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HARDY SPACES, BMO, AND BOUNDARY VALUE PROBLEMS FOR THE LAPLACIAN ON A SMOOTH DOMAIN IN $\mathbf{R}^N$

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Abstract. We study two different local H^p spaces, $0 < p \leq 1$, on a smooth domain in $\mathbf{R}^n$, by means of maximal functions and atomic decomposition. We prove the regularity in these spaces, as well as in the corresponding dual BMO spaces, of the Dirichlet and Neumann problems for the Laplacian.

0. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$, with smooth boundary. The L^p regularity of elliptic boundary value problems on Ω, for $1 < p < \infty$, is a classical result in the theory of partial differential equations (see e.g. [ADN]). In the situation of the whole space without boundary, i.e. where Ω is replaced by $\mathbf{R}^n$, the results for L^p, $1 < p < \infty$, extend to the Hardy spaces H^p when $0 < p \leq 1$ and to BMO. Thus it is a natural question to ask whether the L^p regularity of elliptic boundary value problems on a domain Ω has an H^p and BMO analogue, and what are the H^p and BMO spaces for which it holds.

This question was previously studied in [CKS], where partial results were obtained and were framed in terms of a pair of spaces, $h^p_r(\Omega)$ and $h^p_z(\Omega)$. These spaces, variants of those defined in [M] and [JSW], are, roughly speaking, the "largest" and "smallest" h^p spaces that can be associated to a domain Ω.

Our purpose here is to substantially extend the previous results by determining those h^p spaces on Ω which are particularly applicable to boundary value problems. These spaces allow one to prove sharp results (preservation of the appropriate h^p spaces) for all values of p, $0 < p \leq 1$, as well as the preservation of corresponding spaces of BMO functions.

0.1. **Motivation and statement of results.** There are two approaches to defining the appropriate Hardy spaces on Ω. Recall that the spaces $H^p(\mathbf{R}^n)$, for $p < 1$, are spaces of distributions. Thus one approach is to look at the problem from the point of view of distributions on Ω. If we denote by $\mathcal{D}(\Omega)$ the space of smooth functions with compact support in Ω, and by $\mathcal{D}'(\Omega)$ its dual, we can consider the space of distributions in $\mathcal{D}'(\Omega)$ which are the restriction to Ω of distributions in $H^p(\mathbf{R}^n)$ (or in $h^p(\mathbf{R}^n)$, the local Hardy spaces defined in [G].) These spaces were studied in [M] (for arbitrary open sets) and in [CKS] (for Lipschitz domains), where they were denoted $h^p_r(\Omega)$ (the r stands for "restriction".)

While one is able to prove regularity results for the Dirichlet problem for these spaces when p is near 1 (see [CKS]), these spaces are no longer appropriate when p

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is small, nor for the Neumann problem. This is illustrated for the Dirichlet problem by the following example. Let x be a point on $\partial\Omega$, and denote by f the distribution which is the normal derivative of the delta function at x. Such a distribution is in the local Hardy space $h^p(\mathbf{R}^n)$ when $p < \frac{n}{n+1}$. Furthermore, it is possible to take a sequence of L^2 functions a_j (if fact h_r^p atoms) such that $a_j \to f$ as distributions. Since f vanishes on Ω, this means $a_j \to 0$ in $\mathcal{D}'(\Omega)$. Now consider the Dirichlet problem for the Laplacian on Ω, defined for smooth functions φ by

$$\Delta u = \varphi \text{ on } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

Let $\mathbf{G}$ be Green's operator for the Dirichlet problem, i.e. $u = \mathbf{G}(\varphi)$. By the L^2 theory, we can solve this problem for each a_j, and since $\mathbf{G}$ is self-adjoint, we have, for every $\varphi \in \mathcal{D}(\Omega)$,

$$\langle \mathbf{G}(a_j), \varphi\rangle = \langle a_j, \mathbf{G}(\varphi)\rangle \to \frac{\partial}{\partial \vec{n}}\mathbf{G}(\varphi)|_x$$

as $j \to \infty$. Note that for the Dirichlet problem, the normal derivative of the solution need not vanish on the boundary. Thus as distributions in $\mathcal{D}'(\Omega)$, $\mathbf{G}(a_j) \not\to 0$. This shows that the problem is not well-defined in $\mathcal{D}'(\Omega)$. In essence, this is because the space of test functions, $\mathcal{D}(\Omega)$, is not preserved by the solution of the Dirichlet problem.

To remedy this situation, and define a space of distributions appropriate to the Dirichlet problem, we change our space of test functions from $\mathcal{D}(\Omega)$ to $\mathcal{C}_d^\infty(\overline{\Omega})$, consisting of functions $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$ (the d stands for Dirichlet). Note that this space is preserved under the solution to the Dirichlet problem. Thus if we let $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ be the dual space, we can define the solution to the Dirichlet problem for an element f of $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ in the sense of distributions. Moreover, if f happens to be a function which is smooth up to the boundary, or a function in L^p, this solution agrees with $\mathbf{G}(f)$.

We then define the Hardy spaces $h_d^p(\overline{\Omega})$ to consist of those elements of $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ satisfying the expected maximal function conditions; here the maximal functions are fashioned out of test functions taken from $\mathcal{C}_d^\infty(\overline{\Omega})$. For these spaces we get the following regularity result:

Result 0.1. *The operators* $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$*, defined in the sense of distributions, are bounded from* $h_d^p(\overline{\Omega})$ *to* $h_d^p(\overline{\Omega})$*, for all* p*,* $0 < p \le 1$*.*

This is proved by means of an atomic decomposition for elements of $h_d^p(\overline{\Omega})$, where atoms supported near the boundary are required to satisfy fewer cancellation conditions than those supported away from the boundary. From this atomic decomposition it can be seen that $h_d^p(\overline{\Omega})$ is the same as $h_r^p(\Omega)$ when $\frac{n}{n+1} < p \le 1$; hence the regularity result is an extension to small p of the $h_r^p(\Omega)$ regularity result in [CKS].

A second approach to defining Hardy spaces on Ω is to consider the closure of Ω, $\overline{\Omega}$, and the distributions in $h^p(\mathbf{R}^n)$ which are supported on $\overline{\Omega}$. We shall call the spaces formed by these distributions $h_z^p(\overline{\Omega})$, where the z denotes the fact that these distributions are zero outside $\overline{\Omega}$. These spaces are the same as those defined in [JSW] (for certain closed sets). A variant of these spaces, $h_z^p(\Omega)$, formed by taking a quotient of $h_z^p(\overline{\Omega})$ in order to make it a subspace of $\mathcal{D}'(\Omega)$, was defined in [CKS] (for Lipschitz domains). By the same reasoning used in the example above, one sees that this quotient space is not appropriate for small p because it eliminates all distributions supported on the boundary.

The spaces $h^p_z(\overline{\Omega})$ are useful because elements of $h^p_z(\overline{\Omega})$ have an atomic decomposition into h^p atoms supported in $\overline{\Omega}$. Moreover, they are applicable to both the Dirichlet and Neumann problems. Using the atomic decomposition, we can define the operators $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ and $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ (where $\widetilde{\mathbf{G}}$ is the solution operator of the Neumann problem for the Laplacian) on $h^p_z(\overline{\Omega})$, and prove the following regularity result:

Result 0.2. *The operators* $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ *and* $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ *extend to bounded operators from* $h^p_z(\overline{\Omega})$ *to* $h^p_z(\overline{\Omega})$*, for all* $0 < p \leq 1$.

A weaker version of this result, namely the boundedness from $h^p_z(\Omega)$ to $h^p_r(\Omega)$, is in [CKS]. Note, however, that while the proof given there is valid for atoms, it does not hold for the quotient space $h^p_z(\Omega)$, since the quotient space norm may be much smaller than the one given by the atomic decomposition.

Once we have the appropriate definitions and regularity results for the H^p spaces, when $p = 1$, we can consider the corresponding dual BMO spaces. In this case, the dual spaces to $h^1_d(\overline{\Omega})$ and $h^1_z(\overline{\Omega})$ are the spaces $\mathrm{bmo}_z(\Omega)$ and $\mathrm{bmo}_r(\Omega)$, defined in [M], [JSW] and [C]. Using some additional arguments, one can convert the h^1_d and h^1_z regularity results to the following:

Result 0.3. *The operators* $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ *are bounded on* $\mathrm{bmo}_z(\Omega)$ *and on* $\mathrm{bmo}_r(\Omega)$*. Furthermore, the operators* $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ *are bounded on* $\mathrm{bmo}_r(\Omega)$*.*

We should remark that while the results in this paper are stated only for the Laplacian, one can generalize the proofs to any second order elliptic operator, given that the same kind of estimates hold for the various Green's operators.

0.2. **Organization of the paper.** In Section 1, we define the spaces $h^p_d(\overline{\Omega})$ and $h^p_z(\overline{\Omega})$. The atomic decompositions for these spaces are given in Section 2. The proof of the atomic decomposition for $h^p_d(\overline{\Omega})$ uses the maximal function definition and follows the lines of the proof given in [S2] of the atomic decomposition for $H^p(\mathbf{R}^n)$.

In Section 3 we prove the h^p_d regularity of the Dirichlet problem, and in Section 4 we prove the h^p_z regularity of the Dirichlet and Neumann problems. Besides the atomic decompositions, the proofs of both results make use of the Sobolev estimates for $\mathbf{G}$ and $\widetilde{\mathbf{G}}$, and some Calderón-Zygmund type estimates on the kernels of these operators and their derivatives.

Section 5 contains a different proof of the regularity results when $p = 1$, which gives the atomic decomposition directly from a cancellation property inside the domain, without using the maximal function. We then use the h^1 regularity to prove the bmo regularity. This requires an additional argument involving commutations of vector fields with the Green's operators $\mathbf{G}$ and $\widetilde{\mathbf{G}}$.

In the last section we turn from regularity problems to some more analysis of the Hardy spaces themselves. The results in Section 6 illustrate the various relations among the spaces $h^p_d(\overline{\Omega})$, $h^p_z(\overline{\Omega})$, and the spaces $h^p_r(\Omega)$ and $h^p_z(\Omega)$ defined in [CKS].

1. Definition of spaces

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. In this section we will define two Hardy spaces on Ω.

We first recall the definition of the local Hardy spaces $h^p(\mathbf{R}^n)$, introduced by Goldberg (see [G]). One can define these spaces by means of a "grand" maximal function. We call a smooth function ϕ on $\mathbf{R}^n$ a *normalized bump function* if it is supported in a ball B of radius R, and

$$|\partial^\alpha \phi| \leq R^{-n-|\alpha|}$$

for all $|\alpha| \leq N_p + 1$. Here $N_p = [n(1/p - 1)]$, the greatest non-negative integer in $n(1/p-1)$.

Definition 1.1. For $f \in \mathcal{S}'(\mathbf{R}^n)$, define the local grand maximal function $m(f)$ at a point $x \in \mathbf{R}^n$ by

$$m(f)(x) = \sup |\langle f, \varphi \rangle|,$$

where the supremum is taken over all normalized bump functions supported in balls of radii $R \leq 1$ containing x.

For $0 < p < \infty$, the space $h^p(\mathbf{R}^n)$ is defined as the space of tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ for which $m(f) \in L^p(\mathbf{R}^n)$, with

$$\|f\|_{h^p(\mathbf{R}^n)} \stackrel{\text{def}}{=} \|m(f)\|_{L^p(\mathbf{R}^n)}.$$

We now restrict ourselves to $p \leq 1$, and consider a subspace of $h^p(\mathbf{R}^n)$ specific to Ω.

Definition 1.2. For $0 < p \leq 1$, the space $h^p_z(\overline{\Omega})$ is defined to be the subspace of $h^p(\mathbf{R}^n)$ consisting of those elements which are supported on $\overline{\Omega}$, i.e.

$$h^p_z(\overline{\Omega}) = \{f \in h^p(\mathbf{R}^n) : f = 0 \text{ on } \mathbf{R}^n \setminus \overline{\Omega}\},$$

with

$$\|f\|_{h^p_z(\overline{\Omega})} \stackrel{\text{def}}{=} \|f\|_{h^p(\mathbf{R}^n)}.$$

Remarks. 1. This is a special case of the space $h^p(F)$ considered in [JSW] for a closed "d-set" F satisfying the Markov property (here $F = \overline{\Omega}$, $d = n$ and μ is just the restriction of Lebesgue measure to $\overline{\Omega}$).

2. For $p < 1$, this is not the same as the space $h^p_z(\Omega)$ introduced in [CKS], since that space is the quotient space

$$h^p_z(\Omega) = h^p_z(\overline{\Omega})/\{f \in h^p_z(\overline{\Omega}) : f = 0 \text{ on } \Omega\},$$

consisting of those distributions on Ω which have extensions to elements of $h^p_z(\overline{\Omega})$. See Section 6 for more discussion of this space.

3. Following the maximal function definition of $h^p(\mathbf{R}^n)$, one can also define $h^p_z(\overline{\Omega})$ by means of a grand maximal function (see [JSW], for example). In that case, the space of test functions is $\mathcal{C}^\infty(\overline{\Omega})$, i.e. smooth functions up to the boundary, and the elements of $h^p_z(\overline{\Omega})$ can be considered as linear functionals on that space.

For the second space, $h^p_d(\overline{\Omega})$, we want to restrict our space of test functions to those smooth functions on $\overline{\Omega}$ which vanish on $\partial\Omega$. Denoting this space by $\mathcal{C}^\infty_d(\overline{\Omega})$, we have

$$\mathcal{C}^\infty_d(\overline{\Omega}) = \{\varphi \in \mathcal{C}^\infty(\overline{\Omega}) : \varphi|_{\partial\Omega} = 0\}.$$

We take the topology of $\mathcal{C}_d^\infty(\overline{\Omega})$ to be that inherited from $\mathcal{C}^\infty(\overline{\Omega})$, and let $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ denote the dual space.

We call a function $\varphi \in \mathcal{C}_d^\infty$ a *normalized* $\mathcal{C}_d^\infty$ *bump function* if it is the restriction to $\overline{\Omega}$ of a normalized bump function ϕ on $\mathbf{R}^n$, with $\phi|_{\partial\Omega} = 0$.

Definition 1.3. For $f \in \mathcal{C}_d^{\infty\prime}(\overline{\Omega})$, define the maximal function $m_d(f)$ at a point x in Ω by

$$m_d(f)(x) = \sup |\langle f, \varphi\rangle|,$$

where the supremum is taken over all normalized $\mathcal{C}_d^\infty$ bump functions supported in balls of radii $R \leq 1$ containing x.

For $0 < p \leq 1$, set

$$h_d^p(\overline{\Omega}) = \{f \in \mathcal{C}_d^{\infty\prime}(\overline{\Omega}) : m_d(f) \in L^p(\Omega)\},$$

and for $f \in h_d^p(\overline{\Omega})$,

$$\|f\|_{h_d^p(\overline{\Omega})} \stackrel{\text{def}}{=} \|m_d(f)\|_{L^p(\Omega)}.$$

Remarks. 1. In the definition of the maximal function m_d, we could have taken the supremum over all normalized $\mathcal{C}_d^\infty$ bump functions supported in balls of radii $R \leq \delta$, for some fixed $\delta > 0$. This new maximal function, m_d^δ, is equivalent to m_d in the sense that they define the same space h_d^p, although the norms differ by a constant depending on δ.

2. Convergence in the "$h_d^p(\overline{\Omega})$-norm" implies convergence in $\mathcal{C}_d^{\infty\prime}(\Omega)$. To see this, it suffices to show that for $f \in h_d^p(\overline{\Omega})$ and $\varphi \in \mathcal{C}_d^\infty(\overline{\Omega})$,

$$|\langle f, \varphi\rangle| \leq C_\varphi \|f\|_{h_d^p(\overline{\Omega})},$$

where the constant may depend on φ but not on f.

2. Atomic decomposition

We want to have a characterization of our spaces by means of atomic decomposition. We begin by giving a variant of the definition of atoms for the local Hardy spaces $h^p(\mathbf{R}^n)$.

Notation. In what follows, the word "cube" shall mean a cube with sides parallel to the axes, and if A is a constant, AQ shall denote the dilated cube, meaning the cube with the same center as Q with sidelength multiplied by A.

Definition 2.1. Let $0 < p \leq 1$, $N_p = [n(1/p - 1)]$, and $\nu_p = (N_p + 1)/n + 1 - 1/p$. A function a will be called an $\underline{h^p(\mathbf{R}^n) \text{ atom}}$ if it is supported in a cube $Q \subset \mathbf{R}^n$ and satisfies the size condition

$$\|a\|_\infty \leq |Q|^{-1/p}, \tag{2.1}$$

and, when $|Q| < 1$, the approximate moment conditions

$$\left| \int a(x)(x - x_Q)^\alpha dx \right| \leq |Q|^{\nu_p} \tag{2.2}$$

for all multi-indices α with $|\alpha| \leq N_p$. Here x_Q is the center of the cube Q.

Remarks. 1. Atoms supported in cubes Q with $|Q| < 1$ and satisfying zero moment conditions up to order N_p (i.e. h^p atoms as defined in [G]) satisfy conditions 2.2 trivially for $|\alpha| \leq N_p$. Conversely, as shown in [D], an atom a which satisfies conditions 2.2 for any $\nu > 0$ (not necessarily ν_p) is in $h^p(\mathbf{R}^n)$

with $\|a\|_{h^p} \le C_\nu$. Thus a distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ is in $h^p(\mathbf{R}^n)$ if for some $\nu > 0$ it can be decomposed as $\sum \lambda_j a_j$, where $\sum |\lambda_j|^p < \infty$ and the a_j are atoms satisfying conditions 2.2 for that ν. This shows that the atomic spaces corresponding to different choices of ν in conditions 2.2 are in fact all equivalent to $h^p(\mathbf{R}^n)$, except for the dependence of the norm on ν. As will be seen below, it is most convenient to take $\nu = \nu_p = (N_p+1)/n - 1/p + 1$.

