GOLUBEV SERIES FOR SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. Let $P$ be an elliptic system with real analytic coefficients on an open set $X \subset \mathbb{R}^n$, and let $\Phi$ be a fundamental solution of $P$. Given a locally connected closed set $\sigma \subset X$, we fix some massive measure $m$ on $\sigma$. Here, a non-negative measure $m$ is called massive, if the conditions $s \subset \sigma$ and $m(s) = 0$ imply that $\sigma \setminus s = \sigma$. We prove that, if $f$ is a solution of the equation $Pf = 0$ in $X \setminus \sigma$, then for each relatively compact open subset $U$ of $X$ and every $1 < p < \infty$ there exist a solution $f_e$ of the equation in $U$ and a sequence $f_\alpha (\alpha \in \mathbb{N}_0^n)$ in $L^p(\sigma \cap U, m)$ satisfying $\|f_\alpha\|_{L^p(\sigma \cap U, m)} \to 0$ such that $f(x) = f_e(x) + \sum_{\alpha} \int_{\sigma \cap U} D^\alpha \Phi(x, y) f_\alpha(y) dm(y)$ for $x \in U \setminus \sigma$. This complements an earlier result of the second author on representation of solutions outside a compact subset of $X$.

1. Introduction and statement of the main results

1.1. Let $P$ be a $(k \times k)$-matrix of scalar partial differential operators with real analytic coefficients on an open set $X \subset \mathbb{R}^n$. Suppose further that $P$ has a fundamental solution $\Phi$ which is real analytic outside the diagonal $\Delta$ of $X \times X$. By definition, $\Phi(x, y)$ is a $(k \times k)$-matrix of distributions on $X \times X$ satisfying

$$
\begin{align*}
(P(x, D_x)) \Phi(x, y) &= \delta(x - y) I_k, \\
(P'(y, D_y)) \Phi(x, y) &= \delta(x - y) I_k
\end{align*}
$$

where $P'$ is the transposed operator to $P$, and $I_k$ is the identity $(k \times k)$-matrix.

Recall that, according to a theorem of Malgrange, every elliptic differential operator with real analytic coefficients on $X$ has a fundamental solution with the desired properties.

1.2. If $U$ is an open subset of $X$, then denote by $S_P(U)$ the vector space of all weak solutions of the system $Pf = 0$ on $U$. Note that because of the analytic hypoellipticity of $P$, the solutions in $S_P(U)$ are actually real analytic functions in $U$. For a closed subset $\sigma$ of $X$, solutions $f \in S_P(X \setminus \sigma)$ will be said to have singularities on $\sigma$.

In this article, we are interested in representations of solutions of the equation $Pf = 0$ in $X$ having singularities on a closed subset $\sigma$ of $X$. Before stating our principal result, we must first introduce one technical definition.

A (nonnegative) measure $m$ on $\sigma$ is said to be massive, if the two conditions $s \subset \sigma$ and $m(s) = 0$ imply that $\sigma \setminus s = \sigma$. In other words, every subset of $\sigma$ of
The local connectedness of the compact set $K$ we look at is a very delicate point in the literature. In fact it is related to the problem of extension of analytic functions on a neighborhood of $K$. (See Havin [8], Varfolomeev [17] and Rogers/Zame [12].)

In this paper, we prove the result by generalizing the ideas used in [15] in an appropriate way. Since the article [15] is in Russian and does not seem to be easily available, we have decided to present the paper in a self-contained way and do not use [15] as a reference.

1.4. The converse statement to Theorem 1.1 is quite easy to prove.

**Lemma 1.1.** Let $K$ be a relatively compact subset of $\sigma$, and $1 \leq p < \infty$. For every sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^k} \subset [L^p(K,m)]^k$, satisfying $\|\alpha!c_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$, the series $\sum_\alpha \int_K D^\alpha_y \Phi(x,y)c_\alpha(y)dm(y)$ converges for $x \in X \setminus K$ and defines an element in $S_P(X \setminus K)$. 

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**Example 1.1.** Let $\{y_j\}_{j \in \mathbb{N}}$ be a sequence of points of $K$, which is dense as a set in $\sigma$. Choose a sequence of positive numbers $\{\mu_j\}$ such that $\sum \mu_j < \infty$. For a set $s \subset \sigma$, we define $m(s) = \sum_{y_j \in s} \mu_j$. Then $m$ is a massive measure on $\sigma$.

Let us fix some massive measure $m$ on $\sigma$. Our main result is the following:

**Theorem 1.1.** Assume that $K$ is a locally connected compact subset of $\sigma$, and $1 < p < \infty$. Then for each solution $f \in S_P(X \setminus \sigma)$ there exist both a solution $f_\sigma \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and a sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^k} \subset [L^p(K,m)]^k$ such that

$$f(x) = f_\sigma(x) + \sum_{\alpha \in \mathbb{N}_0^k} \int_K D^\alpha_y \Phi(x,y)c_\alpha(y)dm(y)$$

holds for all $x \in X \setminus \sigma$. Furthermore, $\|\alpha!c_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$.

We emphasize that $\overset{\circ}{K}$ is the interior of $K$ on $\sigma$, i.e., in the induced topology of $\sigma$.
Proof. First note that for \( x \in X \setminus K \) we have
\[
P(x, D) \int_K D_y^a \Phi(x, y)c_\alpha(y)dm = \int_K D_y^a \{ P(x, D)\Phi(x, y)\}c_\alpha(y)dm = 0.
\]
Thus the proof will be complete if we show that the series we look at converges uniformly on compact subsets of \( X \setminus K \). It is well-known that a \( C^\infty \) function \( g \) on an open set \( U \subset \mathbb{R}^n \) is real analytic if and only if for every compact set \( K \subset U \) there are constants \( a = a(g, K) \) and \( c = c(g, K) \) such that
\[
sup_{y \in K} |D^a g(y)| \leq c \cdot a^{|\alpha|} |\alpha|! \text{ for all } \alpha \in \mathbb{N}_0^n.
\]
Now fix a compact set \( \tilde{K} \subset X \setminus K \). Since the fundamental solution \( \Phi \) is real analytic in a neighborhood of \( \tilde{K} \times K \), there exist constants \( a \) and \( c \), depending on \( \Phi \) and \( \tilde{K} \), such that
\[
\sup_{(x, y) \in \tilde{K} \times K} \| D^a_\Phi(x, y) \| \leq c \cdot a^{|\alpha|} |\alpha|! \text{ for all } \alpha \in \mathbb{N}_0^n.
\]
Using (2), for \( \alpha \in \mathbb{N}_0^n \) we get
\[
\sup_{x \in K} \int_K |D^a_\Phi(x, y)c_\alpha(y)dm(y)| \leq c \cdot a^{|\alpha|} |\alpha|! \int_K |c_\alpha(y)dm(y) |
\leq c \cdot a^{|\alpha|} |\alpha|! \| c_\alpha \|_{L^q(K, m)}m(K)^{1/2},
\]
with \( p^{-1} + q^{-1} = 1 \). Therefore
\[
\sum_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \int_K |D^a_\Phi(x, y)c_\alpha(y)dm| \leq c \cdot m(K)^{1/2} \sum_{\alpha \in \mathbb{N}_0^n} a^{|\alpha|} |\alpha|! \| c_\alpha \|_{L^p(K, m)}
= c \cdot m(K)^{1/2} \sum_{j=0}^{\infty} a^j \sup_{|\alpha| = j} |\alpha|! \| c_\alpha \|_{L^p(K, m)} \left( \sum_{|\alpha| = j} \frac{|\alpha|!}{|\alpha|!} \right)
= c \cdot m(K)^{1/2} \sum_{j=0}^{\infty} (a \cdot n) \| c_\alpha \|_{L^p(K, m)}^{1/|\alpha|} j,
\]
where we used that \( \sum_{|\alpha| = j} \frac{|\alpha|!}{|\alpha|!} = n^j \), \( n = \dim \mathbb{R}^n \).

