GROUP EXTENSIONS AND TAME PAIRS

MICHAEL L. MIHALIK

Abstract. Tame pairs of groups were introduced to study the missing boundary problem for covers of compact 3-manifolds. In this paper we prove that if $1 \to A \to G \to B \to 1$ is an exact sequence of infinite finitely presented groups or if $G$ is an ascending HNN-extension with base $A$ and $H$ is a certain type of finitely presented subgroup of $A$, then the pair $(G, H)$ is tame.

Also we develop a technique for showing certain groups cannot be the fundamental group of a compact 3-manifold. In particular, we give an elementary proof of the result of R. Bieri, W. Neumann and R. Strebel:

A strictly ascending HNN-extension cannot be the fundamental group of a compact 3-manifold.

1. Introduction

We introduced the idea of a tame pair $H < G$ of groups in [M1]. The original motivation was to establish a geometric group theoretic approach to attack a well known problem (the missing boundary problem for covers of compact 3-manifolds) in 3-dimensional topology. A 3-manifold $M$ is a missing boundary manifold if $M$ is embedded in a compact manifold $M_1$ such that $M_1 - M$ is a subset of the boundary of $M_1$. It is conjectured that for any compact $P_2$-irreducible 3-manifold $M$ and finitely generated subgroup $H < \pi_1(M)$, the cover of $M$ with fundamental group $H$ is a missing boundary manifold. In [M1], we show that if the pair $(\pi_1(M), H)$ is tame, then the cover of $M$ with fundamental group $H$ is a missing boundary manifold. In [M1], we consider very general combings of groups (almost prefix closed combings) and show that subgroups that are rational (quasi-convex) with respect to these combings define tame pairs of groups. Results in [B] and [E] show that the fundamental group of a closed 3-manifold satisfying Thurston’s geometrization conjecture has an almost prefix closed combing. A consequence of the main theorem of [M1] is:

Theorem [M1]. If $H$ is a rational subgroup of the automatic group $G$, then the pair $(G, H)$ is tame.

Hence if $M$ is a compact $P_2$-irreducible 3-manifold with automatic fundamental group and $H$ is rational with respect to the automatic structure then the cover of $M$ with fundamental group $H$ is a missing boundary manifold.

As general combings and rational subgroups lead to tame pairs, one wonders what other general classes of pairs of groups are tame. Suppose $M$ is a compact 3-manifold and there is a short exact sequence of infinite finitely generated groups $1 \to A \to \pi_1(M) \to B \to 1$. When $A \neq \mathbb{Z}$, the structure of this exact sequence and

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the structure of $M$ is determined by J. Hempel and W. Jaco in [HJ]. In this case it is straightforward to see that if $H$ is a finitely generated subgroup of $A$, then the cover of $M$ corresponding to $H$ is a missing boundary manifold. Hence, a natural question to ask is:

“If $1 \to A \to G \to B \to 1$ is an exact sequence of infinite finitely presented groups, which subgroups $H$ of $A$ are such that $(G, H)$ is tame?”

**Theorem 1.** Let $1 \to A \to G \to B \to 1$ be a short exact sequence of infinite finitely presented groups, and $H$ a finitely generated subgroup of $A$ of infinite index in $A$. Then $(G, H)$ is tame.

If the pair $(G, 1)$ is tame, then $G$ has a tame combing in the sense of [MT]. If $G$ has a tame combing, then $G$ is quasi-simply-filtrated (see [BM1]) by Theorem 3 of [MT]. We thus have the following generalization of the main theorem of [BM2].

**Corollary.** Let $1 \to A \to G \to B \to 1$ be a short exact sequence of infinite finitely presented groups, then $G$ is tame combable.

In the special case of $B \approx \mathbb{Z}$, Theorem 2 (below) shows that for any finitely generated subgroup $H$ of $A$, $(G, H)$ is tame.

**Theorem 2.** Suppose $A$ is a finitely presented group and $f : A \to A$ is a monomorphism. Let $G = \langle A, t : t^{-1}a^{-1}t = f(a) \rangle$ be the corresponding ascending HNN-extension. If $B$ is any finitely generated subgroup of $N(A)$ ($\equiv$ the normal closure of $A$ in $G$), then the pair $(G, B)$ is tame.

An interesting situation arises in the case of Theorem 2; when $G$ is strictly ascending (i.e. when $f : A \to A$ is not an epimorphism), the pair $(G, A)$ is easily shown to be not semistable at infinity (see §4), even though $(G, A)$ is tame. But if $G$ were the fundamental group of a compact $P_2$-irreducible 3-manifold, then we would have that the cover of $M$ with fundamental group $A$ would be a missing boundary manifold. It is straightforward to show that missing boundary manifolds are semistable at infinity. We thus have an elementary proof that a strictly ascending HNN-extension cannot be the fundamental group of a compact 3-manifold, a result first established by R.Bieri, W.Neumann and R.Strebel in [BNS].

This observation opens the possibility of showing a given group $G$ is not a compact 3-manifold group by finding a subgroup $H$ such that $(G, H)$ is tame but not semistable at infinity.

The paper is organized as follows: In §2 we make the relevant definitions and describe the spaces in which we construct certain homotopies. In §3 we prove Theorem 1 and in §4 we prove Theorem 2.

### 2. Preliminaries

Let $P = \langle g_1, \ldots, g_n : r_1, \ldots, r_m \rangle$ be a presentation for the group $G$.

**Definition.** The standard 2-complex corresponding to $P$, denoted $X_P$, has one vertex $*$, a directed loop at $*$ labeled by $g_i$ for each $i$ and a 2-cell attached to the loop with label $r_i$ for each $i$.