2. Due to the choice of ν_p, the size condition for an $h^p(\mathbf{R}^n)$ atom a automatically gives conditions 2.2 for moments of order $|\alpha| = N_p + 1$, since
$$\left|\int a(x)(x-x_Q)^\alpha dx\right| \le (\mathrm{diam}(Q))^{N_p+1}|Q|^{-1/p+1} = C_{n,p}|Q|^{\nu_p}.$$
By expanding any $\psi \in \mathcal{C}^\infty$ in a Taylor expansion of order N_p around x_Q, we can use conditions 2.2 to get
$$\left|\int a(x)\psi(x)dx\right| \le C_{n,p}\|\psi\|_{\mathcal{C}^{N_p+1}(Q)}|Q|^{\nu_p},$$
where
$$\|\psi\|_{\mathcal{C}^N(Q)} = \sum_{|\alpha|\le N} \sup_{x\in Q} \frac{1}{|\alpha|!}\left|\psi^{(\alpha)}(x)\right|.$$
Requiring this for all $\psi \in \mathcal{C}^\infty$ is in fact equivalent to the moment conditions 2.2.

3. Remark 2 show that the conditions imposed on an atom a are invariant under a smooth change of coordinates, in the following sense: if we make a smooth change of variables $y = \Phi(x)$, set $\tilde{a}(y) = a(\Phi^{-1}(y))$, and let $\widetilde{Q}$ be the smallest cube containing $\Phi(Q)$, we have
$$\|\tilde{a}\|_\infty \le C_\Phi|\widetilde{Q}|^{-1/p}$$
and
$$\left|\int \tilde{a}(y)(y-y_{\widetilde{Q}})^\alpha dy\right| \le C_\Phi|\widetilde{Q}|^{\nu_p},$$
where the constant C_Φ depends on Φ and its derivatives up to order N_p+1. This follows from Remark 2 by letting $\psi(x) = (\Phi(x) - y_{\widetilde{Q}})^\alpha|J_\Phi(x)|$.

We now define some atoms specific to the domain Ω. First:

Definition 2.2. An $\underline{h^p_z \text{ atom}}$ is an $h^p(\mathbf{R}^n)$ atom supported in a cube $Q \subset \overline{\Omega}$.

Next, we turn to h^p_d atoms and define two types of atoms, depending on the position of their supporting cubes with respect to the domain. This definition uses some constants arising from the geometry of the domain.

Definition 2.3. Let A_Ω, B_Ω, and C_Ω be constants, with $A_\Omega > 1$, $B_\Omega > 0$, and $C_\Omega > 0$.

A cube will be called a type (a) cube if the dilated cube $A_\Omega Q$ is contained in Ω.

A function a will be called a type (a) h^p_d atom if it is an $h^p(\mathbf{R}^n)$ atom supported in a type (a) cube, with the modification that it need only satisfy the approximate moment conditions 2.2 when $|Q| < C_\Omega$.

A cube will be called a type (b) cube if
$$A_\Omega Q \cap \partial\Omega \ne \emptyset,$$

but

$$|Q \cap \overline{\Omega}| > B_\Omega |Q|.$$

A function a will be called a type (b) h^p_d atom if it is supported in a type (b) cube, satisfies the size condition 2.1, and, when $|Q| < C_\Omega$, the approximate moment conditions

$$\left| \int a(x)\psi(x)dx \right| \le \|\psi\|_{\mathcal{C}^{N_p+1}(Q)} |Q|^{\nu_p}, \tag{2.3}$$

for all $\psi \in \mathcal{C}^\infty_d(\overline{\Omega})$.

Remarks. 1. The conditions on a type (a) cube guarantee that the cube is "far away" from the boundary, relative to its size. Thus type (a) h^p_d atoms will be invariant under diffeomorphisms of Ω, in the sense of the remarks above, with the possible change of the constants A_Ω and C_Ω.

If in addition the diffeomorphism can be extended to a neighborhood of $\overline{\Omega}$, then type (b) h^p_d atoms will also be invariant, again up to a change of constants.

2. The conditions on a type (b) cube guarantee that the cube is "near" the boundary relative to its size, and that a significant portion of the cube is inside the domain. We do not require that the cube be completely included in the domain because we want to allow type (b) h^p_d atoms to be supported up to the boundary; hence their supporting cubes cannot be inside the domain unless the boundary is flat and parallel to the axes. We will see, however, that we can assume this locally under an appropriate change of variables.
3. Using the constants B_Ω and C_Ω, one may define a modified version of h^p_z atoms. We will call a function a a modified h^p_z atom if it is supported in a cube Q with $|Q \cap \overline{\Omega}| > B_\Omega |Q|$, satisfies the size condition 2.1, and satisfies the approximate moment conditions 2.2 when $|Q| < C_\Omega$. In fact, this is exactly a type (a) or a type (b) h^p_d atom with full moment conditions. Unlike the standard h^p_z atoms, these atoms are invariant under a diffeomorphism of $\overline{\Omega}$, up to a change of constants.

Note that the atoms we have defined indeed belong to the appropriate spaces, with uniformly bounded norms. This is obviously true for h^p_z atoms (including the modified version), since they are in $h^p(\mathbf{R}^n)$ with bounded norm, and vanish outside $\overline{\Omega}$.

For h^p_d atoms, we have the following

Lemma 2.4. *If a is an h^p_d atom (of either type), then $a \in h^p_d(\overline{\Omega})$ and*

$$\|a\|_{h^p_d(\overline{\Omega})} = \|m_d(a)\|_{L^p(\Omega)} \le C_p,$$

where C_p is independent of a.

The proof follows the standard arguments and so is omitted.

We now recall the atomic decomposition in $h^p_z(\overline{\Omega})$.

Theorem 2.5 ([JSW], [CKS])**.** *Let Ω be a bounded domain in $\mathbf{R}^n$, with $\mathcal{C}^\infty$ boundary, and $0 < p \le 1$. A distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ is in $h^p_z(\overline{\Omega})$ if and only if it has a decomposition*

$$f = \sum \lambda_l a_l$$

in $\mathcal{S}'(\mathbf{R}^n)$, where the a_l are h^p_z atoms and the λ_l are complex numbers satisfying

$$\sum |\lambda_l|^p < \infty.$$

Furthermore,

$$\|f\|^p_{h^p_z} \approx \inf \left(\sum |\lambda_l|^p \right),$$

where the infimum is taken over all such decompositions, and the constants of proportionality are independent of f.

Remarks. 1. Jonsson, Sjögren, and Wallin (see [JSW], Theorems 3.1 and 3.2) prove this decomposition in the case of a closed "d-set" F in $\mathbf{R}^n$ satisfying the Markov property (see Remark 1 following Definition 1.2). In their proof, the h^p_z atoms are actually the modified h^p_z atoms (see Remark 3 above), i.e. they are supported in F, and their supports are contained in balls with centers in F, but the balls need not be contained in F.

2. In ([CKS], Theorem 3.2), this decomposition is stated for a bounded Lipschitz domain Ω, not for the space $h^p_z(\overline{\Omega})$, but for the quotient space $h^p_z(\Omega)$, without the bounds on the norms. However, the proof (via the square function) only assumes that $f \in h^p(\mathbf{R}^n)$ and f is supported in $\overline{\Omega}$, so it in fact holds for $f \in h^p_z(\overline{\Omega})$, and in that case one gets the bounds on the norms by noting that the $h^p_z(\overline{\Omega})$ norm of f is the same as its $h^p(\mathbf{R}^n)$ norm, which bounds the L^p norm of the square function, and which in turn bounds $\left(\sum |\lambda_l|^p\right)^{1/p}$. Moreover, by a slight modification of their argument it can be shown that all the atoms may be taken to be of type (a), i.e. with support Q such that $A_\Omega Q$ is still in Ω. It should be added that at the time the paper [CKS] was written, the authors were unfortunately unaware of the earlier work of [JSW].

We are now ready to state the main result of this section.

Theorem 2.6. *Let Ω be a bounded domain in $\mathbf{R}^n$, with $\mathcal{C}^\infty$ boundary. Then there are constants A_Ω, B_Ω, and C_Ω (as in Definition 2.3) such that the following holds:*

A distribution $f \in \mathcal{C}^{\infty\prime}_d(\overline{\Omega})$ is in $h^p_d(\overline{\Omega})$ if and only if it has a decomposition

$$f = \sum \lambda_l a_l + \sum \mu_m a_m$$

in $\mathcal{C}^{\infty\prime}_d$, where the first sum is taken over type (a) *h^p_d atoms, the second sum is taken over type* (b) *h^p_d atoms, and λ_l, μ_m are complex numbers satisfying*

$$\sum |\lambda_l|^p + \sum |\mu_m|^p < \infty.$$

Furthermore,

$$\|f\|^p_{h^p_d} \approx \inf \left(\sum |\lambda_l|^p + \sum |\mu_m|^p \right),$$

where the infimum is taken over all such decompositions, and the constants of proportionality are independent of f.

Proof. The easy part of the proof is the "if" part, that is, assuming f has such a decomposition in $\mathcal{C}^{\infty\prime}_d(\Omega)$. For then, if the sum is finite,

$$\langle f, \varphi \rangle = \sum \lambda_l \langle a_l, \varphi \rangle + \sum \mu_m \langle a_m, \varphi \rangle$$

for all normalized $\mathcal{C}^\infty_d$ bump functions φ, and so

$$m_d(f)(x) \le \sum |\lambda_l| m_d(a_l)(x) + \sum |\mu_m| m_d(a_m)(x);$$

hence

$$\begin{aligned}\|m_d(f)\|^p_{L^p(\Omega)} &\le \sum |\lambda_l|^p \|m_d(a_l)\|^p_{L^p(\Omega)} + \sum |\mu_m|^p \|m_d(a_m)\|^p_{L^p(\Omega)} \\ &\le C^p_\nu \left(\sum |\lambda_l|^p + \sum |\mu_m|^p\right)\end{aligned}$$

by the lemma. This gives the convergence of the sum in the h^p_d norm, and so $f \in h^p_d(\overline{\Omega})$.

As for the "only if" part, let us first state the result for a special case. We will then use this to prove the general case.

Lemma 2.7. *Let D be a smoothly bounded domain contained in the upper half-space $\mathbf{R}^n_+$ and containing the open upper half-ball $B_+(0,10) = B(0,10) \cap \mathbf{R}^n_+$. Suppose $f \in h^p_d(\overline{D})$ and f is supported in $B(0,1)$. Then f has a decomposition*

$$f = \sum \lambda_i a_i$$

in $\mathcal{C}^\infty_d{}'(\overline{D})$. Here the λ_i are complex numbers satisfying

$$C_1 \|f\|^p_{h^p_d(\overline{D})} \le \sum |\lambda_i|^p \le C_2 \|f\|^p_{h^p_d(\overline{D})}$$

for some constants C_1 and C_2 depending only on p, and the a_i are as follows.

Each a_i is a function supported in a cube Q_i which is contained in the half-open upper half-ball $B(0,3) \cap \overline{\mathbf{R}^n_+}$, and

$$\|a_i\|_\infty \le \|Q_i\|^{-1/p}.$$

Furthermore, there exists a constant C_n (depending only on the dimension) such that when $|Q_i| < C_n$, a_i satisfies the following cancellation conditions. If $2Q_i \subset \mathbf{R}^n_+$, then

$$\int_{Q_i} a_i(x) x^\alpha dx = 0$$

for all monomials x^α of degree $|\alpha| \le N_p$. If $2Q_i \cap \partial\mathbf{R}^n_+ \ne \emptyset$, then

$$\int_{Q_i} a_i(x) x^\alpha dx = 0$$

for all monomials x^α of degree $|\alpha| \le N_p$ with $\alpha_n \ge 1$.

Assuming the lemma, we continue with the proof of Theorem 2.6.

We take a partition of unity for $\overline{\Omega}$ as follows. Let η_j, $j = 0, \dots, k$, be $\mathcal{C}^\infty$ functions, $\sum \eta_j = 1$ on $\overline{\Omega}$. For $j = 0$, η_0 has compact support in Ω. For $1 \le j \le k$, η_j has compact support inside an open set U_j with $U_j \cap \partial\Omega \ne \emptyset$. Furthermore, each U_j is contained in a larger open set V_j such that there is a $\mathcal{C}^\infty$ diffeomorphism Φ_j between V_j and the ball $B(0,10)$ which maps $V_j \cap \Omega$ onto $B_+(0,10) = B(0,10) \cap \mathbf{R}^n_+$, $V_j \cap \partial\Omega$ into $\partial\mathbf{R}^n_+$, and U_j into $B(0,1)$.

Write $f = \sum(\eta_j f)$. Consider first $j = 0$. Set

$$\delta = \frac{1}{2}\operatorname{dist}(\operatorname{supp}(\eta_0), \partial\Omega),$$

and let m^δ be the local grand maximal function defined as in Definition 1.1 but with bump functions supported in balls of diameter bounded by δ. Notice that $\eta_0 f \in \mathcal{S}'$, and

$$m^\delta(\eta_0 f)(x) = 0$$

for $x \notin \Omega$. For $x \in \Omega$,

$$m^{\delta}(\eta_0 f)(x) \leq C_0 m_d(f)(x),$$

where the constant C_0 only depends on the derivatives of η_0 up to order $N_p + 1$. Thus

$$\|m^{\delta}(\eta_0 f)\|_{L^p(\mathbf{R}^n)} \leq C_0 \|m_d(f)\|_{L^p(\Omega)} < \infty,$$

so $\eta_0 f \in h^p(\mathbf{R}^n)$.

Since the support of $m^{\delta}(\eta_0 f)$ lies inside Ω, and at least distance δ from the boundary, we can get an atomic decomposition

$$\eta_0 f = \sum \lambda_i a_i,$$

where a_i are $h^p(\mathbf{R}^n)$ atoms supported in cubes $Q_i \subset \Omega$ with $\operatorname{dist}(Q_i, \partial\Omega) \geq \delta$ for all i. If $2Q_i \subset \Omega$, then a_i is a type (a) h^p_d atom. If $2Q_i \cap \partial\Omega \neq \emptyset$, then a_i is a type (b) h^p_d atom because a_i satisfies the size condition and, when $|Q| < 1$, the full moment conditions a_i satisfies as an $h^p(\mathbf{R}^n)$ atom imply

$$\left| \int a_i(x)\varphi(x)dx \right| \leq \|\varphi\|_{\mathcal{C}^{N_p+1}(Q)} |Q_j|^{\nu_p}$$

for all $\varphi \in \mathcal{C}^\infty$ (see Remark 2 following Definition 2.1,) hence for all $\varphi \in \mathcal{C}^\infty_d(\overline{\Omega})$.

Now fix j, $1 \leq j \leq k$, and write η for η_j, U for U_j, $V = V_j$, and $\Phi = \Phi_j$. Let $\tau = \eta \circ \Phi^{-1}$. Then τ is a $\mathcal{C}^\infty$ function supported in $B(0,1)$. Note that if φ is a $\mathcal{C}^\infty$ function supported in $\overline{B_+(0,10)}$ and $\varphi|_{\partial\mathbf{R}^n_+} = 0$, then $\varphi \circ \Phi \in \mathcal{C}^\infty_d(\Omega)$. Define g acting on such φ by

$$\langle g, \varphi \rangle = \langle \eta f, \varphi \circ \Phi \rangle = \langle f, (\tau\varphi) \circ \Phi \rangle.$$

If φ is supported outside $B(0,1)$, this gives 0. Thus we can extend g to act on any $\varphi \in \mathcal{C}^\infty_d(D)$ by setting it to be zero on $D \setminus B(0,1)$. The continuity of f implies that of g, so $g \in \mathcal{C}^{\infty\prime}_d(D)$ and g is supported in $B(0,1)$.

In order to use the lemma, we want to show $g \in h^p_d(\overline{D})$. So suppose $x \in B(0,10)$ and φ^x_t is the restriction to $\mathbf{R}^n_+$ of a normalized $\mathcal{C}^\infty$ bump function, supported in a ball radius $t \leq 1$ containing x, and vanishing on $\partial\mathbf{R}^n_+$. Then $\psi = (\tau\varphi^x_t)\circ\Phi \in \mathcal{C}^\infty_d(\overline{\Omega})$ and is supported in a ball of radius δ containing $\Phi^{-1}(x)$, where δ depends only on Φ. We can write ψ as a constant multiple of a normalized $\mathcal{C}^\infty_d$ bump function supported in a ball of radius δ, the constant depending only on η and Φ. Thus

$$|\langle g, \varphi^x_t \rangle| = |\langle f, \psi \rangle| \leq C m^{\delta}_d(f)(\Phi^{-1}(x)),$$

so by Remark 1 following Definition 1.3,

$$\|m_d(g)\|_{L^p(D)} \leq C \|m_d(f)\|_{L^p(\Omega)},$$

and $g \in h^p_d(\overline{D})$.

Thus g satisfies the hypotheses of Lemma 2.7, and we get a decomposition

$$g = \sum \lambda_i a_i.$$

Let $\widetilde{a_i} = a_i \circ \Phi$, and let $\widetilde{Q_i}$ be the smallest cube containing $\Phi^{-1}(Q_i)$, where Q_i is the cube in which a_i is supported. We want to show that the $\widetilde{a_i}$ are multiples of either type (a) or type (b) atoms.

Since the a_i satisfy the size condition, we get that

$$\|\widetilde{a_i}\|_\infty = \|a_i\|_\infty \leq |Q_i|^{-1/p} \leq C_{\Phi,\Omega} |\widetilde{Q_i}|^{-1/p}.$$

Since the Q_i are all contained in $\overline{B_+(0,3)}$, the constant $C_{\Phi,\Omega}$ only depends on the bounds on the derivatives of Φ in a compact set, as well as on the geometry of Ω (and of course p and n). Thus we can divide all the $\widetilde{a_i}$ by this constant, and they will satisfy the size condition.