Now, since \( \sup_{|\alpha| = j} |\alpha|! c_\alpha \|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0 \) when \( j \rightarrow \infty \), the last sum can be majorized by a geometric sum. Hence
\[
\sum_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \int_K |D^a_\Phi(x, y)c_\alpha(y)dm| \leq c(K, \tilde{K}) < \infty.
\]
\[\square\]

1.5. Let us distinguish the principal difficulty in the proof of Theorem 1.1.

Lemma 1.2. Let \( K \) be a locally connected compact subset of \( X \), \( m \) be a massive measure on \( K \) and \( 1 < p < \infty \). Then for every solution \( f \in S_p(X \setminus K) \) there are a solution \( f_\epsilon \in S_p(X) \) and a sequence \( \{ c_\alpha \}_{\alpha \in \mathbb{N}_0^n} \subset [L^p(K, m)]^k \) such that
\[
f(x) = f_\epsilon(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_K D^a_\Phi(x, y)c_\alpha(y)dm(y)
\]
holds for all \( x \in X \setminus K \). Furthermore, \( |\alpha| c_\alpha \|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0 \) when \( |\alpha| \rightarrow \infty \).
As Baernstein showed in [2], even for $P = \partial/\partial x$ Lemma 1.2 is false for arbitrary compact $K$.

We now turn to the proof of Theorem 1.1. Let $U \subset X$ be a relatively compact open set such that $U \cap \sigma = \hat{K}$ and the set $K' = \partial U \cup \hat{K}$ is locally connected. Fix some massive measure $m'$ on $K'$ whose restriction to $K$ is $m$. The existence of such a measure follows from Example 1.1. Given a solution $f \in S_{P}(X \setminus \sigma)$, we consider the function $f'$ which equals $f$ in $U \setminus \sigma$ and is $0$ in $X \setminus \hat{U}$. Then $f'$ is a solution of the system $Pf' = 0$ with singularities on $K'$. Hence by Lemma 1.2 there exist a solution $f'_e \in S_{P}(X)$ and a sequence $\{c'_\alpha\}_{\alpha \in \mathbb{N}_0} \subset \{L^{p}(K', m')\}^{k}$, satisfying $\|\alpha!c'_\alpha\|_{L^{p}(K', m')} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, such that

$$f'(x) = f'_e(x) + \sum_{\alpha \in \mathbb{N}_0} \int_{K'} D^\alpha \Phi(x, y) c'_\alpha(y) dm'(y) \quad (x \in X \setminus K').$$

Set $c_\alpha := c'_\alpha |_{K}$, $\alpha \in \mathbb{N}_0$. Since $\|\alpha!c_\alpha\|_{L^{p}(K', m')} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, the function $f_e$ defined by

$$f_e(x) = f(x) - \sum_{\alpha \in \mathbb{N}_0} \int_{K} D^\alpha \Phi(x, y) c_\alpha(y) dm(y) \quad (x \in X \setminus \sigma)$$

belongs to $S_{P}(X \setminus \sigma)$ because of Lemma 1.1. Moreover, this function satisfies the equation $Pf_e = 0$ also in a neighborhood of each interior point of $K$, since we have

$$f_e(x) = f'_e(x) + \sum_{\alpha \in \mathbb{N}_0} \int_{K' \setminus K} D^\alpha \Phi(x, y) c'_\alpha(y) dm'(y) \quad \text{for } x \in U \setminus \sigma.$$ 

Thus $f_e \in S_{P}(\{(X \setminus \sigma) \cup \hat{K}\})$, as was to be proved. \hfill $\square$

The proof of Lemma 1.2 needs some preparation which we give in the following section by studying more thoroughly the topology on $S_{P}(K)$. For the sake of simplicity, we restrict the following considerations to the case $k = 1$.

2. Equivalent topologies in $S_{P}(K)$

2.1. Let $K$ be any compact set in $X$. In this section, we study various topologies on $S_{P}(K)$, where $P^\prime$ is the transposed operator to $P$. Define the space $S_{P}(K)$ as follows. The function $g$ belongs to $S_{P}(K)$ if there exists an open set $U \supset K$ such that $g$ is a solution of the equation $P^\prime g = 0$ in $U$. If two such functions agree on some neighborhood of $K$, we identify them as elements in $S_{P}(K)$.

For each $U$ as above, let $S_{P}(U)$ denote the space of solutions of the equation $P^\prime g = 0$ in $U$ with the topology of uniform convergence on compact subsets, i.e., the topology induced from $C(U)$. There is a natural map from $S_{P}(U)$ into $S_{P}(K)$, and we endow $S_{P}(K)$ with the finest locally convex topology for which all these maps are continuous. We denote this topology by $\tau$. Alternatively, the space $(S_{P}(K), \tau)$ can be described as the inductive limit of the spaces $S_{P}(U_{\nu})$, where $\{U_{\nu}\}$ is any decreasing sequence of open sets containing $K$ such that each neighborhood of $K$ contains some $U_{\nu}$, and such that each component of each $U_{\nu}$ meets $U_{\nu+1}$. 


Remark 2.1. The space \((S_p(K), \tau)\) is separated, a subset of this space is bounded iff it is contained and bounded in some \(S_p(U_p)\), and each closed bounded subset is compact. Proofs could be given by the same methods as in Koethe [9], p.379.

