ON THE DIOPHANTINE EQUATION
\[ \frac{x^3 - 1}{x - 1} = \frac{y^n - 1}{y - 1} \]

MAOHUA LE

Abstract. In this paper we prove that the equation \( \frac{x^3 - 1}{x - 1} = \frac{y^n - 1}{y - 1} \), \( x, y, n \in \mathbb{N} \), \( x > 1, y > 1, n > 3 \), has only the solutions \( (x, y, n) = (5, 2, 5) \) and \( (90, 2, 13) \) with \( y \) is a prime power. The proof depends on some new results concerning the upper bounds for the number of solutions of the generalized Ramanujan-Nagell equations.

1. Introduction

Let \( \mathbb{Z}, \mathbb{N}, \mathbb{Q} \) be the sets of integers, positive integers and rational numbers respectively. For any positive integer \( N \) with \( N > 2 \), let \( s(N) \) denote the number of solutions \( (x, m) \) of the equation
\[ N = \frac{x^m - 1}{x - 1}, \quad x, m \in \mathbb{N}, \quad x > 1, \quad m > 2. \] Eighty years ago, Ratat [13] and Goormaghtigh [4] observed that \( s(31) = 2 \) and \( s(8191) = 2 \), respectively. Simultaneously, they conjectured that if \( N \not\in \{31, 8191\} \), then \( s(N) \leq 1 \). The problem can be written in the following form

Conjecture A. The equation
\[ \frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \quad x, y, m, n \in \mathbb{N}, \quad x > 1, \quad y > 1, \quad n > m > 2, \]
has only the solutions \( (x, y, m, n) = (5, 2, 3, 5) \) and \( (90, 2, 3, 13) \).

In addition, Bateman (see [5, Problem B25]) asked the following problem.

Problem B. Is \( (x, y, m, n) = (5, 2, 3, 5) \) the only solution of (2) for which \( x, y \) and \( (x^m - 1)/(x - 1) \) are all prime powers?

These are two rather difficult questions. By the results of Baker, Davenport, Lewis, Schinzel, Shorey and Tijdeman (see the references of [14]), we see that (2) has only finitely many solutions \( (x, y, m, n) \) if any two out of the four variables \( x, y, m \) and \( n \) are fixed. However, to this day we do not know whether (2) has only finitely many solutions \( (x, y, m, n) \) if any one out of \( x, y, m \) and \( n \) is fixed.

In this paper we discuss the finiteness of solutions of (2) by the means of the generalized Ramanujan-Nagell equations. Let \( D_1, D_2 \) be coprime positive integers,
and let $p$ be a prime with $p \nmid D_1D_2$. Further let $N(D_1, D_2, p)$ and $N'(D_1, D_2, p)$ denote the numbers of the solutions $(r, s)$ and $(r', s')$ of equations

\[(3) \quad D_1r^2 + D_2 = p^s, \quad r, s \in \mathbb{N}, \]

and

\[(4) \quad D_1r'^2 + D_2 = 2p^{s'}, \quad r', s' \in \mathbb{N}, \]

respectively. In this paper we prove the following two general results.

**Theorem 1.** $N(D_1, D_2, p) \leq 2$ except for $N(1, 7, 2) = 5$ and $N(3, 5, 2) = 3$.

**Theorem 2.** If $2 \nmid D_1D_2$ and $p$ is an odd prime, then $N'(D_1, D_2, p) \leq 2$.

On applying the above-mentioned theorems to the equation (2), we shall deduce the following result.

**Corollary.** The equation (2) has only the solutions $(x, y, m, n) = (5, 2, 3, 5)$ and $(90, 2, 3, 13)$ such that $m = 3$ and $y$ is a prime power.

## 2. Proof of Theorem 1

First we may assume that $D_1$ is squarefree. In [1] and [2], Apéry proved that $N(1, D_2, p) \leq 2$ except for $N(1, 7, 2) = 5$. In [7] and [8], the author proved that $N(D_1, 1, p) \leq 2$ and $N(D_1, D_2, 2) \leq 2$ except for $N(1, 7, 2) = 5$ and $N(3, 5, 2) = 3$, respectively. Therefore, we may assume that $D_1 > 1$, $D_2 > 1$ and $p$ is an odd prime.

**Lemma 1 ([10, Formula 3.76]).** For any positive integer $t$ and any complex numbers $\alpha$ and $\beta$, we have

\[
\alpha^t + \beta^t = \sum_{i=0}^{\lfloor t/2 \rfloor} (-1)^i \binom{t}{i} (\alpha + \beta)^{t-2i}(\alpha\beta)^i,
\]

where $\lfloor t/2 \rfloor$ is the largest integer which does not exceed $t/2$,

\[
\binom{t}{i} = \frac{(t - i - 1)!t}{(t - 2i)!i!}, \quad i = 0, 1, \ldots, \left\lfloor \frac{t}{2} \right\rfloor,
\]

are positive integers.

