HOMOGENEOUS PROJECTIVE VARIETIES WITH DEGENERATE SECANTS

HAJIME KAJI

Dedicated to Professor Satoshi Arima on the occasion of his 70th birthday

Abstract. The secant variety of a projective variety $X$ in $\mathbb{P}$, denoted by $\text{Sec } X$, is defined to be the closure of the union of lines in $\mathbb{P}$ passing through at least two points of $X$, and the secant deficiency of $X$ is defined by $\delta := 2 \dim X + 1 - \dim \text{Sec } X$. We list the homogeneous projective varieties $X$ with $\delta > 0$ under the assumption that $X$ arise from irreducible representations of complex simple algebraic groups. It turns out that there is no homogeneous, non-degenerate, projective variety $X$ with $\text{Sec } X \neq \mathbb{P}$ and $\delta > 8$, and the $E_6$-variety is the only homogeneous projective variety with largest secant deficiency $\delta = 8$. This gives a negative answer to a problem posed by R. Lazarsfeld and A. Van de Ven if we restrict ourselves to homogeneous projective varieties.

Introduction

The secant variety of a projective variety $X$ in $\mathbb{P}$, denoted by $\text{Sec } X$, is defined to be the closure of the union of lines in $\mathbb{P}$ passing through at least two points of $X$, and the secant deficiency of $X$ is defined by

$$\delta := 2 \dim X + 1 - \dim \text{Sec } X.$$ 

In 1979, F. L. Zak proved a significant inequality,

$$3 \dim X + 4 \leq 2 \dim \mathbb{P}$$

for a smooth, non-degenerate $X$ with $\text{Sec } X \neq \mathbb{P}$, which had been conjectured by R. Hartshorne [Ht, Conjecture 4.2] (see also [FL], [LV], [Z]). From the viewpoint of Zak’s inequality, projective varieties $X$ which attain the equality, namely Severi varieties, were studied actively, and Zak finally found that there are exactly four Severi varieties (see [FR], [T], [LV], [Z]): It turns out that those varieties are all homogeneous and have $\delta = 1, 2, 4, 8$. For the extremal case of odd dimensional $X$, in which $3 \dim X + 5 = 2 \dim \mathbb{P}$, T. Fujita [F] gave a classification for 3-dimensional $X$ and M. Ohno [O] recently gave classifications for 5-dimensional $X$ and for 7-dimensional $X$ under a certain condition, where those $X$ of dimension 3,5,7 have $\delta = 1, 2, 3$, respectively. Thus several authors have studied projective varieties $X$ with $\delta > 0$.

The purpose of this article is to list the homogeneous projective varieties $X$ with $\delta > 0$ under the assumption that $X$ arise from irreducible representations of complex simple algebraic groups. Zak already obtained a table of those $X$ in case

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of \(2 \dim X \geq \dim \mathbb{P}\). But we work without any dimensional condition. Although we as well as Zak need another step to investigate which \(X\) has \(\Sec X \neq \mathbb{P}\), our strategy to pick up the candidates of \(X\) with \(\delta > 0\) (not necessarily \(\Sec X \neq \mathbb{P}\)) is different and quite simple, as we will see below.

Let \(G\) be a complex simple algebraic group with Lie algebra \(\mathfrak{g}\), let \(R\) be the root system of \(\mathfrak{g}\), and fix a base \(\Delta\) of \(R\). Let \(\lambda\) be a dominant weight of \(\mathfrak{g}\) with respect to \(\Delta\), \(\rho : G \to GL(V)\) an irreducible, finite-dimensional representation of \(G\) with highest weight \(\lambda\), and \(\nu_\lambda\) a maximal vector in \(V\) with weight \(\lambda\). In this article we discuss projective varieties \(X\) in \(\mathbb{P}_*(V)\) which is an orbit of the subspace spanned by \(\nu_\lambda\) under the action of \(G\), where \(\mathbb{P}_*(V)\) denotes the 1-dimensional subspaces of \(V\). Denote by \(\omega_i\) the \(i\)-th fundamental weight as in [B].

The result is

**Theorem.** \(X\) in \(\mathbb{P}_*(V)\) has \(\delta > 0\) if and only if the type of \(\mathfrak{g}\) and \(\lambda\) is one of the following:

- \((A_1)\) \(\omega_1; 2\omega_1\)
- \((A_2)\) \(\omega_1; \omega_2; 2\omega_1, 2\omega_2; \omega_1 + \omega_2\)
- \((A_3)\) \(\omega_1; \omega_3; \omega_2; 2\omega_1, 2\omega_3; \omega_1 + \omega_3\)
- \((A_{l \geq 4})\) \(\omega_1, \omega_l; \omega_2, \omega_{l-1}; 2\omega_1, 2\omega_2; \omega_1 + \omega_l\)
- \((B_2)\) \(\omega_1; \omega_2; 2\omega_2\)
- \((B_{l=3,4})\) \(\omega_1, \omega_2, \omega_l; \omega_1, \omega_2, 2\omega_2\)
- \((B_{l \geq 5})\) \(\omega_1; \omega_l; \omega_2\)
- \((C_{l \geq 3})\) \(\omega_1; \omega_2; 2\omega_2\)
- \((D_{l=4,5})\) \(\omega_1, \omega_2, \omega_{l-1}, \omega_l; \omega_1, \omega_2\)
- \((D_{l \geq 6})\) \(\omega_1; \omega_2\)
- \((E_6)\) \(\omega_1; \omega_6; \omega_2\)
- \((E_7)\) \(\omega_1; \omega_8\)
- \((E_8)\) \(\omega_8\)
- \((F_4)\) \(\omega_1; \omega_4\)
- \((G_2)\) \(\omega_1; \omega_2\)

From this result one obtains the following table of homogeneous projective varieties with degenerate secants (see, for details, §3).