2.2. We will embed \(S_p(K)\) algebraically in a space \(L^q\) whose topological dual consists of sequences of functions from \(L^p(K, m)\). Lemma 1.2 follows from the Hahn-Banach Theorem once we show that the topology of \(L^q\) restricted to \(S_p(K)\) is finer than the topology \(\tau\). To do this, we have first to study some Banach spaces.

Definition 2.1. Given positive numbers \(q\) and \(r\), the space \(l^q(r)\) is defined to consist of all sequences \(\{\eta_\alpha\}_{\alpha \in \mathbb{N}_0^q} \subseteq \mathbb{C}\) with \((\sum_{\alpha \in \mathbb{N}_0^q} |\eta_\alpha|^q)^{1/q} < \infty\).

If \(K\) is an arbitrary compact subset of \(X\) and \(m\) is an arbitrary measure on \(K\), then we denote by \(l^q(r)^K\) the space of all functions \(\eta(\cdot) = \{\eta_\alpha(\cdot)\}_{\alpha \in \mathbb{N}_0^q}\) on \(K\) with values in \(l^q(r)\) such that \(\eta_\alpha(\cdot) \in L^q(K, m)\) for every \(\alpha \in \mathbb{N}_0^q\) and

\[
\left(\sum_{\alpha \in \mathbb{N}_0^q} \|\eta_\alpha\|_{L^q(K, m)}^q r^{q|\alpha|}\right)^{1/q} < \infty.
\]

Lemma 2.1. For \(q \in [1, \infty]\), the functional

\[
\|\{\eta_\alpha\}\|_{l^q(r)^K} = \left(\sum_{\alpha \in \mathbb{N}_0^q} \|\eta_\alpha\|_{L^q(K, m)}^q r^{q|\alpha|}\right)^{1/q}
\]

defines a norm on \(l^q(r)^K\).

Proof. The proof is an easy exercise from functional analysis. \(\square\)

Equipped with the norm (1), the space \(l^q(r)^K\) is a Banach space, provided \(q \in [1, \infty]\). Instead of proving this directly, we proceed by the following

Lemma 2.2. Let \(r > 0, q \geq 1\) be arbitrary real numbers, and let \(p \in [1, \infty]\) be the conjugate exponent to \(q\). We have an isometrical isomorphism

\[
(l^q(r)^K)' \cong l^p\left(\frac{1}{r}\right)^K.
\]

Proof. Assume that \(q > 1\). Fix some \(\theta = \{\theta_\alpha\}_{\alpha \in \mathbb{N}_0^q} \in l^p\left(\frac{1}{r}\right)^K\). Then \(\theta\) defines a linear functional on \(l^q(r)^K\) via \(\langle \theta, \eta \rangle = \sum_{\alpha \in \mathbb{N}_0^q} \int_K \theta_\alpha(y) \eta_\alpha(y) dm(y)\), for \(\eta = \{\eta_\alpha\} \in l^q(r)^K\). Since

\[
|\langle \theta, \eta \rangle| \leq \sum_{\alpha \in \mathbb{N}_0^q} \left(\|\theta_\alpha\|_{L^p(K, m)} r^{-1|\alpha|}\right) \left(\|\eta_\alpha\|_{L^q(K, m)} r^{1|\alpha|}\right)
\]

(2) \[
\leq \|\theta\|_{l^p(\frac{1}{r})^K} \cdot \|\eta\|_{l^q(r)^K},
\]

this functional is continuous. Conversely, let \(F \in (l^q(r)^K)\). Given a multi-index \(\alpha \in \mathbb{N}_0^q\), denote by \(e_\alpha\) the element in \(l^q(r)\) which is 1 in the \(\alpha\)-th entry and 0 in all other entries. On \(L^q(K, m)\), we may define a functional by juxtaposition \(g \mapsto F(g e_\alpha)\) for \(g \in L^q(K, m)\). Since \(F\) is continuous, this functional is continuous, too. By duality, there is a function \(\theta_\alpha \in L^p(K, m)\) such that \(F(g e_\alpha) = \int_K \langle \theta_\alpha(y), g(y) \rangle dm(y)\) for all \(g \in L^q(K, m)\). Since for an element \(\eta = \{\eta_\alpha\} \in l^q(r)^K\) we have \(\eta = \sum_{\alpha \in \mathbb{N}_0^q} \eta_\alpha e_\alpha\) and the series converges in the norm of \(l^q(r)^K\), it follows that

\[
F(\eta) = \sum_{\alpha \in \mathbb{N}_0^q} F(\eta_\alpha e_\alpha) = \sum_{\alpha \in \mathbb{N}_0^q} \int_K \langle \theta_\alpha(y), \eta_\alpha(y) \rangle dm(y).
\]
Put $\theta := \{\theta_\alpha\}_{\alpha \in \mathbb{N}_0}$. To complete the proof, it remains to show that $\theta$ is in $l^p(\frac{1}{r})^K$.

To this end, we consider the sequence $\{\eta_\alpha\}_{\alpha \in \mathbb{N}_0}$ of measurable functions on $K$ given by

$$
\eta_\alpha := \begin{cases}
|\theta_\alpha|^{p-2} \theta_\alpha \frac{r^{-p}|\alpha|}{r}, & \theta_\alpha \neq 0, \\
0, & \theta_\alpha = 0.
\end{cases}
$$

Since $|\eta_\alpha|^q = |\theta_\alpha|^p r^{-p|\alpha|}$ each function $\eta_\alpha(\cdot)$ is in $L^q(K, m)$. Hence it follows

$$
\sum_{|\alpha| \leq N} \|\theta_\alpha\|^p_{L^p(K, m)} \left(\frac{1}{r}\right)^{p|\alpha|} = \|F\left( \sum_{|\alpha| \leq N} \eta_\alpha e_\alpha \right)\| \leq \|F\|_{L^{q}(r^\alpha K)} \sum_{|\alpha| \leq N} \eta_\alpha e_\alpha \|\nu(r)^{K}\|
$$

$$
= \|F\|_{L^{q}(r^\alpha K)} \left( \sum_{|\alpha| \leq N} r^{-p|\alpha|} \|\theta_\alpha\|^p_{L^p(K, m)} \right)^{1/q},
$$

Thus $\left( \sum_{|\alpha| \leq N} r^{-p|\alpha|} \|\theta_\alpha\|^p_{L^p(K, m)} \right)^{1/p} \leq \|F\|_{L^{q}(r^\alpha K)}$, for every positive integer $N$.

Together with (2) it follows that

$$
\|\theta\|_{l^p(\frac{1}{r})^K} = \|F\|_{L^{q}(r^\alpha K)},
$$

as was to be proved.