**Lemma 2 ([9, Lemma 4]).** If the equation

\[(5) \quad D_1X^2 + D_2Y^2 = p^z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0,
\]

has solutions $(X, Y, Z)$, then it has a unique solution $(X_1, Y_1, Z_1)$ such that $X_1 > 0$, $Y_1 > 0$ and $Z_1 \leq Z$, where $Z$ runs through all solutions $(X, Y, Z)$ of (5). $(X_1, Y_1, Z_1)$ is called the least solution of (5). Further, every solution $(X, Y, Z)$ of (5) can be expressed as

\[
Z = Z_1t, \quad X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1(X_1\sqrt{D_1} + \lambda_2Y_1\sqrt{-D_2})^t,
\]

$t \in \mathbb{N}, t \nmid D_1, \lambda_1, \lambda_2 \in \{-1, 1\}$.

**Lemma 3.** If $N(D_1, D_2, p) > 2$, then (3) has two solutions $(r_1, s_1)$ and $(r_2, s_2)$ such that

\[(6) \quad s_1 = Z_1t_1, \quad s_2 = Z_1t_2, \quad 1 < t_1 < t_2,
\]
where \((X_1, Y_1, Z_1)\) is the least solution of (5), \(t_1\) and \(t_2\) are odd primes satisfying

\[
\sum_{i=0}^{(t_j-1)/2} \left[ \frac{t_j}{i} \right] (-4D_2)^{t_j-1/2-i}p^{Z_ji} = (-1)^{(t_j-1)(p^{Z_1})/4}, \quad j = 1, 2,
\]

\[
\frac{\pi}{2 \arcsin(D_2/p^{Z_1})} < t_2
\]

and

\[
\left| \arcsin \left( \frac{D_2}{p^{Z_1}} \right)^{1/2} - \frac{k\pi}{t_j} \right| < \frac{\pi}{2t_jp^{Z_1(t_j-1)/2}}, \quad j = 1, 2,
\]

where \(k\) is a positive integer satisfying \(k \leq (t_j - 1)/2\).

Proof. We see that if (3) has a solution \((r, s)\), then (5) has a solution \((X, Y, Z) = (r, 1, s)\). It follows from Lemma 2 that the least solution \((X_1, Y_1, Z_1)\) of (5) satisfies \(Y_1 = 1\). Then \((r, s) = (X_1, Z_1)\) is a solution of (3). Further, let

\[
\varepsilon = X_1 \sqrt{D_1} + \sqrt{-D_2}, \quad \bar{\varepsilon} = X_1 \sqrt{D_1} - \sqrt{-D_2}.
\]

By the proof of [9, Theorem 1], if \((r, s)\) is a solution of (3) with \((r, s) \neq (X_1, Z_1)\), then there exists a positive integer \(t\) satisfying \(t > 1, 2 \nmid t\),

\[
s = Z_1t
\]

and

\[
\left| \frac{\varepsilon^t - \bar{\varepsilon}^t}{\varepsilon - \bar{\varepsilon}} \right| = 1.
\]

Therefore, by (11), if \(N(D_1, D_2, p) > 2\), then (3) has two solutions \((r_1, s_1)\) and \((r_2, s_2)\) satisfying (6). Moreover, we may assume that (3) has no solutions \((r, s)\) satisfying

\[
Z_1 < s < s_1, \quad s_1 < s < s_2.
\]

Since \(\varepsilon - \bar{\varepsilon} = 2\sqrt{-D_2}\) and \(\varepsilon \bar{\varepsilon} = p^{Z_1}\), by Lemma 1, we get from (12) that

\[
\frac{\varepsilon^t - \bar{\varepsilon}^t}{\varepsilon - \bar{\varepsilon}} = \sum_{i=0}^{(t-1)/2} (-1)^i \left[ \frac{t}{i} \right] (\varepsilon - \bar{\varepsilon})^{t-2i-1}(-\varepsilon \bar{\varepsilon})^i = \sum_{i=0}^{(t-1)/2} \left[ \frac{t}{i} \right] (-4D_2)^{(t-1)/2-i}p^{Z_1} \]

\[= \pm 1.
\]

Since \(t = (-1)^{(t-1)/2}(mod 4)\) and \(p^{Z_1} \equiv (-1)(p^{Z_1-1})/2 (mod 4)\), we obtain (7) from (14).

