CONJUGACY CLASSES OF $SU(h, O_S)$ IN $SL(2, O_S)$

DONALD G. JAMES

ABSTRACT. Let $K$ be a quadratic extension of a global field $F$, of characteristic
not two, and $O_S$ the integral closure in $K$ of a Dedekind ring of $S$-integers
$\mathfrak{D}_S$ in $F$. Then $PSL(2, O_S)$ is isomorphic to the spinorial kernel $O'(L)$ for an
indefinite quadratic $\mathfrak{D}_S$-lattice $L$ of rank 4. The isomorphism is used to study
the conjugacy classes of unitary groups $PSU(h, O_S)$ of primitive odd binary
hermitian matrices $h$ under the action of $PSL(2, O_S)$.

1. Introduction

Let $O_d$ be the ring of integers in $\mathbb{Q}((\sqrt{-d})$, where $d$ is a square-free integer. It
was shown in Theorem 2.1 of James and Maclachlan [4] that the Bianchi group
$PSL(2, O_d)$, for $d > 0$ and $d \equiv 1, 2 \mod 4$, is isomorphic to the spinorial kernel
$O'(L)$ of an integral orthogonal group $O(L)$. Here

$$L = \mathbb{Z}r \perp \mathbb{Z}s \perp (\mathbb{Z}u + \mathbb{Z}v)$$

is a lattice on the quadratic space $V$ with quadratic form $q : V \to \mathbb{Q}$ and associated
bilinear form $f(x, y) = q(x + y) - q(x) - q(y)$, with $q(r) = 1$, $q(s) = d$, and $u$
and $v$ isotropic with $f(u, v) = d$. The extended Bianchi group $B_d$ is isomorphic to
$PSO(L)$. For $d \equiv 3 \mod 4$, $L$ must be replaced by $L + \mathbb{Z}2^{-1}(r - s)$.

Much of the proof in [4] remains valid when $d < 0$. In particular, there is a
homomorphism $\Phi$ from the Hilbert modular group $SL(2, O_d)$ into the group $O'(L)$
with kernel the center $\pm I$. In [4] this map was shown surjective only for $d > 0$ by
using the extended Bianchi group as the maximal discrete extension of $PSL(2, O_d)$
in $PSL(2, \mathbb{C})$. We now give a local-global number theoretic treatment in the more
general setting of a quadratic extension of global fields $K/F$ with $O_d$ replaced by
a ring of integers $O_S$ in $K$. Here $O_S$ is the integral closure in $K$ of a Dedekind
ring $\mathfrak{D}_S$ of $S$-integers in $F$ (see [7]). We prove $PSL(2, O_S)$ is isomorphic to the
spinorial kernel $O'(L)$ for a suitable $\mathfrak{D}_S$-lattice $L$ on a quadratic space $V$ over
$F$. When $\mathfrak{D}_S = \mathbb{Z}$ and $d \equiv 1, 2 \mod 4$, $L$ is the $\mathbb{Z}$-lattice given in (1). For
$F = \mathbb{F}(X)$ a function field over a finite field, of characteristic not two, $\mathfrak{D}_S = \mathbb{F}[X]$ 
and $K = F(\sqrt{-d})$ with $d$ a square-free polynomial, $L$ is the corresponding $\mathbb{F}[X]$-
lattice. However, in general, only the localizations $L_p$ are explicitly determined.

The results in [4] also gave a classification of the non-elementary maximal Fuch-
sian subgroups of the Bianchi group up to conjugacy. A Fuchsian subgroup stabilizes
a circle in the complex plane. The conjugacy classes of the projective special
unitary groups $PSU(h, O_S)$ of primitive binary hermitian matrices $h$ over $O_S$ are

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classified in the final sections. Relating \( h \) to a circle in the complex plane then gives a geometric connection between the two problems for the Bianchi groups (see also [5], [6], [9] and [10]). Some examples from cyclotomic fields are also given.

2. \( SL(2, K) \) AND QUADRATIC FORMS

In this section, the relationship between \( SL(2, K) \) and the orthogonal group of the related quadratic form is summarized when \( K \) is the quadratic extension of a field \( F \) with characteristic not two. Let \( K = F(\sqrt{d}) \) where \( -d \in F \) (we keep the negative sign to match the notation in [4]). Let \( \bar{a} \) denote the conjugate of \( a \in K \) under the non-trivial galois automorphism of \( K \) fixing \( F \).

Let \( A \) denote the quaternion algebra \( \mathbb{M}(2, K) \) with standard basis \( I, i, j, ij \) where

\[
i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then \( A \) admits a conjugate-linear involution \( \tau \) defined by

\[
\tau(a_0I + a_1i + a_2j + asi) = a_0I - a_1i - a_2j - a_3ij
\]

whose fixed point set \( V \) is a 4-dimensional space over \( F \). With the restriction of the norm form, denoted by \( q \), \( V \) is a regular quadratic space with orthogonal group \( O(V) \). Let \( f : V \times V \to F \) denote the associated symmetric bilinear form. In \( V \), fix a basis \( \{r, s, u, v\} \) with \( q(r) = 1, q(s) = d, q(u) = q(v) = 0 \) and \( f(u, v) = d \) by choosing \( r = I, s = (\sqrt{d})j, u = \frac{1}{2}((\sqrt{d})-i)j \) and \( v = \frac{1}{2}(\sqrt{d})(i + ij) \).