For $q = 1$, the proof follows the same lines with the obvious modifications. \(\square\)

Since the dual space to a normed space is a Banach space, Lemma 2.2 implies the following

**Corollary 2.1.** Let $r > 0$ and $q > 1$. Then $l^q(r)^K$ is a reflexive Banach space.

2.3. Note that if $r' > r'' > 0$, we have a continuous embedding $l^q(r')^K \hookrightarrow l^q(r''^K)$.

Now let $\{r_\nu\}_{\nu \in \mathbb{N}}$ be some decreasing sequence of positive numbers tending to zero. The space $L^{(q)}$ is defined to be the inductive limit of the spaces $l^q(r_\nu)^K$. The space $L^{(q)}$ is separated. Each bounded set is contained and bounded in one of the $l^q(r_\nu)^K$. Moreover, $L^{(q)}$ is a (DF)-space, because it is the separated inductive limit of a sequence of normed, hence (DF)-, spaces (see Théorème 9 of Grothendieck [6]).

Our aim is to show that $S_{r'}(K)$ is topologically isomorphic to a subspace of $L^{(q)}$.

Thus we proceed by constructing an embedding $S_{r'}(K) \hookrightarrow L^{(q)}$. More precisely, for each solution $g \in S_{r'}(K)$ we define

$$
\left(\frac{D^\alpha g}{\alpha!} \right)_{\alpha \in \mathbb{N}_0}.
$$

**Lemma 2.3.** For every $g \in S_{r'}(K)$, the sequence $j(g)$ is in $L^{(q)}$, and the mapping $j : S_{r'}(K) \hookrightarrow L^{(q)}$ is continuous and injective.

**Proof.** Let $g \in S_{r'}(K)$. Then there is a neighborhood $U$ of $K$ in $X$ such that $g \in S_{r'}(U)$. Now choose a function $\varphi \in \mathcal{D}(X)$ which is equal to 1 in a neighborhood of $K$. Since $\Phi$ is a left fundamental solution of $P$, we get $g = \Phi P'(\varphi g)$ in a neighborhood of $K$.

The function $P'(\varphi g)$ is supported by the closure of the set of those points $x \in U$ such that $\nabla \varphi(x) \neq 0$. Let us denote this closure by $\sigma$. Then $\sigma$ is a compact subset of $U \setminus K$, so there is a function $\psi \in \mathcal{D}(U \setminus K)$ which equals 1 in a neighborhood of $\sigma$.

Since $P'(\varphi g) = \psi P'(\varphi g)$, we have $g = \Phi^j(\psi P'(\varphi g))$ in a neighborhood of $K$.

Hence it follows for each multi-index $\alpha$ that

$$
D^\alpha g(y) = \int P(x, D)(\psi(x)D^\alpha_\psi \Phi(x, y)) \cdot (\varphi(x)g(x))dx \quad (y \in K).
$$
Using estimate (2) with \( \tilde{K} = \text{supp} \, \psi \), we get
\[
\sup_{y \in K} |D^\alpha g(y)| \leq c' a^{|\alpha| + \text{order} P} (|\alpha| + \text{order} P)! \sup_{x \in \text{supp} \varphi} |g(x)|
\leq c'' (a')^{|\alpha|} |\alpha|! \sup_{x \in \text{supp} \varphi} |g(x)|,
\]
where \( a' \) is any number larger than \( a \), and the constant \( c'' \) does not depend on \( g \in S_{P'}(U) \) and \( \alpha \). It now follows that
\[
\sum_{\alpha \in \mathbb{N}_0^n} \frac{|D^\alpha g|}{\alpha!} \|L_{s(K,m)}^{q|x|}\|_{\nu}^{q|\alpha|} \leq (c'' q m(K)) \left( \sum_{\alpha \in \mathbb{N}_0^n} ((a')^{|\alpha|} r_{\nu}^{|\alpha|} |\alpha|! \alpha!^{q}) \sup_{x \in \text{supp} \varphi} |g(x)| \right)
= (c'' q m(K)) \left( \sum_{j=0}^{\infty} (n a' r_{\nu})^{qj} \right) \sup_{x \in \text{supp} \varphi} |g(x)|.
\]

Choose \( \nu_0 \) large enough, such that \( n a r_{\nu_0} < 1 \). Then (4) shows that \( j(g) \in l^q(r_{\nu_0})^K \) as well as the continuity of the mapping \( j : S_{P'}(U) \to l^q(r_{\nu_0})^K \).

Since a linear operator from \( S_{P'}(K) \) into a locally convex space is continuous if and only if its restriction to each \( S_{P'}(U) \) is continuous (for a proof cf. Bourbaki [3]), it follows that the mapping \( j : S_{P'}(K) \to L^q \) is continuous.

To show that \( j \) is injective let \( g \in S_{P'}(K) \) be such that \( j(g) = 0 \). This means that \( D^\alpha g |_{K} \equiv 0 \) in \( K \) for all \( \alpha \in \mathbb{N}_0^n \), and hence, since \( g \) is real analytic, it follows \( g \equiv 0 \) in a neighborhood of \( K \).

2.4. Now put
\[ S^{(q)}_{P'} := j(S_{P'}(K)) \subseteq L^q. \]
We endow this space with the topology induced by \( L^q \). We want to show

**Lemma 2.4.** Let \( K \) be a locally connected compact subset of \( X \), and \( q > 1 \). Then \( S^{(q)}_{P'} \) is a closed subspace of \( L^q \).

For the proof of Lemma 2.4 we shall use the following result:

**Lemma 2.5.** Assume that \( \{ L_{\nu} \} \) is a sequence of reflexive Banach spaces, such that \( L_{\nu} \) is continuously embedded in \( L_{\nu+1} \) for all \( \nu \), and \( L \) is the inductive limit of the sequence. Then a vector subspace \( \Sigma \) of \( L \) is closed if and only if for all \( \nu \) the intersection \( \Sigma \cap L_{\nu} \) is closed in \( L_{\nu} \).

**Proof.** See Makarov [11].

**Proof (of Lemma 2.4).** Using Lemma 2.5 it is sufficient to show that for each \( \nu \) the subspace \( S^{(q)}_{P'} \cap (l^q(r_{\nu})^K) \) is closed in \( l^q(r_{\nu})^K \).