For any positive integer \(t\) with \(2 \nmid t\), let

\[
X_t = \left| \frac{\varepsilon^t + \bar{\varepsilon}^t}{2\sqrt{D_1}} \right|, \quad Y_t = \left| \frac{\varepsilon^t - \bar{\varepsilon}^t}{\varepsilon - \bar{\varepsilon}} \right|.
\]

By Lemma 1, \(X_t\) and \(Y_t\) are positive integers satisfying

\[
D_1X_t^2 + D_2Y_t^2 = p^{Z_1t}.
\]

Further, by Lemma 2, we see from (16) that \((r, s)\) is a solution of (3) satisfying (11) if and only if \(Y_1 = 1\). From (7), we get

\[
Y_1 = Y_2 = 1.
\]
If $t_1$ is not an odd prime, then $t_1 = k_1k_2$, where $k_1$ and $k_2$ are positive integers satisfying $k_1 > 1$, $k_2 > 1$ and $2 \nmid k_1k_2$. On applying Lemma 1, we find from (17) that
\begin{equation}
1 = Y_{t_1} = Y_{k_1} \left| \frac{e^{(k_1)k_2} - \overline{e^{(k_1)k_2}}}{e^{k_1} - \overline{e^{k_1}}} \right| = Y_{k_1} \left| \sum_{i=0}^{(k_2-1)/2} \frac{k_2}{i} (-4D_2Y_{k_1}^{2})^{(k_2-1)/2-i}p^{Z_1k_1i} \right|.
\end{equation}

Hence, we get from (18) that $Y_{k_i} = Y_{t_1}/Y_{k_i} = 1$. It implies that (3) has a solution $(r, s) = (X_{k_1}, Z_{k_1})$ satisfying $Z_1 < s < s_1$, which contradicts (13). Thus $t_1$ must be an odd prime.

By the same argument, if $t_2$ is not an odd prime, then from (13) we get $t_2 = t_1^2$ and $Y_{t_2}/Y_{t_1} = 1$. In this case, by Lemma 1, we have
\begin{equation}
\frac{Y_{t_2}}{Y_{t_1}} = \left| \frac{\varphi_2^2 - \overline{\varphi_2^2}}{\varphi_1^2 - \overline{\varphi_1^2}} \right| = \left| \sum_{i=0}^{(t_1-1)/2} \frac{t_1}{i} (-4D_2)^{(t_1-1)/2-i}p^{Z_1t_1i} \right| = 1.
\end{equation}

Since $(-1)^{(t_1-1)(p^{2Z_1}+1)/4} = (-1)^{(t_1-1)(p^{2Z_1}+1)/4}$, we get from (19) that
\begin{equation}
\sum_{i=0}^{(t_1-1)/2} \frac{t_1}{i} (-4D_2)^{(t_1-1)/2-i}p^{Z_1t_1i} = (-1)^{(t_1-1)(p^{2Z_1}+1)/4} = 0 \pmod{p^{3Z_1}}.
\end{equation}

Further, since $t_1 \geq 3$, we see from (20) that
\begin{equation}
(-4D_2)^{(t_1-1)/2} - (-1)^{(t_1-1)(p^{2Z_1}+1)/4} \equiv 0 \pmod{p^{3Z_1}}.
\end{equation}

On the other hand, by (7), we have
\begin{equation}
\left((-4D_2)^{(t_1-1)/2} - (-1)^{(t_1-1)(p^{2Z_1}+1)/4}\right) + \left[t_1 \atop 1\right] (-4D_2)^{(t_1-3)/2}p^{Z_1}
\end{equation}
\begin{equation}
+ \left[t_1 \atop 2\right] (-4D_2)^{(t_1-5)/2}p^{2Z_1} \equiv 0 \pmod{p^{3Z_1}}.
\end{equation}

The combination of (21) and (22) yields
\begin{equation}
t_1 \left(4D_2 - \left(t_1 - \frac{3}{2}\right)p^{Z_1} \right) \equiv 0 \pmod{p^{2Z_1}}.
\end{equation}

Since $p \nmid D_2$ and $t_1$ is an odd prime, (23) is impossible. Thus $t_2$ is an odd prime too.