To check the cancellation conditions, we must specify the constants used in Definition 2.3. We will do this first just for the map $\Phi = \Phi_j$. There exists a constant $A_\Phi > 1$ (depending only on the maximum and minimum of the derivatives of Φ on a compact set) such that for any cube $Q \subset B_+(0,3)$, if $\widetilde{Q}$ is the smallest cube containing $\Phi^{-1}(Q)$, and if $A_\Phi \widetilde{Q} \subset \Omega$, then $2Q \subset \mathbf{R}^n_+$. Furthermore, there exists a constant $B_\Phi > 0$, again depending only on Φ and the geometry of Ω, such that if Q is a cube in $\overline{B_+(0,3)}$, and $2Q \cap \overline{\mathbf{R}^n_+} \neq 0$, then the smallest cube $\widetilde{Q}$ containing $\Phi^{-1}(Q)$ satisfies $|\widetilde{Q} \cap \Omega| \geq B_\Phi |\widetilde{Q}|$. Finally, there exists a constant $C_\Phi > 0$ (in fact a constant multiple of the minimum for the Jacobian determinant $|J_\Phi|$ on $\Phi^{-1}(\overline{B_+(0,3)})$) such that if Q and $\widetilde{Q}$ are as above, and $|\widetilde{Q}| < C_\Phi$, then $|Q| < C_n$ (with C_n as in Lemma 2.7).

Now note that since there are only finitely many Φ_j, we can choose these constants uniformly. (Here we also take into account the implicit choices made for $j = 0$, namely $A_\Phi = 2$, $B_\Phi = 1$ and $C_\Phi = 1$.) Thus we will replace A_Φ, B_Φ and C_Φ by A_Ω, B_Ω and C_Ω, respectively. We are now ready to prove the cancellation conditions.

First suppose $A_\Omega \widetilde{Q_i} \subset \Omega$. Since all the cubes $Q_i \subset \overline{B_+(0,3)}$, we have, by the choice of A_Ω, that $2Q_i \subset \mathbf{R}^n_+$. In that case a_i in an $h^p(\mathbf{R}^n)$ atom, so if in addition $|\widetilde{Q_i}| < C_\Omega$, then $|Q_i| < C_n$, and

$$\int_{Q_i} a_i(y) y^\alpha dy = 0$$

whenever $|\alpha| \leq N_p$. From Remarks 1 and 3 following Definition 2.1, we get

$$\left| \int_{\widetilde{Q_i}} \widetilde{a_i}(x)(x - x_{\widetilde{Q_i}})^\alpha dx \right| \leq C |\widetilde{Q_i}|^{\nu_p},$$

where $x_{\widetilde{Q_i}}$ is the center of $\widetilde{Q_i}$ and the constant C depends on Φ and the geometry of Ω, but is independent of $\widetilde{a_i}$. Dividing $\widetilde{a_i}$ by C, we get that $\widetilde{a_i}$ is a type (a) atom.

Next, suppose $A_\Omega \widetilde{Q_i} \cap \partial\Omega \neq \emptyset$. Then we have by the choice of B_Ω that

$$|\widetilde{Q_i} \cap \overline{\Omega}| \geq B_\Omega |\widetilde{Q_i}|.$$

So we want to show that $\widetilde{a_i}$ satisfies the cancellation conditions for a type (b) h^p_d atom. Let $\psi \in \mathcal{C}^\infty_d(\overline{\Omega})$. By using a cut-off function, we may assume $\operatorname{supp}(\psi) \subset V$. Define a function φ on $B(0,10) \cap \mathbf{R}^n_+$ by

$$\varphi(y) = \psi\left(\Phi^{-1}(y)\right) |J_\Phi^{-1}(y)|.$$

Then by a change of variables $y = \Phi(x)$, we have

$$\int_{\widetilde{Q_i}} \widetilde{a_i}(x)\psi(x) dx = \int_{Q_i} a_i(y)\varphi(y) dy.$$

Let y^* be the projection of the center of Q_i onto $\partial\mathbf{R}^n_+$. Since $\psi = 0$ on $\partial\Omega$, we have that $\varphi = 0$ on $\partial\mathbf{R}^n_+$, so all the tangential derivatives of φ at y^* vanish. Thus the

Taylor expansion of φ around y^* is

$$\varphi(y) = \sum_{|\alpha| \leq N_p, \alpha_n \geq 1} (\partial^\alpha \varphi)(y^*)(y-y^*)^\alpha + R_{N_p+1}(\varphi, y, y^*).$$

Plugging this into the integral, we can use the moment conditions on a_i (noting that $(y-y^*)^\alpha$ contains a factor of y_n of order $\alpha_n \geq 1$) to get

$$\begin{aligned}
\left|\int_{Q_i} a_i(y)\varphi(y)dy\right| &= \left|\int_{Q_i} a_i(y) R_{N_p+1}(\varphi, y, y^*)dy\right| \\
&\leq \sum_{|\alpha|=N_p+1} \frac{1}{(N_p+1)!} \sup_{y \in Q_i} |\partial^\alpha(\varphi(y))| \int |a_i(y)||y-y^*|^{(N_p+1)}dy \\
&\leq \|\varphi\|_{\mathcal{C}^{N_p+1}(Q_i)} |Q_i|^{1-1/p} \operatorname{diam}(Q_i)^{(N_p+1)} \\
&\leq C\|\psi\|_{\mathcal{C}^{N_p+1}(\widetilde{Q}_i)} |\widetilde{Q}_i|^{\nu_p},
\end{aligned}$$

where the constant C depends on Φ, the cutoff function, and the geometry of Ω, but is independent of $\widetilde{a}_i$. Again, we can divide $\widetilde{a}_i$ by the constant to get a type (b) h^p_d atom.

We have thus shown that in $\mathcal{C}^{\infty\prime}_d(\overline{\Omega})$,

$$\eta f = g \circ \Phi = \sum \lambda_i a_i \circ \Phi = \sum \lambda_i \widetilde{a}_i = \sum (C\lambda_i)(\widetilde{a}_i/C),$$

where $\widetilde{a}_i/C$ are either type (a) h^p_d atoms or type (b) h^p_d atoms. Furthermore,

$$\sum |C\lambda_i|^p \leq C'\|g\|^p_{h^p_d(\overline{D})} \leq C''\|f\|^p_{h^p_d(\overline{\Omega})}.$$

Getting back to the global picture, we have, in $\mathcal{C}^{\infty\prime}_d(\overline{\Omega})$,

$$f = \sum_{j=0}^{k} \eta_j f = \sum_{j=0}^{k} \sum_i \lambda^j_i a^j_i,$$

which is an atomic decomposition that satisfies the conditions of the theorem. $\square$

In order to prove Lemma 2.7, we will follow the proof in [S2], Chapter III, Section 2, (which is itself adapted from several earlier arguments going back to the work of Latter and others). We begin with the following proposition.

Proposition 2.8. *Let D be as in Lemma 2.7. Suppose that $f \in \mathcal{C}^{\infty\prime}_d(\overline{D})$, $m_d(f) \in L^p(D)$, and f is supported in $B(0,1)$. Then for $\gamma \geq 0$, there is a decomposition $f = g + b$, $b = \sum b_k$, and a collection of cubes $\{Q^*_k\}$, so that*

1. *the $\{Q^*_k\}$ are contained in $B(0,3) \cap \overline{\mathbf{R}^n_+}$, have the bounded intersection property, and*

$$\text{interior}\left(\bigcup_k Q^*_k\right) = \{x \in D : m_d(f)(x) > \gamma\};$$

2. *each $b_k \in \mathcal{C}^{\infty\prime}_d(\overline{D})$ is supported in Q^*_k and satisfies*

$$\int_D m_d(b_k)^p dx \leq c \int_{Q^*_k} m_d(f)^p dx;$$

3. *if* Q_k^* *is such that* $2Q_k^* \subset \mathbf{R}^n_+$, *and* $|Q_k^*| < C_n$ *(with* C_n *as in Lemma 2.7), then* b_k *satisfies*
$$\langle b_k, q\rangle = 0$$
for all polynomials q *of degree up to* N_p*; otherwise, if* $2Q_k^* \cap \partial\mathbf{R}^n_+ \neq \emptyset$, *and* $|Q_k^*| < C_n$, *then* b_k *satisfies*
$$\langle b_k, q\rangle = 0$$
for all polynomials of degree up to N_p *which are divisible by* x_n*;*
4. $g \in \mathcal{C}_d^{\infty\prime}(\overline{D})$ *and*
$$m_d(g) \leq C m_d(f)(x)\chi_F + C\gamma \sum_k \frac{l_k^{n+N_p+1}}{(l_k + |x - x_k|)^{n+N_p+1}},$$
where F *is the closure of the set* $\{x \in D : m_d(f)(x) \leq \gamma\}$, *and* l_k *and* x_k *are the side-length and center of* Q_k^*.

Proof. Let us outline the modifications needed for the proof in [S2], Chapter III, Section 2.2, to apply in this situation.

Let F be the closure of the set $\{x \in D : m_d(f)(x) \leq \gamma\}$ in $\mathbf{R}^n$. Note that since f is supported in $B(0,1)$, and the maximal function m_d is defined using bump functions supported in balls of radii not greater than 1, we must have that $m_d(f)$ is supported in $B(0,3)$, and hence $(\overline{D} \setminus F) \subset B(0,3) \cap \overline{\mathbf{R}^n_+}$.

Consider the dyadic Whitney decomposition for $\mathbf{R}^n \setminus F$ (see [S1], Ch. VI, Section 1.1). From this decomposition, we take only those cubes lying in $\overline{D}$. This is possible since any cube in the dyadic decomposition lies on one side or the other of $\partial\mathbf{R}^n_+$, and furthermore $\partial D \subset F \cup \partial\mathbf{R}^n_+$, so any cube in the decomposition lies either entirely in $\overline{D}$ or entirely outside of it.

Thus we can get closed cubes $Q_k \subset B(0,3) \cap \overline{\mathbf{R}^n_+}$ whose interiors are mutually disjoint, with
$$\operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, F) \leq 4\operatorname{diam}(Q_k)$$
and
$$\bigcup_k Q_k = \overline{D} \setminus F \subset B(0,3) \cap \overline{\mathbf{R}^n_+}.$$

Let $\widetilde{Q_k} = \frac{3}{2}Q_k \cap \overline{\mathbf{R}^n_+}$ and $Q_k^* = 2Q_k \cap \overline{\mathbf{R}^n_+}$. While Q_k^* may no longer be a cube (in case $\operatorname{dist}(Q, \partial\mathbf{R}^n_+)$ is smaller than half the sidelength of Q), we can enlarge it in the x_n direction (by no more than half the sidelength of Q) to make it a cube. Even if that is the case, we still have that, for all $y \in Q_k^*$,
$$\operatorname{dist}(y, Q_k) \leq \frac{\sqrt{n+3}}{2\sqrt{n}}\operatorname{diam}(Q_k) < \operatorname{diam}(Q_k)$$
(since $n > 1$), so as we assumed that $\operatorname{dist}(Q_k, F) \geq \operatorname{diam}(Q_k)$, we have
$$\operatorname{dist}(Q_k^*, F) > 0.$$
Combined with the fact that $Q_k^* \subset \overline{\mathbf{R}^n_+}$, and recalling that $\partial D \subset F \cup \partial\mathbf{R}^n_+$, this implies
$$Q_k^* \subset \overline{D} \setminus F \subset B(0,3) \cap \overline{\mathbf{R}^n_+}.$$
As we already have $\bigcup Q_k = \overline{D} \setminus F$, we get that
$$\bigcup_k Q_k^* = \overline{D} \setminus F$$

and hence

$$\text{interior}\left(\bigcup_k Q_k^*\right) = \{x \in D : m_d(f)(x) > \gamma\}.$$

In constructing the partition of unity $\{\eta_k\}$ for $\overline{D}\setminus F$, we first follow the construction in [S2], namely we take a partition of unity subordinate to the cubes $\frac{3}{2}Q_k$, but then we restrict the functions to $\overline{\mathbf{R}^n_+}$. Thus η_k will be smooth when $\widetilde{Q_k} \subset \mathbf{R}^n_+$, and otherwise η_k will be the restriction of a smooth function to $\overline{\mathbf{R}^n_+}$.

We set

$$b_k = (f - c_k)\eta_k,$$

where the polynomial c_k is picked as follows.

Let $C_n = (3\sqrt{n})^{-n}$. If $|Q_k^*| \geq C_n$, we let $c_k = 0$.

If $|Q_k^*| < C_n$ and Q_k^* is a type (a) cube, namely $2Q_k^* \subset \mathbf{R}^n_+$, then c_k is the unique polynomial for which

$$\langle f, q\eta_k \rangle = \int c_k q \eta_k dx$$

for all polynomials q of degree $\leq N_p$. Note that we can write

$$c_k(x) = \langle f, P(x, \cdot)\eta_k \rangle,$$

where $P(x, y)$ is the kernel of the projection operator from the L^2 space with weight function η_k onto its subspace $\mathcal{H}_{k,N_p}$ consisting of polynomials of degree $\leq N_p$.

If $|Q_k^*| < C_n$ and Q_k^* is a type (b) cube, i.e. $2Q_k^* \cap \overline{\mathbf{R}^n_+} \neq \emptyset$, pick c_k to be the unique polynomial for which

$$\langle f, x^\alpha \eta_k \rangle = \int c_k x^\alpha \eta_k dx$$

for all monomials of degree $|\alpha| \leq N_p$ with $\alpha_n \geq 1$.

We claim that

$$|c_k \eta_k| \leq c\gamma$$

and

$$|c_k \eta_k| \leq c m_d(f)(x)$$

for any x in Q_k^*. This is obvious when $|Q_k^*| \geq C_n$, so assume below that $|Q_k^*| < C_n$.

For a type (a) cube, this follows from the proof as in [S2].

For type (b) cubes, we can modify the proof by considering instead of $P(x, y)$ the kernel $\widetilde{P}(x, y)$ of the projection onto the subspace of $\mathcal{H}_{k,N_p}$ consisting of all polynomials divisible by x_n. This kernel must vanish for $y \in \partial \mathbf{R}^n_+$, so $\widetilde{P}(x, y)\eta_k(y)$ will also be a normalized $\mathcal{C}^\infty_d$ bump function in y, and the same argument as above shows that it lies in a ball of radius ≤ 1 containing a point of F.

Since f belongs to $\mathcal{C}^{\infty\prime}_d(\overline{D})$, so does b_k, and it is supported in Q_k^*. It remains to show that

$$\int_D m_d(b_k)^p dx \leq c \int_{Q_k^*} m_d(f)^p dx.$$

We restrict ourselves to "small" cubes, where the main difficulty lies. One notes first that

$$m_d(f\eta_k)(x) = \sup_{\varphi^x_t, t \leq 1} |\langle f, \eta_k \varphi^x_t \rangle| \leq C_{\eta_k} m_d(f)(x),$$

and since we already have $|c_k \eta_k| \leq C m_d(f)(x)$, we get

$$m_d(b_k)(x) \leq C m_d(f)(x)$$

for any x in Q_k^*. Integrating, we get

$$\int_{Q_k^*} m_d(b_k)^p dx \le C \int_{Q_k^*} m_d(f)^p dx.$$

Looking at $D \setminus Q_k^*$ and the case when $|Q_k^*| < C_n$, we can write

$$\langle b_k, \varphi_t^x \rangle = \langle b_k, \varphi_t^x - q \rangle = \langle f, \eta_k(\varphi_t^x - q) \rangle.$$

Here

$$q(y) = \sum_{|\alpha| \le N_p} \partial^\alpha \varphi(x_k)(y - x_k)^\alpha,$$

where x_k is the center of Q_k^* if Q_k^* is a type (a) cube, and the projection of the center of Q_k^* onto $\partial \mathbf{R}_+^n$ if Q_k^* is a type (b) cube. Note that in the latter case, $\partial^\alpha \varphi_t^x(x_k) = 0$ if $\alpha_n = 0$, since φ_t^x vanishes identically on $\partial \mathbf{R}_+^n$. Thus the only non-zero terms in the sum above involve positive powers of y_n, against which the integral of b_k vanishes.

In both cases, we still have $|x - y| \simeq |x - x_k|$ when $x \notin Q_k^*$ and $y \in \widetilde{Q_k}$, and the radius t of the support of φ_t^x still satisfies $t \ge c|x - x_k|$. Furthermore, $\eta_k(\varphi_t^x - q)$ is supported in $\widetilde{Q_k}$, which (since $|Q_k^*| < C_n$) is contained in a ball of radius ≤ 1 containing a point of F (see above). Thus the rest of the estimates in [S2] go through unchanged, and we get

$$m_d(b_k)(x) \le c\gamma \frac{\operatorname{diam}(Q_k)^{n+N_p+1}}{|x - x_k|^{n+N_p+1}}, \tag{2.4}$$

if $x \notin Q_k^*$. Integrating, and using the definition of N_p, we have that

$$\int_{D \setminus Q_k^*} m_d(b_k)^p dx \le c\gamma^p |Q_k^*| \le c \int_{Q_k^*} m_d(f)^p dx.$$

This proves part 2 of the proposition. We now let $b = \sum b_k$, $g = f - b$. Then $g \in \mathcal{C}_d^{\infty\prime}(\overline{\mathbf{R}_+^n})$, so it remains to prove the estimate in part 4 of the proposition. But this is just a straightforward adaptation of the argument on pp. 111-112 of [S2], using the maximal function m_d instead of $\mathcal{M}_0$, and incorporating the modifications on pp. 104-105 for small values of p. Note that in this argument we can ignore the cubes Q_k with $|Q_k^*| \ge C_n$, because for those we have that $g = 0$ in Q_k.

This completes the proof of the proposition. $\square$

Proof of Lemma 2.7. Again we will follow the proof in [S2], pp. 107-112. There are a few changes that need to be noted.

First we want to assume that f is locally integrable. This is justified by the fact that $h_d^p(\overline{D}) \cap L^1(D)$ is dense in $h_d^p(\overline{D})$. To see this it suffices to know that for every $\gamma > 0$, the distribution g in the proposition is in fact in $L^1(D)$, since letting $\gamma \to \infty$ we have that $\|f - g\|_{h_d^p} = \|b\|_{h_d^p} \to 0$. Using part 4 of the proposition, we can show (see p. 112 of [S2]) that $m_d(g) \in L^1(\mathbf{R}_+^n)$. Now for every $x \in \mathbf{R}_+^n$,

$$m_d(g)(x) \ge \sup \langle g, \varphi \rangle$$

where this time the supremum is taken over all smooth normalized bump functions φ with compact support inside D. But then $g \in L^1$.