Assume that for a solution \( g \in S_{P'}(K) \) the image \( j(g) \) is in \( l^q(r_{\nu})^K \). Then for all points \( y \in K \), except perhaps for a set of zero measure \( m \), we have
\[
\left( \sum_{\alpha \in \mathbb{N}_0^n} \frac{|D^\alpha g(y)|}{\alpha!} |q_{\nu} q^{(|\alpha|)} r_{\nu}^{1/q}\right) < \infty.
\]

Since the measure \( m \) is supposed to be massive, this inequality holds for a set \( \sigma_g \) of points \( y \in K \) which is dense in \( K \). So
\[
\limsup_{|\alpha| \to \infty} \frac{|D^\alpha g(y)|}{\alpha!} |1/|\alpha| | \leq \frac{1}{r_{\nu}} \quad \text{for all } y \in \sigma_g.
\]
We shall construct a complex neighborhood $U_\nu$ of $K$ into which all the elements of $j^{-1}(l^q(r_\nu)^K)$ have (single valued) holomorphic extensions. This is the only place where we use the local connectedness of $K$.

For each $y \in K$ choose a neighborhood $O_y$ in $\mathbb{C}^n$ such that $O_y \subset \Delta(y, r_\nu)$ and such that $K \cap O_y$ is connected. This is possible, since $K$ is assumed to be locally connected. Here $\Delta(y, r) = \{z \in \mathbb{C}^n : |z_i - y_i| < r \ (i = 1, \ldots, n)\}$ is the polydisk in $\mathbb{C}^n$ with center $y$ and radius $r$. Choose $r_\nu$ such that $\Delta(y, 2r_\nu) \subset O_y$. Define $U_\nu = \bigcup_{y \in K} \Delta(y, r_\nu)$. Then $U_\nu$ is a neighborhood of $K$ in $\mathbb{C}^n$.

Let $g \in j^{-1}(l^q(r_\nu)^K)$ and $z \in U_\nu$. Define $\tilde{g}(z) = \sum_\alpha \frac{D^\alpha g(y)}{\alpha!} (z - y)^\alpha$ where $y$ is any point of $\sigma_\alpha$ such that $z \in \Delta(y, r_\nu)$. The series converges, since $|z_i - y_i| < \frac{1}{2r_\nu}$ for all $i = 1, \ldots, n$. We have to show that $\tilde{g}(z)$ does not depend on $y$.

Suppose that $z \in \Delta(y', r_{y'}) \cap \Delta(y'', r_{y''})$, where $y', y'' \in \sigma_\alpha$. Let $r_{y'''} \leq r_{y''}$. Then $|y'_i - y'_i| < r_{y'''} + r_{y''} \leq 2r_{y''}$ for all $i = 1, \ldots, n$; hence $y'' \in \Delta(y', 2r_{y''}) \subset O_{y''}$. We conclude that both $y'$ and $y''$ belong to the connected set $K \cap O_{y''}$. Let $U$ be an open set in $\mathbb{C}^n$ containing $K$, into which $g$ has a (single valued) holomorphic extension. Then $K \cap O_{y''} \subset U \cap \Delta(y', r_{y''})$, and we denote by $O$ the component of the set on the right which contains $y'$. Obviously, $y''$ is in $O$, too. The equation $g(z) = \sum_\alpha \frac{D^\alpha g(y)}{\alpha!} (z - y)^\alpha$ is valid for all $z \in O$. Hence the series

$$\tilde{g}(z) = \sum_\alpha \frac{D^\alpha g(y'')}{\alpha!} (z - y'')^\alpha$$

is a rearrangement of the series

$$g(z) = \sum_\alpha \frac{D^\alpha g(y')}{\alpha!} (z - y')^\alpha$$

about $y'$, and uniqueness of $\tilde{g}(z)$ follows.

It is obvious that $\tilde{g}$ is holomorphic in $U_\nu$. Moreover, it is easily verified that $\tilde{g}$ and $g$ agree on $U_\nu \cap U$. We may assume that the coefficients of the differential operator $P$ have holomorphic extensions to $U_\nu$. Then $P^* \tilde{g} \equiv 0$ in $U_\nu$, since the function $P^* \tilde{g}$ is holomorphic in $U_\nu$ and vanishes on an open subset of each component of $U_\nu$.

Thus every solution $g \in j^{-1}(l^q(r_\nu)^K)$ has a (single valued) extension to the complex neighborhood $U_\nu$ of $K$. Now, let $\{\eta^{(j)}\}$ be a sequence in $S^q_{\nu} \cap l^q(r_\nu)^K$ which converges to an element $\eta = \{\eta_\alpha\}$ in $l^q(r_\nu)^K$. We would like to prove that $\eta$ is in $S^q_{\nu} \cap l^q(r_\nu)^K$, too. By definition of $S^q_{\nu}$, for every $j = 1, 2, \ldots$ there is a $g_j \in S^q_{\nu}(K)$ such that $\eta^{(j)}_\alpha = \frac{D^\alpha g_j}{\alpha!} |_K \ (\alpha \in \mathbb{N}_0^n)$. Moreover, as was already proved, each element $g_j$ is represented by a holomorphic function $g_j(z)$ in the complex neighborhood $U_\nu$ of $K$ satisfying $P^* g_j = 0$ there.

The convergence $\eta^{(j)} \to \eta$ in $l^q(r_\nu)^K$ means that

$$\lim_{j \to \infty} \left( \int_K \sum_{\alpha \in \mathbb{N}_0^n} r^q_{\nu} |D^\alpha g_j(y)|^{1/q} dm(y) \right)^{1/q} = 0.$$

Hence it follows that there exists a subsequence $\{g_{j_k}\}$ such that for all points $y \in K$, except for a set of zero measure $m$, we have

$$\lim_{j_k \to \infty} \left( \sum_{\alpha \in \mathbb{N}_0^n} r^q_{\nu} |D^\alpha g_{j_k}(y)|^{1/q} \right) = 0.$$
Since the measure $m$ is massive, equality holds for a set $\sigma$ of points $y \in K$ which is dense in $K$. We now use compactness of $K$ to conclude the following. There are a finite number of points $y^{(1)}, \ldots, y^{(n)}$ in $\sigma$ and a positive $r < r_\nu$ such that $K$ is contained in the union $U = \Delta(y^{(1)}, r) \cup \ldots \cup \Delta(y^{(n)}, r)$, and $U \subset U_\nu$. Our purpose is to show that the sequence $\{g_{y,r} \}$ converges to some function $g$ in $S_{P^*}(U)$. Since the space $S_{P^*}(U)$ is complete, it suffices to prove that this sequence is a Cauchy sequence in $S_{P^*}(U)$, i.e., in each of the spaces $C(k)$, where $k$ is a compact subset of $U$. Obviously, we may restrict ourselves to compact sets $k$ lying in one of the polydisks $\Delta(y^{(1)}, r), \ldots, \Delta(y^{(n)}, r)$.