By (10), we have
\begin{equation}
\epsilon = p^{Z_1/2}e^{\theta \sqrt{-1}}, \quad z = p^{Z_1/2}e^{-\theta \sqrt{-1}},
\end{equation}
where $\theta$ is a real number satisfying
\begin{equation}
\sin \theta = \frac{\epsilon - z}{2p^{Z_1/2} \sqrt{-1}} = \left( \frac{D_2}{p^{Z_1}} \right)^{1/2}.
\end{equation}

Since $0 < D_2/p^{Z_1} < 1$, we may assume that $\theta$ satisfies
\begin{equation}
0 < \theta < \frac{\pi}{2}.
\end{equation}

Further, since $Y_{t_1} = Y_{t_2} = 1$ by (17), we get from (24) and (25) that
\begin{equation}
\sin t_j \theta = (-1)^{(t_j-1)(p^{2Z_1}+1)/4} \frac{\sin \theta}{p^{Z_1(t_j-1)/2}}, \quad j = 1, 2.
\end{equation}
Hence
\[(28)\quad t_j \theta = k \pi + (-1)^{k+t_j-1} Z^{1/4} \phi_j, \quad j = 1, 2,\]
where \(k\) is an integer, \(\phi_j (j = 1, 2)\) are positive numbers satisfying
\[(29)\quad \sin \phi_j = \frac{\sin \theta}{p^Z Z^{1/2}}, \quad 0 < \phi_j < \frac{\pi}{2}, \quad j = 1, 2.\]
Notice that \(0 < \theta < \pi/2\) by (26). Since \(0 < \phi_j < \theta < \pi/2\) \((j = 1, 2)\) by (26) and (29), we see from (28) that \(k\) satisfies \(1 \leq k \leq (t_j - 1)/2\). Thus, by (25), (28) and (29), we get
\[\left| \arcsin \left( \frac{D_2}{p^Z} \right)^{1/2} - \frac{k \pi}{t_j} \right| = \left| \theta - \frac{k \pi}{t_j} \right| = \frac{\phi_j}{t_j} = \frac{\arcsin((\sin \theta)/p^Z (Z^{1/2}))}{t_j} < \frac{\pi}{2 t_j p^Z (Z^{1/2})}, \quad j = 1, 2.\]
The inequality (9) is proved.

From (25) and (27), we get
\[(30)\quad |\sin t_1 \theta| = \left( \frac{D_2}{p^Z} \right)^{1/2}, \quad |\sin t_2 \theta| = \left( \frac{D_2}{p^Z} \right)^{1/2}.\]
By (30), there exist suitable nonnegative integers \(k_1\) and \(k_2\) such that
\[(31)\quad |k_1 \pi - t_1 \theta| = \arcsin \left( \frac{D_2}{p^Z} \right)^{1/2}, \quad |k_2 \pi - t_2 \theta| = \arcsin \left( \frac{D_2}{p^Z} \right)^{1/2}.\]
Since \(s_1 < s_2\) and \(\arcsin(D_2/p^Z)^{1/2} > \arcsin(D_2/p^Z)^{1/2}\), we get from (31) that
\[(32)\quad 0 < \left| \frac{k_1}{t_1} - \frac{k_2}{t_2} \right| \leq \frac{1}{t_1} \arcsin \left( \frac{D_2}{p^Z} \right)^{1/2} + \frac{1}{t_2} \arcsin \left( \frac{D_2}{p^Z} \right)^{1/2} < \frac{2}{t_1} \arcsin \left( \frac{D_2}{p^Z} \right)^{1/2}.\]
Notice that \(|k_1/t_1 - k_2/t_2| \geq 1/t_1 t_2\) if \(k_1/t_1 \neq k_2/t_2\). We obtain (8) from (32). The lemma is proved.

**Lemma 4.** If (3) has two solutions \((r_1, s_1)\) and \((r_2, s_2)\) satisfying (6) with \(t_1 = 3\), then we have
\[(33)\quad 4D_2 = 3p^{Z_1} + (-1)(p^{Z_1-1})^{1/2}\]
and
\[(34)\quad t_2 \geq 2p^{2Z_1} + 3.\]

**Proof.** Since \(t_1 = 3\), we get (33) from (7). Let \(\delta = (-1)(p^{Z_1+1})^{1/2}\). We get from (17) and (33) that
\[(-3p^{Z_1} + \delta) (t_2-1)/2 + t_2(-3p^{Z_1} + \delta) (t_2-3)/2) (3p^{Z_1}) \equiv \delta (t_2-1)/2 \pmod{p^{2Z_1}},\]
whence we obtain \((t_2 - 3)/2 \equiv 0 \pmod{p^{Z_1}}\). By (7) again, we get
\[(35)\quad -(t_2 - 3)/2 + \left( \frac{(t_2 - 1)/2}{2} \right) - 3 \left[ t_2 \left( \frac{(t_2 - 3)/2}{2} \right) + \left[ t_2 \right] \right] \delta p^{Z_1} \equiv 0 \pmod{p^{2Z_1}}.\]
Notice that
\[
\left(\frac{(t_2 - 1)/2}{2}\right) \equiv \left(\frac{(t_2 - 3)/2}{1}\right) \equiv \left[\frac{t_2}{2}\right] \equiv 0 \pmod{\frac{t_2 - 3}{2\lambda}},
\]
where \(\lambda = 1\) or \(2\) according to whether \(t_2 \equiv 1 \pmod{4}\) or not. We find from (35) that \((t_2 - 3)/2 \equiv 0 \pmod{p^{2\lambda}}\). Since \(t_2 > 3\), it implies (34). The lemma is proved.