Second, convergence is now in the "$h_d^p(\overline{D})$-norm", i.e. controlled by $\|m_d(\cdot)\|_{L^p(D)}^p$. As discussed in the remarks following Definition 1.3, this implies convergence in $\mathcal{C}_d^{\infty\prime}(\overline{D})$.

Third, when defining the polynomials $c_{k,l}$, one needs to distinguish between the different kinds of cubes. If any of the cubes involved have volume $\geq C_n$, we set the corresponding projection to be zero. Otherwise, if Q_l^{j+1*} is a type (a) cube and $|Q_l^{j+1*}| < C_n$, the projection P_l^{j+1} remains as it is in the proof in [S2], i.e. the projection onto the whole space of polynomials $\mathcal{H}_{l,N_p}$. However, if Q_l^{j+1*} is a type (b) cube (with $|Q_l^{j+1*}| < C_n$), we need to replace P_l^{j+1} by $\widetilde{P_l^{j+1}}$, the projection onto the subspace of $\mathcal{H}_{l,N_p}$ consisting of polynomials divisible by x_n.

This guarantees that if Q_k^{j*} and all of the cubes Q_l^{j+1*} intersecting Q_k^j are type (a) cubes of volume $< C_n$, the atom A_k^j will have the full moment conditions. On the other hand, if all the cubes are of volume $< C_n$ but either Q_k^j or any of the Q_l^{j+1} intersecting it are type (b) cubes, the atom A_k^j will only have the partial moment conditions corresponding to monomials x^α with $\alpha_n \geq 1$. It remains to show that in this case A_k^j will be supported in a type (b) cube.

As in the proof in [S2], we know that A_k^j is supported in a fixed dilate cQ_k^j of the cube Q_k^{j*}. (Actually, the proof in [S2] uses a ball B_k^j, but a cube will do just as well.) Certainly if Q_k^{j*} is a type (b) cube, so is cQ_k^j (here $c \geq 1$, of course). Similarly, if one of the Q_l^{j+1*} is a type (b) cube, i.e. $2Q_l^{j+1*} \cap \partial\mathbf{R}^n_+ \neq \emptyset$, then since $Q_l^{j+1*} \subset cQ_k^j$ implies $2Q_l^{j+1*} \subset 2cQ_k^j$, so is cQ_k^j.

Finally, if Q_k^{j*} or one of the cubes Q_l^{j+1} intersecting it have volume $\geq C_n$, then certainly $|cQ_k^j| \geq C_n$, and so A_k^j need not have any moment conditions.

The rest of the details are the same as in [S2], and this concludes the proof of the lemma. $\square$

3. The h_d^p regularity of the Dirichlet problem

We now want to study regularity for the Dirichlet problem in the context of the spaces $h_d^p(\overline{\Omega})$. We begin by looking at the problem in the sense of distributions in $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary, as above, and let $\mathbf{G}$ be the Green's operator for the Dirichlet problem on Ω, i.e. when $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$, $u = \mathbf{G}(\varphi)$ is the solution of the Dirichlet problem

$$\Delta u = \varphi \text{ on } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

By the classical theory, we know that for $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ we have $\mathbf{G}(\varphi) \in \mathcal{C}^\infty(\overline{\Omega})$, hence $\mathbf{G}(\varphi) \in \mathcal{C}_d^\infty(\overline{\Omega})$. Moreover, we can consider, for $1 \leq j, l \leq n$, the functions $\frac{\partial^2\varphi}{\partial x_j \partial x_l}$. These are also in $\mathcal{C}^\infty(\overline{\Omega})$, and hence $\mathbf{G}\left(\frac{\partial^2\varphi}{\partial x_j \partial x_l}\right)$ will be in $\mathcal{C}_d^\infty(\overline{\Omega})$. Furthermore, by the Sobolev estimates on $\mathbf{G}$, we know that this map

$$\mathbf{G} \circ \frac{\partial^2}{\partial x_j \partial x_l} : \mathcal{C}_d^\infty(\overline{\Omega}) \to \mathcal{C}_d^\infty(\overline{\Omega})$$

is continuous.

This allows us to define a continuous operator on $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$, as follows:

Definition 3.1. For $1 \leq j, l \leq n$, define $T_{j,l} : \mathcal{C}_d^{\infty\prime}(\overline{\Omega}) \to \mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ by

$$\langle T_{j,l}(f), \varphi \rangle = \left\langle f, \mathbf{G}\left(\frac{\partial^2\varphi}{\partial x_j \partial x_l}\right)\right\rangle$$

for all $f \in \mathcal{C}_d^{\infty\prime}(\overline{\Omega})$, $\varphi \in \mathcal{C}_d^\infty(\overline{\Omega})$.

When f is a smooth function, $T_{j,l}(f)$ takes on a familiar form:

Lemma 3.2. *For $f \in \mathcal{C}^\infty(\overline{\Omega})$,*

$$T_{j,l}(f) = \frac{\partial^2 \mathbf{G}(f)}{\partial x_j \partial x_l}$$

in $\mathcal{C}_d^\infty(\overline{\Omega})$.

Proof. Let $f \in \mathcal{C}^\infty(\overline{\Omega})$. Then $\frac{\partial^2 \mathbf{G}(f)}{\partial x_j \partial x_l} \in \mathcal{C}^\infty(\overline{\Omega})$ and for $\varphi \in \mathcal{C}_d^\infty(\overline{\Omega})$,

$$\left\langle \frac{\partial^2 \mathbf{G}(f)}{\partial x_j \partial x_l}, \varphi \right\rangle = \int_\Omega \frac{\partial^2 \mathbf{G}(f)}{\partial x_j \partial x_l}\, \varphi \, dV.$$

Since $\varphi|_{\partial\Omega} = 0$, we can integrate by parts without boundary terms to get

$$-\int_\Omega \frac{\partial \mathbf{G}(f)}{\partial x_l} \frac{\partial \varphi}{\partial x_j} dV.$$

But now since $\mathbf{G}(f)|_{\partial\Omega} = 0$, we can integrate by parts again, and we have

$$\int_\Omega \mathbf{G}(f) \frac{\partial^2 \varphi}{\partial x_j \partial x_l} dV.$$

The Green's operator for the Dirichlet problem is symmetric, so this is the same as

$$\int_\Omega f \mathbf{G}\left(\frac{\partial^2 \varphi}{\partial x_j \partial x_l} \right) dV,$$

which is by definition

$$\langle T_{j,l}(f), \varphi \rangle.$$

□

Thus we can consider $T_{j,l}$ as an extension to $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ of the operator $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$, initially defined on $\mathcal{C}^\infty(\overline{\Omega})$. In particular, since $L^2(\Omega) \subset \mathcal{C}_d^{\infty\prime}(\overline{\Omega})$, this operator is defined on L^2, and we have the following:

Lemma 3.3. *For $1 \le j, l \le n$, $T_{j,l}$ is a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$.*

This follows by the argument above since the mapping $\psi \to \partial_j \partial_\ell \mathbf{G}(\psi)$ extends to a bounded operator on $L^2(\Omega)$.

We now come to the main result of this section.

Theorem 3.4. *With Ω as above, and $1 \le j, l \le n$, the extension $T_{j,l}$ of $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ to $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$ is a bounded operator from $h_d^p(\overline{\Omega})$ to $h_d^p(\overline{\Omega})$.*

Proof. Let $f \in h_d^p(\overline{\Omega})$. By the atomic decomposition, we can write

$$f = \sum \lambda_k a_k$$

in $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$, where the a_k are either type (a) h_d^p atoms or type (b) h_d^p atoms (with respect to some constants A_Ω, B_Ω and C_Ω as in Definition 2.3 and Theorem 2.6 of Section 2.) By the continuity of $T_{j,l}$,

$$T_{j,l}(f) = \sum \lambda_k T_{j,l}(a_k)$$

in $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$. If we can show that, for every k, $T_{j,l}(a_k) \in h_d^p(\overline{\Omega})$ and $\|T_{j,l}(a_k)\|_{h_d^p} \le C$ independently of k, then we will have that $\sum \lambda_k T_{j,l}(a_k)$ converges in $h_d^p(\overline{\Omega})$ and

$$\|T_{j,l}(f)\|_{h_d^p} \le C \left(\sum |\lambda_k|^p\right)^{1/p}.$$

Since this would be true for any atomic decomposition, we will have

$$\|T_{j,l}(f)\|_{h_d^p} \le C\|f\|_{h_d^p}.$$

□

Therefore, it suffices to prove the following:

Lemma 3.5. *If a is an h_d^p atom (of either type), then $T_{j,l}(a) \in h_d^p(\overline{\Omega})$ and*

$$\|T_{j,l}(a)\|_{h_d^p} \le C,$$

with C independent of a.

Proof of Lemma. Let $g = T_{j,l}(a)$. We want to show $m_d(g) \in L^p(\Omega)$. For $x \in \Omega$, write

$$m_d(g)(x) = \sup\langle g, \varphi_t^x\rangle,$$

where the supremum is taken over all $\mathcal{C}_d^\infty$ bump functions φ_t^x supported in balls of radii $t \le 1$ containing x. Let Q be the supporting cube of a.

<u>Case I: $x \in \Omega \cap 2Q$ or $|Q| \ge C_\Omega$.</u> Here we will use the L^2 estimate from Lemma 3.3, namely $g \in L^2$ and $\|g\|_{L^2} \le C\|a\|_{L^2}$. By extending each $\mathcal{C}_d^\infty$ bump function to a bump function in $\mathbf{R}^n$, we see that

$$m_d(g)(x) = \sup_{\varphi_t^x \in \mathcal{C}_d^\infty} \langle g, \varphi_t^x\rangle \le \sup_{\psi_t^x \in \mathcal{D}(\mathbf{R}^n)} \langle g, \psi_t^x\rangle = m(g)(x),$$

where m is the local grand maximal function in $\mathbf{R}^n$. Since m is bounded on L^2, we have

$$\|m_d(g)\|_{L^2} \le \|m(g)\|_{L^2} \le C\|g\|_{L^2} \le C'\|a\|_{L^2} \le C'|Q|^{1/2-1/p},$$

so

$$\begin{aligned}\int_{2Q\cap\Omega} m_d(g)^p(x)dx &\le \|m_d(g)\|_{L^2}^p |2Q|^{1-p/2} \\ &\le C_p\left(|Q|^{1/2-1/p}\right)^p |2Q|^{1-p/2} \\ &= C_p'.\end{aligned}$$

When $|Q| \ge C_\Omega$, we can use the estimate above to bound the integral of $m_d(g)^p$ over all of Ω, which proves the lemma for the case where $|Q| \ge C_\Omega$.

<u>Case II: $|Q| < C_\Omega$ and $x \in \Omega \setminus 2Q$.</u> Fix a $\mathcal{C}_d^\infty$ bump function φ_t^x with $t \le 1$. By definition of $T_{j,l}$, we have

$$\begin{aligned}\langle g, \varphi_t^x\rangle &= \langle a, \mathbf{G}(\partial_j\partial_l\varphi_t^x)\rangle \\ &= \int_\Omega a(y)\mathbf{G}(\partial_j\partial_l\varphi_t^x)(y)dy.\end{aligned}$$

Let

$$K_{\varphi_t^x}(y) = \mathbf{G}(\partial_j\partial_l\varphi_t^x)(y).$$

Claim 3.6. The function $K_{\varphi_t^x}$ is smooth on $\overline{\Omega}$, vanishes identically on the boundary (i.e. $K_{\varphi_t^x}(y) = 0$ for $y \in \partial\Omega$), and satisfies the estimates

$$\left|\frac{\partial^{|\alpha|}}{\partial y^\alpha} K_{\varphi_t^x}(y)\right| \le \frac{C_{|\alpha|}}{|x-y|^{n+|\alpha|}}$$

for all multi-indices α, $|\alpha| \le N_p + 1$. Here the constants $C_{|\alpha|}$ are independent of the choice of bump function φ_t^x.

Assuming the conclusions of the claim, let us proceed with the proof of the lemma. Again there are two cases

<u>Case II(a): a is a type (a) atom with $|Q| < C_\Omega$.</u> In this case $Q \subset \Omega$ and we may assume 0 is the center of Q, $Q = [-\delta, \delta]^n$ for some $\delta < {C_\Omega}^{1/n}$.

By Remark 2 following Definition 2.1, and the bounds on the derivatives of $K_{\varphi_t^x}$ (given by the claim), we have

$$\begin{aligned}
\left|\int_Q a(y) K_{\varphi_t^x}(y) dy\right| &\le C_{n,p} \|K_{\varphi_t^x}\|_{\mathcal{C}^{N_p+1}(Q)} |Q|^{\nu_p} \\
&= C_{n,p} \sum_{|\alpha| \le N_p+1} \frac{1}{|\alpha|!} \sup_{y\in Q} \left|\partial_y^\alpha K_{\varphi_t^x}(y)\right| |Q|^{\nu_p} \\
&\le \sum_{|\alpha| \le N_p+1} \sup_{y \in Q} \frac{C'_{|\alpha|}}{|x-y|^{n+|\alpha|}} |Q|^{\nu_p} \\
&\le \sum_{|\alpha| \le N_p+1} \frac{C''_{|\alpha|}}{|x|^{n+|\alpha|}} |Q|^{\nu_p} \\
&\le C \frac{|Q|^{\nu_p}}{|x|^{n+N_p+1}}.
\end{aligned}$$

For the next-to-last inequality we used the fact that when $x \notin 2Q$, $|x - y| \ge C|x|$ for all $y \in Q$, while for the last inequality we used the fact that $|x|$ is bounded.

Since this bound does not depend on the choice of bump function φ_t^x, we get, for $x \in \Omega \setminus 2Q$,

$$m_d(g)(x) = \sup_{\varphi_t^x} \left|\int_Q a(y) K_{\varphi_t^x}(y) dy\right| \le C \frac{|Q|^{\nu_p}}{|x|^{n+N_p+1}}.$$

Taking p-th powers and integrating over $\Omega \setminus 2Q$, we see that

$$\begin{aligned}
|Q|^{p\nu_p} \int_{x\in\Omega\setminus 2Q} |x|^{-(n+N_p+1)p} dx &\le C\delta^{(N_p+1)p-n+np} \int_{2\delta \le |x| \le A} |x|^{-(n+N_p+1)p} dx \\
&\le C
\end{aligned}$$

since $(N_p + 1)p - n + np = (n + N_p + 1)p - n > [n + n(1/p - 1)]p - n = 0$. This shows

$$\int_{x\in\Omega\setminus 2Q} m_d(g)(x)^p dx \le C,$$

which proves the lemma for the case of a type (a) atom with $|Q| < C_\Omega$.

<u>Case II(b): a is a type (b) atom with $|Q| < C_\Omega$.</u> Note that by the claim, $K_{\varphi_t^x} \in \mathcal{C}_d^\infty(\overline{\Omega})$, so by the moment conditions on a (see Definition 2.3),

$$\left|\int_Q a(y) K_{\varphi_t^x}(y) dy\right| \le \|K_{\varphi_t^x}\|_{\mathcal{C}^{N_p+1}(Q)} |Q|^{\nu_p}$$

and we can proceed exactly as in case II(a) above to show that

$$\int_{x\in\Omega\setminus 2Q} m_d(g)(x)^p dx \le C.$$

This proves the lemma for a type (b) atom with $|Q| < C_\Omega$. □

Now that we have completed the proof of the lemma, we can get back to the

Proof of Claim 3.6. The smoothness of $K_{\varphi_t^x}$ on $\overline{\Omega}$ and its vanishing on the boundary follow from the fact that it is the solution to the Dirichlet problem with smooth data $\partial_j\partial_l\varphi_t^x$.

To prove the estimates on the derivatives of $K_{\varphi_t^x}(y)$, we will first consider the situation when $|x-y| < 4t$. Let

$$u(y) = K_{\varphi_t^x}(y) = \mathbf{G}(\partial_j\partial_l\varphi_t^x)(y).$$

Then it suffices to show

$$|\partial^\alpha u| \le Ct^{-n-|\alpha|}$$

for all multi-indices α. We will do this by using the Sobolev embedding theorem.

First note that

$$\|\partial^\alpha u\|_{L^2(\Omega)} \le Ct^{-n/2-|\alpha|}.$$

For $\alpha = 0$, this follows from:

$$\|u\|_{L^2(\Omega)} = \|\mathbf{G}(\partial_j\partial_l\varphi_t^x)\|_{L^2(\Omega)} \le C\|\varphi_t^x\|_{L^2(\Omega)} \le Ct^{-n/2},$$

since φ_t^x is a bump function. For $|\alpha| \ge 2$, this follows from the regularity of $\mathbf{G}$:

$$\|\partial^\alpha u\|_{L^2(\Omega)} \le \|u\|_{|\alpha|} \le C\|\partial_j\partial_l\varphi_t^x\|_{|\alpha|-2} \le Ct^{-n/2-|\alpha|}.$$

Here $\|\cdot\|_s$ denotes the Sobolev s norm. Finally, for $|\alpha| = 1$, we have, since u satisfies the Dirichlet boundary conditions,

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\Omega)} &= \int_\Omega \nabla u \cdot \overline{\nabla u} dV \\
&= -\int_\Omega (\Delta u)\overline{u} dV \\
&= -\int_\Omega \partial_j\partial_l\varphi_t^x \overline{u} dV \\
&\le \|\partial_j\partial_l\varphi_t^x\|_{L^2}\|u\|_{L^2} \\
&\le Ct^{-n/2-2}t^{-n/2} \\
&= Ct^{-n-2}.
\end{aligned}$$

Now fix a multi-index α with $|\alpha| \le N_p + 1$. We want to use the Sobolev embedding theorem to bound the L^∞ norm of $\partial^\alpha u$ by the L^2 norms of u and its derivatives in balls of radius t. Near the boundary, however, such balls may not be contained in $\overline{\Omega}$. Therefore we must first cover a tubular neighborhood of $\partial\Omega$, of radius 1, by a collection of open sets U_i, such that each U_i is contained in a larger open set V_i, and in each V_i there is a diffeomorphism Φ_i which takes $V_i \cap \Omega$ onto the upper half unit ball $B_+(0,2)$ and $U_i \cap \Omega$ onto $B_+(0,1)$. For each i, we can extend $u \circ \Phi_i^{-1}$ to a smooth function on the whole ball $B(0,2)$ while maintaining its Sobolev norms (see [S1], Chapter VI, Section 3). These are essentially the Sobolev norms of u, up to a constant depending on Φ_i.