Define the group \( A_F^* \) by

\[
A_F^* = \{ \beta \in A^* \mid \det \beta \in F^* \}.
\]

For \( \beta \in A_F^* \), define \( \phi_\beta : V \to V \) by \( \phi_\beta(t) = (\det \beta)^{-1} \beta t \tau(\beta) \). Then \( \phi_\beta \in O(V) \).

Setting \( \Phi(\beta) = \phi_\beta \) defines a homomorphism

\[
\Phi : A_F^* \to O(V).
\]

As in [4] we give a description of \( \Phi \) in terms of the basis \( \{r, s, u, v\} \). Thus if \( \beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \), then \( \Phi(\beta) \) is \( (\det \beta)^{-1} \) times the \( 4 \times 4 \) matrix

\[
\begin{pmatrix}
\mathcal{R}(xw - y\bar{z}) & -d\mathcal{I}(xw + y\bar{z}) & -d\mathcal{I}(xz) & d\mathcal{I}(y\bar{w}) \\
\mathcal{I}(xw - y\bar{z}) & \mathcal{R}(xw + y\bar{z}) & \mathcal{R}(xz) & -\mathcal{R}(y\bar{w}) \\
2\mathcal{I}(xy) & 2\mathcal{R}(xy) & x\bar{x} & -y\bar{y} \\
2\mathcal{I}(w\bar{z}) & -2\mathcal{R}(w\bar{z}) & -z\bar{z} & w\bar{w}
\end{pmatrix}.
\]

The notation here is that, if \( \alpha = a + b\sqrt{-d} \) with \( a, b \in F \), then \( \mathcal{R}(\alpha) = a \) and \( \mathcal{I}(\alpha) = b \). It follows that the kernel of \( \Phi \) is \( F^*I \).

Let

\[
\theta : SO(V) \to F^*/F^{*2}
\]

denote the spinor norm, with kernel \( O(V) \). This group is also the commutator subgroup of \( SO(V) \), and also the subgroup generated by all Eichler transformations (see [2]). Since \( q(u) = 0 \), we can define for each \( t \in V \) with \( f(u, t) = 0 \) the Eichler transformation \( E(u, t) \) by

\[
E(u, t)(w) = w - f(u, w)t + f(t, w)u - q(t)f(u, w)u.
\]

Let \( \beta \in SL(2, K) \) have the form \( \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \) where \( \alpha = a + b\sqrt{-d} \). Then, as in [4],

\[
\Phi(\beta) = E(u, ad^{-1}s - br).
\]
Also $\Phi(\beta^t) = E(v, -ad^{-1}s - br)$. Since $SL(2, K)$ is generated by all $\beta, \beta^t$ it follows that $\Phi(SL(2, K)) \subseteq O'(V)$. In fact, since $E(u, t)$ and $E(v, t)$ generate $O'(V)$, the following sequence is exact:

$$I \rightarrow \{\pm I\} \rightarrow SL(2, K) \xrightarrow{\Phi} O'(V) \rightarrow I.$$  

Now let $\beta = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \in F^\ast$. Then $\Phi(\beta)$ fixes $r, s$ and maps $u \mapsto au$, $v \mapsto a^{-1}v$. Thus $\Phi(\beta) \in SO(V)$ and $\vartheta(\Phi(\beta)) = aF^\ast$. It follows that $\Phi(A_F^\ast) = SO(V)$, and the kernel of $\Phi$ consists of $aI$ with $a \in F^\ast$. Hence, for $\beta \in A_F^\ast$, the spinor norm of $\Phi(\beta)$ is $(\det \beta)F^\ast$. 

### 3. S-lattices and integral groups

Now let $K = F(\sqrt{-d})$ be a quadratic extension of a global field $F$ with characteristic not two, where $d$ is an algebraic integer in $F$. Let $S$ be a Dedekind set of primes for $F$ (see [7]), $\mathcal{O}_S$ the corresponding ring of integers in $F$, and $\mathcal{O}_S$ its integral closure in $K$. We show that