Let \(\alpha\) be an algebraic number of degree \(d\) with conjugates \(\sigma_1 \alpha, \sigma_2 \alpha, \ldots, \sigma_d \alpha\) and the minimal polynomial
\[
a_0 z^d + a_1 z^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (z - \sigma_i \alpha) \in \mathbb{Z}[z], \quad a_0 > 0.
\]
Then
\[
h(\alpha) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \max(1, |\sigma_i \alpha|) \right)
\]
is called the logarithmic absolute height of \(\alpha\).

**Lemma 5 ([6, Théorème 3]).** Let \(\alpha\) be an algebraic number with \(|\alpha| = 1\). Let \(b_1, b_2\) be positive integers, and let \(\Lambda = b_1 \log \alpha - b_2 \pi \sqrt{-1}\). If \(\alpha\) is not a root of unity, then we have
\[
\log |\Lambda| \geq -8.87AB^2,
\]
where \(A = \max(20, 10.98 |\log \alpha| + rh(\alpha)), \; B = \max(17, r^{1/10}, 5.03 + 2.35r + r \log(b_1/68.9 + b_2/2A)), \; r = \mathbb{Q}(\alpha): \mathbb{Q}/2\).

**Lemma 6.** If (3) has two solutions \((r_1, s_1)\) and \((r_2, s_2)\) satisfying (6), then we have
\[
t_2 < 2 + 2563.43 \left(1 + \frac{10.98\pi}{\log p^{2\lambda/2}}\right).
\]

**Proof.** Under the assumption, we get from (15) and (17) that
\[
2p^{Z_1/2} > 2D_2^{1/2} |\varepsilon - \varepsilon'| = |\varepsilon^{t_2} - \varepsilon^{t_2}|.
\]
Let \(\alpha = \varepsilon/\varepsilon\). Then from (37) we get
\[
\log 2p^{Z_1/2} > \log |\varepsilon^{t_2}| + \log |\alpha^{t_2} - 1| = t_2 \log p^{Z_1/2} + \log |\alpha^{t_2} - 1|.
\]
Since \(t_2 \geq 5\), if \(|\alpha^{t_2} - 1| \geq 1\), then (38) is impossible. Therefore, we have
\[
|\alpha^{t_2} - 1| \geq |t_2 \log \alpha - k \pi \sqrt{-1}|
\]
for some positive integers \(k\) with \(k \leq t_2\). Let \(\Lambda = t_2 \log \alpha - k \pi \sqrt{-1}\). From (38) and (39), we get
\[
\log 2p^{Z_1/2} - \log |\Lambda| > t_2 \log p^{Z_1/2}.
\]
We see from (10) that \(\alpha\) satisfies
\[
p^{Z_1} \alpha^2 - 2(D_1 X_1^2 - D_2) \alpha + p^{Z_1} = 0.
\]

It implies that \( \mathbb{Q}(\alpha) : \mathbb{Q} = 2 \), \( h(\alpha) = \log p^{Z_1/2} \) and \( \alpha \) is not a root of unity. Further, by (24) and (26), we have \( |\alpha| = 1 \) and \( |\log \alpha| = 2\theta < \pi \). Since \( k \leq t_2 \), by Lemma 5, we get

\[
\log |\Lambda| \geq -8.87(10.98\pi + \log p^{Z_1/2}) \times \left( \max \left(17, 7.38 + \log \left( \frac{t_2}{68.9} + \frac{t_2}{2(10.98\pi + \log p^{Z_1/2})} \right) \right) \right)^2.
\]

If

\[
7.38 + \log \left( \frac{t_2}{68.9} + \frac{t_2}{2(10.98\pi + \log p^{Z_1/2})} \right) \geq 17,
\]

then we have

\[
\log 0.0288015t_2 \geq 9.62,
\]

whence we conclude that

\[
t_2 > 523063.
\]

On the other hand, by (40), (41) and (42), we get

\[
3 + 565.8842(7.38 + \log 0.0288015t_2)^2 > 1 + \frac{10.98\pi}{\log p^{Z_1/2}} + 8.87 \left( 1 + \frac{10.98\pi}{\log p^{Z_1/2}} \right) \left( 7.38 + \log \left( \frac{t_2}{68.9} + \frac{t_2}{2(10.98\pi + \log p^{Z_1/2})} \right) \right)^2 > t_2.
\]

We calculate from (44) that \( t_2 < 150000. \) It contradicts with (43). So we have

\[
7.38 + \log \left( \frac{t_2}{68.9} + \frac{t_2}{2(10.98\pi + \log p^{Z_1/2})} \right) < 17.
\]

Thus, by (40), (41) and (45), we obtain (36). The lemma is proved.