Fixing $y \in \overline{\Omega}$, since $t \leq 1$, we have either $B(y,t) \subset \overline{\Omega}$, or $y \in U_i$ for some i. In the latter case, by making a change of variables, we may assume $y \in B_+(0,1)$, hence $B(y,t) \subset B(0,2)$. Replacing u by $u \circ \Phi_i^{-1}$, and extending to $B(0,2)$, we may assume that u is defined in $B(y,t)$ and its Sobolev norms are bounded as above.

For simplicity, make a translation so that $y = 0$. Thus we have

$$\|\partial^\beta u\|_{L^2(B(0,t))} \leq Ct^{-n/2-|\beta|}.$$

Now define v in $B(0,1)$ by $v(z) = u(tz)$. Then it is easy to see that

$$\|\partial^\beta v\|_{L^2(B(0,1))} = \|\partial^\beta u\|_{L^2(B(0,t))} t^{|\beta|-n/2} \leq Ct^{-n}.$$

Let k be an integer greater than $n/2$. By the Sobolev embedding theorem, applied to the function $\partial^\alpha v$, we can write

$$\sup_{B(0,1/2)} \|\partial^\alpha v\| \leq C \sum_{|\gamma| \leq k} \|\partial^\gamma(\partial^\alpha v)\|_{L^2(B(0,1))} \leq Ct^{-n}$$

(see [S1], pp. 124-130). But $\partial^\alpha v(z) = t^{|\alpha|}(\partial^\alpha u)(tz)$ so

$$|\partial^\alpha u(0)| \leq t^{-|\alpha|} \sup_{B(0,1/2)} \|\partial^\alpha v\| \leq Ct^{-n-|\alpha|}.$$

Recalling that $\partial^\alpha u(0) = \frac{\partial^{|\alpha|}}{\partial y^\alpha} K_{\varphi_t^x}(y)$, we see that we are done with the case $|x-y| \leq 4t$.

When $|x - y| > 4t$, we write

$$K_{\varphi_t^x}(y) = \int_\Omega G(y,z) \frac{\partial^2 \varphi_t^x}{\partial z_j \partial z_l}(z) dz.$$

Here y is outside the support of φ_t^x, so the integral is taken only over the region where $G(y,z)$ is smooth in z. In addition, both φ_t^x and G vanish on the boundary, so we can integrate by parts to get

$$K_{\varphi_t^x}(y) = \int_\Omega \partial_{z_j} \partial_{z_l} G(y,z) \varphi_t^x(z) dz.$$

Differentiating in y, this gives

$$\frac{\partial^{|\alpha|}}{\partial y^\alpha} K_{\varphi_t^x}(y) = \int_\Omega \partial_y^\alpha \partial_{z_j} \partial_{z_l} G(y,z) \varphi_t^x(z).$$

Furthermore, for z in the support of φ_t^x we have

$$|x-y| \leq |x-z| + |z-y| \leq 2t + |z-y| < |x-y|/2 + |z-y|;$$

hence $|x-y|/2 < |y-z|$. Thus if we can show

$$|\partial_y^\alpha \partial_{z_j} \partial_{z_l} G(y,z)| \leq C|y-z|^{-n-|\alpha|} \tag{3.1}$$

for all $z \in \Omega \setminus \{y\}$, we would get

$$\begin{aligned} \left| \frac{\partial^{|\alpha|}}{\partial y^\alpha} K_{\varphi_t^x}(y) \right| &\leq \left| \int_\Omega \partial_y^\alpha \partial_{z_j} \partial_{z_l} G(y,z) \varphi_t^x(z) dz \right| \\ &\leq \int_\Omega \frac{C}{|y-z|^{n+|\alpha|}} |\varphi_t^x(z)| dz \\ &\leq C|x-y|^{-n-|\alpha|}, \end{aligned}$$

which is the desired estimate.

Proof of estimate (3.1). We will follow the general outline of the proofs in [CKS], Section 5, but without using the reflection mapping. We start by writing the Green's operator $\mathbf{G}$, acting on some $f \in \mathcal{C}^\infty(\overline{\Omega})$, as

$$\mathbf{G}(f) = \mathbf{E}(f) + \mathbf{H}(f).$$

Here $\mathbf{E}(f) = E * f$ is convolution with the Newtonian potential E, while $-\mathbf{H}(f) = \mathbf{P}(E * f|_{\partial\Omega})$ is the Poisson integral of the restriction of $\mathbf{E}(f)$ to the boundary. Thus we can write the Green's function as

$$G(y, z) = E(y, z) + H(y, z),$$

where

$$E(y, z) = c_n |y - z|^{-(n-2)}$$

and $H(y, z)$ is the kernel of the operator $\mathbf{H}$. Now E is symmetric in y and z and its second derivatives form a Calderón-Zygmund kernel, so the inequality (3.1) holds with E instead of G. Thus we need only concentrate on H.

In order to estimate the derivatives of $H(y, z)$, we write, for $z \in \Omega \setminus \{y\}$,

$$\begin{aligned}\frac{\partial^2}{\partial z_j \partial z_l} H(y, z) &= \frac{\partial^2}{\partial z_j \partial z_l} \int_{\partial\Omega} P(y, \zeta) E(\zeta, z) d\sigma(\zeta) \\ &= \int_{\partial\Omega} P(y, \zeta) \frac{\partial^2}{\partial z_j \partial z_l} E(\zeta, z) d\sigma(\zeta),\end{aligned}$$

where $P(y, \zeta)$ is the Poisson kernel in Ω, and $d\sigma$ is surface measure on $\partial\Omega$. We differentiate the integral with respect to y and separate into local and global parts by means of a cut-off function $\phi : (0, \infty) \to \mathbf{R}$ such that $\phi(t) = 1$ for $t < 1/4$ and $\phi(t) = 0$ for $t > 1/2$:

$$\begin{aligned}&\frac{\partial^{|\alpha|}}{\partial y^\alpha} \int_{\partial\Omega} P(y, \zeta) \frac{\partial^2}{\partial z_j \partial z_l} E(\zeta, z) d\sigma(\zeta) \\ &= \int_{\partial\Omega} \partial_y^\alpha P(y, \zeta) \phi\left(\frac{|\zeta - z|}{|y - z|}\right) \partial_{z_j} \partial_{z_l} E(\zeta, z) d\sigma(\zeta) \\ &\quad + \int_{\partial\Omega} \partial_y^\alpha P(y, \zeta) \left[1 - \phi\left(\frac{|\zeta - z|}{|y - z|}\right)\right] \partial_{z_j} \partial_{z_l} E(\zeta, z) d\sigma(\zeta) \\ &= I_1 + I_2.\end{aligned}$$

We want to estimate the local part, I_1 (where $|\zeta - z| \le |y - z|/2$), by means of integration by parts. In order to do this, we first have to convert the z derivatives of E into ζ derivatives. Note that because of the symmetric form of E, we have

$$\partial_{z_j} \partial_{z_l} E(\zeta, z) = \partial_{\zeta_j} \partial_{\zeta_l} E(\zeta, z).$$

Next, for z, y fixed and ζ in a neighborhood of $\partial\Omega$, with $|\zeta - z| \le |y - z|/2$, we choose tangential and normal coordinates (ζ', ρ) in ζ. Converting the old derivatives to the new ones, we get

$$\frac{\partial}{\partial \zeta_j} = \sum_{k=1}^{n-1} a_{j,k} \frac{\partial}{\partial \zeta'_k} + b_j \frac{\partial}{\partial \rho}$$

for some smooth functions $a_{j,k}$, b_j, and therefore

$$\frac{\partial^2}{\partial \zeta_j \partial \zeta_l} E(\zeta, z) = \sum_{|\beta| + k \le 2} c_{\beta,k} \frac{\partial^{|\beta|+k}}{\partial(\zeta')^\beta \partial \rho^k} E(\zeta, z).$$

By choosing ρ to be the signed geodesic distance to $\partial\Omega$, we can rewrite the equation $\Delta_\zeta E(\zeta,z)=0$ in the new coordinates as

$$\frac{\partial^2}{\partial\rho^2}E(\zeta,z)=\sum_{1\le i,m\le n-1} d_{i,m}\frac{\partial^2}{\partial\zeta_i'\partial\zeta_m'}E(\zeta,z)+\text{ first order terms.}$$

This reduces the number of ρ derivatives by one, and we remain with an expression of the form

$$\sum_{|\beta|+k\le 2,k=0,1} g_{\beta,k}(\zeta)\frac{\partial^{|\beta|+k}}{\partial(\zeta')^\beta\partial\rho^k}E(\zeta,z)$$

for the second derivatives $\partial_j\partial_l$ of $E(\zeta,z)$. Inserting this into I_1, we get

$$\sum_{|\beta|+k\le 2,k=0,1}\int_{\partial\Omega}\partial_y^\alpha P(y,\zeta)\phi\left(\frac{|\zeta-z|}{|y-z|}\right)g_{\beta,k}(\zeta)\frac{\partial^{|\beta|+k}}{\partial(\zeta')^\beta\partial\rho^k}E(\zeta,z)d\sigma(\zeta).$$

We are now ready to integrate by parts in the tangential variables ζ'. The derivatives can fall either on $\partial_y^\alpha P(\eta,\zeta)$, on $\phi(\frac{|\zeta-z|}{|y-z|})$, or on $g_{\beta,k}(\zeta)$, and we use the following estimates:

$$\begin{aligned}|\partial_{\zeta'}^\gamma\partial_y^\alpha P(y,\zeta)| &\le C|\zeta-y|^{-(n-1+|\gamma|+|\alpha|)}\\ &\le C|y-z|^{-(n-1+|\gamma|+|\alpha|)}\end{aligned}$$

since $|\zeta-y|\ge|y-z|-|\zeta-z|\ge|y-z|/2$,

$$\left|\partial_{\zeta'}^\gamma\phi\left(\frac{|\zeta-z|}{|y-z|}\right)\right|\le C|y-z|^{-|\gamma|},$$

and

$$|\partial_{\zeta'}^\gamma g_{\beta,k}(\zeta)|\le C.$$

Combining, and using the fact that we can always choose the largest negative exponent of $|y-z|$, since it is bounded above, we have

$$\begin{aligned}|I_1| &\le \frac{C_1}{|y-z|^{n+1+|\alpha|}}\int_{|\zeta-z|\le|y-z|/2}|E(\zeta,z)|d\sigma(\zeta)\\ &\quad+\frac{C_2}{|y-z|^{n+|\alpha|}}\int_{|\zeta-z|\le|y-z|/2}\left|\frac{\partial}{\partial\rho}E(\zeta,z)\right|d\sigma(\zeta)\\ &\le \frac{C}{|y-z|^{n+1+|\alpha|}}\int_{|\zeta-z|\le|y-z|/2}|\zeta-z|^{-(n-2)}d\sigma(\zeta)\\ &\quad+\frac{C}{|y-z|^{n+|\alpha|}}\int_{\partial\Omega}|\partial_{\nu_\zeta}E(\zeta,z)|d\sigma(\zeta)\\ &\le C|y-z|^{-(n+|\alpha|)}.\end{aligned}$$

The last estimate (bounding the integral of the normal derivative of the Newtonian potential by a constant) can be found in [F] (Lemma 3.20, p. 165).

The global part of the integral, I_2, can be estimated in the same way as was done in [CKS], in the proof of Lemma 5.3 (pp. 327-331), i.e. by using integration by parts to transfer derivatives from the Poisson kernel to the Newtonian potential. Here we must use the estimate

$$\begin{aligned}|\partial_{\zeta'}^\gamma E(\zeta,z)| &\le C|\zeta-z|^{-(n-2+|\gamma|)}\\ &\le C|y-z|^{-(n-2+|\gamma|)}\end{aligned}$$

for $|\zeta - z| \geq |y-z|/4$. Since the largest $|\gamma|$ possible is $|\alpha|+2$, we have

$$|I_2| \leq C|y-z|^{-(n+|\alpha|)},$$

and so all in all

$$\left|\frac{\partial^{|\alpha|}}{\partial y^\alpha}H(y,z)\right| = \left|\frac{\partial^{|\alpha|}}{\partial y^\alpha}\int_{\partial\Omega} P(y,\zeta)\frac{\partial^2}{\partial z_j \partial z_l}E(\zeta,z)d\sigma(\zeta)\right| \leq C|y-z|^{-(n+|\alpha|)}.$$

Combined with the equivalent estimates on $E(y,z)$, we get estimate (3.1). $\square$

Now the proof of Claim 3.6 is complete. $\square$

This also concludes the proof of Theorem 3.4, the h^p_d regularity of the Dirichlet problem.

4. The h^p_z regularity of the Dirichlet and Neumann problems

In this section we will study the regularity of the Dirichlet and Neumann problems in the context of the spaces $h^p_z(\overline{\Omega})$. We shall concentrate on the Neumann problem, since it involves the more difficult analysis.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary, and let $\widetilde{\mathbf{G}}$ be a solution operator for the Neumann problem, defined on $f \in \mathcal{C}^\infty(\overline{\Omega})$ with $\int_\Omega f = 0$ by $\widetilde{\mathbf{G}}(f) = u$, where

$$\Delta u = f \text{ on } \Omega,$$

$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega,$$

and $\int_\Omega u = 0$. Here $\vec{n}$ is the outward unit normal vector field on $\partial\Omega$.

If $f \in \mathcal{C}^\infty(\overline{\Omega})$ with $\int_\Omega f = 0$, then $\widetilde{\mathbf{G}}(f) \in \mathcal{C}^\infty(\overline{\Omega})$, so we can apply the operators $\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$, $j,l = 1,\ldots,n$, to f. We want to extend these operators to $h^p_z(\overline{\Omega})$. To do this, we will proceed through the L^2 theory.

As is well known, $\widetilde{\mathbf{G}}$ extends to a bounded operator from $L^2(\Omega)$ to the Sobolev space $H_2(\Omega)$, so for $j,l = 1,\ldots,n$, the operators $\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$, extend to bounded operators on $L^2(\Omega)$. Thus we can define $\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(a)$ for an h^p_z atom a. We shall prove the following

Theorem 4.1. *With Ω as above, and $1 \leq j,l \leq n$, there is an extension $\widetilde{T}_{j,l}$ of $\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ to a bounded operator from $h^p_z(\overline{\Omega})$ to $h^p_z(\overline{\Omega})$.*

Before proceeding with the proof of this theorem, we shall state the corresponding result for the Dirichlet problem. As above, one can also define the operator $\frac{\partial^2\mathbf{G}}{\partial x_j \partial x_l}$ on h^p_z atoms, where $\mathbf{G}$ is the solution operator of the Dirichlet problem for the Laplacian, as in Section 3. Thus we have

Theorem 4.2. *With Ω as above, and $1 \leq j,l \leq n$, there is an extension of $\frac{\partial^2\mathbf{G}}{\partial x_j \partial x_l}$ to a bounded operator from $h^p_z(\overline{\Omega})$ to $h^p_z(\overline{\Omega})$.*

The proof of this result is just a minor modification of the proof of Theorem 4.1, so we will omit it. In Section 5, we will give a different proof for the case $p = 1$.

The essence of the proof of Theorem 4.1 is contained in the following

Lemma 4.3. *If a is an h^p_z atom, then $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(a) \in h^p_z(\overline{\Omega})$ and*

$$\left\| \frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(a) \right\|_{h^p_z} \leq C,$$

with C independent of a.

Proof of lemma. Let $g = \frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(a)$. Note that by definition, we are taking $g = 0$ on $\mathbf{R}^n \setminus \overline{\Omega}$. Thus to show $g \in h^p_z(\overline{\Omega})$, we only have to show $g \in h^p(\mathbf{R}^n)$. Let m be the local grand maximal function, as in Definition 1.1. Then we want to show $m(g) \in L^p(\mathbf{R}^n)$. In fact, we only need to show $m(g) \in L^p(\widetilde{\Omega})$, where $\widetilde{\Omega}$ is some bounded set containing all points of distance at most 2 from $\overline{\Omega}$.

Let Q be the supporting cube of a.

Case I: $x \in 2Q$ or $|Q| > 1$. Here we will use an L^2 estimate. Since both $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ and m are bounded on L^2, we have

$$\|m(g)\|_{L^2} \leq C\|g\|_{L^2} \leq C\|a\|_{L^2} \leq C|Q|^{1/2-1/p}.$$

Thus

$$\begin{aligned} \int_{2Q} m(g)^p(x)dx &\leq \|m(g)\|^p_{L^2}|2Q|^{1-p/2} \\ &\leq C\left(|Q|^{1/2-1/p}\right)^p |2Q|^{1-p/2} \\ &= C. \end{aligned}$$

When $|Q| > 1$, we can use the estimate above to bound the integral of $m(g)^p$ over all of $\widetilde{\Omega}$, thus proving the lemma for that case.

Case II: $|Q| < 1$ and $x \notin 2Q$. Fix a $\mathcal{C}^\infty$ bump function φ^x_t with $t \leq 1$. Then we have

$$\begin{aligned} \langle g, \varphi^x_t \rangle &= \int_\Omega \frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(a)(x)\varphi^x_t(x)dx \\ &= \int_\Omega \left\{ \int_\Omega \frac{\partial^2}{\partial x_j \partial x_l} \widetilde{G}(x,y)a(y)dy \right\} \varphi^x_t(x)dx \\ &= \int_\Omega \left\{ \int_\Omega \frac{\partial^2}{\partial x_j \partial x_l} \widetilde{G}(x,y)\varphi^x_t(x)dx \right\} a(y)dy \\ &= \int_\Omega \widetilde{K_{\varphi^x_t}}(y)a(y)dy, \end{aligned}$$

where

$$\widetilde{K_{\varphi^x_t}}(y) = \int_\Omega \frac{\partial^2}{\partial z_j \partial z_l} \widetilde{G}(z,y)\varphi^x_t(z)dz,$$

i.e. the dual operator $(\partial_j \partial_l \widetilde{\mathbf{G}})^*$ applied to φ^x_t.