$$\Phi(SL(2, \mathcal{O}_S)) = O'(L)$$

for a suitably defined $\mathcal{O}_S$-lattice $L$ in $V$. Put

$$H = \mathcal{O}_S u + \mathcal{O}_S v.$$  

For $p \in S$, denote by $\mathcal{O}_p$ the localization of $\mathcal{O}_S$ at $p$ (without completion). We also denote by $p$ a prime element of $\mathcal{O}_S$. If $p$ does not split in $K$, let $\mathcal{O}_p$ denote the localization of $\mathcal{O}_S$ at the unique extension of $p$ to $K$. Then $\mathcal{O}_p = \mathcal{O}_S + \omega_p \mathcal{O}_p$ for some $\omega_p \in \mathcal{O}_p$. In fact, whenever $2d$ is a unit or a non-dyadic prime in $\mathcal{O}_p$, we can take $\omega_p = \sqrt{-d}$. In this case put

$$L_p = \mathcal{O}_p r \perp \mathcal{O}_p s \perp H_p,$$

an $\mathcal{O}_p$-lattice on $V$. Then $\Phi(SL(2, \mathcal{O}_p)) \subseteq O'(L_p)$, since in (2) all the matrix entries are in $\mathcal{O}_p$.

When $p \in S$ splits in $K$, let $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$ denote the localizations of $\mathcal{O}_S$ at the two conjugate extensions $p_1$ and $p_2$ of $p$ to $K$. Then, for $p$ non-dyadic and with $d$ a unit in $\mathcal{O}_p$, we have $\mathcal{O}_{p_1} \cap \mathcal{O}_{p_2} = \mathcal{O}_p[\sqrt{-d}]$, a semilocal ring. Again take $L_p$ as in (4). Then, from (2),

$$\Phi(SL(2, \mathcal{O}_p[\sqrt{-d}])) \subseteq O'(L_p).$$

Hence, for all but a finite number of $p \in S$, $L_p$ has been chosen as the localization of

$$L' = \mathcal{O}_S r \perp \mathcal{O}_S s \perp H.$$

For a non-dyadic prime $p$ where ord$_d p \geq 2$, take $\mu_p \in p\mathcal{O}_p$ such that $d\mu_p^{-2}$ is either a unit or a prime in $\mathcal{O}_p$. If $p$ does not split in $K$, put $\omega_p = \mu_p^{-1}\sqrt{-d}$ so that $\mathcal{O}_p = \mathcal{O}_p[\omega_p]$. Now take

$$L_p = \mathcal{O}_p r \perp \mathcal{O}_p \mu_p^{-1}s \perp \mu_p^{-1}H_p$$

so that, in essence, $d$ has been replaced by $\mu_p^{-2}d$. Then it again follows that $\Phi(SL(2, \mathcal{O}_p)) \subseteq O'(L_p)$. The non-dyadic split case is similar with $\mathcal{O}_{p_1} \cap \mathcal{O}_{p_2} = \mathcal{O}_p[\mu_p^{-1}\sqrt{-d}]$. Define $\mathcal{O}_p = \mathcal{O}_{p_1} \cap \mathcal{O}_{p_2}$ in all the split cases. Of course, (5) includes (4) by putting $\mu_p = 1$. 

It remains to consider dyadic primes \( p \in S \). Let \( e = \text{ord}_p \ 2 \). There are four possibilities (see [1, §5]).

1. The dyadic prime \( p \in S \) has two conjugate extensions \( p_1 \) and \( p_2 \) to \( K \)—the split case. Then \(-d\mu_p^{-2} \equiv 1 \mod 4p\) for some \( \mu_p \in \mathfrak{D}_p \). Here \( \mathfrak{D}_p = \mathfrak{D}_{p_1} \cap \mathfrak{D}_{p_2} = \mathfrak{D}_p[\omega_p] \) where \( \omega_p = (1 + \mu_p^{-1}/\sqrt{-d})/2 \).

2. The extension \( K/F \) is unramified at \( p \). Now, for some \( \mu_p \in \mathfrak{D}_p \), \(-d\mu_p^{-2} \equiv 1 + 4\delta \mod 4p\) with \( \delta \in \mathfrak{D}_p \) a unit. Then \( \mathfrak{D}_p = \mathfrak{D}_p[\omega_p] \) where \( \omega_p = (1 + \mu_p^{-1}/\sqrt{-d})/2 \).

3. The extension \( K/F \) is ramified at \( p \) with \( \text{ord}_p \ d = 2m + 1 \) odd—the ramified prime case. Then \( \mathfrak{D}_p \) is generated over \( \mathfrak{D}_p \) by 1 and \( \omega_p = p^{-m}/\sqrt{-d} \).

4. The extension \( K/F \) is ramified at \( p \) and \( \text{ord}_p \ d \) is even—the ramified unit case. Then \(-d\mu_p^{-2} \equiv 1 - p^{2k+1}\delta \mod 4p\) for some \( \mu_p \in \mathfrak{D}_p \), unit \( \delta \in \mathfrak{D}_p \), and rational integer \( k \) with \( 0 \leq k < e \). Now \( \mathfrak{D}_p \) is generated over \( \mathfrak{D}_p \) by 1 and \( \omega_p = (1 + \mu_p^{-1}/\sqrt{-d})p^{-k} \).