**Lemma 7** ([3]). *The equation*

\[
X^3 + X^2Y - 2XY^2 - Y^3 = 1, \quad X, Y \in \mathbb{Z}
\]

*has only the solutions* \((X, Y) = (1, 0), (0, -1), (-1, 1), (2, -1), (-1, 2), (5, 4), (4, -9)\) *and* \((-9, 5).\)

**Proof of Theorem 1.** We now suppose that \( N(D_1, D_2, p) > 2. \) By Lemma 3, then (3) has two solutions \((r_1, s_1)\) and \((r_2, s_2)\) satisfying (6).

By Lemmas 4 and 6, if \( t_1 = 3, \) then we have

\[
2p^{2Z_1} + 3 \leq t_2 < 2 + 2563.43 \left( 1 + \frac{10.98\pi}{\log p^{Z_1/2}} \right),
\]

whence we conclude that \( p^{Z_1} \leq 137. \) Recall that \( D_1 > 1, D_2 > 1 \) and \( D_1 \) is square free. By (33), we only need to consider the cases \((D_1, D_2, p^{Z_1}) = (2, 5, 7), (2, 7, 9), (3, 8, 11), (3, 10, 13), (5, 14, 19), (6, 17, 23), (6, 19, 25), (7, 20, 27), (7, 22, 29), (2, 23, 31), (10, 31, 41), (11, 32, 43), (3, 35, 47), (3, 37, 49), (13, 40, 53), (15, 44, 59), (15, 46, 61), (17, 50, 67), (2, 53, 71), (2, 55, 73), (5, 59, 79), (5, 61, 81), (21, 62, 83), (22, 67, 89), (6, 73, 97), (35, 77, 109), (2, 83, 103), (9, 89, 113), (7, 91, 115), (2, 97, 117), (10, 101, 127), (25, 107, 137), (22, 109, 139), (4, 119, 143), (22, 127, 151), (12, 131, 157), (11, 139, 163), (6, 143, 167), (22, 151, 179), (14, 157, 181), (2, 159, 187), (37, 163, 197), (2, 173, 199), (10, 181, 209), (16, 187, 211), (64, 197, 227), (25, 199, 233)\).
If the equation

\[
D_1 X^2 + D_2 Y'^2 = 2p^{Z_1}, \quad X', Y', Z' \in \mathbb{Z}, \quad \gcd(X', Y') = 1, \quad Z' > 0,
\]
has solutions \((X', Y', Z')\), then it has a unique solution \((X'_1, Y'_1, Z'_1)\) such that \(X'_1 > 0, Y'_1 > 0\) and \(Z'_1 \leq Z'\), where \(Z'\) runs through all solutions \((X', Y', Z')\) of (50). \((X'_1, Y'_1, Z'_1)\) is called the least solution of (50). Moreover, every solution \((X', Y', Z')\) of (50) can be expressed as

\[
Z' = Z'_1 t', \quad \frac{X' \sqrt{D_1} + Y' \sqrt{-D_2}}{\sqrt{2}} = \lambda_1 \left( \frac{X'_1 \sqrt{D_1} + \lambda_2 Y'_1 \sqrt{-D_2}}{\sqrt{2}} \right)^{t'},
\]

where \(t' \in \mathbb{N}, \ 2 \nmid t', \lambda_1, \lambda_2 \in \{-1, 1\}\).
Lemma 9. The equation (4) has solutions \((r', s')\) if and only if (50) has solutions \((X', Y', Z')\) and its least solution \((X'_1, Y'_1, Z'_1)\) satisfies \(Y'_1 = 1\). Moreover, if \(N'(D_1, D_2, p) > 2\), then (4) has two solutions \((r'_1, s'_1)\) and \((r'_2, s'_2)\) such that

\[
X'_1 = Z'_1 r'_1, \quad Y'_1 = Z'_1 r'_2, \quad 1 < t'_1 < t'_2,
\]

where \(t'_1\) and \(t'_2\) are odd primes satisfying

\[
(t'_j - 1)/2 \leq \frac{\pi}{2 \arcsin(D_2/2p^{Z'_1}Z'_1)^{1/2}} < t'_j,
\]

and

\[
\left| \arcsin \left( \frac{D_2}{2p^{Z'_1}Z'_1} \right)^{1/2} - \frac{k\pi}{t'_j} \right| < \frac{\pi}{2t'_j p^{Z'_1}Z'_1(t'_j - 1)/2}, \quad j = 1, 2,
\]

where \(k\) is a positive integer satisfying \(k \leq (t'_j - 1)/2\).