Claim 4.4. The function $\widetilde{K_{\varphi^x_t}}$ is smooth on $\overline{\Omega}$ and satisfies the estimates

$$\left| \frac{\partial^{|\alpha|}}{\partial y^\alpha} \widetilde{K_{\varphi^x_t}}(y) \right| \leq \frac{C_{|\alpha|}}{|x-y|^{n+|\alpha|}}$$

for all multi-indices α, $|\alpha| \leq N_p + 1$. Here the constants $C_{|\alpha|}$ are independent of the choice of bump function φ^x_t.

Assuming the conclusions of the claim, let us continue with the proof of Case II of the lemma. Recall that $Q \subset \overline{\Omega}$. We may assume 0 is the center of Q, and $Q = [-\delta, \delta]^n$ for some $\delta < 1$. Using the moment conditions on a and the bounds on the derivatives of $\widetilde{K_{\varphi_t^x}}$ given by the claim, we can proceed exactly as in Case II(a) of the proof of Lemma 3.5 to get that

$$\begin{aligned} m(g)(x) &= \sup_{\varphi_t^x} \left| \int_Q a(y) \widetilde{K_{\varphi_t^x}}(y) dy \right| \\ &\leq \frac{C|Q|^{\nu_p}}{|x|^{n+N_p+1}}. \end{aligned}$$

and

$$\begin{aligned} \int_{x \in \widetilde{\Omega} \setminus 2Q} m(g)(x)^p dx &\leq C\delta^{pn\nu_p} \int_{2\delta \leq |x| \leq \tilde{A}} |x|^{-(n+N_p+1)p} dx \\ &\leq C. \end{aligned}$$

This proves Case II of the lemma, and the proof of the lemma is complete. □

Proof of Claim 4.4. We will first prove that the operator $(\partial_j \partial_l \widetilde{\mathbf{G}})^*$ is bounded from the Sobolev space $H_k(\Omega)$ to itself, $k = 0, 1, 2, \ldots$. Specifically, we want to show that if $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$, then $(\partial_j \partial_l \widetilde{\mathbf{G}})^*(\varphi) \in H_k(\Omega)$ for all k, with

$$\|(\partial_j \partial_l \widetilde{\mathbf{G}})^*(\varphi)\|_k \leq C\|\varphi\|_k, \tag{4.1}$$

where $\|\cdot\|_k$ denotes the Sobolev k norm on Ω. Since $\varphi_t^x \in \mathcal{C}^\infty(\overline{\Omega})$, this will show that $\widetilde{K_{\varphi_t^x}} = (\partial_j \partial_l \widetilde{\mathbf{G}})^*(\varphi_t^x)$ is smooth up to the boundary.

To prove 4.1, we first take a smooth partition of unity $\{\eta_\mu\}$, $\mu = 0, \ldots, M$, where as usual η_0 has compact support in Ω, and for $\mu \geq 1$, η_μ has compact support in an open set U_μ which comes equipped with a system of tangential and normal coordinates (ζ', ρ), ρ being the geodesic distance to $\partial\Omega$. Write

$$(\partial_j \partial_l \widetilde{\mathbf{G}})^*(\varphi) = \sum_{\mu=0}^{M} (\partial_j \partial_l \widetilde{\mathbf{G}})^*(\eta_\mu \varphi).$$

We want to prove 4.1 for each $\eta_\mu \varphi$.

When $\mu = 0$, since $\eta_0 \varphi$ has compact support in Ω, we can integrate by parts, and since $\widetilde{\mathbf{G}}$ is self-dual, we have that

$$(\partial_j \partial_l \widetilde{\mathbf{G}})^*(\eta_0 \varphi) = \widetilde{\mathbf{G}}(\partial_j \partial_l (\eta_0 \varphi)).$$

Thus we can use the classical Sobolev estimates for the Neumann problem to get (4.1) for $\eta_0 \varphi_t^x$. See Section 3, proof of Claim 3.6 for the analogous argument in the case of the Dirichlet problem.

Now fix $\mu \geq 1$, and let $U = U_\mu$, $\eta = \eta_\mu$. Making the change of variables $\Phi(z) = (\zeta', \rho) = \zeta$, and letting $\tau = [(\eta\varphi) \circ \Phi^{-1}] J_\Phi^{-1}$, we have

$$(\partial_j \partial_l \widetilde{\mathbf{G}})^*(\eta\varphi)(y) = \int_{U \cap \Omega} \frac{\partial^2}{\partial z_j \partial z_l} \widetilde{G}(z, y) \eta(z) \varphi(z) dz \tag{4.2}$$

$$= \sum_{1 \leq i,m \leq n} \int_{\mathbf{R}^n_+} a_{i,m}^{j,l}(\zeta) \frac{\partial^2}{\partial \zeta_i \partial \zeta_m} \widetilde{G}(\Phi^{-1}(\zeta), y) \tau(\zeta) d\zeta \tag{4.3}$$

$$+ \sum_{1 \leq m \leq n} \int_{\mathbf{R}^n_+} b_m^{j,l}(\zeta) \frac{\partial}{\partial \zeta_m} \widetilde{G}(\Phi^{-1}(\zeta), y) \tau(\zeta) d\zeta. \tag{4.4}$$

We will deal with each term in the sums individually, and distinguish between tangential and normal derivatives.

Case I: Tangential derivatives. In this case, we want to integrate by parts to transfer the derivatives from $\widetilde{G}$ to φ, and use the Sobolev estimates for $\widetilde{G}$. Because of the change of coordinates, there are certain details to keep track of.

Consider the function $f \in L^2(\Omega)$ defined by

$$f(y) = \int_{\mathbf{R}^n_+} a(\zeta) \frac{\partial^2}{\partial \zeta_i \partial \zeta_m} \widetilde{G}(\Phi^{-1}(\zeta), y)\tau(\zeta) d\zeta,$$

with $i, m \leq n-1$, where $a = a^{j,l}_{i,m}$ is smooth. Testing f against some $\psi \in \mathcal{C}^\infty(\overline{\Omega})$, we can integrate by parts to get

$$\begin{aligned}
\int_\Omega f(y)\psi(y)dy &= \int_\Omega \left\{ \int_{\mathbf{R}^n_+} a(\zeta) \frac{\partial^2}{\partial \zeta_i \partial \zeta_m} \widetilde{G}(y, \Phi^{-1}(\zeta))\tau(\zeta) d\zeta \right\} \psi(y) dy \\
&= \int_{\mathbf{R}^n_+} \left\{ \int_\Omega \frac{\partial^2}{\partial \zeta_i \partial \zeta_m} \widetilde{G}(\Phi^{-1}(\zeta), y)\psi(y) dy \right\} a(\zeta)\tau(\zeta) d\zeta \\
&= \int_{\mathbf{R}^n_+} \frac{\partial^2}{\partial \zeta_i \partial \zeta_m} \widetilde{\mathbf{G}}(\psi)(\Phi^{-1}(\zeta)) a(\zeta)\tau(\zeta) d\zeta \\
&= \int_{\mathbf{R}^n_+} \widetilde{\mathbf{G}}(\psi)(\Phi^{-1}(\zeta)) \frac{\partial^2}{\partial \zeta_i \partial \zeta_m} \left[a(\zeta)\tau(\zeta) \right] d\zeta \\
&= \int_\Omega \widetilde{\mathbf{G}}(\psi)(z)\phi(z) J_\Phi(z) dz \\
&= \int_\Omega \psi \widetilde{\mathbf{G}}(\phi J_\Phi),
\end{aligned}$$

where

$$\phi = \frac{\partial^2}{\partial \zeta_i \partial \zeta_m}(a\tau) \circ \Phi.$$

This shows $f = \widetilde{\mathbf{G}}(\phi J_\Phi)$ in $L^2(\Omega)$, so the Sobolev estimates for $\widetilde{G}$, combined with the fact that

$$\|\phi\|_k \leq C_\Phi \|\varphi\|_{k+2}$$

for all $k \geq 0$, give the desired estimate 4.1 for the terms in the sum 4.3 with $i, m \leq n-1$. A similar argument gives an even better estimate for the terms in the sum 4.4 with $m \leq n-1$.

Case II: One normal derivative. This is the case of a term in the sum 4.3 or 4.4 with $m = n$. If in addition we have a tangential derivative, we may integrate by parts as above. Thus what remains is to show that the operator $\widetilde{\mathbf{G}}^*_\rho$, defined on $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ by

$$\widetilde{\mathbf{G}}^*_\rho(\varphi)(y) = \int_\Omega \frac{\partial}{\partial \rho_z} \widetilde{G}(y, z)\varphi(z) dz,$$

is bounded from $H_k(\Omega)$ to $H_{k+1}(\Omega)$. Recall (see [CKS], Section 7) that we can write the solution operator $\widetilde{\mathbf{G}}$ to the Neumann problem as

$$\widetilde{\mathbf{G}} = \mathbf{E} + \widetilde{\mathbf{H}} + S,$$

where $\mathbf{E}$ is convolution with the Newtonian potential E (followed by restriction to $\overline{\Omega}$),

$$\widetilde{\mathbf{H}} = -\mathbf{P}\left(Q\mathbf{R}\frac{\partial}{\partial\rho}\mathbf{E}\right), \tag{4.5}$$

and S is a smoothing error. Here Q is smoothing of order 1 on the boundary.

Since $\mathbf{E}$ maps $H_k(\Omega)$ to $H_{k+2}(\Omega)$, its derivatives are bounded from $H_k(\Omega)$ to $H_{k+1}(\Omega)$. Furthermore, denoting the kernel by $E(y,z)$, we have

$$\frac{\partial}{\partial\rho_z}E(y,z) = \sum c_m(z)\frac{\partial}{\partial z_m}E(y,z) = -\sum c_m(z)\frac{\partial}{\partial y_m}E(y,z), \tag{4.6}$$

for some smooth functions c_m. Thus the operator $\mathbf{E}^*_\rho$ defined by

$$\mathbf{E}^*_\rho(\varphi)(y) = \int_\Omega \frac{\partial}{\partial\rho_z}E(y,z)\varphi(z)dz$$

also maps $H_k(\Omega)$ to $H_{k+1}(\Omega)$.

Now let us look at $\widetilde{\mathbf{H}}^*_\rho$, again defined by

$$\widetilde{\mathbf{H}}^*_\rho(\varphi)(y) = \int_\Omega \frac{\partial}{\partial\rho_z}\widetilde{H}(y,z)\varphi(z)dz,$$

where $\widetilde{H}$ is the kernel of $\widetilde{\mathbf{H}}$. Looking at 4.5, we see that since we are differentiating with respect to the variable of integration, the derivative falls on the innermost operator, namely $\mathbf{E}$, so that

$$\widetilde{\mathbf{H}}^*_\rho = -\mathbf{P}\left(Q\mathbf{R}\frac{\partial}{\partial\rho}\mathbf{E}^*_\rho\right).$$

Using 4.6, we see that

$$\frac{\partial}{\partial\rho_z}E(y,z) + \frac{\partial}{\partial\rho_y}E(y,z) = \sum[c_m(y) - c_m(z)]\frac{\partial}{\partial y_m}E(y,z),$$

which shows that the operator

$$\mathbf{E}^*_\rho - \frac{\partial}{\partial\rho}\mathbf{E}$$

is smoothing of order 2 on $\overline{\Omega}$. Thus

$$\widetilde{\mathbf{H}}^*_\rho = \mathbf{P}\left(Q\mathbf{R}\frac{\partial^2}{\partial\rho^2}\mathbf{E}\right) + \widetilde{\mathbf{H}}',$$

where $\widetilde{\mathbf{H}}'$ has the same regularity properties as $\widetilde{\mathbf{H}}$, namely it maps $H_k(\Omega)$ to $H_{k+2}(\Omega)$.

But now note that since E is harmonic, we can write $\frac{\partial^2}{\partial\rho^2}\mathbf{E}$ in terms of purely tangential derivatives, and at most one normal derivative. The tangential derivatives commute with the operators $\mathbf{R}$, Q and $\mathbf{P}$ up to lower order terms, so that in the end we remain with operators of the form $\partial_j\partial_l\widetilde{\mathbf{H}}$, which we know have the desired regularity (see [CKS]), plus operators which are smoothing to at least order 1. As for the remaining normal derivative, it gives an operator which is essentially $\widetilde{\mathbf{H}}$, up to multiplication by a smooth function. Thus we see that we can write $\widetilde{\mathbf{H}}^*_\rho$ as a sum of operators, all of which map $H_k(\Omega)$ to $H_{k+1}(\Omega)$.

Finally, looking at the smoothing operator S, since its kernel can be taken to be as smooth as we like, applying $\frac{\partial}{\partial\rho_z}$ to it still gives a kernel which is smoothing.

Thus we have shown that $\widetilde{\mathbf{G}}_\rho^*$ maps $H_k(\Omega)$ to $H_{k+1}(\overline{\Omega})$, which proves the Sobolev estimates 4.1 for any term containing one normal derivative.

Case III: Two normal derivatives. Here we look at the operator with kernel $\frac{\partial^2}{\partial\rho_z^2}\widetilde{G}(y,z)$. We use the fact that the Neumann function $\widetilde{G}(y,z)$ is harmonic in both y and z, so again we can write this in terms of purely tangential derivatives and at most one normal derivative, which cases were dealt with above.

Putting everything together, we see that the estimates (4.1) are satisfied for each of the terms in the sums 4.3 and 4.4; hence they are satisfied for $\eta_\mu\varphi$ for each μ, and we have shown that the operator $(\partial_j\partial_l\widetilde{\mathbf{G}})^*$ is bounded from $H_k(\Omega)$ to $H_k(\Omega)$, and in particular $\widetilde{K_{\varphi_t^x}}$ is smooth up to the boundary.

To prove the estimates on the derivatives of $\widetilde{K_{\varphi_t^x}}(y)$, we will first consider the situation when $|x-y|<4t$. Then it suffices to show

$$|\partial^\alpha \widetilde{K_{\varphi_t^x}}| \leq Ct^{-n-|\alpha|}$$

for all multi-indices α. This can be done by using the Sobolev embedding theorem, as in the case of the Dirichlet problem (see Section 3, proof of Claim 3.6). We only need the estimates

$$\|\partial^\alpha \widetilde{K_{\varphi_t^x}}\|_{L^2(\Omega)} \leq Ct^{-n/2-|\alpha|},$$

which follow from the regularity of $\left(\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j\partial x_l}\right)^*$ discussed above:

$$\|\partial^\alpha \widetilde{K_{\varphi_t^x}}\|_{L^2(\Omega)} \leq \|\widetilde{K_{\varphi_t^x}}\|_{|\alpha|} \leq C\|\varphi_t^x\|_{|\alpha|} \leq Ct^{-n/2-|\alpha|}.$$

When $|x-y|>4t$, we write

$$\widetilde{K_{\varphi_t^x}}(y) = \int_\Omega \frac{\partial^2}{\partial z_j\partial z_l}\widetilde{G}(y,z)\varphi_t^x(z)dz.$$

Here y is outside the support of φ_t^x, so the integral is taken only over the region where $\widetilde{G}(y,z)$ is smooth in z. Differentiating in y, this gives

$$\frac{\partial^{|\alpha|}}{\partial y^\alpha}\widetilde{K_{\varphi_t^x}}(y) = \int_\Omega \partial_y^\alpha\partial_{z_j}\partial_{z_l}\widetilde{G}(y,z)\varphi_t^x(z)dz.$$

Furthermore, for z in the support of φ_t^x we have

$$|x-y| \leq |x-z|+|z-y| \leq 2t+|z-y| < |x-y|/2+|z-y|;$$

hence $|x-y|/2<|y-z|$. Thus if we can show

$$|\partial_y^\alpha\partial_{z_j}\partial_{z_l}\widetilde{G}(y,z)| \leq C|y-z|^{-n-|\alpha|} \tag{4.7}$$

for all $z\in\Omega\setminus\{y\}$, we would get

$$\begin{aligned}
\left|\frac{\partial^{|\alpha|}}{\partial y^\alpha}\widetilde{K_{\varphi_t^x}}(y)\right| &\leq \left|\int_\Omega \partial_y^\alpha\partial_{z_j}\partial_{z_l}\widetilde{G}(y,z)\varphi_t^x(z)dz\right| \\
&\leq \int_\Omega \frac{C}{|y-z|^{n+|\alpha|}}|\varphi_t^x(z)|dz \\
&\leq C|x-y|^{-n-|\alpha|},
\end{aligned}$$

which is the desired estimate.

So it remains to prove the estimate (4.7) for the Neumann function $\widetilde{G}$. Again we write

$$\widetilde{\mathbf{G}} = \mathbf{E}+\widetilde{\mathbf{H}}+S.$$

The Newtonian potential $E(y,z)$ is symmetric in y and z and its second derivatives form a Calderón-Zygmund kernel, so the inequality (4.7) holds with E instead of $\widetilde{G}$. Furthermore, the kernel for S can be taken to be as smooth as desired. Thus we need only concentrate on the kernel $\widetilde{H}(y,z)$ of $\widetilde{\mathbf{H}}$.

In order to estimate the derivatives of $\widetilde{H}(y,z)$, again write

$$\widetilde{\mathbf{H}} = -\mathbf{P}\left(Q\mathbf{R}\frac{\partial}{\partial\rho}\mathbf{E}\right).$$

Let $T = Q\mathbf{R}\frac{\partial}{\partial\rho}\mathbf{E}$. Since Q is smoothing of order 1 on $\partial\Omega$, T is a reverse Poisson operator of the same order as $\mathbf{RE}$. When we consider $\partial_y^\alpha\partial_{z_j}\partial_{z_l}\widetilde{H}(y,z)$, the y derivatives fall on the Poisson kernel P, while the z derivatives fall on the kernel of T. If $K(\zeta,z)$ is this kernel ($\zeta\in\partial\Omega$, $z\in\overline{\Omega}$), then

$$\partial_y^\alpha\partial_{z_j}\partial_{z_l}\widetilde{H}(y,z) = \int_{\partial\Omega}\partial_y^\alpha P(y,\zeta)\partial_{z_j}\partial_{z_l}K(\zeta,z).$$

We can now proceed as in the proof of Claim 3.6 (for the Dirichlet problem), with $K(\zeta,z)$ instead of the kernel $E(\zeta,z)$ of $\mathbf{RE}$. Namely, we use integration by parts to transfer derivatives between K and P. In the "local" case (denoted by I_1 in the proof of Claim 3.6), we can use the harmonicity of E and integration by parts to express $\partial_{z_j}\partial_{z_l}K(\zeta,z)$ as

$$\sum_{|\beta|\le 2}\partial_\zeta^{|\beta|}K_\beta(\zeta,z),$$

where the derivatives in ζ are tangential, and the operators associated to the kernels K_β are of the same (or lower) order as T. Thus we can integrate by parts to move the derivatives onto P. For the "global" case (I_2), we transfer the derivatives from P onto K.