The only-if-part is the main contribution of this work, while the if-part follows from well-known facts, results of Zak, and a recent result of M. Ohno, O. Yasukura and the author (see §3). Denote by \(\bar{\alpha}\) the highest root of \(\mathfrak{g}\), by \(\mu\) the lowest weight of \(\rho\), and by \((\ast, \ast)\) the inner product defined by the Killing form. The key to prove the only-if-part is a simple

**Criterion.**

\[(\lambda - \mu, \lambda - \bar{\alpha}) > 0 \Rightarrow \delta = 0.\]

It turns out, after proving the Theorem, that the converse is also true.

Using a result of Zak [Z, III, Corollary 1.7], we obtain from our table the following results for arbitrary homogeneous projective varieties \(X\) such that \(G\) is not necessarily simple. The first yields a partial answer to a problem posed by R. Lazarsfeld and A. Van de Ven [LV, §1f, Problem]:

**Corollary 1.** There is no homogeneous, non-degenerate, projective variety \(X\) with \(\Sec X \neq \mathbb{P}\) and \(\delta > 8\). Furthermore, the \(E_8\)-variety is the only homogeneous projective variety with largest secant deficiency \(\delta = 8\).
### Table of homogeneous projective varieties with degenerate secants

<table>
<thead>
<tr>
<th>type</th>
<th>weight λ</th>
<th>representation</th>
<th>δ</th>
<th>X</th>
<th>dim P + l</th>
<th>Sec X ⊆ P</th>
<th>−ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>ω₁</td>
<td>standard</td>
<td>2</td>
<td>P¹</td>
<td>2</td>
<td>= 1/4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2ω₁, 2nd symm.</td>
<td>3</td>
<td></td>
<td>v₂(P¹)</td>
<td>3</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td>A2</td>
<td>ω₁, ω₂</td>
<td>standard</td>
<td>3</td>
<td>P²</td>
<td>3</td>
<td>= 1/6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2ω₁, 2ω₂, 2nd symm.</td>
<td>1</td>
<td></td>
<td>v₂(P²)</td>
<td>6</td>
<td>≠ 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ω₁ + ω₂</td>
<td>adjoint</td>
<td>1</td>
<td>P(T₄) = P² × P² ∩ (1)</td>
<td>8</td>
<td>≠ 0</td>
<td></td>
</tr>
<tr>
<td>A₃</td>
<td>ω₁, ω₃</td>
<td>standard</td>
<td>4</td>
<td>P³</td>
<td>4</td>
<td>= 1/8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ω₂</td>
<td>2nd ext.</td>
<td>4</td>
<td>G(2, 4)</td>
<td>6</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2ω₁, 2ω₃</td>
<td>2nd symm.</td>
<td>1</td>
<td>v₂(P³)</td>
<td>10</td>
<td>≠ 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ω₁ + ω₃</td>
<td>adjoint</td>
<td>1</td>
<td>P(T₄) = P³ × P³ ∩ (1)</td>
<td>15</td>
<td>≠ 0</td>
<td></td>
</tr>
<tr>
<td>A₄</td>
<td>ω₁, ω₄</td>
<td>standard</td>
<td>2l - 1</td>
<td>Q^{2l-1}</td>
<td>2l + 1</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ω₂</td>
<td>spin</td>
<td>6</td>
<td>S₂ = P²</td>
<td>2²</td>
<td>= 1/20, 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2ω₂, adjoint</td>
<td>1</td>
<td></td>
<td>v₂(P²)</td>
<td>10</td>
<td>≠ 0</td>
<td></td>
</tr>
<tr>
<td>B₁</td>
<td>ω₁</td>
<td>standard</td>
<td>2l + 1</td>
<td>Q^{2l+1}</td>
<td>2l</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ω₂</td>
<td>adjoint</td>
<td>1</td>
<td>P(T₄) = P³ × P³ ∩ (1)</td>
<td>15</td>
<td>≠ 0</td>
<td></td>
</tr>
<tr>
<td>C₁</td>
<td>ω₁</td>
<td>standard</td>
<td>2l + 2</td>
<td>Q^{2l+2}</td>
<td>2l</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ω₂</td>
<td>adjoint</td>
<td>1</td>
<td>P(T₄) = P³ × P³ ∩ (1)</td>
<td>15</td>
<td>≠ 0</td>
<td></td>
</tr>
</tbody>
</table>

**NOTATION:** v₂ denotes the Veronese embedding, Qⁿ a quadric hypersurface of dimension n, G(k, m) the Grassmann variety of k-dimensional subspaces of an m-dimensional vector space, Fₘ(Qⁿ) the Fano variety of m-planes in Qⁿ, S₁ the spinor variety, that is, an irreducible component of Fₘ(Q²k) embedded via a “square root” of the Plücker embedding, ∩(1) cutting by a general hyperplane, and ε := (λ − μ, λ − α).