The universal cover of a space $X$ is denoted $\tilde{X}$. The 1-skeleton of $\tilde{X}_P$ is the Cayley graph of $G$ with respect to the generating set $\{g_1, \ldots, g_n\}$. (Hence the vertices of $\tilde{X}_P$ are the elements of $G$ and the edges of $\tilde{X}_P$ are directed and labeled by the elements of $\{g_1, \ldots, g_n\}$.)
We work in covering spaces of standard 2-complexes. If $X$ is such a space and $Y$ is a subcomplex of $X$, then $\text{St}(Y)$ has as 1-skeleton all edges that intersect $Y$. A 2-cell is in $\text{St}(Y)$ if its boundary is contained in $\text{St}(Y)$. Inductively let $\text{St}^N(Y) \equiv \text{St}^{N-1}(\text{St}(Y))$ for $N \geq 1$ ($\text{St}^0(Y) \equiv Y$).

**Definition.** Suppose $P$ is a finite presentation of $G$, $H$ is a finitely generated subgroup of $G$ and $*$ is a vertex of $X_P$. The pair $(G, H)$ is tame if for each integer $N$ there is an integer $M$ such that for any edge path $\alpha$ in $\text{Cl}(\tilde{X} - \text{St}^N(H*))$ with $\alpha(0), \alpha(1) \in \text{St}^N(H*)$, $\alpha$ is homotopic rel $\{0, 1\}$ to an edge path $\beta$ in $\text{St}^M(H*)$, by a homotopy in $\text{Cl}(\tilde{X} - \text{St}^N(H*))$.

In [M1], this definition is shown to be independent of presentation $P$, for $G$ and Corollary 3 there states:

**Theorem [M1].** If $M$ is a compact $P^2$-irreducible 3-manifold and $H$ is a finitely generated subgroup of $\pi_1(M)$, then $H/\tilde{X}$ is a missing boundary manifold if and only if $(\pi_1(M), H)$ is tame.

The following definition is used in §4:

**Definition.** A locally finite CW-complex $X$ is semistable at infinity if for any proper ray $r:[0, \infty) \rightarrow X$ and compact set $C \subset X$ there exists a compact set $D$ such that for any loop $\alpha$ based on $r$ in $X - D$, and compact set $E$, $\alpha$ is homotopic rel $r$ to a loop in $X - E$ by a homotopy in $X - C$.

### 3. The proof of Theorem 1

Let $P \equiv \langle a_1, \ldots, a_n, h_1, \ldots, h_k, b_1, \ldots, b_m : r_1, \ldots, r_q, s_1, \ldots, s_t \rangle$ be a presentation for $G$ where $(a_1, \ldots, a_n, h_1, \ldots, h_k : r_1, \ldots, r_q)$ is a presentation for the subgroup $A$, $h_1, \ldots, h_k$ are generators of $H$ and for each $b \in \{b_1^{\pm1}, \ldots, b_m^{\pm1}\}$ and $r \in \{a_1^{\pm1}, \ldots, a_n^{\pm1}, h_1^{\pm1}, \ldots, h_k^{\pm1}\}$ the conjugation relation $b^{-1}ab(a, b)$, for $w(a, b)$ a word in the letters $\{a_1^{\pm1}, \ldots, a_n^{\pm1}, h_1^{\pm1}, \ldots, h_k^{\pm1}\}$, is one of the relations $s_i$.

Let $X \equiv X_P$ and let $Y$ be the subcomplex of $X$ consisting of the loops and 2-cells corresponding to $a_1, \ldots, a_n, h_1, \ldots, h_k$ and $r_1, \ldots, r_q$ respectively. Let $\tilde{X} \rightarrow X$ be the universal cover of $X$. Observe that $q^{-1}(Y)$ is a disjoint union of copies of the universal cover of $Y$, one for each element of $B$.

The edges of $X$ are directed and labeled, one for each generator of $P$. Take each edge of $\tilde{X}$ to have the label and direction of the edge of $X$ that $q$ maps it to. Let $\tilde{X} \rightarrow Z$ be the quotient by the action of $A$ on $\tilde{X}$. Observe that $Z$ is an infinite, locally finite 2-complex.

We prove the following result which is equivalent to Theorem 1.

**Theorem A.** For any integer $N$ there is an integer $S$ such that if $\alpha$ is an edge path in $\text{Cl}(\tilde{X} - \text{St}^S(H))$ with $\alpha(0), \alpha(1) \in \text{St}^N(H)$, then $\alpha$ is homotopic rel $\{0, 1\}$ to a path in $\text{St}^S(H)$ by a homotopy in $\text{Cl}(\tilde{X} - \text{St}^N(H))$.

The proof is an easy consequence of four lemmas.

**Lemma 4.** If $\alpha$ is an edge path in $\tilde{X}$ with $p(\alpha(0)) = p(\alpha(1))$, and $\text{im}(\alpha) \cap p(\text{St}^N(H)) = \emptyset$, then any edge path $\beta$, in $A$-edges from $\alpha(0)$ to $\alpha(1)$, is homotopic rel $\{0, 1\}$ to $\alpha$ in $\tilde{X} - \text{St}^N(H)$.
Proof. Since \( \text{im}(pa) \cap p(\text{St}^N(H)) = \emptyset \), the copies of \( \tilde{Y} \) in \( \tilde{X} \) that intersect \( \alpha \) do not intersect \( \text{St}^N(H) \). Let \( \tilde{Y}_0 \) be the copy of \( \tilde{Y} \) in \( \tilde{X} \) containing \( \alpha(0) \) and \( \tilde{Y}_* \) be the copy of \( \tilde{Y} \) containing \( H \).