Lemma 10. If (4) has two solutions \((r'_1, s'_1)\) and \((r'_2, s'_2)\) satisfying (51) with \(t'_1 = 3\), then we have

\[
2D_2 = 3p^{Z'_1} + (-1)(p^{Z'_1} + 1)/2
\]

and

\[
t'_2 \geq 2p^{2Z'_1} + 3.
\]

Lemma 11. If (4) has two solutions \((r'_1, s'_1)\) and \((r'_2, s'_2)\) satisfying (51), then we have

\[
t'_2 < 3 + 2563.43 \left( 1 + \frac{10.98\pi}{\log p^{Z'_1}Z'_1} \right).
\]

Proof of Theorem 2. We now suppose that \(N'(D_1, D_2, p) > 2\). By Lemma 9, then (4) has two solutions \((r'_1, s'_1)\) and \((r'_2, s'_2)\) satisfying (51).

By Lemmas 10 and 11, if \(t'_1 = 3\), then we have

\[
2p^{2Z'_1} + 3 \leq t'_2 < 3 + 2563.43 \left( 1 + \frac{10.98\pi}{\log p^{Z'_1}Z'_1} \right),
\]

whence we calculate that \(p^{Z'_1} \leq 137\). Hence, by (55), we only need to consider the cases \((D_1, D_2, p^{Z'_1})\) such that \(p^{Z'_1} \leq 137\) and \(2D_2 = 3p^{Z'_1} + (-1)(p^{Z'_1} + 1)/2\). On the other hand, by (54) and (55), we get

\[
\left| \arcsin \left( \frac{3}{4} + \frac{(-1)(p^{Z'_1} + 1)/2}{4p^{Z'_1}} \right)^{1/2} - \frac{k\pi}{t'_2} \right| < \frac{\pi}{2t'_j p^{Z'_1}(t'_j - 1)/2}
\]

for some positive integers \(k\) with \(k \leq (t'_2 - 1)/2\). However, for the above-mentioned cases, if \(t'_2\) satisfies (58), then (59) is impossible.

If \(t'_1 = 5\), then from (52) we get

\[
4D_2^2 - 10D_2 p^{Z'_1} + 5p^{2Z'_1} = -1.
\]
It implies that

\[(4D_2 - 5p^{Z_1})^2 - 5p^{2Z_1} = -4.\]

For any nonnegative integer \(m\), let \(F_m\) and \(L_m\) denote the \(m\)th Fibonacci number and the \(m\)th Lucas number, respectively. Notice that all solutions \((X,Y)\) of the equation

\[X^2 - 5Y^2 = -4, \quad X, Y \in \mathbb{N}, \quad \gcd(X, Y) = 1\]

are given by \((X,Y) = (L_{6l}, F_{6l} + 1)\) and \((L_{6l+5}, F_{6l+5})\), where \(l\) runs through all nonnegative integers. We see from (60) that

\[(61) \quad (D_2, p^{Z_1}) = \left(\frac{1}{2}L_{6l}, F_{6l} + 1\right) \text{ or } \left(\frac{1}{2}L_{6l+6}, F_{6l+5}\right).\]

On the other hand, by (53) and (57), we have

\[(62) \quad p^{2Z_1} < \left(\frac{2p^{Z_1}}{D_2}\right)^{1/2} < \frac{\pi}{2 \arcsin(D_2/2p^{Z_1})^{1/2}} < t'_2 < 3 + 2563.43 \left(1 + \frac{10.98\pi\log p^{Z_1/2}}{\log p^{Z_1/2}}\right),\]

whence we calculate that \(p^{Z_1} < 200\). Therefore, we find from (61) that \((D_1, D_2, p^{Z_1}) = (1, 9, 5), (17, 9, 13), (17, 161, 89)\). Further, by (54), we get

\[(63) \quad \left|\arcsin\left(\frac{D_2}{2p^{Z_1}}\right)^{1/2} - \frac{k\pi}{t'_2}\right| < \frac{\pi}{2t'_2p^{Z_1}(t'_2 - 1)/2}\]

for some positive integers \(k\) with \(k \leq (t'_2 - 1)/2\). However, if \(t'_2\) satisfies (62), then (63) is impossible.