#### The second is

**Corollary 2 (Cf. [R]).** Let X be a homogeneous, non-degenerate, projective variety in ℙᴺ, and let νᵈ be the d-uple embedding of ℙᴺ. If d ≥ 2 and X ≠ ℙᴺ, then νᵈ(X) has non-degenerate secants.

(see Acknowledgements)
1. A Proof of the Criterion

The criterion follows from two lemmas below.

Let \( \mathfrak{h} \) be a Cartan subalgebra of the Lie algebra \( \mathfrak{g} \) of \( G \), denote by \( \mathfrak{h}^+ \) the real vector space spanned by the roots \( R \) in the dual space \( \mathfrak{h}^\ast \). By means of the Killing form on \( \mathfrak{g} \), one can consider \( \mathfrak{h}^+ \) as an Euclidean space with inner product \( (\ast, \ast) \) such that the action of the Weyl group on \( \mathfrak{h}^+ \) is orthogonal. Denote by \( R^+ \) the set of positive roots in \( \mathfrak{h}^+ \). Let \( \alpha \) be the highest root of \( \mathfrak{g} \), and let \( \mu \) be the lowest weight of the representation \( \rho \).

Let \( W \) be the Weyl chamber, that is, \( W := \{ \omega \in \mathfrak{h}^+ | \alpha \in R^+ \Rightarrow (\omega, \alpha) \geq 0 \} \), and denote by \( w_0 \) the involution on \( \mathfrak{h}^+ \) such that \( W \) maps to \( -W \) (see \([B, VI, \S 1, n^6, Cor. 3]\)): We have \( -\tilde{\alpha} = w_0(\tilde{\alpha}) \).

For an element \( \alpha \) and a subset \( S \) of \( \mathfrak{h}^+ \), denote by \( \alpha + S \) the set \( \{ \alpha + \beta \in \mathfrak{h}^+ | \beta \in S \} \), and by \( (\alpha, S) \) the set \( \{ (\alpha, \beta) \in \mathbb{R} | \beta \in S \} \). For example, \( \max(\alpha, S) \) means \( \max\{ (\alpha, \beta) \in \mathbb{R} | \beta \in S \} \).

**Lemma 1.**

\[(\lambda - \mu, \lambda - \tilde{\alpha}) > 0 \Rightarrow (\lambda + R) \cap (\mu + R) = \emptyset.\]

**Proof.** We have that \( w_0 \) is orthogonal, \( w_0(\lambda) = \mu \) and \( w_0(R) = R \). So it follows that \( (\lambda - \mu, R) = -(\lambda - \mu, R) \) and \( (\lambda - \mu, \mu) = -(\lambda - \mu, \lambda) \), hence \( (\lambda - \mu, \mu + R) = -(\lambda - \mu, \lambda + R) \). Thus,
\[
\min(\lambda - \mu, \lambda + R) > 0 \Rightarrow (\mu + R) \cap (\lambda + R) = \emptyset.
\]

On the other hand, since \( -w_0(W) = W \) and \( \lambda \in W \), we have \( -\mu = -w_0(\lambda) \in W \), hence \( \lambda - \mu \in W \). Therefore it follows from the definition of \( W \) that if \( \alpha \in R^+ \), then \( (\lambda - \mu, \alpha) \geq 0 \). Hence, \( \max(\lambda - \mu, R) \) is attained by the highest root \( \tilde{\alpha} \) (see, for example, \([B, VI, \S 1, n^6, Proposition 25]\)), and
\[
\min(\lambda - \mu, \lambda + R) = (\lambda - \mu, \lambda - \tilde{\alpha}).
\]

\[\Box\]

**Lemma 2.**

\[\delta \leq \#\{(\lambda + R) \cap (\mu + R)\} .\]

In particular,
\[
(\lambda + R) \cap (\mu + R) = \emptyset \Rightarrow \delta = 0
\]

**Proof.** According to \([LV, p. 14]\), the deficiency \( \delta \) in characteristic zero is equal to the dimension of the intersection \( \mathfrak{g} \cdot v_\lambda \cap \mathfrak{g} \cdot v_\mu \) in \( V \):
\[
\delta = \dim(\mathfrak{g} \cdot v_\lambda \cap \mathfrak{g} \cdot v_\mu),
\]

where \( v_\lambda, v_\mu \) are weight vectors corresponding to \( \lambda \) and \( \mu \), respectively, and \( \cdot \) means the action of \( \mathfrak{g} \) on \( V \) via the differential \( d\rho \).