Let $k$ be a compact subset of $\Delta(y, r)$ where $\Delta(y, r)$ is one of the polydisks previously mentioned. Denote by $d$ the distance from $k$ to the $n$-skeleton of $\Delta(y, r)$, i.e., $\partial_k \Delta(y, r) = \{ \zeta \in \mathbb{C}^n : |\zeta - y_i| = r \ (i = 1, \ldots, n) \}$. The distance is taken in the polydisk-norm.

We may regard some branch of $(g_{y,r}(z) - g_{y,r}(z))^q$ in $\Delta(y, r)$ to yield a holomorphic function there. By Cauchy’s Theorem we have for all $z \in \Delta(y, r)$:

\[
(6) \quad (g_{y,r}(z) - g_{y,r}(z))^q = \frac{1}{(2\pi)^n} \int_{\partial_k \Delta(y, r)} (g_{y,r}^{(\zeta)}(\zeta) - g_{y,r}^{(\zeta)}(\zeta))^q d\zeta_1 \land \ldots \land d\zeta_n.
\]

The Taylor-series expansion for $(g_{y,r}^{(\zeta)}(\zeta) - g_{y,r}^{(\zeta)}(\zeta))$, centered at $y$, converges uniformly in the closure of $\Delta(y, r)$. So (6) implies for $z \in k$:

\[
|g_{y,r}(z) - g_{y,r}(z)| \leq \left( \frac{1}{(2\pi)^n} \int_{\partial_k \Delta(y, r)} |g_{y,r}^{(\zeta)}(\zeta) - g_{y,r}^{(\zeta)}(\zeta)||d\zeta_1| \land \ldots \land |d\zeta_n| \right)^{1/q}.
\]

Using Hölder’s inequality and taking into account that $r < r_\nu$, we get

\[
\sup_{z \in k} |g_{y,r}(z) - g_{y,r}(z)|
\]

\[
\leq \left( \frac{1}{(2\pi)^n} \int_{\partial_k \Delta(y, r)} \left( \sum_{\alpha \in \mathbb{N}^n} \left| \frac{D^\alpha (g_{y,r}(z) - g_{y,r}(z))}{\alpha!} \right| \frac{|d\zeta_1| \land \ldots \land |d\zeta_n|}{r_\nu^{\alpha|q|}} \right)^{1/q}.
\]

\[
= \left( \frac{r}{d} \right)^n/q \left( \sum_{\alpha \in \mathbb{N}^n} \frac{r_\nu^{\alpha|q|}}{\alpha!} \left( \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha g_{y,r}(z) - D^\alpha g_{y,r}(z)}{\alpha!} \right)^{1/q}.
\]

\[
\leq \left( \frac{r}{d} \right)^n/q \left( \sum_{\alpha \in \mathbb{N}^n} \frac{r_\nu^{\alpha|q|}}{\alpha!} \left( \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha g_{y,r}(z) - D^\alpha g_{y,r}(z)}{\alpha!} - \eta_\alpha(y) |q\right)^{1/q
\}
\]

\[
+ \left( \sum_{\alpha \in \mathbb{N}^n} \frac{r_\nu^{\alpha|q|}}{\alpha!} \left( \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha g_{y,r}(z) - D^\alpha g_{y,r}(z)}{\alpha!} - \eta_\alpha(y) |q\right)^{1/q
\}.
\]

By (5) it follows that $\sup_{z \in k} |g_{y,r}(z) - g_{y,r}(z)| \to 0$ when both $j_x$ and $j_t$ tend to infinity. This is just what we wanted to prove. Thus, there is a solution $g \in S_{P^*}(U)$ such that $g_{y,r} \to g$ in $S_{P^*}(U)$. Because of Lemma 2.3, we obtain $\eta = j(g)$. Hence $\eta \in S_{P^*}(U)$, as was to be proved.

The main result of this section consists of the following.
Theorem 2.1. Assume that $K$ is a locally connected compact subset of $X$, and $q > 1$. Then the mapping $j^{-1}: S^{(q)}_P \to S_P(K)$ is continuous.

Proof. The assertion follows from Lemma 2.4 and a version of the Open Mapping Theorem, but we prefer the direct proof. As was already mentioned, the mapping $j^{-1}: S^{(q)}_P \to S_P(K)$ is continuous, if each restriction $j^{-1}: S^{(q)}_P \cap L^q(r_\nu) \to K$ is continuous (see Bourbaki [3]). Let $\{g_j\}$ be a sequence of $S_P(K)$ such that the sequence $\{D^\alpha g_j\}_{\alpha \in \mathbb{N}^q}$ converges to zero in $L^q(r_\nu)^K$. By the same way as we proceeded in the proof of Lemma 2.4, we find a complex neighborhood $U_\nu$ of $K$ such that every element $g_j$ is represented by a holomorphic function $g_j(z)$ in $U_\nu$, satisfying $P^q g_j = 0$ there.

Choose a positive $r < r_\nu$ such that the set $U = \bigcup_{y \in K} \Delta(y, r)$ is contained in $U_\nu$ together with its closure. Then we claim that $\{g_j\}$ tends to zero uniformly on compact subsets of $U$. In fact, otherwise there would exist a compact set $K \subset U$, an $\varepsilon > 0$ and a subsequence $\{g_{j_n}\}$ such that $\sup_{z \in K} |g_{j_n}(z)| \geq \varepsilon$ for all $j_n$. But then it follows just in the same way as in the proof of Lemma 2.4 that some subsequence of $\{g_{j_n}\}$ should tend to zero uniformly on compact subsets of $U$. This contradiction implies our statement. Hence $g_j \to 0$ in $S_P(K)$, as was to be proved. □

Combining Theorem 2.1 and Lemma 2.3, we obtain the

Corollary 2.2. Under the conditions of Theorem 2.1, the mapping $j: S_P(K) \to S^{(q)}_P$ is a topological isomorphism of the space $(S_P(K), \tau)$ onto the space $S^{(q)}_P$, equipped with the topology induced by $L^{(q)}$.

3. Proof of the main lemma and remarks

3.1. In order to prove Lemma 1.2, we shall use the fact that each solution $f \in S_P(X \setminus K)$ may be written as the sum of a solution in $S_P(X)$ and a solution in $S_P(X \setminus K)$ which is regular at infinity. The latter notion can be introduced as follows:

Denote by $\hat{X}$ the one point compactification of $X$, i.e., the union of $X$ and the symbolic point $\infty$. The topology in $\hat{X}$ is defined by the following system of neighborhoods: If $x \in X$, then we take the usual neighborhood basis, and if $x = \infty$, then we take the family of complements of all compact subsets in $X$. Let $U$ be a neighbourhood of $\infty$. A function $f \in S_P(U)$ which has the representation (in a neighborhood of $\infty$, possibly smaller than $U$) $f = \Phi(F)$, for some distribution $F$ with compact support, in $K$, is called regular at infinity. Here $\Phi(F)$ is the value of the pseudo-differential operator $\Phi$ on $F$. For smooth functions $F$ with compact support $\Phi(F)$ is defined by $\Phi(F) = \int_{\mathbb{R}^n} \Phi(\cdot, y) F(y) dy$. For distributions $F$ with compact support, $\Phi(F)$ is defined by duality.