Note that the normality of \( A \) in \( G \) implies:

If \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) are copies of \( \tilde{Y} \), \( y \) is a vertex of \( \tilde{Y}_1 \) and \( d(y, \tilde{Y}_2) = n \) (here \( d(y, \tilde{Y}_2) \) is the length of a minimal edge path from \( y \) to a vertex of \( \tilde{Y}_2 \)), then for every vertex \( v \) of \( \tilde{Y}_1 \), \( d(v, \tilde{Y}_2) = n \).

Let \( \beta \) be an edge path in \( \tilde{Y}_0 \) from \( \alpha(0) \) to \( \alpha(1) \). Let \( K \) be an integer such that the loop \( \langle \alpha, \beta^{-1} \rangle \) is homotopically trivial in \( \text{St}^K(x) \) for any vertex \( x \) of \( \langle \alpha, \beta^{-1} \rangle \).

As \( H \) has infinite index in \( A \), there are vertices of \( \tilde{Y}_* \), arbitrarily far from \( H \) (and hence from \( \text{St}^N(H) \)) when measured in \( \tilde{X} \). By the above note there are vertices of \( \tilde{Y}_0 \) arbitrarily far from \( \text{St}^N(H) \).

Let \( \gamma \) be an edge path in \( \tilde{Y}_0 \) from \( \alpha(0) \) to a vertex \( x \) such that \( \text{St}^K(x) \cap \text{St}^N(H) = \emptyset \). The translate of \( \langle \alpha, \beta^{-1} \rangle \) to \( x \) is homotopically trivial by a homotopy missing \( \text{St}^N(H) \).

Say \( \alpha = (e_1, e_2, \ldots, e_n) \). Using the 2-cells corresponding to the conjugation relations we see that \( \langle e_1^{-1}, \gamma, xe_1 \rangle \) is homotopic rel\( \{0, 1\} \) to an edge path \( \gamma_1 \) (in A-edges), by a homotopy in \( \tilde{X} - \text{St}^N(H) \). (In fact the image under \( p \) of this homotopy does not intersect \( p(\text{St}^N(H)) \).) (See Figure 1.)

Inductively \( \langle e_i^{-1}, \gamma_i, xe_{i+1} \rangle \) is homotopic rel\( \{0, 1\} \) to the edge path \( \gamma_{i+1} \) (in A-edges) by a homotopy in \( \tilde{X} - \text{St}^N(H) \). The loop \( \langle \beta, \gamma_n, (x \beta)^{-1}, \gamma^{-1} \rangle \) is a loop in \( \tilde{Y}_0 \) and hence is homotopically trivial in \( \tilde{Y}_0 \). Patching together these homotopies as in Figure 1 gives the desired homotopy of \( \alpha \) to \( \beta \). \( \square \)
Remark. The edge path $\beta$ is homotopic to the $A$-edge path $\langle \gamma, x \beta, \gamma_{n-1} \rangle$ by a homotopy in $\bar{X} - St^N(H)$, and this fact only depends upon $A$ being finitely generated (as opposed to $A$ being finitely presented).

Next we list integers and certain finite subcomplexes of $\bar{X}$ used extensively in the remainder of the proof.

Choose $M$ so that for any two vertices $v, w \in St(p(St^N(H)))$, there is an edge path of length $\leq M$ from $v$ to $w$. Observe that $p(H)$ is a single vertex of $Z$.

Choose $M' > M$ such that if $x, y$ are vertices of $St(p(St^N(H))) - p(St^N(H))$, in the same component of $Z - p(St^N(H))$, then there is an edge path of length $\leq M'$ from $x$ to $y$ in $Z - p(St^N(H))$.

Choose $L$ such that if $\alpha$ is an edge path of length $\leq 2M' + 1$ such that $\alpha(0)$ and $\alpha(1)$ are in the same component of $\gamma$ with respect to $\tau$, then there exists an edge path in $A$-edges from $\alpha(0)$ to $\alpha(1)$ of length $\leq L$.

Let $Q$ be an integer such that any edge loop $\gamma$ in $\bar{X}$ of length $\leq 2M' + L + 1$ is homotopically trivial in $St^{Q}(w)$ for any vertex of $w$ of $\gamma$.

For each vertex $v \in Bd(St^Q + N(H))$ such that $p(v) \in Z - p(St^N(H))$ take $\alpha_v$ to be a shortest edge path from $v$ to a vertex of $H$. Let $\beta_v$ be the shortest subpath of $\alpha_v$ beginning at $v$ such that $p(\beta_v(1)) \in St(p(St^N(H)))$. Then $\beta_v$ is an edge path of length $\leq Q$ such that $\beta_v(0) = v$, $im(\beta_v) \cap p(St^N(H)) = \emptyset$, and $\beta_v(1) \in St(p(St^N(H)))$ and $\im(\beta_v) \subset St^{Q+N}(H)$.

**Lemma 5.** If $\alpha$ is an edge path in $C(\bar{X} - St^{Q+N}(H))$ with $\alpha(0), \alpha(1) \in St^{Q+N}(H)$, then $\alpha$ is homotopic rel$\{0, 1\}$, by a homotopy in $\bar{X} - St^N(H)$, to an edge path $\langle \beta_1, \tau, \beta_2 \rangle$ where for each vertex $w \in St(p(St^N(H)))$, and $im(\beta_i) \subset St^{Q+N}(H)$ for $i \in \{1, 2\}$. (i.e. $\beta_1$ is “close” to $H$ and $\tau$ is “close” to $p(H)$.)