In the ramified prime case take \( L_p \) as in (5) above with \( \mu_p = p^m \). In the three remaining cases take

\[
L_p = (\mathfrak{D}_p r + \mathfrak{D}_p p^{-k}(r - \mu_p^{-1}s)) \perp \mu_p^{-1}H_p
\]

where \( k = e \) in the split and unramified cases (so \( p^k \) is essentially 2). This case is the same as (5) when \( k = 0 \). In the split and unramified cases, \( q(L_p) = \mathfrak{D}_p \) and \( L_p \) is an even unimodular \( \mathfrak{D}_p \)-lattice. In the ramified unit case, \( q(L_p) = \mathfrak{D}_p \) but \( L_p \) is not unimodular. Again, after a computation using (2),

\[
\Phi(SL(2, \mathfrak{D}_p[\omega_p])) \subset O'(L_p).
\]

By [7, 81:14], there now exists an \( \mathfrak{D}_S \)-lattice \( L \) on \( V \) that localizes to the chosen \( L_p \) at each \( p \in S \). When \( \mathfrak{D}_S = \mathfrak{D}_S[\sqrt{-d}] \), we have \( L = L' \). Moreover, in all cases, \( \Phi(SL(2, \mathfrak{D}_S)) \subset O'(L) \), since if \( \beta \in SL(2, \mathfrak{D}_S) \) with \( \Phi(\beta) = \phi_\beta \), then \( \phi_\beta(L_p) = L_p \) for all \( p \in S \) (including those \( p \) that split in \( K \)). Hence \( \phi_\beta(L) = L \) by [7, 101:6]. In the next section we show that \( \Phi(SL(2, \mathfrak{D}_S)) = O'(L) \).

### 4. Generators for \( O'(L_p) \) and \( O'(L) \)

Let \( E \) denote the subgroup of \( O'(L) \) generated by the integral Eichler transformations \( E(u, t) \) and \( E(v, t) \), and let \( E_p \) be the corresponding local subgroup in \( O'(L_p) \). For the lattice \( L_p \) in (4), \( E(u, t) \) is integral when \( t \in \mathfrak{D}_p r \perp \mathfrak{D}_p r^{-1}s \). See [2] for many relations involving these transformations. In particular, for \( q(t) \neq 0 \),

\[
\Psi(t)\Psi(u - v) = T(-dq(t))E(v, t)E(u, (q(t)d)^{-1}t)E(v, t)
\]

where \( T(c) \) is the isometry fixing \( r \) and \( s \) while sending \( u \) to \( cu \) and \( v \) to \( c^{-1}v \), and \( \Psi(t) \) is the symmetry \( x \rightarrow x - f(x, t)q(t)^{-1}t \). Taking \( t = d^{-1}s \) it follows that \( \Psi(s) \in O(H_p)E_p \) when \( L_p \) is as in (4). Similarly, \( \Psi(r) \in O(H_p)E_p \) when \( d \) is a unit in \( \mathfrak{D}_p \).

For \( p \in S \), let

\[
J_p = \{ x \in L_p \mid q(x) \in d\mu_p^{-2}\mathfrak{D}_p \}
\]

and

\[
M_p = \{ x \in L_p \mid f(x, J_p) \subset 2d\mu_p^{-2}\mathfrak{D}_p \}.
\]

For \( L_p \) as in (5), we have

\[
\mu_p J_p = \mathfrak{D}_p d\mu_p^{-1}r \perp \mathfrak{D}_p s \perp H_p.
\]
suffices to consider with 2

Proof. Only the surjectivity of \( \Phi \) remains to be shown in the sequence. We already have the exact sequence

\[
I \to \{ \pm 1 \} \to SL(2, K) \xrightarrow{\Phi} O'(V) \to I.
\]

Then \( \Phi(SL(2, O_p)) = \mathcal{E}_p = O'(L_p) \), since, by (3), each integral Eichler transformation is the image of an integral elementary matrix. Fix \( \phi \in O'(L) \). Then \( \phi \) can be extended to \( L_p \), and hence there exist exactly two isometries \( \pm \sigma \in SL(2, O_p) \subseteq \)
with \( \Phi(\pm \sigma) = \phi \). Letting \( p \) vary over all the extensions of \( p \in S \) to \( K \), since \( \bigcap_p \mathcal{O}_p = \mathcal{O}_S \) and \( \pm \sigma \) cannot change, it follows that \( \sigma \in SL(2, \mathcal{O}_S) \).

By Vaserstein [8], \( SL(2, \mathcal{O}_S) \) is generated by integral elementary matrices except when \( F = \mathbb{Q} \) and \( d > 0 \). Hence, from (3), \( O'(L) \) is generated by integral Eichler transformations.

\[ \square \]

5. Unitary groups

The non-elementary maximal Fuchsian subgroups of a Bianchi group were shown in [4, §3] to be in one-one correspondence with certain stabilizer subgroups of \( O'(L) \).