On the other hand, for a root \( \alpha \) of \( \mathfrak{g} \) we have \( \dim \mathfrak{g}_\alpha \cdot v_\lambda \leq 1 \) and \( \mathfrak{g}_\alpha \cdot v_\lambda \subseteq V_{\lambda + \alpha} \). From the Cartan decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \) we obtain \( \mathfrak{g} \cdot v_\lambda = \mathbb{C} \cdot v_\lambda \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta \cdot v_\lambda \) in \( V \) since \( \mathfrak{h} \cdot v_\lambda = \mathbb{C} \cdot v_\lambda \). Hence, we see that if \( \dim(\mathfrak{g}_\alpha \cdot v_\lambda \cap \mathfrak{g}_\beta \cdot v_\mu) = 1 \), then \( \lambda + \alpha = \mu + \beta \). Therefore we have
\[
\dim(\mathfrak{g} \cdot v_\lambda \cap \mathfrak{g} \cdot v_\mu) \leq \#\{(\lambda + R) \cap (\mu + R)\}.
\]

\[\Box\]
2. Candidates

**Proposition.** For a dominant weight $\lambda$ of a complex simple Lie algebra $\mathfrak{g}$, $(\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0$ if and only if the type of $\mathfrak{g}$ and $\lambda$ is one of the weights listed in Theorem.

To show this proposition, realize $\mathfrak{h}^*_R$ in a real vector space as in [B]. Then for a given $\lambda = \sum_{i=1}^l b_i \omega_i$ with non-negative integers $b_i$, one can compute the coordinates of $\lambda$ and the corresponding lowest $\mu$ in $\mathfrak{h}^*_R$ by virtue of $w_0(\lambda) = \mu$, hence those of $\lambda - \mu$ explicitly, where $\omega_1, \ldots, \omega_l$ are the fundamental weights. One can also compute those of the highest root $\tilde{\alpha}$. Thus for a weight $\lambda = \sum_{i=1}^l b_i \omega_i$, setting

$$\varepsilon := (\lambda - \mu, \lambda - \tilde{\alpha}),$$

one can write down the value $\varepsilon$ in terms of the integers $b_i$. We compute below the set of non-trivial solutions $(b_i)$ of non-negative integers $b_i$ for an inequality,

$$\varepsilon \leq 0$$

in each type of $\mathfrak{g}$.

We denote by $e_i$ the $i$-th canonical basis of $\mathbb{R}^m$ with $1 \leq i \leq m$, and consider $\sum_{i=a}^b$ void unless $a \leq b$.

**Lemma A.** For any $\lambda$ in case of type $A_l$ with $l \geq 1$, $(\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0$ if and only if $\lambda$ is one of the following: $\omega_1, 2\omega_1$ in case of $l = 1$; $\omega_1, \omega_2, 2\omega_1, 2\omega_2, \omega_1 + \omega_2$ in case of $l = 2$; $\omega_1, \omega_2, \omega_3, 2\omega_1, 2\omega_2, \omega_1 + \omega_2$ in case of $l = 3$; $\omega_1, \omega_2, \omega_{l-1}, \omega_l, 2\omega_1, 2\omega_2, \omega_1 + \omega_l$ in case of $l \geq 4$.

**Proof.** In this case, $\mathfrak{h}^*_R = \{(x_i) \in \mathbb{R}^{l+1} | \sum_{i=1}^{l+1} x_i = 0 \} \subseteq \mathbb{R}^{l+1}$ and $\tilde{\alpha} = \omega_1 + \omega_l = e_1 - e_{l+1}$. We have

$$\lambda = \sum_{i=1}^l b_i \left( \sum_{k=1}^i e_k - \frac{i}{l+1} \sum_{j=1}^{l+1} e_j \right).$$

Let $W_0$ be a linear transformation on $\mathbb{R}^{l+1}$ such that $e_i$ maps to $e_{l+2-i}$ with $1 \leq i \leq l+1$. We see from [B] that the restriction to $\mathfrak{h}^*_R$ of $W_0$ gives the involution $w_0$, and we have

$$\lambda - \mu = \sum_{k=1}^{l+1} \left( \sum_{i=k}^l b_i - \sum_{j=l+2-k}^l b_j \right) e_k.$$

It follows that

$$2(l+1)(\lambda - \mu, \lambda) = \sum_{k=1}^{l+1} \left( \sum_{i=k}^{l+1} b_i \right)^2,$$

$$2(l+1)(\lambda - \mu, \tilde{\alpha}) = 2 \sum_{k=1}^l b_k,$$

and

$$2(l+1)\varepsilon = \left( \sum_{i=1}^l b_i - 1 \right)^2 + \sum_{k=2}^{l+1} \left( \sum_{i=k}^{l+1} b_i \right)^2 - 1.$$
Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is: \\{(1), (2)\} if \(l = 1\); \\{(10), (01), (20), (02), (11)\} if \(l = 2\); \\{(100), (001), (010), (200), (002), (101)\} if \(l = 3\); \\{(10 \cdots 0), (0 \cdots 01), (010 \cdots 0), (0 \cdots 010), (20 \cdots 0), (0 \cdots 02), (10 \cdots 01)\} if \(l \geq 4\).

**Lemma B.** For any \(\lambda\) in case of type \(B_l\) with \(l \geq 2\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda\) is one of the following: \(\omega_1, \omega_2, 2\omega_2\) in case of \(l = 2\); \(\omega_1, \omega_2, \omega_l\) in case of \(l = 3\), 4; \(\omega_1, \omega_2\) in case of \(l \geq 5\).