Of course, this notion depends on our particular choice of the fundamental solution $\Phi$, while the space of solutions regular at infinity does not depend on $\Phi$ on the whole.

Let us denote by $S^{(r)}_P(X \setminus K)$ the subspace of $S_P(X \setminus K)$ consisting of the solutions regular at infinity.

Lemma 3.1. For each compact set $K \subset X$, it follows that

$$S_P(X \setminus K) = S_P(X) \oplus S^{(r)}_P(X \setminus K).$$

The sum on the right is topological.
Proof. Let $G_P$ be a Green operator for $P$, i.e., a bidifferential operator of order $\text{ord}(P) = 1$ on $X$ with the property that $dG_P(g, f) = \langle (g, Pf) \rangle_x - \langle P^Tg, f \rangle_x$ for all $g$ and $f$, which are smooth enough in $X$. Here $dx = dx_1 \wedge \ldots \wedge dx_n$. Given a solution $f \in S_P(X \setminus K)$, we define the functions $f_e$ and $f_r$ in the following way. Let $x \in X$. Choose an open set $U \subset X$ with piecewise smooth boundary such that $K \subset U$ and $x \in U$. Set $f_e(x) = -\int_{\partial U} G_P(\Phi(x, \cdot), f)$. It follows from the Green formula that $f_e(x)$ does not depend on the particular choice of $U$. Obviously, $f_e \in S_P(X)$. Now let $x \in X \setminus K$. Let $U \subset X$ be an open set with piecewise smooth boundary such that $K \subset U$ and $x \notin U$. Set $f_r(x) = \int_{\partial U} G_P(\Phi(x, \cdot), f)$. Again, $f_r$ does not depend on the choice of $U$. It is clear that $f_r \in S_P^r(K \setminus K)$. By the Green formula we get $f = f_e + f_r$. The rest of the proof is obvious.

Thus, every solution $f \in S_P(X \setminus K)$ may be written in the form $f = f_e + f_r$, with $f_e \in S_P(X)$ and $f_r \in S_P^r(K \setminus K)$, and this representation is unique.

3.2. Given a solution $f \in S_P(X \setminus K)$, we define a linear functional $F_f$ on $S_P^r(K)$ as follows. Let $g \in S_P^r(K)$. This means that there is a neighborhood $U$ of $K$ such that $g \in S_P^r(U)$. Choose a new neighborhood $U_g$ of $K$ such that $U_g \subset U$ and the boundary of $U_g$ is piecewise smooth. Put

$$
(1) \quad \langle F_f, g \rangle = \int_{\partial U_g} G_P(g, f) \quad (g \in S_P^r(K)).
$$

It follows from the Green formula, that the value $\langle F_f, g \rangle$ does not depend on the particular choice of $U_g$. Moreover, $F_f$ is a continuous linear functional on $S_P^r(K)$.

Lemma 3.2. If $f \in S_P(X \setminus K)$, then

$$
(2) \quad \langle F_f, \Phi(x, \cdot) \rangle = f_r(x) \quad \text{for} \quad x \in X \setminus K.
$$

Proof. In fact, if $x \in X \setminus K$, then $\Phi(x, \cdot)$ satisfies $P^T\Phi(x, \cdot) = 0$ in the neighborhood $X \setminus \{x\}$ of $K$. So the left-hand side of (2) is well-defined. To finish the proof, it only remains to look at the proof of Lemma 3.1. □

3.3. We proceed now by applying Theorem 2.1. Therefore, we are interested in a representation of functionals $F \in (L^q)^\prime$, where $1 < q < \infty$.

Lemma 3.3. Let $1 < q < \infty$ and $p$ be the conjugate exponent to $q$. To each continuous linear functional $F$ on $(L^q)$ there is a sequence $f = \{f_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ in $L^p(K, m)$ such that $\|f_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \to 0$ when $|\alpha| \to \infty$, such that

$$
\langle F, \eta \rangle = \sum_{\alpha \in \mathbb{N}_0^n} \int_K \langle f_\alpha(y), \eta_\alpha(y) \rangle dm(y) \quad \text{for all} \quad \eta = \{\eta_\alpha\} \in L^q(K).
$$

Proof. Let $\eta \in l^q(r)$. Then $\eta = \sum_{\alpha \in \mathbb{N}_0^n} \eta_\alpha e_\alpha$, and the series converges with respect to the norm of $l^q(r)^K$. Since $F$ is a continuous functional on $L^q(K)$, its restriction to each of the $l^q(r)^K$ is continuous, too. Therefore, we have $\langle F, \eta \rangle = \sum_{\alpha} \langle F, \eta_\alpha e_\alpha \rangle$ for all $\eta = \{\eta_\alpha\} \in L^q(K)$. For a fixed multi-index $\alpha$, we consider the linear functional on $L^q(K, m)$ defined by $g \mapsto \langle F, ge_\alpha \rangle \quad (g \in L^q(K, m))$. This functional is obviously continuous, so by duality there is a function $f_\alpha \in L^p(K, m)$ such that $\langle F, ge_\alpha \rangle = \int_K \langle f_\alpha(y), g(y) \rangle dm(y)$ for all $g \in L^q(k, m)$. Hence $\langle F, \eta \rangle = \sum_{\alpha} \int_K \langle f_\alpha(y), \eta_\alpha(y) \rangle dm(y)$ for all $\eta \in L^q(K)$. The expression on the right hand side of this equality is a continuous linear functional on $L^q(K)$, and thus on each of the
spaces \( l^q(r_\nu)^K \). Hence it follows by Lemma 2.2 that \( \{f_\alpha\} \in l^p(\frac{1}{r_\nu})^K \) for every \( \nu \).

Then \( \sum_\alpha (\|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \frac{1}{r_\nu})^{p|\alpha|} < \infty \), showing that \( \limsup_{|\alpha| \to \infty} \|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \leq r_\nu \)
for all \( \nu \). Since \( r_\nu \to \infty \), the assertion follows.