**Proof.** Let $x = \alpha(0)$ and $y = \alpha(1)$. If $p(x)(p(y))$ is in $St(p(St^N(H)))$, then $\beta_1(\beta_2)$ is the constant path. Otherwise let $\beta_1(\beta_2)$ be $\beta_g(\beta_{g^{-1}})$. We consider the case $\beta_1$ and $\beta_2$ non-trivial, as the others are completely analogous. Partition the consecutive vertices of $\langle \beta_1^{-1}, \alpha, \beta_2 \rangle$ as $v_1, \ldots, v_{n(1)}, w_{n(1)} + 1, \ldots, w_{n(2)}, v_{n(2)} + 1, \ldots, v_{n(k)}$, where $p(v_i) \notin p(St^N(H))$ and $p(w_i) \in p(St^N(H))$.

Define $n(0)$ to be 0.

Observe that for even $i$, $p(v_n(i+1)), p(v_n(i+1)) \in St(p(St^N(H))) - p(St^N(H))$ and they lie in the same component of $Z - p(St^N(H))$. Hence there is an edge path $\gamma_{n(i+1)}$ from $p(v_n(i+1))$ to $p(v_n(i+1))$ of length $\leq M'$ in $Z - p(St^N(H))$. Lift $\gamma_{n(i+1)}$ to the vertex $v_n(i+1)$ and call the resulting path $\gamma_n(i+1)$ (see Figure 2).

For all $i, p(w_i) \in p(St^N(H))$. So for odd $i$ there is a path $\gamma_1(i+1) \subset St(p(St^N(H)))$ from $p(w_{n(i)} + 1)$ to $p(v_{n(i) - 1} + 1)$, of length $\leq M$. Lift $\gamma_{n(i) + 1}$ to $w_{n(i) + 1}$ and call the resulting path $\gamma_n(i+1)$.

Observe that for odd $i$, $v_{n(i-1) + 1}$ and the end points of $\gamma_n(i)$ and $\gamma_n(i+1)$ lie in the same copy of $\tilde{Y}$. Furthermore $p$ maps each of these points to $p(v_{n(i-1) + 1}) \in St(p(St^N(H))) - p(St^N(H))$, so this copy of $\tilde{Y}$ does not intersect $St^N(H)$. For even $i$, let $\delta_n(i) \subset St(p(St^N(H)))$ for all $i$, and for odd $i$, $p(\delta_n(i)) \subset St(p(St^N(H)))$. For odd $i$, let the subpath of $\alpha$ between $w_{n(i)} + 1$ and $v_{n(i) + 1}$ be $\alpha_n(i)$, the subpath
Proof. Let \( v_1 \) be a vertex of \( \text{St}(H) \) and \( v_2 \) be a vertex of \( \text{St}(H) \) such that \( \alpha(v_1, v_2) \) is an integer such that the length of \( \alpha(v_1, v_2) \) is \( M + N + Q \). Then there is a path in \( \text{St}(H) \) between \( v_1 \) and \( v_2 \). (See Figure 3.)

Recall the conjugation relations \( b^{-1}abw(a, b) \) for \( a \in \{ a_1^\pm, \ldots, a_n^\pm, h_1^\pm, \ldots, h_k^\pm \} \), \( b \in \{ b_1^\pm, \ldots, b_m^\pm \} \) and \( w(a, b) \) a word in the letters \( \{ a_1^\pm, \ldots, a_n^\pm, h_1^\pm, \ldots, h_k^\pm \} \). If \( R \) is an integer such that the length of \( w(a, b) \) is less than \( R \) for all \( a, b \), then there is an \( A \)-edge path between the end points of the path \( \langle a_1, e_1, a_1^{-1} \rangle \) of length \( \leq R^{|a_1|} \leq R^{M+N+Q} \) for each \( i \in \{ 1, \ldots, n \} \). As the end points of each \( e_i \) are in \( H \), there is an edge path in \( A \)-edges from \( v_1 \) to \( v_3 \) (\( \equiv \) the end point of
Proof. Choose $T$ such that if an element of $A$ has length $\leq 2(M + N + Q)$ in $\tilde{X}$, then it has length $\leq T$ in the $A$-generators. There is an $A$-edge path between $v_3$ and $v_2$ of length $\leq T$. Let $S = \max\{M + N + Q + R^{M+N+Q}, M + N + Q + T\}$. \hfill $\square$

Lemma 7. If $\lambda$ is an edge path in $\tilde{X} - St^N(H)$ such that $\{\lambda(0), \lambda(1)\} \subset St^{N+Q}(H)$ and for each vertex $w$ of $\lambda, p(w) \in St(p(St^N(H)))$, then $\lambda$ is homotopic rel$\{0,1\}$ to a path in $St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$.

Proof. The path $\lambda$ can be partitioned as $\langle \tau_1, \xi_1, \tau_2, \xi_2, \ldots, \tau_{n-1}, \xi_{n-1}, \tau_n \rangle$ where $\tau_i$ has image in $St^S(H)$, $\xi_i$ has image in $Cl(\tilde{X} - St^{Q+N}(H))$, $\{\xi_i(0), \xi_i(1)\} \subset Bd(St^{Q+N}(H))$ and some vertex of $\xi_i$ is in $\tilde{X} - St^S(H)$. If $\lambda = \tau_1$, we are finished. Otherwise it suffices to show that $\xi_i$ is homotopic rel$\{0,1\}$ to a path in $St^S(H)$ by a homotopy in $\tilde{X} - St^N(H)$. Say the vertices of $\xi_i$ are $w_0, w_1, \ldots, w_n$ and the edge of $\xi_i$ connecting $w_j$ and $w_{j+1}$ is $e_{j+1}$.