**Proof.** In this case, \(h^*_R = \mathbb{R}^l, \tilde{\alpha} = e_1 + e_2\) and \(w_0 = -1\). We have
\[
\lambda - \mu = 2\lambda = \sum_{k=1}^{l} \left( \sum_{i=k}^{l} b_i - b_l \right) e_k.
\]
It follows that
\[
2(2l - 1)(\lambda - \mu, \lambda) = \frac{1}{2} \sum_{k=1}^{l} \left( \sum_{i=k}^{l} b_i - b_l \right)^2,
\]
and
\[
2(2l - 1)(\lambda - \mu, \tilde{\alpha}) = \sum_{k=1}^{l} \left( \sum_{i=k}^{l} b_i \right)^2 - 2.
\]
For the case \(l = 2\), we have
\[
12\varepsilon = (2b_1 + b_2 - 1)^2 + (b_2 - 1)^2 - 2.
\]
For any \(l \geq 3\), we have
\[
4(2l - 1)\varepsilon = \left( \sum_{i=1}^{l-1} b_i + b_l - 1 \right)^2 + \left( \sum_{i=2}^{l-1} b_i + b_l - 1 \right)^2
+ \sum_{k=3}^{l-1} \left( \sum_{i=k}^{l-1} b_i + b_l \right)^2 + b_l^2 - 2.
\]
Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is: \\{(10), (01), (02)\} if \(l = 2\); \\{(10 \cdots 0), (010 \cdots 0), (0 \cdots 01)\} if \(l = 3\), 4; \\{(10 \cdots 0), (010 \cdots 0)\} if \(l \geq 5\).

**Lemma C.** For any \(\lambda\) in case of type \(C_l\) with \(l \geq 3\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda = \omega_1, \omega_2\) or \(2\omega_1\).

**Proof.** In this case, \(h^*_R = \mathbb{R}^l, \tilde{\alpha} = 2e_1\) and \(w_0 = -1\). We have
\[
\lambda - \mu = 2\lambda = \sum_{k=1}^{l} \left( \sum_{i=k}^{l} b_i \right) e_k.
\]
It follows that
\[
4(l + 1)(\lambda - \mu, \lambda) = 2 \sum_{k=1}^{l} \left( \sum_{i=k}^{l} b_i \right)^2,
\]
\[
4(l + 1)(\lambda - \mu, \tilde{\alpha}) = 4 \sum_{i=1}^{l} b_i.
\]
and

\[ 2(l + 1)\varepsilon = \left( \sum_{i=1}^{l} b_i - 1 \right)^2 + \sum_{k=2}^{l} \left( \sum_{i=k}^{l} b_i \right)^2. \]

Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is \{ \((10\cdots0), \ (010\cdots0), \ (20\cdots0)\) \}. \(\Box\)

**Lemma D.** For any \(\lambda\) in case of type \(D_l\) with \(l \geq 4\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda\) is one of the following: \(\omega_1, \omega_2, \omega_{l-1}, \omega_l\) in case of \(l = 4, 5\); \(\omega_1, \omega_2\) in case of \(l \geq 6\).

**Proof.** In this case, \(H^{\alpha}_l = \mathbb{R}^l\) and \(\tilde{\alpha} = e_1 + e_2\). We have

\[ \lambda = \sum_{k=1}^{l} \left( \sum_{i=k}^{l} b_i - \frac{b_{l-1} + b_l}{2} \right) e_k. \]

In case of even \(l\) with \(l \geq 4\), we have \(w_0 = -1\) and

\[ \lambda - \mu = 2\lambda = \sum_{k=1}^{l} \left( 2 \sum_{i=k}^{l} b_i - b_{l-1} - b_l \right) e_k. \]

In case of odd \(l\) with \(l \geq 5\), we see from [B] that \(w_0\) is equal to a linear transformation of \(\mathbb{R}^l\) such that \(e_i\) maps to \(-e_i\) with \(1 \leq i \leq l-1\) and \(e_l\) maps to \(e_l\), and we have

\[ \lambda - \mu = \sum_{k=1}^{l-1} \left( 2 \sum_{i=k}^{l} b_i - b_{l-1} - b_l \right) e_k. \]

For any \(l \geq 4\) we have

\[ 4(l - 1)(\lambda - \mu, \lambda) = \frac{1}{2} \sum_{k=1}^{2\lfloor l/2 \rfloor} \left( 2 \sum_{i=k}^{l} b_i - b_{l-1} - b_l \right)^2, \]

and

\[ 4(l - 1)(\lambda - \mu, \tilde{\alpha}) = \sum_{k=1}^{2} \left( 2 \sum_{i=k}^{l} b_i - b_{l-1} - b_l \right). \]