If \(t'_2 = 7\), then from (52) we get

\[(64) \quad (2p^{Z_1} - 2D_2)^3 + 2(2p^{Z_1} - 2D_2)^2p^{Z_1} - 2(2p^{Z_1} - 2D_2)p^{2Z_1} - p^{3Z_1} = (-1)(p^{Z_1+1})^2.\]

By Lemma 7, we obtain from (64) that \((D_1, D_2, p^{Z_1}) = (7, 11, 9)\). For this case, by (53), (54) and (57), we get

\[(65) \quad 729 < t'_2 < 83099\]

and

\[(66) \quad \left|\arcsin\left(\frac{11}{18}\right)^{1/2} - \frac{k\pi}{t'_2}\right| < \frac{\pi}{2 \cdot 3t'_2 - 1t'_2}\]

for some positive integers \(k\) with \(k \leq (t'_2 - 1)/2\). However, (66) is impossible if \(t'_2\) satisfies (65).

By (53) and (57), if \(t'_1 > 7\), then \(t'_1 \geq 11\) and \(p^{Z_1} < 9\). Except for the already considered cases, we have \((D_1, D_2, p^{Z_1}) = (5, 1, 3), (1, 9, 5), (1, 13, 7), (5, 9, 7), (9, 5, 7), (11, 3, 7), (13, 1, 7), (1, 17, 9), (11, 7, 9), (13, 5, 9)\) or \((17, 1, 9)\). For these cases, (52) is false when \(t'_1 = 11, 13, 17\) and \(19\). So we have \(t'_1 \geq 23\). Then, by (53) and (57) again, we get

\[177147 \leq p^{11Z_1} < p^{Z_1(t'_1-1)/2} < \frac{\pi}{2 \arcsin(D_2/2p^{Z_1})^{1/2}} < t'_2 < 3 + 2563.43 \times \left(1 + \frac{10.98\pi\log 3^{1/2}}{\log 3^{1/2}}\right) < 153544,\]

a contradiction. The theorem is proved.
4. Proof of the Corollary

Lemma 12 ([11]). The equation

\[ z^2 + 4 = y^n, \quad x, y, n \in \mathbb{N}, \quad 2 \nmid y, \quad n > 3, \]

has no solution \((z, y, n)\).

Proof of Corollary. Let \((x, y, m, n)\) be a solution of (2) such that \(m = 3\) and \(y\) is a prime power. Then we have

\[(y - 1)(2x + 1)^2 + (3y + 1) = 4y^n, \quad n > 3.\]  

(67)

If \(y = 2\), then from (67) we get

\[(2x + 1)^2 + 7 = 2^{n+2}, \quad n > 3.\]  

(68)

By [12], we find from (68) that \((x, y, m, n) = (5, 2, 3, 5)\) and \((90, 2, 3, 13)\).

If \(y = 2^k\), where \(k\) is a positive integer with \(k > 1\), then

\[(2^{k-1})(2x + 1)^2 + (3 \cdot 2^k + 1) = 2^{kn+2}, \quad n > 3.\]  

(69)

Let \(D_1 = 2^{k-1}, \quad D_2 = 3 \cdot 2^k + 1\) and \(p = 2\). We see from (69) that \((r, s) = (2x + 1, kn)\) is a solution of (3) with \(s > 3k + 2\). However, by [8], then (3) has exactly two solutions \((r, s) = (1, k + 2)\) and \((2^{k+1} + 1, 3k + 2)\). Therefore, (69) is impossible.

If \(2 \nmid y\), then \(y = p^k\), where \(p\) is an odd prime and \(k\) is a positive integer.

If \(y = 5\), then from (67) we get

\[(2x + 1)^2 + 4 = 5^n, \quad n > 3.\]  

(70)

By Lemma 12, (70) is impossible.

If \(y \equiv 1 \pmod{4}\) and \(y > 5\), then we have

\[ \left(\frac{p^k - 1}{4}\right)(2x + 1)^2 + \left(\frac{3p^k + 1}{4}\right) = p^kn, \quad n > 3.\]  

(71)

Let \(D_1 = (p^k - 1)/4\) and \(D_2 = (3p^k + 1)/4\). We see from (71) that \((r, s) = (2x + 1, kn)\) is a solution of (3) with \(s > 3k\). Notice that (3) has two solutions \((r, s) = (1, k)\) and \((2p^k + 1, 3k)\) in this case. Therefore, by Theorem 1, (71) is impossible.

Similarly, if \(y \equiv 3 \pmod{4}\), then we have

\[ \left(\frac{p^k - 1}{2}\right)(2x + 1)^2 + \left(\frac{3p^k + 1}{2}\right) = 2p^kn, \quad n > 3.\]  

(72)

Let \(D_1 = (p^k - 1)/2\) and \(D_2 = (3p^k + 1)/2\). Since (4) has two solutions \((r', s') = (1, k)\) and \((2p^k + 1, 3k)\) in this case, by Theorem 2, (72) is impossible. Now, the corollary is proved.

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Department of Mathematics, Zhanjiang Teachers College, Postal Code 524048, Zhanjiang, Guangdong, P. R. China