Let $\gamma_j$ be an edge path of length $\leq M$ from $w_j$ to a vertex of $\tilde{Y}_*$. Let $\delta_j$ be an edge path of length $\leq L$ in $A$-edges from $\gamma_{j-1}(1)$ to $\gamma_j(1)$. As $\langle \gamma_{j-1}, \delta_j, \gamma_j^{-1}, e_{j}^{-1} \rangle$ is a loop of length $\leq 2M + L + 1$ containing a vertex of $Cl(\tilde{X} - St^{Q+N}(H))$, it is homotopically trivial in $\tilde{X} - St^N(H)$. (See Figure 4.)

Hence $\xi_i$ is homotopic rel$\{0,1\}$ to the path $\langle \gamma_0, \delta_1, \delta_2, \ldots, \delta_n, \gamma_n^{-1} \rangle$ by a homotopy missing $St^N(H)$. As $\delta_1(0)$ and $\delta_n(1)$ are vertices of $St^{M+Q+N}(H) \cap \tilde{Y}_*$, Lemma 6 gives an edge path $\beta$, in $St^S(H) \cap \tilde{Y}_*$ from $\delta_1(0)$ to $\delta_n(1)$. Now $\langle \delta_1, \delta_2, \ldots, \delta_n, \beta^{-1} \rangle$ is an edge loop in $\tilde{Y}_*$ and so is homotopically trivial by a homotopy in $\tilde{Y}_*$. In
particular, this homotopy misses \( \text{St}^N(H) \). We have \( \xi_i \) homotopic rel \( \{0, 1\} \) to the path \( \langle \gamma_0, \beta, \gamma_n^{-1} \rangle \) (which has image in \( \text{St}^S(H) \)) by a homotopy in \( \tilde{X} - \text{St}^N(H) \).

To finish the proof of Theorem A (and Theorem 1) let \( \langle \delta_0, \alpha_1, \delta_1, \alpha_2, \delta_2, \ldots, \delta_{n+1} \rangle \) be a partition of \( \alpha \), where \( \text{im}(\delta_i) \subset \text{St}^S(H) \), \( \alpha_i(0), \alpha_i(1) \in \text{Bd}(\text{St}^{N+Q}(H)) \), and \( \text{im}(\alpha_i) \subset \text{Cl}(\tilde{X} - \text{St}^{N+Q}(H)) \). Applying Lemmas 5 and 7 to \( \alpha_i \) shows that \( \alpha_i \) is homotopic rel \( \{0, 1\} \) to an edge path in \( \text{St}^S(H) \), by a homotopy in \( \tilde{X} - \text{St}^N(Q) \).

4. THE PROOF OF THEOREM 2

Before beginning this proof it is convenient to slightly change our definition of \( \text{St} \). If \( P \) is a finite presentation of a group and \( X_P \) then for any subcomplex \( Y \) of \( X \), \( \text{St}(Y) \) is defined to be the union of \( Y \) and all (closed) 2-cells that intersect \( Y \).

As a first step we consider the case when \( B \) is a finitely generated subgroup of \( A \).

Proof. Let \( Q = \{a_1, \ldots, a_n, b_1, \ldots, b_m\} \) be a set of generators for \( A \) where \( \{b_1, \ldots, b_m\} \) generates \( B \) and \( \langle Q : R \rangle \) is a presentation for \( A \). For each \( i \) and \( j \) let \( w(a_i) \) and \( w(b_j) \) be a word in the alphabet \( Q \) representing \( f(a_i) \) and \( f(b_j) \) respectively. Let \( P \) be the following presentation of \( G \): \( \langle \{t\} \cup Q : R, t^{-1}a_it = w(a_i), t^{-1}b_jt = w(b_j) \rangle \) for each \( i \) and \( j \). Let \( X = X_P \). The 1-skeleton of \( \tilde{X} \) is the Cayley graph of the presentation \( P \) of \( G \). So the vertices of \( \tilde{X} \) are the elements of \( G \). Let \( h : G \to \mathbb{Z} \) be the homomorphism that kills the normal closure of \( A \). We say that an element \( g \) of \( G \) (i.e. a vertex of \( \tilde{X} \)) is in level \( L \) if \( h(g) = L \). Hence each vertex of the coset \( xA \) is in level \( h(x) \), and \( \alpha \) is any word in the generators of \( P \), representing \( x \), then \( h(x) \) is the exponent sum of \( t \) in \( \alpha \). The groups \( A \) and \( B \) are in level 0. The 2-cells corresponding to the conjugation.
relations of \( P \) can be used to slide an \( A \) or \( B \) edge to an edge path in the next level up. Any \( A \) or \( B \) edge \( e \) can be slid up \( L \) levels by a homotopy in \( St^L(e) \). I.e. \( e \) is homotopic rel\( \{0,1\} \) to a path \( \langle t^L, \lambda, t^{-L} \rangle \) by a homotopy in \( St^L(e) \) where \( \lambda \) is a path in the level, \( L \) levels above the level containing \( e \).

Now we need a lemma.

**Lemma 8.** If \( \gamma \) is an edge path in levels \( N + 1 \) and above of \( \tilde{X} \) such that the end points of \( \gamma \) are in \( St^L(B) \), then \( \gamma \) is homotopic rel\( \{0,1\} \) to a path in \( St^{2L+N+1}(B) \) by a homotopy in \( \tilde{X} - St^N(B) \).