Therefore, for even \(l\) we have

\[ 8(l - 1)\varepsilon = \left( 2 \sum_{i=1}^{l-2} b_i + b_{l-1} + b_l - 1 \right)^2 + \left( 2 \sum_{i=2}^{l-2} b_i + b_{l-1} + b_l - 1 \right)^2 \]

\[ + \sum_{k=3}^{l-2} \left( 2 \sum_{i=k}^{l-2} b_i + b_{l-1} + b_l \right)^2 + 2b_{l-1}^2 + 2b_l^2 - 2, \]

and for odd \(l\) we have

\[ 8(l - 1)\varepsilon = \left( 2 \sum_{i=1}^{l-2} b_i + b_{l-1} + b_l - 1 \right)^2 + \left( 2 \sum_{i=2}^{l-2} b_i + b_{l-1} + b_l - 1 \right)^2 \]

\[ + \sum_{k=3}^{l-2} \left( 2 \sum_{i=k}^{l-2} b_i + b_{l-1} + b_l \right)^2 + (b_{l-1} + b_l)^2 - 2. \]
Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is: \(\{(10 \cdots 0), (010 \cdots 0), (0 \cdots 010), (0 \cdots 01)\}\) if \(l = 4, 5\); \(\{(10 \cdots 0), (010 \cdots 0)\}\) if \(l \geq 6\).

Lemma \(E_6\). For any \(\lambda\) in case of type \(E_6\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda = \omega_1, \omega_2\) or \(\omega_6\).

Proof. In this case,
\[
\mathfrak{h}_k^* = \{(x_i) \in \mathbb{R}^8 | x_6 = x_7 = -x_8 \} \subseteq \mathbb{R}^8
\]
and
\[
\tilde{\alpha} = \frac{1}{2} \left( \sum_{i=1}^5 e_i - e_6 - e_7 + e_8 \right).
\]

We have
\[
\lambda = \frac{1}{2}(b_2 - b_3) e_1 + \frac{1}{2}(b_2 + b_3) e_2 + \left( \frac{1}{2}(b_2 + b_3) + b_4 \right) e_3
\]
\[
+ \left( \frac{1}{2}(b_2 + b_3) + b_4 + b_5 \right) e_4
\]
\[
+ \left( \frac{1}{2}(b_2 + b_3) + b_4 + b_5 + b_6 \right) e_5
\]
\[
+ \left( \frac{2}{3} b_1 + \frac{1}{2} b_2 + \frac{5}{6} b_3 + b_4 + \frac{2}{3} b_5 + \frac{1}{3} b_6 \right) (-e_6 - e_7 + e_8).
\]

Let \(W_0\) be a linear transformation on \(\mathbb{R}^8\) defined by a matrix
\[
-\frac{1}{2} \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix},
\]
where we set
\[
W := \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.
\]

We see from [B] that the restriction to \(\mathfrak{h}_k^*\) of \(W_0\) gives the involution \(w_0\) (To obtain this form of matrix \(W_0\) representing \(w_0\), impose an extra condition that the linear transformation leaves \(e_5 + e_7\) and \(e_6 + e_8\) invariant). Using \(W_0\), we have
\[
\lambda - \mu = \left( b_2 - \frac{1}{2}(b_3 + b_5) \right) e_1 + \left( b_2 + \frac{1}{2}(b_3 + b_5) \right) e_2
\]
\[
+ \left( b_2 + 2b_4 + \frac{1}{2}(b_3 + b_5) \right) e_3 + \left( b_2 + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) e_4
\]
\[
+ \left( b_2 + (b_1 + b_6) + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) (e_5 - e_6 - e_7 + e_8).
\]
It follows that
\[
24(\lambda - \mu, \lambda) = \left( b_2 - \frac{1}{2}(b_3 + b_5) \right) \left( \frac{1}{2}(b_2 - b_3) \right) \\
+ \left( b_2 + \frac{1}{2}(b_3 + b_5) \right) \left( \frac{1}{2}(b_2 - b_3) \right) \\
+ \left( b_2 + 2b_4 + \frac{1}{2}(b_3 + b_5) \right) \left( \frac{1}{2}(b_2 + b_3) + b_4 \right) \\
+ \left( b_2 + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) \left( \frac{1}{2}(b_2 + b_3) + b_4 + b_5 \right) \\
+ \left( b_2 + (b_1 + b_6) + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) \\
\times \left( \frac{1}{2}(b_2 + b_3) + b_4 + b_5 + b_6 \right) \\
+ 3 \left( b_2 + (b_1 + b_6) + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) \\
\times \left( \frac{2}{3}b_1 + \frac{1}{2}b_2 + \frac{5}{6}b_3 + b_4 + \frac{2}{3}b_5 + \frac{1}{3}b_6 \right) \\
= b_2^2 + \frac{1}{2}(b_3 + b_5)^2 + (b_2 + b_3 + 2b_4 + b_5)^2 \\
+ \frac{1}{2}(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6)^2,
\]

\[
24(\lambda - \mu, \tilde{\alpha}) = \frac{1}{2} \left\{ \left( b_2 - \frac{1}{2}(b_3 + b_5) \right) + \left( b_2 + \frac{1}{2}(b_3 + b_5) \right) \\
+ \left( b_2 + 2b_4 + \frac{1}{2}(b_3 + b_5) \right) + \left( b_2 + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) \\
+ 4 \left( b_2 + (b_1 + b_6) + 2b_4 + \frac{3}{2}(b_3 + b_5) \right) \right\} \\
= b_2 + (b_2 + b_3 + 2b_4 + b_5) + (2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6),
\]

and
\[
24\varepsilon = \left( b_2 - \frac{1}{2} \right)^2 + \frac{1}{2}(b_3 + b_5)^2 + \left( b_2 + b_3 + 2b_4 + b_5 - \frac{1}{2} \right)^2 \\
+ \frac{1}{2}(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 - 1)^2 - 1.
\]

Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is \(\{(100000), (010000), (000001)\}\).