We will now relate the projective special unitary groups in \( PSL(2, \mathcal{O}_S) \) to similar stabilizer subgroups. Let \( \Phi(SL(2, \mathcal{O}_S)) = O'(L) \).

For \( b \in \mathcal{O}_S \) and \( a, c \in \mathcal{O}_S \) with \( D = b^2 - ac \neq 0 \), the matrix

\[
\begin{pmatrix}
a & b \\ b & c
\end{pmatrix}
\]

is hermitian with discriminant \( D \in \mathcal{O}_S \). Call the matrix \( h \) primitive when \( (a, b, c, D) = \mathcal{O}_S \). Let \( SU(h, \mathcal{O}_S) \subset SL(2, \mathcal{O}_S) \) be the special unitary group of \( h \).

There are two types of local hermitian forms at a ramified dyadic prime \( p \). The matrix \( h \) is locally odd at \( p \) when there exists \( g \in \mathcal{O}_p \times \mathcal{O}_p \) with \( ghg^* \) a unit in \( \mathcal{O}_p \); this is equivalent to \( a \) or \( c \) being a unit (since \( trace(\mathcal{O}_p) \subseteq p \mathcal{O}_p \)). Otherwise, \( h \) is even at \( p \). The matrix \( h \) is globally odd when it is odd at all ramified dyadic primes.

In particular,

\[
h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -D \end{pmatrix}
\]

is odd. Let \( \beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL(2, \mathcal{O}_S) \). Then \( \beta \in SU(h_0, \mathcal{O}_S) \) if and only if \( \beta h_0 = h_0 (\beta^t)^{-1} \), or \( x = \bar{w} \) and \( z = D \bar{y} \). Define \( \Phi(\beta) = \phi \in O'(L) \) as in §2, so that

\[
\phi(u) = -dI(x\bar{z})r + R(x\bar{z})s + x\bar{u} - z\bar{v}
\]

and

\[
\phi(v) = dI(yw)r - R(yw)s - y\bar{j}u + w\bar{w}v.
\]

Hence \( \beta \in SU(h_0, \mathcal{O}_S) \) if and only if \( \phi(u + Dv) = u + Dv \). Therefore,

\[ \Phi^* : PSU(h_0, \mathcal{O}_S) \rightarrow O'(L) \]

is the isomorphism induced by \( \Phi \).

For \( t \in V \), define the stabilizer

\[ Stab(L, t) = \{ \phi \in O'(L) \mid \phi(t) = t \} \]

with \( Stab(L_p, t) \) the corresponding local group. Then \( \phi \in Stab(L, t) \) if and only if \( \phi \in Stab(L_p, t) \) for all \( p \in S \). If \( \sigma \in O(L) \), then

\[ \sigma Stab(L, t) \sigma^{-1} = Stab(L, \sigma(t)). \]

For \( a \neq 0 \), put \( \gamma_a = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in A_p^t \), so that \( \Phi(\gamma_a) \in SO(V) \). Let \( b = b_1 + b_2 \sqrt{-d} \) where \( b_1, b_2 \in F \). Computation then gives

\[
\Phi(\gamma_a)(u + Dv) = -db_2r + b_1s + au - cv = t
\]

where \( q(t) = dD \). Since \( \gamma_a h_0 \gamma_a^t = ah \), it follows that

\[ \gamma_a SU(h_0, \mathcal{O}_S) \gamma_a^{-1} \subset SU(h, K), \]
and also, when \( a \neq 0 \),
\[
\Phi(SU(h, K)) = Stab(V, t).
\]

A similar argument with the same \( t \) holds for \( \gamma_c = \begin{pmatrix} b & -1 \\ c & 0 \end{pmatrix} \) when \( c \neq 0 \). Put
\[
\delta_g = \begin{pmatrix} 1 & gb \\ 0 & 1 \end{pmatrix}
\]
with \( g \in O_S \). Then \( \sigma_g = \Phi(\delta_g) \in O'(L) \). When \( a = c = 0 \), put
\[
h' = \delta_1 h \delta_1^t = \begin{pmatrix} 2D & b \\ b & 0 \end{pmatrix}.
\]
Then, from (3), \( \Phi(SU(h', K)) = Stab(V, \sigma_1(t)) \), so that again (9) holds.