**Proof.** Let \( \gamma_1 \), resp. \( \gamma_2 \), be any edge path in \( St^L(B) \), from the initial point of \( \gamma \), resp. from the terminal point of \( \gamma \), to a point of \( B \). Let \( \gamma_3 \) be an edge path in \( B \)-edges from the terminal point of \( \gamma_1 \) to the terminal point of \( \gamma_2 \). As \( St^L(B) \) lies between levels \( -L \) and \( L \), the edges of the path \( \tau = \langle \gamma_1, \gamma_3, \gamma_2^{-1} \rangle \) lie in levels \( -L \) and above. Each edge \( e \) of \( \tau \), that lies below level \( N + 1 \), can be slid up to level \( N + 1 \) by a homotopy with image in \( St^{L+N+1}(e) \subset St^{2L+N+1}(B) \). Hence there is a path \( \gamma_4 \) in levels \( N + 1 \) and above, with the same end points as \( \gamma \), and with image in \( St^{2L+N+1}(B) \). As \( \gamma_4 \) and \( \gamma \) have the same end points and both paths lie in levels \( N + 1 \) and above, the loop \( \gamma \) followed by \( \gamma_4^{-1} \) is homotopically trivial in levels \( N + 1 \) and above. (Slide all of the edges of this loop up to a common level. Any loop in a single level lies in a copy of the universal cover corresponding to \( A \).)

**Remark.** This is the only place in this proof that we use the fact that \( A \) is finitely presented. If \( A \) were merely finitely generated and we still knew that any loop in levels \( K \) and above were homotopically trivial in levels \( K \) and above, then our proof would still work.

Suppose \( \alpha \) is an edge path that begins and ends in \( St^{3N+2}(B) \) and such that the image of \( \alpha \) is a subset of the closure \( Cl[\tilde{X} - St^{3N+2}(B)] \). It suffices to show that \( \alpha \) is homotopic rel\( \{0,1\} \) to a path in \( St^{15N+11}(B) \), by a homotopy in \( \tilde{X} - St^N(B) \). Clearly we can slide any \( A \) or \( B \) edge of \( \alpha \) that lies below level \( -N - 1 \) to level \( -N - 1 \) by a homotopy that does not intersect \( St^N(B) \) (or \( St^N(A) \) for that matter). Suppose \( \alpha = \langle e_1, \ldots, e_k \rangle \). We may assume that each \( A \) and \( B \) edge of \( \alpha \) lies in level \( -N - 1 \) or above, and if \( e \) is an edge of \( \alpha \) not in level \( -N - 1 \), then \( e \) is in \( Cl[\tilde{X} - St^{3N+2}(B)] \). We form a new path \( \beta \), with the same end points as \( \alpha \) by:

1) If \( e \) is an edge of \( \alpha \) in a level from \( -N \) to \( N \), then slide \( e \) to level \( N + 1 \) by a homotopy with image in \( St^{2N+1}(e) \subset \tilde{X} - St^N(B) \). (So \( e \) is replaced by a path of the form \( \langle t^k, \tau, t^{-k} \rangle \) where \( \tau \) has image in level \( N + 1 \).)

2) If \( e \) is an edge of \( \alpha \) in level \( -N - 1 \) and sliding \( e \) to level \( -N \) does not intersect \( St^{3N+2}(B) \), then again slide \( e \) to level \( N + 1 \) by a homotopy with image in \( \tilde{X} - St^N(B) \).

Canceling any pairs of edges of the form \( tt^{-1} \) or \( t^{-1}t \) we see that \( \alpha \) is homeomorphic rel\( \{0,1\} \) to \( \beta \), by a homotopy in \( \tilde{X} - St^N(B) \), where \( \beta \) can have various forms depending upon where the end points of \( \alpha \) lie. In any case, \( \beta = \langle u_0, \beta_1, u_1, \beta_2, \ldots, u_n, \beta_{n+1}, u_{n+1} \rangle \) such that

1) For each \( i \), \( u_i = t^{(r(i)}} \) and for \( i \in \{1, 2, \ldots, n\} \), \( r(i) = \pm(2N + 2) \) where the \( r(i) \) alternate in sign.

2) For \( i \in \{2, \ldots, n\} \), the \( \beta_i \) alternate between edge paths in level \( -N - 1 \) with image in \( St^{3N+3}(B) \) (recall edges in level \( -N - 1 \) not in \( St^{3N+3}(B) \) were slid to level \( N + 1 \) missing \( St^N(B) \)) and edge paths that begin and end in level \( N + 1 \) and
lie in levels $N + 1$ and above. The $\beta_i$ of the second type satisfies the hypothesis of Lemma 8 with $L = 5N + 5$ since the $u_i$ provide paths of length $\leq 2N + 2$ to a point (of a $\beta_i$ of the first type) in $S^3N + 3(B)$.

So at this stage we have:

**Lemma 9.** The subpath $(u_1, \beta_2, \ldots, u_n)$ of $\beta$ is homotopic rel{0,1} to a path in $S^3N + 3(B)$ by a homotopy in $X - St^N(B)$.

Hence we need only deal with the paths $(\alpha, \beta_1)$ and $(\beta_{n+1}, u_{n+1})$ in various special cases.

If the initial point of $\alpha$ is in a level from $-N$ to $N$, then $r(0)$ is an integer in $[-2N - 1, 2N + 1]$, and $\beta_1$ is as in 2) above so the argument goes as above for $(\alpha, \beta_1)$. Similarly for $(\beta_{n+1}, u_{n+1})$ if the terminal point of $\alpha$ is in a level $-N$ to $N$.