**Lemma E7.** For any \(\lambda\) in case of type \(E_7\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda = \omega_1\).
Proof. In this case, \( \mathfrak{b}_8^* = \{ (x_i) \in \mathbb{R}^8 | x_7 + x_8 = 0 \} \subseteq \mathbb{R}^8 \), \( \bar{\alpha} = -e_7 + e_8 \) and \( w_0 = -1 \).

We have

\[
\lambda - \mu = 2\lambda = (b_2 - b_3)e_1 + (b_2 + b_3)e_2 \\
+ (b_2 + b_3 + 2b_4)e_3 + (b_2 + b_3 + 2b_4 + 2b_5)e_4 \\
+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)e_6 \\
+ (2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7)(-e_7 + e_8).
\]

It follows that

\[
36(\lambda - \mu, \lambda) = \frac{1}{2} \{ (b_2 - b_3)^2 + (b_2 + b_3)^2 \\
+ (b_2 + b_3 + 2b_4)^2 + (b_2 + b_3 + 2b_4 + 2b_5)^2 \\
+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)^2 \\
+ 2(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7)^2 \},
\]

and

\[
72\varepsilon = (b_2 - b_3)^2 + (b_2 + b_3)^2 \\
+ (b_2 + b_3 + 2b_4)^2 + (b_2 + b_3 + 2b_4 + 2b_5)^2 \\
+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)^2 \\
+ 2(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7 - 1)^2 - 2.
\]

Thus the set of non-trivial solutions \((b_1)\) for \( \varepsilon \leq 0 \) is \( \{(1000000)\} \).

\[ \square \]

Lemma \( E_8 \). For any \( \lambda \) in case of type \( E_8 \), \( (\lambda - \mu, \lambda - \bar{\alpha}) \leq 0 \) if and only if \( \lambda = \omega_8 \).

Proof. In this case, \( \mathfrak{b}_8^* = \mathbb{R}^8 \), \( \bar{\alpha} = e_7 + e_8 \) and \( w_0 = -1 \). We have

\[
\lambda - \mu = 2\lambda = (b_2 - b_3)e_1 + (b_2 + b_3)e_2 \\
+ (b_2 + b_3 + 2b_4)e_3 + (b_2 + b_3 + 2b_4 + 2b_5)e_4 \\
+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)e_6 \\
+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7 + 2b_8)e_7 \\
+ (4b_1 + 5b_2 + 7b_3 + 10b_4 + 8b_5 + 6b_6 + 4b_7 + 2b_8)e_8.
\]
It follows that

\[ 60(\lambda - \mu, \lambda) = \frac{1}{2}((b_2 - b_3)^2 + (b_2 + b_3)^2) \]
\[ + (b_2 + b_3 + 2b_1)^2 + (b_2 + b_3 + 2b_4 + 2b_5)^2 \]
\[ + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6)^2 \]
\[ + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)^2 \]
\[ + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7 + 2b_8)^2 \]
\[ + (4b_1 + 5b_2 + 7b_3 + 10b_4 + 8b_5 + 6b_6 + 4b_7 + 2b_8)^2 \}, \]
\[ 60(\lambda - \mu, \tilde{\alpha}) = (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7 + 2b_8) \]
\[ + (4b_1 + 5b_2 + 7b_3 + 10b_4 + 8b_5 + 6b_6 + 4b_7 + 2b_8), \]

and

\[ 120\varepsilon = (b_2 - b_3)^2 + (b_2 + b_3)^2 \]
\[ + (b_2 + b_3 + 2b_1)^2 + (b_2 + b_3 + 2b_4 + 2b_5)^2 \]
\[ + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6)^2 + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)^2 \]
\[ + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7 + 2b_8 - 1)^2 \]
\[ + (4b_1 + 5b_2 + 7b_3 + 10b_4 + 8b_5 + 6b_6 + 4b_7 + 2b_8 - 1)^2 - 2. \]

Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is \(\{(000000001)\}. \quad \square \)

**Lemma F.** For any \(\lambda\) in case of type \(F_4\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda = \omega_1\) or \(\omega_2\).

**Proof.** In this case, \(h^*_R = \mathbb{R}^4\), \(\tilde{\alpha} = e_1 + e_2\) and \(w_0 = -1\). We have

\[ \lambda - \mu = 2\lambda = (2b_1 + 4b_2 + 3b_3 + 2b_4)e_1 + (2b_1 + 2b_2 + b_3)e_2 + (2b_2 + b_3)e_3 + b_3e_4. \]

It follows that

\[ 18(\lambda - \mu, \lambda) = \frac{1}{2}\{(2b_1 + 4b_2 + 3b_3 + 2b_4)^2 + (2b_1 + 2b_2 + b_3)^2 + (2b_2 + b_3)^2 + b_3^2\}, \]
\[ 18(\lambda - \mu, \tilde{\alpha}) = (2b_1 + 4b_2 + 3b_3 + 2b_4) + (2b_1 + 2b_2 + b_3), \]

and

\[ 36\varepsilon = (2b_1 + 4b_2 + 3b_3 + 2b_4 - 1)^2 + (2b_1 + 2b_2 + b_3 - 1)^2 + (2b_2 + b_3)^2 + b_3^2 - 2. \]

Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is \(\{(1000), (0001)\}. \quad \square \)

**Lemma G.** For any \(\lambda\) in case of type \(G_2\), \((\lambda - \mu, \lambda - \tilde{\alpha}) \leq 0\) if and only if \(\lambda = \omega_1\) or \(\omega_2\).