Alternatively, one can proceed to prove the estimates on the derivatives of the kernel $\widetilde{H}(y,z)$ via the symbolic calculus, as in [CKS] (see pp. 342-346, although in this case one needs to reverse the variables when taking derivatives).

This gives the estimate (4.7) on the derivatives of $\widetilde{G}$, which in turn proves Claim 4.4. $\square$

Now we can finally give the

Proof of Theorem 4.1. Let $f\in h_z^p(\overline{\Omega})$. By the atomic decomposition, we can write

$$f = \sum \lambda_k a_k$$

in $\mathcal{S}'(\mathbf{R}^n)$, where the a_k are h_z^p atoms. By Lemma 4.3, for every k, $\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j\partial x_l}(a_k)\in h_z^p(\overline{\Omega})$, and $\left\|\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j\partial x_l}(a_k)\right\|_{h_z^p}\le C$ independently of a_k, so that $\sum\lambda_k\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j\partial x_l}(a_k)$ converges in the $h_z^p(\overline{\Omega})$ norm. Since $h_z^p(\overline{\Omega})$ is a closed subspace of $h_z^p(\mathbf{R}^n)$, hence complete, this sum is an element of $h_z^p(\overline{\Omega})$, and we can define $\widetilde{T}_{j,l}$ on f by

$$\widetilde{T}_{j,l}(f)\overset{\text{def}}{=}\sum\lambda_k\frac{\partial^2\widetilde{\mathbf{G}}}{\partial x_j\partial x_l}(a_k).$$

To see that this is independent of the choice of atomic decomposition, it suffices to show that $\widetilde{T}_{j,l}$ is continuous on $L^2(\Omega)$ in the distribution topology, i.e. if $\{f_k\}$ is a sequence in $L^2(\Omega)$ with $f_k \to 0$ in $\mathcal{C}^{\infty\prime}(\overline{\Omega})$, then

$$\left\langle \frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(f_k), \varphi \right\rangle \to 0$$

for every $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$. But as in Case II of the proof of Lemma 4.3,

$$\left\langle \frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}(f_k), \varphi \right\rangle = \langle f_k, (\partial_j \partial_l \widetilde{\mathbf{G}})^*(\varphi)\rangle.$$

The right-hand-side converges to zero since $f_k \to 0$ in $\mathcal{C}^{\infty\prime}(\overline{\Omega})$ and the operator $(\partial_j \partial_l \widetilde{\mathbf{G}})^*$ is continuous on $\mathcal{C}^\infty(\overline{\Omega})$, as demonstrated in the proof of Claim 4.4.

Thus the extension $\widetilde{T}_{j,l}$ of $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ to $h^p_z(\overline{\Omega})$ is well defined. Furthermore, we have that

$$\|\widetilde{T}_{j,l}(f)\|_{h^p_z} \le C \left(\sum |\lambda_k|^p\right)^{1/p},$$

with C independent of the atomic decomposition. Since

$$\|f\|_{h^p_z(\overline{\Omega})} \approx \inf \left(\sum |\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all such decompositions, we get

$$\|\widetilde{T}_{j,l}(f)\|_{h^p_z} \le C' \|f\|_{h^p_z},$$

i.e. $\widetilde{T}_{j,l}$ is bounded on $h^p_z(\overline{\Omega})$.

We have now proved the h^p_z regularity of the Neumann problem. □

5. H^1 and BMO regularity

In this section we will give a different proof of the h^1_z regularity of the Dirichlet and Neumann problems for the Laplacian. Then we will then state and prove the corresponding results for the appropriate dual space, bmo_r, and for the dual space of $h^1_d(\overline{\Omega})$, bmo_z.

Let us state again the case $p = 1$ of Theorems 4.1 and 4.2.

Theorem 5.1. *If Ω is as above, and $\mathbf{G}$, $\widetilde{\mathbf{G}}$ are the solution operators for the Dirichlet and Neumann problems, respectively, then for $1 \le j, l \le n$, the operators $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ and $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ extend to bounded operators from $h^1_z(\overline{\Omega})$ to $h^1_z(\overline{\Omega})$.*

Instead of using the maximal function, we will prove this by using a cancellation property to obtain the atomic decomposition directly. To illustrate this method, we will first prove an analogue in the upper half-space. Just as for the local spaces $h^p_z(\overline{\Omega})$, we define the space $H^p_z(\overline{\mathbf{R}^n_+})$ to be the subspace of $H^p(\mathbf{R}^n)$ consisting of those distributions supported in $\overline{\mathbf{R}^n_+}$.

Proposition 5.2. *If $\mathbf{G}$ (resp. $\widetilde{\mathbf{G}}$) is the solution operator for the Dirichlet problem (resp. Neumann problem) on the upper half-space $\mathbf{R}^n_+$, then for $1 \le j, l \le n$, $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ (resp. $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$) extends to a bounded operator from $H^1_z(\overline{\mathbf{R}^n_+})$ to $H^1_z(\overline{\mathbf{R}^n_+})$.*

Proof. Let $f \in H^1_z(\overline{\mathbf{R}^n_+})$. Then by an analogue of Theorem 2.5 (see [JSW], Theorem 5.3, and [CKS], Theorem 3.3), f has an atomic decomposition $\sum \lambda_j a_j$, where $\sum |\lambda_j| < \infty$ and the a_j are now H^1 atoms supported in cubes Q_j contained entirely in $\overline{\mathbf{R}^n_+}$. Recall that an H^1 atom must satisfy a size condition, which in this case we take in the L^2 sense:

$$\|a\|_{L^2} \leq |Q|^{-1/2},$$

and an *exact* moment condition:

$$\int a(x)dx = 0.$$

As in the proofs of Theorem 3.4 and Theorem 4.1, it suffices to show that the operators are bounded in the H^1_z norm when acting on atoms.

Consider such an atom a. Then $a \in L^2$, so we can define the solutions to the Dirichlet and Neumann problems for a by $u = \mathbf{G}(a) = E_o * a$ and $u = \widetilde{\mathbf{G}}(a) = E_e * a$, respectively. Here E is the Newtonian potential, and E_o, E_e are its odd and even parts. Let

$$F_{j,l} = \frac{\partial^2 u}{\partial x_j \partial x_l}.$$

We want to show that for any $1 \leq j, l \leq n$, $F_{j,l}|_{\overline{\mathbf{R}^n_+}} \in H^1_z(\overline{\mathbf{R}^n_+})$ with norm bounded by a constant.

We begin by proving a cancellation property for $F_{j,l}$, namely that

$$\int_{\mathbf{R}^n_+} F_{j,l} = 0. \tag{5.1}$$

A priori, we know $F_{j,l}$ exists locally in the L^2 sense, and is therefore locally integrable. We will show that it is actually integrable at infinity, so $F_{j,l} \in L^1(\mathbf{R}^n_+)$. Note that

$$F_{j,l} = \frac{\partial^2 E_o}{\partial x_j \partial x_l} * a$$

for the Dirichlet problem, or

$$F_{j,l} = \frac{\partial^2 E_e}{\partial x_j \partial x_l} * a$$

for the Neumann problem. Let Q be the supporting cube of a, with center y_Q. Then by the moment condition on a, and the size of the derivatives of the Newtonian potential, we have, for $x \in \mathbf{R}^n \setminus 2Q$:

$$\begin{aligned}
|F_{j,l}(x)| &\leq \left| \int_Q \frac{\partial^2 E_o}{\partial x_j \partial x_l}(x-y)a(y)dy \right| \\
&= \left| \int_Q \left\{ \frac{\partial^2 E_o}{\partial x_j \partial x_l}(x-y) - \frac{\partial^2 E_o}{\partial x_j \partial x_l}(x-y_Q) \right\} a(y)dy \right| \\
&\leq \frac{C}{|x-y_Q|^{n+1}} \int_Q |y-y_Q||a(y)|dy \\
&\leq \frac{C|Q|^{1/n}}{|x-y_Q|^{n+1}}.
\end{aligned}$$

Exactly the same estimate holds for the case of the Neumann problem.

Now that we have the decay and integrability at infinity, we immediately get that

$$\int_{\mathbf{R}^n_+} \frac{\partial^2 u}{\partial x_j \partial x_l} dx = 0$$

whenever either $j < n$ or $l < n$. Furthermore, since $\Delta u = a$ on $\mathbf{R}^n_+$, and a has vanishing integral, we have

$$\int_{\mathbf{R}^n_+} \frac{\partial^2 u}{\partial x_n^2} dx = \int_{\mathbf{R}^n_+} \left\{ a - \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} \right\} dx = 0.$$

Thus $\int_{\mathbf{R}^n_+} F_{j,l} = 0$ for all j, l.

Now fix j and l and let $F = F_{j,l}$. We will show $F|_{\mathbf{R}^n_+}$ has an atomic decomposition with atoms supported entirely in $\overline{\mathbf{R}^n_+}$.

Again looking at the supporting cube Q of a, we define Q_k, $k \geq 1$, to be the smallest cube such that

$$(2^k Q) \cap \overline{\mathbf{R}^n_+} \subset Q_k \subset \overline{\mathbf{R}^n_+}.$$

Set

$$F_{Q_k} = \frac{1}{|Q_k|} \int_{Q_k} F(x) dx$$

for $k \geq 1$,

$$g_1 = F\chi_{Q_1} - F_{Q_1}\chi_{Q_1},$$

and

$$g_k = F\chi_{Q_k \setminus Q_{k-1}} + F_{Q_{k-1}}\chi_{Q_{k-1}} - F_{Q_k}\chi_{Q_k}$$

for $k \geq 2$. Then $\int g_1 = 0$ and for $k \geq 2$,

$$\int g_k = \int_{Q_k \setminus Q_{k-1}} F + \left\{ \frac{1}{|Q_{k-1}|} \int_{Q_{k-1}} F \right\} |Q_{k-1}| - \left\{ \frac{1}{|Q_k|} \int_{Q_k} F \right\} |Q_k| = 0.$$

We claim

$$\sum_{i=1}^{k} g_i = F\chi_{Q_k} - F_{Q_k}\chi_{Q_k}.$$

This is true for $k = 1$, and if we assume it is true for $k - 1$, we get

$$\begin{aligned} \sum_{i=1}^{k} g_i &= \sum_{i=1}^{k-1} g_i + g_k \\ &= F\chi_{Q_{k-1}} - F_{Q_{k-1}}\chi_{Q_{k-1}} \\ &\quad + F\chi_{Q_k \setminus Q_{k-1}} + F_{Q_{k-1}}\chi_{Q_{k-1}} - F_{Q_k}\chi_{Q_k} \\ &= F\chi_{Q_k} - F_{Q_k}\chi_{Q_k}. \end{aligned}$$

This, combined with the fact that $\int_{\mathbf{R}^n_+} F = 0$, gives

$$\begin{aligned} \int_{\mathbf{R}^n_+} |F - \sum_{i=1}^{k} g_i| &\leq \int_{\mathbf{R}^n_+} |F - F\chi_{Q_k}| + |F_{Q_k}||Q_k| \\ &= \int_{\mathbf{R}^n_+ \setminus Q_k} |F| + \left| \int_{Q_k} F \right| \\ &= \int_{\mathbf{R}^n_+ \setminus Q_k} |F| + \left| \int_{\mathbf{R}^n_+ \setminus Q_k} F \right| \\ &\leq 2 \int_{\mathbf{R}^n_+ \setminus Q_k} |F| \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Thus

$$F|_{\mathbf{R}^n_+} = \sum_{i=1}^{\infty} g_i$$

in L^1.

Now we want to estimate the L^2 norms of the g_k. Note that for $k = 1$, we have, by the L^2 boundedness,

$$\begin{aligned} \|g_1\|_{L^2} &\leq \left\{ \int_{Q_1} |F|^2 \right\}^{1/2} + |F_{Q_1}||Q_1|^{1/2} \\ &\leq 2 \left\{ \int_{Q_1} |F|^2 \right\}^{1/2} \\ &\leq 2 \left\| \frac{\partial^2 u}{\partial x_j \partial x_l} \right\|_{L^2} \\ &\leq C\|a\|_{L^2} \\ &\leq C_1 |Q_1|^{-1/2}. \end{aligned}$$

For $k \geq 2$ we can again write

$$\|g_k\|_{L^2} \leq \left\{ \int_{Q_k \setminus Q_{k-1}} |F|^2 \right\}^{1/2} + |F_{Q_{k-1}}||Q_{k-1}|^{1/2} + |F_{Q_k}||Q_k|^{1/2}.$$

To estimate the first term, recall that, for $x \in \mathbf{R}^n_+ \setminus Q_1$,

$$|F(x)| \leq \frac{C|Q|^{1/n}}{|x - y_Q|^{n+1}}.$$

Therefore, with δ being one-half the sidelength of Q, we have

$$\begin{aligned} \left\{ \int_{Q_k \setminus Q_{k-1}} |F|^2 \right\}^{1/2} &\leq \frac{C\delta}{(2^{k-1}\delta)^{n+1}} |Q_k|^{1/2} \\ &= C 2^{-k(n/2+1)} \delta^{-n/2} \\ &= C 2^{-k} |Q_k|^{-1/2}. \end{aligned}$$

As for the second and third terms, we can use the fact that $\int_{\mathbf{R}^n_+} F = 0$, and the estimate for $F(x)$ in $\mathbf{R}^n_+ \setminus Q_1$, to get:

$$\begin{aligned} |F_{Q_k}| &= \frac{1}{|Q_k|} \left| \int_{Q_k} F(x) dx \right| \\ &= \frac{1}{|Q_k|} \left| \int_{\mathbf{R}^n_+ \setminus Q_k} F(x) dx \right| \\ &\leq \frac{C|Q|^{1/n}}{|Q_k|} \int_{|x - y_Q| \geq 2^k \delta} \frac{1}{|x - y_Q|^{n+1}} dx \\ &\leq C \frac{\delta}{(2^k \delta)^{n+1}}, \end{aligned}$$

so

$$\begin{aligned} |F_{Q_k}||Q_k|^{1/2} &\leq C\frac{\delta}{(2^k\delta)^{n+1}}|Q_k|^{1/2} \\ &= C2^{-k}|Q_k|^{-1/2}. \end{aligned}$$

The estimate for $k-1$ is the same up to a constant. Thus we see that

$$\|g_k\|_{L^2} \leq C2^{-k}|Q_k|^{-1/2}.$$

Now let $\lambda_k = C2^{-k}$ for $k \geq 2$, $\lambda_1 = C_1$, and $a_k = \lambda_k^{-1} g_k$. Then each a_k is supported in $Q_k \subset \overline{\mathbf{R}}^n_+$, and satisfies $\int a_k = 0$ and $\|a_k\|_{L^2} \leq |Q_k|^{-1/2}$, so it is an $H^1_z(\overline{\mathbf{R}}^n_+)$ atom. Also $F = \sum \lambda_k a_k$, and

$$\sum_{k=1}^{\infty} |\lambda_k| = C_1 + C\sum_{k=2}^{\infty} 2^{-k} = A < \infty.$$

Here A is independent of a.

This completes the proof of Proposition 5.2. □

Proof of Theorem 5.1. We are going to give a local version of the proof of Proposition 5.2. We will give the proof for the case of the Dirichlet problem, and indicate where changes must be made for the Neumann problem. Again, it suffices to consider an h^1_z atom a. Let $u = \mathbf{G}(a)$, fix j, l, and set

$$F = \frac{\partial^2 u}{\partial x_j \partial x_l}.$$

We will give an atomic decomposition for F in terms of modified h^1_z atoms (see Remark 3 following Definition 2.3).

We must distinguish between the case where a is supported "away from the boundary", and the case where a is supported "near the boundary". Cover $\overline{\Omega}$ by open sets $U_0, \ldots, U_M$, where $\overline{U_0} \subset \Omega$, $\partial\Omega \subset \bigcup_{i=1}^M U_i$, and in each U_i, $i = 1, \ldots, M$, we have a system of tangential and normal coordinates $(t_1, \ldots, t_n)$. Let $C_\Omega > 0$ be such that if B is a ball with $|B| < C_\Omega$, then $B \cap \overline{\Omega} \subset U_i$ for some i. If Q is the supporting cube of a, and $|Q| < C_\Omega$, let K be the largest integer such that $|2^K Q| < C_\Omega$, and set

$$Q_K = 2^K Q \cap \overline{\Omega}.$$

Then one of the following cases must hold:

<u>Case 0: $|Q| \geq C_\Omega$.</u> In this case, we treat F as a multiple of a large modified h^1_z atom, supported in all of $\overline{\Omega}$. Since $|\overline{\Omega}| \geq C_\Omega$, we only need to check the size condition, which follows from the L^2 estimates:

$$\|F\|_{L^2} = \left\|\frac{\partial^2 u}{\partial x_j \partial x_l}\right\|_{L^2} \leq C\|a\|_{L^2} \leq CC_\Omega{}^{-1/2}.$$

This shows that the h^1_z norm of F is bounded by a constant independent of a.

<u>Case 1: $|Q| < C_\Omega$ and $Q_K \subset U_0$.</u> By rotating and translating, we can assume $Q = [-\delta, \delta]^n$ for some $\delta > 0$.