**Theorem 5.1.** The group \( SU(h, O_S) \) is commensurable in \( GL(2, K) \) to a conjugate of \( SU(h_0, O_S) \). Moreover, with \( h \) primitive and odd, and \( t \) as in (8),
\[
\Phi^*(PSU(h, O_S)) = Stab(L, t).
\]

**Proof.** Assume \( a \neq 0 \); let \( SU(h_0, aO_S) \) be the congruence subgroup of \( SU(h_0, O_S) \) consisting of those \( \beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \equiv I \mod aO_S \). Thus \( x - w, y, z \in aO_S \), and then \( \gamma_a \beta \gamma_a^{-1} \) is integral. Hence, modifying [5], we obtain
\[
SU(h, a^2O_S) \subseteq \gamma_a SU(h_0, aO_S) \gamma_a^{-1} \subseteq SU(h, O_S)
\]
and \( \gamma_a SU(h_0, O_S) \gamma_a^{-1} \) and \( SU(h, O_S) \) are commensurable subgroups of \( GL(2, K) \). Also, when \( a \) is a unit in \( O_p \), we have \( \gamma_a \in GL(2, O_p) \) and hence \( \gamma_a SU(h_0, O_p) \gamma_a^{-1} = SU(h, O_p) \). Therefore,
\[
\Phi(SU(h, O_p)) = \Phi(\gamma_a) Stab(L_p, u + Dv) \Phi(\gamma_a)^{-1} = Stab(L_p, t).
\]

A similar argument with the same \( t \) holds for \( \gamma_c \) when \( c \neq 0 \).

It remains to show that \( \Phi(SU(h, O_p)) = Stab(L, t) \) for all \( p \in S \) with \( a, c \in pO_p \) and consequently \( b \) is a unit in \( O_p \). Since \( h \) is assumed odd, \( p \) is not ramified dyadic. The \( (1, 1) \)-entry in \( h' = \delta_1 h \delta_1^t \) is congruent to \( (g + \bar{g})b \bar{d} \mod p \). Hence, as above, if \( g + \bar{g} \) is a unit, then \( \Phi(SU(h, O_p)) = Stab(L_p, \sigma_g(t)) \) and \( \Phi(SU(h, O_p)) = Stab(L_p, t) \). Take \( g = (1 + \mu_p^{-1} \sqrt{-d})/2 \) in the unramified and split dyadic cases, and when \( 2 \) is a unit in \( O_p \), take \( g = 1 \). \( \square \)

We analyse \( t \) more carefully. For \( L_p \) as in (5) and \( b = b_1 + b_2 \omega_p \) with \( b_1, b_2 \in O_p \) and \( (a, b, b, c) = (a, b_1, b_2, c) = O_p \), it follows from (8) that
\[
t = -d \mu_p^{-1} b_2 r + b_1 s + au - cv \in \mu_p J_p
\]
and \( f(t, \mu_p M_p) = 2dO_p \). For dyadic \( L_p \) as in (6) and \( b = b_1 + b_2 \omega_p \), we have
\[
t = -d \mu_p^{-1} b_2 p^{-k} r + (b_1 + b_2 p^{-k}) s + au - cv \in \mu_p J_p
\]
and \( f(t, \mu_p M_p) = 2dO_p \). Hence \( t \in L \) for all \( p \in S \), so that \( t \in L \).

Define \( L(D) \) to be the set of all \( t \in L \) with \( q(t) = dD \), and \( t \in \mu_p J_p \) and \( f(t, \mu_p M_p) = 2dO_p \) for all \( p \in S \). This is a generalization of the definition of \( L(D) \) given in [4]. The group \( O'(L) \) acts on \( L(D) \) and we set \( N(L, D) \) to be the number of orbits under this action. We have now shown

**Theorem 5.2.** The map \( \Phi^* \) induces an injection from the conjugacy classes of the projective special unitary groups \( PSU(h, O_S) \), of odd primitive hermitian matrices \( h \in M(2, O_S) \) with discriminant \( D \neq 0 \), under the action of \( PSL(2, O_S) \), into the orbits in \( L(D) \) under the action of \( O'(L) \).
Note, for \( \gamma \in SL(2, \mathcal{O}_S) \) and \( \gamma SU(h, \mathcal{O}_S) \gamma^{-1} = SU(h', \mathcal{O}_S) \), where \( h' = \gamma h \gamma^t \), it does not follow that \( h' \) is also primitive (for example, \( \mathcal{O}_S = \mathbb{Z}, d = 5, h = \begin{pmatrix} 1 & b \\ b & 0 \end{pmatrix} \) with \( b = 1 + \sqrt{-5} \), and \( \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)). Each conjugacy class considered in Theorem 5.2 need only involve at least one primitive hermitian matrix. However, the restriction that \( h \) is odd and primitive means that \( h \) locally represents a unit at each \( p \in S \), and \( h' \) inherits this key property, which could have been used as a conjugacy invariant definition of primitivity.

We now study the image of the induced injective map \( \Phi^* \) in the set of orbits in \( L(D) \) under \( O'(L) \). Denote by \( n(D) \) the size of this image. Then \( n(D) \leq N(L, D) \).

Let \( t = dbx + b_1s + au - cv \in L(D) \). Then \( t \in \mu_p \mathcal{L}_p \) so that \( a, c \in \mathcal{O}_p \) for all \( p \in S \), and hence \( a, c \in \mathcal{O}_S \). Put \( b = b_1 - b_2 \sqrt{-d} \). Then \( bb = D + ac \in \mathcal{O}_S \).