If the initial point of $\alpha$ is in level $N + 1$ or above, then $r(0)$ is 0 and $\beta_1$ will be an edge path in levels $N + 1$ and above, that ends in level $N + 1$. (This does include the “awkward” case that $\beta_1$ is a power of $t$.) In this case we have that the initial point of $\alpha$ (and hence the initial point of $\beta_1$) is in $S^3N + 2(B)$ and $u_1$ is a path from the terminal point of $\beta_1$ to a point of $S^3N + 3(B)$. Hence $\beta_1$ satisfies the hypothesis of Lemma 8, again with $L = 5N + 5$. Similarly for $\beta_{n+1}$ if the terminal point of $\alpha$ is in level $N + 1$ or above.

Note also that if $n = 0$ (i.e. $\beta_1 = \beta_{n+1}$), then Lemma 8 again applies to $\beta_1$, with $L \leq 5N + 5$.

Finally we consider the case that the initial point of $\alpha$ is in a level below level $-N$. As $S^3N + 2(B)$ lies between levels $-3N - 2$ and $3N + 2$, $r(0)$ (the length of $u_0$) is $\leq 4N + 3$. Now either $\beta_1$ is in level $-N - 1$ (in which case $\beta_1$ is in $S^3N + 3(B)$ and $r(0)$ is in level 2 + 1) so that $(\alpha, \beta_1)$ is in $S^3N + 3(B)$ or $u_0$ is in $S^3N + 3(B)$ and $\beta_1$ satisfies the hypothesis of Lemma 8 with $L = 7N + 5$. In all cases, $\alpha$ is homotopic rel{0,1} to a path in $S^3N + 3(B)$, by a homotopy in $X - St^N(B)$.

This finishes the case of $B$ a finitely generated subgroup of $A$.

To finish the proof of Theorem 2, suppose $(a_1, \ldots, a_n : R)$ is a presentation for $A$. Let $(a_1, \ldots, a_n, t : R, t^{-1}a_i = w_i)$ be a presentation for $G$. The Tietze move that adds a generator $h = ta_1t^{-1}$ gives the presentation $Q = (a_1, \ldots, a_n, h, t : R, t^{-1}a_i = w_i, t^{-1}ht = a_j)$ and we see that $G$ is an ascending HNN-extension with base group, the subgroup $H$, of $G$ generated by $\{a_1, \ldots, a_n, h\}$. The group $H$ need not be finitely presented (see the example following this proof), but if $X$ is the universal cover of the finite 2-complex corresponding to the presentation $Q$ and $\alpha$ is any loop in the levels $K$ and above of $X$, then by sliding all of the edges of $\alpha$ up to a common level we obtain a loop in the edges with labels in $\{a_i^{\pm 1}, a_j^{\pm 1}, h\}$. Sliding up one more level gives a loop in the edges with labels in $\{a_i^{\pm 1}, a_j^{\pm 1}\}$, which is trivial in that level. Hence (see the above remark), if $B$ is a finitely generated subgroup of $H$, then $(B, G)$ is tame. Now let $B$ be a finitely generated subgroup of $\langle A \rangle$ the normal closure of $A$ in $G$. Say $b_1, \ldots, b_m$ are words in $F$ (equiv the free group on $\{a_1, \ldots, a_n, t\}$ representing a generating set of $B$. The exponent sum of $t$ in each $b_i$ is zero. Hence there is an integer $N \geq 0$ such that $B \leq \langle a_1, a_2, \ldots, a_n, t = b_1, a_2, a_2^{-1}, \ldots, a_n, t, t^{-1}, \ldots, t^{-N}a_i, a_i^{t-N} \rangle \leq G$.

If we let $a_{ij} = t^ia^{-1}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{0, 1, \ldots, N\}$, then using Tietze moves (as above) we obtain a presentation for $G$:

$$Q = (a_{10}, \ldots, a_{1N}, a_{20}, \ldots, a_{2N}, \ldots, a_{n0}, \ldots, a_{nN}, t : R, t^{-1}a_0t = w_i, \text{ for } i \in \{1, \ldots, n\}, t^{-1}a_{ij} = a_{i(j-1)} \text{ for } i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, N\})$$.
Hence if $H$ is the subgroup of $G$ generated by $\{a_{10}, \ldots, a_{1N}, \ldots, a_{n0}, \ldots, a_{nN}\}$, then $A \leq H$, $G$ is an ascending HNN-extension of $H$ and if $\tilde{X}$ is the universal cover of the finite 2-complex corresponding to $Q$, then any edge loop $\alpha$ in levels $K$ and above can be slid up to a common level. Sliding up $N$ more levels gives a loop in the edges labeled $a_{10} = a_1, \ldots, a_{n0} = a_n$. This loop is homotopically trivial in this level. Hence by the above Remark, we are finished.

The following example (due to J. Stallings [S] and alluded to in the above proof) is an ascending HNN extension $G$ with base a finitely presented group $A$ so that the subgroup of $G$ generated by $A$ and $tat^{-1}$ (for some $a \in A$) is not finitely presented. (This example shows that Theorem 2 is not a restatement of the first case considered.)

Let $A = (\mathbb{Z}_{p}\ast\mathbb{Z}_{q}) \times (\mathbb{Z}_{x}\ast\mathbb{Z}_{y})$, (where $\mathbb{Z}_{k}$ is the infinite cyclic group with generator $k$). So $A$ has presentation $(p, q, x, y : [p, x], [p, y], [q, x], [q, y])$.

The subgroup $K$ of $A$ with generating set $\{x, p, qy^{-1}\}$ is normal in $A$ and not finitely presented (see [P] or [M2] for instance).