\( \square \)

3.4. We now turn to the

Proof (of Lemma 1.2). Assume that \( f \in S_P(X \setminus K) \). We consider the continuous linear functional \( F_f \) on \( S_P(K) \) given by formula (1). The composition \( F = F_f \circ j^{-1} \)
defines a linear functional on the space \( S_P^{(q)} \), as follows from Lemma 2.3. Because of Theorem 2.1, the functional \( F \) is continuous. By the Hahn-Banach Theorem, \( F \) can be continuously extended to the whole space \( L^q \). According to Lemma 3.3, there exists a sequence \( \{f_\alpha\}_{\alpha \in \mathbb{N}_0^n} \in L^p(K,m) \), satisfying \( \|f_\alpha\|_{L^p(K,m)}^{1/|\alpha|} \to 0 \) when \( |\alpha| \to \infty \), such that

\[
\langle F,j(g) \rangle = \sum_\alpha \int_K \langle f_\alpha(y), \frac{D^\alpha g(y)}{\alpha!} \rangle dm(y) \quad \text{for all } g \in S_P(K).
\]

Now putting \( g = \Phi(x,\cdot) \), where \( x \) is a fixed point of \( X \setminus K \), and using Lemma 3.2 we derive the assertion of Lemma 1.2 with \( c_\alpha = f_\alpha/\alpha! \) (\( \alpha \in \mathbb{N}_0^n \)), since

\[
\langle F,j(\Phi(x,\cdot)) \rangle = \langle F,f(x,\cdot) \rangle = f_r(x) = f(x) - f_e(x).
\]

\( \square \)

3.5. When \( K \) is a single point, the representation asserted by Lemma 1.2 is just the Laurent expansion of \( f \).

Corollary 3.1. Let \( y_0 \) be a fixed point of \( X \). Then for every solution \( f \in S_P(X \setminus \{y_0\}) \) there exist a solution \( f_e \in S_P(X) \) and a sequence \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subset \mathbb{C}^k \), satisfying \( |\alpha| c_\alpha 1/|\alpha| \to 0 \) when \( |\alpha| \to \infty \), such that

\[
f(x) = f_e(x) + \sum_\alpha D^\alpha g(x,y_0) c_\alpha \quad (x \in X \setminus \{y_0\}).
\]

Proof. The assertion follows by using \( m(y_0) = 1 \) as a massive measure on \( K = \{y_0\} \).

The coefficients \( \{c_\alpha\} \) will not be uniquely determined by \( f \), since

\[
P'(y_0,D_y)\Phi(x,y_0) = \delta(x - y_0)I_k
\]
becomes zero off \( y_0 \).

The Laurent-series expansions for solutions of general elliptic equations were first studied by Lopatinskii [10].

3.6. If \( O \subset X \) is an open set whose boundary is locally connected, then each solution \( f \) of \( Pf = 0 \) in \( O \) has a representation (1) for \( x \in O \) with \( K = \partial O \). The only thing we have to do is to construct a massive measure \( m \) on \( \partial O \), and to extend \( f \) to a function satisfying the equation in the complement of \( \partial O \). The assertion follows by Lemma 1.2.
3.7. Theorem 1.1 implies that arbitrary singularities of solutions of elliptic equations may be locally separated into atomic (i.e., one-point) singularities.

**Corollary 3.2.** Assume that \( K \) is a locally connected compact subset of \( \sigma \), and \( \{y_\nu\} \) is a dense sequence of points of \( K \). Then every solution \( f \in S_P(X \setminus \sigma) \) can be written in the form \( f = f_e + \sum\nu f_\nu \), where \( f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{\sigma}) \) and \( f_\nu \in S_P(X \setminus \{y_\nu\}) \), and the series converges in the topology of \( S_P(X \setminus K) \).

**Proof.** We use the massive measure \( m \) on \( K \) constructed in Example 1.1. By Theorem 1.1

\[
 f(x) = f_e(x) + \sum_\alpha \left( \sum_\nu D_\alpha^y \Phi(x, y_\nu) c_\alpha(y_\nu) \mu_\nu \right) \text{ for } x \in X \setminus \sigma,
\]

where \( f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{\sigma}) \) and \( \lim_{|\alpha| \to \infty} (\sum_\nu |\alpha!c_\alpha(y_\nu)|^p \mu_\nu)^{1/(p|\alpha|)} = 0 \). The last condition allows to rearrange the summations and to derive \( f = f_e + \sum\nu f_\nu \) with

\[
 f_\nu = \sum_\alpha D_\alpha^y \Phi(x, y_\nu) c_\alpha(y_\nu) \mu_\nu,
\]

as was to be proved.

\[ \square \]

3.8. For the Laplace operator we obtain the following result (which seems to be new).

**Corollary 3.3.** Let \( K \subset \sigma \) be a locally connected compact set, and \( 1 < p < \infty \). Then every harmonic function \( f \) in \( X \setminus \sigma \) has the form

\[
 f(x) = f_e(x) + \sum_{j=0}^\infty \int_K \frac{h_j(y, x - y)}{|x - y|^{n+2j-1}} dm(y) \quad (x \in X \setminus \sigma)
\]

where \( f_e \) is a harmonic function in \((X \setminus \sigma) \cup \overset{\circ}{K}\), and \( h_j(y, z) \) are homogeneous harmonic polynomials of degree \( j \) in \( z \) with coefficients in \( \mathcal{L}^p(K, m) \) with respect to \( y \), such that \( \lim_{j \to \infty} \frac{1}{j!} \int_K |h_j(y, D_z) h_j(y, z)|^{p/2} dm(y) \right)^{1/p} = 0 \).

**Proof.** It suffices to transform formula (1) by means of the Hecke identity (cf. Stein [14]).

\[ \square \]

3.9. We finish this section by mentioning one more aspect of Theorem 1.1. It is a natural question to ask whether a given solution \( f \in S_P(X \setminus \{y_0\}) \) admits a representation (3) with a finite number of summands. This is obviously the case iff \( f \) has a finite order of growth near \( y_0 \), i.e., \(|f(x)| \leq c|x - y_0|^{-\gamma} \) in some deleted neighborhood of \( y_0 \). In other words, \( y_0 \) has to be a pole of \( f \). Therefore, the solutions \( f \in S_P(X \setminus K) \) for which the expansions (1) have only a finite number of terms are analogues of solutions with poles in general. Such solutions can be characterized as follows.

**Theorem 3.1.** Let \( K \) be an arbitrary compact set in \( X \), \( m \) be a massive measure on \( K \), and \( 1 < p < \infty \). A solution \( f \in S_P(X \setminus K) \) has a representation (1) with a finite number of terms iff the functional \( F_f \) given by (1) is continuous on \( S_P(K) \) with respect to the topology defined by the family of seminorms \( \|D_\alpha^y g\|_{\mathcal{L}^p(K, m)} \; (\alpha \in \mathbb{N}_0^n) \).

**Proof.** See Tarkhanov [15].

\[ \square \]
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