If we show that \( b + \bar{b} = 2b_1 \in \mathcal{O}_S \), it then follows that \( b \in \mathcal{O}_S \) since \( \mathcal{O}_S \) is the integral closure of \( \mathcal{O}_S \) in \( K \). For type (5) we have \( b_1 \in \mathcal{O}_p \). For type (6) it follows from \( t \in \mu_p \mathcal{L}_p \) that \( 2b_1 \in \mathcal{O}_p \). Hence \( b_1 \in \mathcal{O}_S \). If \( h = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) is primitive, the conjugacy class of \( PSU(h, \mathcal{O}_S) \) then corresponds to the orbit of \( t \). The matrix \( h \) is locally primitive at any \( p \) where \( D \) is locally a unit, and hence it suffices to consider only those \( p \in S \) where \( p \mid D \). However, unlike the corresponding situation in [4], if the orbit of \( t \) lies in the image of \( \Phi^* \), then \( t \) must also be primitive in \( \mu_p \mathcal{L}_p \) for all \( p \mid D \). For \( t \in \mu_p \mathcal{L}_p \), this also holds for all elements in the orbit of \( t \); it follows that \( a, c \in \mathcal{O}_p \) and when \( p \mid D \) all the \( h \) corresponding to elements in the orbit of \( t \) are not primitive. Therefore, when computing \( n(D) \) from the local information about \( N(L, D) \) given in [4], these orbits must be excluded. Similarly, since the condition analogous to \( h \) odd is not assumed in [4], all the ramified dyadic orbits corresponding to \( t \in \mathcal{M}_p \) must also be excluded for the current situation.

We return to the question of the primitivity of the \( h \) constructed above under the additional assumptions that \( t \) is locally primitive in \( \mu_p \mathcal{L}_p \) for all \( p \mid D \), and that either \( a \) or \( c \) is locally a unit for all ramified dyadic \( p \in S \). Since \( f(t, \mu_p \mathcal{M}_p) = 2d \mathcal{O}_p \), it follows that \( (a, b_1, b_2, c)_p = \mathcal{O}_p \) where now we have locally rewritten \( b = b_1 + b_2 \sqrt{-d} \in \mathcal{O}_p[\omega_p] \). Then \( h \) is locally primitive whenever \( a, c \) or \( D \) is a local unit. It remains to consider those \( p \in S \) dividing \( a, c \) and \( D \); in particular, \( p \) is not ramified dyadic. Then \( p \mid bb \) so that \( b_2 \) is necessarily a local unit. Moreover, since \( t \) is now assumed to be primitive in \( \mu_p \mathcal{L}_p \), either \( b_1 \) or \( d \mu_p^{-2} \) is also a unit, and hence both are units when \( p \) is non-dyadic since \( p \mid D \). Using the Strong Approximation Theorem (see [7, 21:2]), take \( gd \in \mathcal{O}_S \) with \( gd \equiv 1 \mod p \) for all non-dyadic \( p \mid a, c, D \), and \( gd \in p \mathcal{O}_p \) for the remaining \( p \mid D \). Also, choose \( g' \in K \) with \( g' \mu_p \in \mathcal{O}_p \) for all \( p \in S \), such that \( g' \mu_p \equiv 1 \mod p \) for all dyadic \( p \mid a, c, D \), and such that \( g' \mu_p \in p \mathcal{O}_p \) for all the remaining \( p \mid D \). Then \( E(u, g'r + gs) \in O'(L) \). Put \( t' = E(u, g'r + gs)(t) \). Either the coefficient of \( u \) or of \( v \) in \( t' \) is now a unit for all \( p \mid D \), and hence the corresponding hermitian matrix \( h' \) primitive. (In fact, it would suffice for the proof above, to find suitable \( t'_p \) and \( h'_p \) for each \( p \mid D \), one at a time.) Therefore the \( O'(L) \)-orbit of \( t \) is in the image of \( \Phi^* \).

6. Quadratic and cyclotomic fields

We now relate Theorem 5.2 above with Theorem 3.1 and other results in [4]. There, using the Strong Approximation Theorem for rotations (see [7, 104:4]), the
Theorem 6.1. Let $K = F(\sqrt{-d})$ where $F = \mathbb{F}(X)$ is a function field and $d$ is a square-free polynomial in $\mathbb{F}[X]$. Assume the Hilbert symbol $(D, -d)_{\infty} = 1$ at the infinite prime. Then there are $n(D) = 2^m$ conjugacy classes of projective special unitary groups $PSU(h, O_S)$, of primitive hermitian $h$ with discriminant $D \neq 0$, under the action of $PSL(2, O_S)$.

Proof. The Witt index condition needed for the Strong Approximation Theorem is equivalent to $(D, -d)_{\infty} = 1$ at the infinite prime. The result then follows by modifying the data in Theorem 5.1 of [4] by excluding the orbits coming from $t$ that are not primitive in $L$. \qed

If $D$ and $d$ are monic polynomials in $\mathbb{F}[X]$, then $(D, -d)_{\infty} = 1$ if and only if $d$ has odd degree, or $D$ has even degree, or $-1 \in \mathbb{F}^{\ast}$.  

