \( \psi = \psi_0 \circ \text{tr}_{K/F} \) (see §2), results in \( \psi(\text{tr}(c_i(z - 1))) = 1 \). Thus \( \chi_i(x) = \chi_i(y) \). As \( y \in E^x H_i \), we may apply results of §6 to evaluate \( \chi_i(y) \).

Let \( \nu \) be as in §3. Note that \( \nu(z) = \nu(y) \) and \( \mu(x) = \mu(y) \). If \( y \) is conjugate to an element of \( E^x H_i \), then by Lemma 6.3 and Corollary 6.5,
\[ \chi_i(x) \text{ is a positive multiple of } \]
\begin{align}
(8.2) \quad & \begin{cases}
-1)^{\nu(\epsilon)} \theta_i(N_{E/E_i}(\mu(x))), & \text{if } c(\epsilon) \text{ is odd, } c(E_{i-1}/(E_{i-1} \cap L)) = 2, \\
\theta_i(N_{E/E_i}(\mu(x))), & \text{otherwise.}
\end{cases}
\end{align}

As shown in Lemma 6.6, the first case in (8.2) can occur if and only if \( c(E/K) \) is odd and \( c(K/F) = 2 \), and then it must occur for exactly one \( i, 1 \leq i \leq r \).

It follows from (8.1), (8.2), Lemma 6.6, and the definition of \( \kappa \) (see §5), that if \( x \in H_0^i \) and \( \chi_\kappa(x) \neq 0 \), then \( \chi_\kappa(x) \) is a positive multiple of
\[ \theta(\mu(x))(-1)^{\nu(\epsilon)}(c(K/F)-1)c(E/K). \]

In particular, if \( x \in (E^x P_{m_1}(r-1) \cdots P_{m_1}(0))^{\overline{\rho}} \), and \( \theta \) is as in (ii), then \( \chi_\kappa(x) > 0 \). Thus, by (4.2), \( \Phi_\kappa(h_k, f_k) > 0, k = 1, 2 \).

As in §4, we let \( G' \) and \( G'' \) be the \( F \)-rational points of the quasi-split unitary groups in \( 2n \) and \( 2n + 1 \) variables, respectively, defined with respect to \( K/F \). Recall that \( P' \) and \( P'' \) are parabolic subgroups of \( G' \) and \( G'' \), respectively, having Levi components isomorphic to \( G \) and \( G \times K^1 \), respectively. Given a character \( \xi \) of \( K^1 \), the supercuspidal representation \( \Pi_\xi \) of \( G \times K^1 \) is defined by \( \Pi_\xi(x, \alpha) = \pi(x) \xi(\det_0(x\alpha(x)) \alpha), x \in G, \alpha \in K^1 \). We can combine Theorem 8.1 and Goldberg’s reducibility criterion (Theorem 4.1) to obtain results concerning reducibility of the representations \( I(\pi) = \text{Ind}_{G''}^G(\pi \otimes 1) \), and \( I(\Pi_\xi) = \text{Ind}_{G''}^G(\Pi_\xi \otimes 1) \).
Theorem 8.3. Suppose that the admissible character $\theta$ satisfies (i) or (ii) of Theorem 8.1. Then $I(\pi)$ is irreducible and $I(\Pi_\xi)$ is reducible (for any $\xi$).

It is likely that the above conditions on $\theta$ are necessary and sufficient for irreducibility of $I(\pi)$ (equivalently, for reducibility of $I(\Pi_\xi)$). See §11 of [MR] for a discussion of the analogous situation for induced representations of split classical groups. In order to show sufficiency, it would be necessary to prove that $\Phi_\eta(h_1, f) = -\Phi_\eta(h_2, f)$ for all choices of matrix coefficients $f$ of $\pi$.

Conjecture 8.4. $I(\pi)$ is irreducible if and only if $\theta | L^\times$ satisfies

$$\theta | L^\times = \begin{cases} 1, & \text{if } f(E/L) = 1, \\ (-1)^{\text{ord}_E(\cdot)}(\epsilon(K/F) - 1)\epsilon(E/K), & \text{if } f(E/L) = 2. \end{cases}$$

Combining Theorem 8.3 and a result of Goldberg, we can get information about reducibility of representations induced from non-unitary supercuspidal representations of $G$ and of $G \times K^1$. Let $| \cdot |_K$ denote the normalized valuation on $K$. For $s$ a non-negative real number, set

$$I(s, \pi) = I(\pi \otimes |\det_0(\cdot)|^{s/2})$$

and

$$I(s, \Pi_\xi) = I(\Pi_\xi \otimes |\det_0(\cdot)|^{s}_K).$$

Corollary 8.5. Suppose that the admissible character $\theta$ satisfies (i) or (ii) of Theorem 8.1. Then $I(s, \pi)$ is reducible if and only if $s = 1$, and $I(s, \Pi_\xi)$ is irreducible for all $s > 0$.

Proof. By Theorem 8.3, $I(\pi) = I(0, \pi)$ is irreducible, and $I(\Pi_\xi) = I(0, \Pi_\xi)$ is reducible. The result then follows from Theorems 3.1 and 6.3 of [G2].

References


Department of Mathematics, University of Toronto, 100 St. George Street, Toronto, Canada, M5S 3G3

E-mail address: fiona@math.toronto.edu

Department of Mathematics, University of Toronto, 100 St. George Street, Toronto, Canada, M5S 3G3

E-mail address: repka@math.toronto.edu