**Proof.** In this case, \(h^*_R = \{(x_i) \in \mathbb{R}^3| \sum_{i=1}^{3} x_i = 0\} \subseteq \mathbb{R}^3\), \(\tilde{\alpha} = -e_1 - e_2 + 2e_3\) and \(w_0 = -1\). We have

\[ \lambda - \mu = 2\lambda = 2\{-b_2e_1 - (b_1 + b_2)e_2 + (b_1 + 2b_2)e_3\}. \]

It follows that

\[ 24(\lambda - \mu, \lambda) = 2\{(-b_2)^2 + ((b_1 + b_2))^2 + (b_1 + 2b_2)^2\}, \]
\[ 24(\lambda - \mu, \tilde{\alpha}) = 2\{b_2 + (b_1 + b_2) + 2(b_1 + 2b_2)\}, \]
and
\[ 12\varepsilon = \left( b_2 - \frac{1}{2} \right)^2 + \left( b_1 + b_2 - \frac{1}{2} \right)^2 + (b_1 + 2b_2 - 1)^2 - \frac{3}{2}. \]

Thus the set of non-trivial solutions \((b_i)\) for \(\varepsilon \leq 0\) is \(\{(10), (01)\}\).

### 3. Proofs of main results

**Proof of Theorem.** If \(X\) corresponding to \(\lambda\) has \(\delta > 0\), then it follows from the Criterion and Proposition that \(\lambda\) is one of the dominant weights listed in the statement of the Theorem.

We show the converse. For the adjoint representation, the required results follow from one of the main theorems in [KOY], which with the same notations as in the Introduction asserts that *if G is simple and of rank \(\geq 2\), and if \(\rho\) is the adjoint representation, then the corresponding variety \(X\) has \(\delta = 1\).* For the other cases, using well-known facts [FH] and results of Zak [LV, Appendix], [Z], one can show that each \(X\) has \(\delta > 0\): for \(\omega_1, \omega_6\) in case of \(E_6\), the corresponding variety \(X\) is well-known as the Severi variety of the largest dimension, and \(X\) for \(\omega_4\) in case of \(F_4\) is its hyperplane section; for the remaining dominant weights, the corresponding variety \(X\) is either a projective space, its Veronese embedding, a quadric hypersurface, a Grassmann variety of lines in a projective space, its hyperplane section, or a spinor variety.

Moreover, using results in [FH], [KOY], [LV], [Z], one can verify that a projective variety \(X\) obtained from each weight listed in the Theorem enjoys the properties stated in our table.

Finally we prove two results for arbitrary homogeneous projective varieties mentioned in the Introduction. Without loss of generality, we may assume for any homogeneous projective variety \(X\) that \(X\) is obtained from an irreducible representation of a semi-simple algebraic group \(G\) (see [Z, III, §1], [FH, Prop. 9.17]).

**Proof of Corollary 1.** Suppose that \(\text{Sec} X \neq \mathbb{P}\) and \(\delta > 8\) for some \(X\). According to a result of Zak [Z, III, Corollary 1.7], if \(G\) were not simple, and if \(\delta > 0\), then \(\delta = 2\); this is a contradiction. Hence \(G\) must be simple. But we see from our table that there is no such \(X\).

**Proof of Corollary 2.** Suppose that \(\nu_d(X)\) has \(\delta > 0\) for some \(X\) and \(d \geq 2\). According to a result of Zak [Z, III, Corollary 1.7], if \(G\) were not simple, and if \(\delta > 0\), then \(X\) would be isomorphic to some Segre product \(\mathbb{P}^a \times \mathbb{P}^b\), hence \(\mathcal{O}_X(d) = \mathcal{O}_{\nu_d(X)}(1) = \mathcal{O}_{\mathbb{P}^a}(1) \boxtimes \mathcal{O}_{\mathbb{P}^b}(1)\); This is a contradiction since the last line bundle is indivisible by \(d \geq 2\). Hence \(G\) is simple.

If \(X\) is corresponding to \(\lambda\), then its \(d\)-uple embedding \(\nu_d(X)\) is corresponding to \(d\lambda\). In our table, dominant weights of the form \(d\lambda\) for some \(d \geq 2\) and for some dominant weight \(\lambda\), are \(2\omega_1, 2\omega_7\) in case of \(A_l\), \(2\omega_2\) in case of \(B_l\), and \(2\omega_4\) in case of \(C_l\), and all \(X\) in those cases are projective spaces.

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**References**


Department of Mathematics, School of Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169, Japan

E-mail address: kaji@mse.waseda.ac.jp