The hermitian matrices $h$ as in (7) are the starting point for Vulakh’s treatment in [9] and [10] of the conjugacy classes of the maximal non-elementary Fuchsian subgroups of Bianchi groups. He relates $h$ to the circle $C$ in the complex plane with discriminant $D$ given by

$$aZ\bar{Z} + bZ + \bar{b}\bar{Z} + c = 0$$

where $Z = X + iY \in \mathbb{C}$. Instead of a primitivity condition, an equivalence relation on rational hermitian $h$ is introduced. A different treatment is given in [4] and [6] where it is shown, using the underlying hyperbolic geometry, that the maximal Fuchsian subgroup $F'$ corresponding to the transformations

$$Z' = (xZ + z)(yZ + w)^{-1}$$

that stabilize $C$ then corresponds to $\{\phi \in O'(L) \mid \phi(t) = \pm t\}$, with $t \in L(D)$ as before, and that the conjugacy classes of these $F'$ with discriminant $D$ are in one-to-one correspondence with the orbits in $L(D)$ under the action of $O'(L)$, the group generated by $O'(L)$ and $-I$. The ambiguity in sign is introduced because $\pm h$, or $\pm t$, both determine the same circle $C$. This problem of Fuchsian subgroups is
closely related to our classification problem here, but distinct since the primitivity condition used for \( h \) is stronger.

Now let \( K = \mathbb{Q}(\zeta) \), where \( \zeta \) is a primitive \( l \)-th root of unity, and let \( F = K \cap \mathbb{R} = \mathbb{Q}(\zeta + \zeta) \). Take \( \mathcal{O} \) to be the ring of algebraic integers in \( F \). For \( l = 4m + 3 \) prime, \( K = F(\sqrt{-l}) \) and \( l \) is totally ramified in \( \mathcal{O} \). Since \(-l \equiv 1 \pmod{4} \), the extension \( K/F \), viewed dyadically, is either split or unramified at each \( p' \), \( 2 \) in \( \mathcal{O} \) prime over 2 in \( F \), which is an even unimodular lattice and \( N(L_p, D) = 1 \) by Theorem 5.3(1) in [4]. Let \( p \) be the unique prime over \( l \) in \( F \). Then \( \mathcal{O}_p = \mathcal{O}[p^{-m} \sqrt{-l}] \) since \( [F : \mathbb{Q}] = 2m + 1 \) and \( l \) is totally ramified. Take \( L_p \) as in (4) with \( q(s) = f(u, v) = lp^{-2m} \) a prime. Then, generalizing Theorem 4.1 in [4], \( N(L, D) = N(L_p, D) \) when \( h \) has a totally positive discriminant \( D \). Excluding the local orbits that are not primitive in Theorem 5.1 in [4], we get

**Theorem 6.2.** Let \( K = \mathbb{Q}(\zeta) \) with \( l = 4m + 3 \) prime, and \( F = K \cap \mathbb{R} \). Then, for \( p \) the unique prime over \( l \), and totally positive \( D \in \mathcal{O} \),

1. \( n(D) = 1 \) when \( D \) is a unit in \( \mathcal{O}_p \).
2. \( n(D) = 2 \) otherwise.

A similar theorem holds for \( l \equiv 1 \pmod{4} \) and prime.

When \( l = 2^n \geq 8 \), 2 is the only prime ramifying in \( \mathcal{O} \) and \( p = \zeta + \zeta \) is the unique prime over 2 in \( F \). Then \( K = F(\sqrt{-1}) = \mathbb{Q}(\zeta) \). For \( p' \) non-dyadic, take \( L_p \) as in (4) with \( d = 1 \). Hence \( N(L, D) = N(L_p, D) \) for \( D > 0 \). Dyadically, \( K/F \) is a ramified unit extension with \( e = 2^{n-2} \geq 2 \) and \( k = e - 1 \) (see Lemma 7.2 in [3]). From (6) with \( \mu_p = 1 \), we take

\[
L_p = (\mathcal{O}_p r + \mathcal{O}_p p^{-k}(r - s)) \perp H_p
\]

where

\[
d = q(s) = f(u, v) = (1 + p^{e/2} + p^{3e/4} + p^{7e/8} + \cdots + p^{(e-1)e/e^2})^2.
\]

For \( l = 8 \), \( p = \pm \sqrt{2} \) and \( d = (1 \pm \sqrt{2})^2 \) is a unit in \( \mathcal{O} \); then

\[
L = (\mathcal{O} r + \mathcal{O} \sqrt{2}^{-1}(r - s)) \perp H.
\]

The dyadic orbits are complicated when 2 is ramified and \( n(D) \) has not been computed.

**References**


DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

E-mail address: james@math.psu.edu