Consider the monomorphism $f : A \rightarrow A$ defined by

$$f(p) = p, \quad f(q) = qpq^{-1}, \quad f(x) = x \quad \text{and} \quad f(y) = xyy^{-1}.$$ 

Let $G$ be the ascending HNN extension of $A$ obtained from $f$, so that $G$ has presentation:

$$\langle t, p, q, u : t^{-1}qt = p, t^{-1}qt = qpq^{-1}, t^{-1}xt = x, t^{-1}yt = yxy^{-1}, [p, x], [p, y], [q, x], [q, y] \rangle.$$

Now $K \leq A \leq G$ and we observe that $K$ is generated by $f(A) \cup \{qy^{-1}\}$. I.e. that $K = \langle p, qpq^{-1}, x, yxy^{-1}, qy^{-1} \rangle$. (This follows since $K$ is generated by $\{x, p, qy^{-1}\}$ and since $K$ is normal in $A$.)

In $G$, the subgroup $K = \langle f(A) \cup \{qy^{-1}\} \rangle = \langle t^{-1}At \cup \{qy^{-1}\} \rangle$ is isomorphic to the subgroup $\langle A \cup \{t(qy^{-1})t^{-1}\} \rangle$. Hence $\langle A \cup \{t(qy^{-1})t^{-1}\} \rangle$ is not finitely generated.

Next we devise a technique to show that a finitely presented group is not the fundamental group of a compact 3-manifold.

First of all, following the ideas in [M1], one can show that the notion of a pair of groups being semistable is well defined. More specifically:

**Proposition 1.** If $X_1$ and $X_2$ are finite simplicial complexes and there is an isomorphism of pairs $(\pi_1(X_1), A)$ to $(\pi_1(X_2), B)$, then $A/X_1$ is semistable at infinity iff $B/X_2$ is semistable at infinity.

The next proposition is shape theoretic in nature and we refer the reader to [MS] as a basic reference.

**Proposition 2.** Any missing boundary 3-manifold is semistable at infinity.

**Proof.** If $M$ is a missing boundary 3-manifold, then say $M$ is a subset of a compact 3-manifold $M_1$ such that $M_1 - M$ is a subset of the boundary of $M_1$. The boundary components of $M_1$ are surfaces and if $S$ is one such surface, then suppose $C$ is a component of the intersection of $S$ with the closure of $M$ in $M_1$ (so that $C$ corresponds to an end of $M$). Now, $C$ is pointed 1-movable. This can be seen by altering K. Borsuk’s proof that every pointed continuum in $\mathbb{R}^2$ is 1-movable (see Theorem 5 Ch. II § 8.1 [MS]) or by appealing directly to [K] or [Mc]. Hence by a theorem of J. Krasinkiewicz (see Theorem 4 Ch. II § 8.1 [MS]), $C$ has the shape of a locally connected continuum. Using regular neighborhoods of $S$, we see that $C$
Proposition 3. Suppose $G$ is a finitely presented group and $A$ is a finitely generated subgroup of $G$ such that the pair $(G, A)$ is tame, but not semistable at infinity. Then $G$ is not the fundamental group of a compact 3-manifold.

Proof. Suppose $M$ were such a 3-manifold. Then the tameness of $(\pi_1(M), A)$ implies that $A/\tilde{M}$ is a missing boundary manifold and by Proposition 2 is semistable at infinity. But this implies that $(G, H)$ is semistable at infinity, the desired contradiction. □

Proposition 4. Suppose $A$ has a presentation $\langle a_1, \ldots, a_n : r_1, \ldots, r_m \rangle$, $f : A \to A$ is a monomorphism but not an epimorphism and $G$ is the strictly ascending HNN-extension with presentation $P \equiv \langle t, a_1, \ldots, a_n : r_1, \ldots, r_m, t^{-1}a_it = f(a_i) \rangle$. Then $\hat{X}_P \equiv A/\hat{X}_P$ is not semistable at infinity (and so $G$ is not the fundamental group of a compact 3-manifold).

The motivating example is $P \equiv \langle t, x : t^{-1}xt = x^2 \rangle$.

Proof. Let $Y$ be the subcomplex of $\hat{X}_P$ consisting of the loops labeled by the $a_i$ union with the 2-cells given by the $r_i$. If $\tilde{X}_P \xrightarrow{f} X_P$ is the universal covering of $X_P$ and $\tilde{X}_P \xrightarrow{p} \hat{X}_P$ is the quotient map, then $f^{-1}(Y)$ is a disjoint union of copies of $Y$.

Let $\tilde{Y}$ be the copy of $Y$ containing the vertex $t^i$, for $i \in \{0, -1, -2, \ldots \}$. We have that $p(\tilde{Y})$ is a copy of $Y$ in $\hat{X}_P$. Furthermore, the copies of $\tilde{Y}_i$ union the 2-cells corresponding to the conjugation relations $t^{-1}a_it = f(a_i)$ where $a_i$ is an edge in one of the $\tilde{Y}_i$ for $i < 0$ are mapped by $p$ to a sort of mapping telescope $T$ in $\tilde{X}_P$. Observe that $T - p(\tilde{Y}_0)$ is a component of $\tilde{X}_P$ minus the compact set $p(\tilde{Y}_0)$.

Pick an edge loop $\alpha$ in $p(\tilde{Y}_1)$ labeled by an element of $A - f(A)$. Then $\alpha$ is not homotopic to an edge loop in $p(\tilde{Y}_j)$ for any $j < i$. Hence $T$ is not semistable at infinity and so $\hat{X}_P$ is not semistable at infinity. □

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Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240
E-mail address: mihalikm@ctrvax.vanderbilt.edu