In order to use get the atomic decomposition for F, we need an analogue of the cancellation property 5.1. We start by getting the size estimates on F and the first derivatives of u. If G is the Green's function for the Dirichlet problem, then

$$F(x) = \int_\Omega \frac{\partial^2 G(x,y)}{\partial x_j \partial x_l} a(y)dy.$$

By interchanging the role of the variables in estimate 3.1 and its proof, we get that

$$|\partial_y^\alpha \partial_{x_j} \partial_{x_l} G(x,y)| \le C|x-y|^{-n-|\alpha|}$$

for $0 \le |\alpha| \le 1$. The same holds for the Neumann function $\widetilde{G}$, following the proof of estimate (4.7). Combined with the moment condition on a, this gives

$$\begin{aligned} |F(x)| &\le \frac{C}{|x|^{n+1}} \int_Q |y||a(y)|dy + \frac{C}{|x|^n} \left| \int_Q a(y)dy \right| \\ &\le \frac{C|Q|^{1/n}}{|x|^{n+1}} + \frac{C|Q|^{\nu_1}}{|x|^n} \\ &\le \frac{C'|Q|^{1/n}}{|x|^{n+1}}. \end{aligned}$$

Note that we were able to absorb the second term because $\nu_1 = 1/n$, and $|x|$ is bounded above. Applying the same estimates to the first order derivatives, and combining terms, we also get

$$\left| \frac{\partial u}{\partial x_l} \right| \le \frac{C|Q|^{1/n}}{|x|^n}.$$

Thus we can integrate F in one variable first and use this estimate to get

$$\left| \int_{Q_K} F(x)dx \right| = \left| \int_{[-2^K\delta, 2^K\delta]^{n-1}} \left\{ \int_{-2^K\delta}^{2^K\delta} \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_l} \right) dx_j \right\} dx' \right| \tag{5.2}$$

$$\le \frac{2C|Q|^{1/n}}{|2^K\delta|^n} \mathrm{Vol}_{n-1}([-2^K\delta, 2^K\delta]^{n-1}) \tag{5.3}$$

$$\le C|Q|^{1/n} \tag{5.4}$$

as $2^K\delta \approx C_\Omega{}^{1/n}$. This is the approximate cancellation condition which replaces equation (5.1).

Now we proceed to define, as in the case of the upper half-space: $Q_k = 2^k Q$ and $F_{Q_k} = \frac{1}{|Q_k|} \int_{Q_k} F(x)dx$ for $1 \le k \le K$,

$$g_1 = F\chi_{Q_1} - F_{Q_1}\chi_{Q_1},$$

and

$$g_k = F\chi_{Q_k \setminus Q_{k-1}} + F_{Q_{k-1}}\chi_{Q_{k-1}} - F_{Q_k}\chi_{Q_k}$$

for $2 \le k \le K$. We also set $Q_{K+1} = \overline{\Omega}$ and

$$g_{K+1} = F\chi_{\Omega \setminus Q_K} + F_{Q_K}\chi_{Q_K}.$$

Then again we have that g_k is supported in Q_k, $\int g_k = 0$ for $1 \le k \le K$, and

$$\sum_{k=1}^{K+1} g_k = F\chi_{Q_K} - F_{Q_K}\chi_{Q_K} + g_{K+1} = F|_\Omega.$$

The estimates for the L^2 norms of the g_k have to be modified to take into account the approximate cancellation conditions. While we still have

$$\|g_1\|_{L^2} \le C\|a\|_{L^2} \le C_1|Q_1|^{-1/2},$$

and for $2 \le k \le K$,

$$\left\{ \int_{Q_k \setminus Q_{k-1}} |F|^2 \right\}^{1/2} \quad \le \quad C 2^{-k} |Q_k|^{-1/2},$$

we no longer have $\int F = 0$, but rather $|\int_{Q_K} F| \le C|Q|^{1/n}$. This gives

$$\begin{aligned} |F_{Q_k}| &= \frac{1}{|Q_k|} \left| \int_{Q_k} F(x) dx \right| \\ &\le \frac{1}{|Q_k|} \left\{ \left| \int_{Q_K \setminus Q_k} F(x) dx \right| + \left| \int_{Q_K} F(x) dx \right| \right\} \\ &\le \frac{C|Q|^{1/n}}{|Q_k|} \left\{ \int_{2^k \delta \le |x| \le c} |x|^{-(n+1)} dx + 1 \right\} \\ &\le C \frac{\delta}{(2^k \delta)^{(n+1)}}. \end{aligned}$$

Note that here we again used the fact that $2^K \delta \approx C_\Omega{}^{1/n}$. Hence

$$|F_{Q_k}| |Q_k|^{1/2} \le C 2^{-k} |Q_k|^{-1/2},$$

as above. Combining these estimates, we once more get, for $2 \le k \le K$,

$$\|g_k\|_L^2 \le 2^{-k} |Q_k|^{-1/2}.$$

For $K+1$, we have

$$\begin{aligned} \|g_{K+1}\|_{L^2} &\le \left\{ \int_{\Omega \setminus Q_K} |F|^2 \right\}^{1/2} + |F_{Q_K}| |Q_K|^{1/2} \\ &\le C_\Omega |Q|^{1/n} \\ &\le C_\Omega. \end{aligned}$$

This shows $g_{K+1} \in h^1_z(\overline{\Omega})$ with norm bounded by a constant, so we only need to consider $F - g_{K+1}$.

Set $\lambda_1 = C_1$ and $\lambda_k = C 2^{-k}$ for $2 \le k \le K$. Then $g_k = \lambda_k a_k$ where each a_k is supported in $Q_k \subset \overline{\Omega}$, $\int_{Q_k} a_k = 0$ when $|Q_k| \le |Q_K| < C_\Omega$, and $\|a_k\|_{L^2} \le |Q_k|^{-1/2}$. Thus each a_k is a modified $h^1_z(\overline{\Omega})$ atom. Also $F - g_{K+1} = \sum \lambda_k a_k$ in Ω, and

$$\sum_{k=1}^{K} |\lambda_k| \le C_1 + C \sum_{k=2}^{K} 2^{-k} \le C.$$

Case 2: $|Q| < C_\Omega$ and $Q_K \subset U_i$ for some $i = 1, \ldots, n.$ From the estimates on the Green's function and the moment condition, we again get

$$|F(x)| \le \frac{C|Q|^{1/n}}{|x - y_Q|^{n+1}},$$

for $x \in \Omega \setminus 2Q$, where y_Q is the center of Q. For such x, we can also bound any first derivatives of u by $|Q|^{1/n} |x - y_Q|^{-n}$, and u itself by $|Q|^{1/n} |x - y_Q|^{-n+1}$. However, this does not immediately give us the estimate 5.4 on the integral of F. In order to do this, we must switch coordinates.

Let $U = U_i$. Recall that in U there exists a system of tangential and normal coordinates $(t_1, \ldots, t_n)$. More specifically, we take $(t_1, \ldots, t_{n-1})$ to be coordinates on $\partial\Omega \cap U$ and t_n to be the signed geodesic distance to $\partial\Omega$ in the Euclidean metric in U. Since a is supported in $\overline{\Omega}$, and its supporting cube is "near" the boundary ($Q_K \subset U$), we may assume that in the new coordinates it is supported in the cube

$$\widetilde{Q} = [-\delta, \delta]^{n-1} \times [0, 2\delta],$$

where δ is proportional to the sidelength of the original supporting cube (the constants of proportionality depending only on the change of coordinates.) We can now set

$$\widetilde{Q_k} = [-2^k\delta, 2^k\delta]^{n-1} \times [0, 2^{k+1}\delta] \subset U_i$$

for $1 \leq k \leq K$, which would allow us to continue as in Case 1, provided we can get the estimate 5.4.

In U, we can rewrite F as

$$F = \sum_{1 \leq i,m \leq n} a_{i,m} \frac{\partial^2 u}{\partial t_i \partial t_m} + \sum_{1 \leq i \leq n} b_i \frac{\partial u}{\partial t_i}$$

for some smooth functions $a_{i,m}$ and b_i. Thus to bound the integral of F, it suffices to bound the integrals of all first and second derivatives of u in the coordinates t_i.

We do this first for the Dirichlet problem. To bound the integral of the first derivatives, we can use the fundamental theorem of calculus and the values of u on the boundary of $\widetilde{Q_K}$ to get

$$\left| \int_{Q_K} \frac{\partial u}{\partial t_i}(t) dt \right| = \left| \int \left\{ \int \frac{\partial u}{\partial t_i} dt_i \right\} dt' \right| \leq C|Q|^{1/n}.$$

Note that when $i = n$, this involves the Dirichlet boundary conditions.

For the second derivatives, we again use the bound on the first derivatives of u on the boundary of $\widetilde{Q_K}$ to get

$$\left| \int_{Q_K} \frac{\partial^2 u}{\partial t_j \partial t_l}(t) dt \right| \leq C|Q|^{1/n}$$

as long as either $j < n$ or $l < n$.

In order to get an estimate for $\frac{\partial^2 u}{\partial t_n^2}$, we need to use the fact that $\Delta u = a$ in Ω. In the new coordinates, since we chose t_n to be the signed geodesic distance to $\partial\Omega$ in the Euclidean coordinates, this translates into

$$\frac{\partial^2 u}{\partial t_n^2} + \sum_{1 \leq i,m \leq n-1} c_{i,m} \frac{\partial^2 u}{\partial t_i \partial t_m} + \sum d_i \frac{\partial u}{\partial t_i} = a,$$

where $c_{i,m}$ and d_i are smooth functions in U. Combined with the approximate moment condition on a, this gives the desired estimate on the integral of $\frac{\partial^2 u}{\partial t_n^2}$, and hence on the integral of F (estimate 5.4).

For the Neumann problem, we are no longer able to estimate the integral of $\frac{\partial u}{\partial t_n}$ on Q_K, as above. However, we can use the Neumann boundary conditions to estimate the integral of $\frac{\partial^2 u}{\partial t_n^2}$, as long as $\frac{\partial}{\partial t_n}|_{\partial\Omega} = -\vec{n}$, where $\vec{n}$ is the outward normal vector on the boundary. Then we can proceed to estimate the integral of $\frac{\partial u}{\partial t_n}$ on Q_K by solving for $\frac{\partial u}{\partial t_n}$ in the equation $\Delta u = a$. We need to make sure that the coefficient of $\frac{\partial}{\partial t_n}$ in the expression for the Laplacian is non-vanishing. This coefficient turns out to be the Euclidean Laplacian of t_n, Δt_n. But this can always be taken to be non-vanishing, if necessary by replacing t_n with $1 - e^{-t_n}$. These modifications give estimate 5.4 for the Neumann problem.

Now we can proceed exactly as in Case 1 to get an atomic decomposition for F with atoms supported entirely in $\overline{\Omega}$. These will be h_z^1 atoms with respect to the new coordinates $(t_1, \dots, t_n)$ in U, and therefore when we pull back to the original coordinates, they will become modified h_z^1 atoms.

Thus in all three cases we have exhibited an atomic decomposition for F with modified h_z^1 atoms, proving that $F \in h_z^1(\overline{\Omega})$ with bounded norm. $\square$

5.1. **BMO regularity.** We now come to the question of BMO regularity. First we have to consider the appropriate BMO spaces. We begin by recalling the definition of the local BMO, as defined in [G] (Section 5). We shall call this space $\mathrm{bmo}(\mathbf{R}^n)$.

Definition 5.3. A locally integrable functions g is said to belong to $\mathrm{bmo}(\mathbf{R}^n)$ if

(5.5)
$$\|g\|_{\mathrm{bmo}} = \sup_{|Q|<1} \frac{1}{|Q|} \int_Q |g(x) - g_Q| dx + \sup_{|Q|\geq 1} \frac{1}{|Q|} \int_Q |g(x)| dx < \infty,$$

where the suprema are taken over all cubes $Q \subset \mathbf{R}^n$ with sides parallel to the axes. Here g_Q denotes the mean value of g over Q. We call $\mathrm{bmo}(\mathbf{R}^n)$ the space of such functions, with norm given by $\| \cdot \|_{\mathrm{bmo}}$.

We can now define a subspace of this space which turns out to be the dual of h_d^1.

Definition 5.4. Let Ω be a bounded domain with smooth boundary. The space $\mathrm{bmo}_z(\overline{\Omega})$ is defined to be the subspace of $\mathrm{bmo}(\mathbf{R}^n)$ consisting of those elements which are supported on $\overline{\Omega}$, i.e.

$$\mathrm{bmo}_z(\overline{\Omega}) = \{g \in \mathrm{bmo}(\mathbf{R}^n) : g = 0 \text{ on } \mathbf{R}^n \setminus \overline{\Omega}\},$$

with

$$\|g\|_{\mathrm{bmo}_z(\overline{\Omega})} \overset{\mathrm{def}}{=} \|g\|_{\mathrm{bmo}(\mathbf{R}^n)}.$$

Theorem 5.5 ([M], [C]). *The space* $\mathrm{bmo}_z(\overline{\Omega})$ *is the dual of* $h_d^1(\overline{\Omega})$.

Remarks. 1. A global version of bmo_z was defined by Miyachi [M], and the local version was defined in [C], both using a different definition which involved distinguishing between type (a) and type (b) cubes. However, the remark following the proof of Theorem 4 in [M], and part (1) of Proposition 2.2 in [C], show that this definition is equivalent.

2. The duality theorem follows from Theorem 2 in [M] and Theorem 2.3 in [C], after observing that the space $h_d^1(\overline{\Omega})$ is in fact the same as the space $h_r^1(\Omega)$ defined in [CKS] (see Proposition 6.4 in Section 6.)

Next, we define the BMO space corresponding to h_z^1.

Definition 5.6. Let Ω be a bounded domain with smooth boundary. A locally integrable functions g is said to belong to $\mathrm{bmo}_r(\Omega)$ if

(5.6)
$$\|g\|_{\mathrm{bmo}_r} = \sup_{|Q|<1} \frac{1}{|Q|} \int_Q |g(x) - g_Q| dx + \sup_{|Q|\geq 1} \frac{1}{|Q|} \int_Q |g(x)| dx < \infty,$$

where the suprema are taken over all cubes $Q \subset \Omega$. Here g_Q denotes the mean value of g over Q.

We call $\mathrm{bmo}_r(\Omega)$ the space of such functions, with norm given by $\| \cdot \|_{\mathrm{bmo}_r}$.

Remarks. 1. This definition was introduced in [C] for a bounded Lipschitz domain Ω. A similar notion is found in [JSW], where the local bmo space $\mathrm{bmo}(F)$ associated to a closed "d-set" F is defined by using essentially the same norm, except the cubes are replaced with balls centered inside the set.

2. This space is a local version of the space $\mathrm{BMO}(\mathcal{D})$ defined by Jones ([J]) for a connected open set $\mathcal{D}$, using the norm
$$\|g\|_{*,\mathcal{D}} = \sup_{Q\subset\mathcal{D}} \frac{1}{|Q|}\int_Q |g(x)-g_Q|dx.$$

3. It should be noted that $\mathrm{bmo}_z(\Omega) \subset \mathrm{bmo}_r(\Omega)$, and simple examples show that this inclusion is strict.

We now state the duality result:

Theorem 5.7 ([JSW], [C])**.** *The space* $\mathrm{bmo}_r(\Omega)$ *is the dual of* $h^1_z(\overline{\Omega})$.

Remarks. 1. Jonsson, Sjögren and Wallin (see [JSW], Theorem 4.2) prove this theorem for a closed "d-set" F having the Markov property, namely they show that $\mathrm{bmo}(F)$ is the dual of the Hardy space $h^1(F)$. As mentioned above (see Remark 1 following Definition 1.2), this is the same as $h^1_z(\overline{\Omega})$ when Ω is a smoothly bounded domain.

2. In ([C], Theorems 2.1, 3.3) it is shown that for a bounded Lipschitz domain in $\mathbf{R}^n$, $\mathrm{bmo}_r(\Omega)$ is the dual of the quotient space $h^1_z(\Omega)$, as defined in [CKS] (see Remark 2 following Definition 1.2). When $p=1$, this quotient space is the same as $h^1_z(\overline{\Omega})$, since there are no nonzero h^1 functions which are supported in $\overline{\Omega}$ and vanish on Ω.

In view of the duality theorems, we want to use our h^1_d and h^1_z regularity results for the Dirichlet and Neumann problems to prove similar regularity results for bmo_z and bmo_r. We first have to define the operators $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ and $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ on these spaces.

Note that the spaces $\mathrm{bmo}_z(\overline{\Omega})$ and $\mathrm{bmo}_r(\Omega)$ are subsets of $L^2(\Omega)$. In fact, by the John-Nirenberg inequality, since $\mathrm{bmo}_z(\overline{\Omega}) \subset \mathrm{bmo}(\mathbf{R}^n) \subset \mathrm{BMO}(\mathbf{R}^n)$ (see [G], Corollary 1 in Section 4) and Ω is bounded,
$$\|g\|_{\mathrm{bmo}_z(\overline{\Omega})} \geq C\|g\|_{L^q(\Omega)}$$
for all $q<\infty$. Similarly, we have
$$\|g\|_{\mathrm{bmo}_r(\Omega)} \geq C\|g\|_{L^q(\Omega)}$$
(see [C], Lemma 1.6, and the proof of Theorem 4.2 in [JSW]). Thus we can define the operators $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ and $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ on $\mathrm{bmo}_z(\overline{\Omega})$ and $\mathrm{bmo}_r(\Omega)$, in the L^2 sense.

We have the following regularity results:

Theorem 5.8. *If* Ω *is as above, and* $\mathbf{G}$ *is the solution operator for the Dirichlet problem, then for* $1\leq j,l\leq n$, $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$ *is a bounded operator from* $\mathrm{bmo}_z(\overline{\Omega})$ *to* $\mathrm{bmo}_z(\overline{\Omega})$ *and from* $\mathrm{bmo}_r(\Omega)$ *to* $\mathrm{bmo}_r(\Omega)$.

Theorem 5.9. *If* Ω *is as above, and* $\widetilde{\mathbf{G}}$ *is the solution operator for the Neumann problem, then for* $1\leq j,l\leq n$, $\frac{\partial^2 \widetilde{\mathbf{G}}}{\partial x_j \partial x_l}$ *is a bounded operator from* $\mathrm{bmo}_r(\Omega)$ *to* $\mathrm{bmo}_r(\Omega)$.

Before proceeding with the proofs, we will state and prove a couple of useful lemmas.

Lemma 5.10. *If $T : L^2(\Omega) \to L^2(\Omega)$ and the kernel $K(x,y)$ of T satisfies*

$$|K(x,y)| \le C|x-y|^{-(n-1)}$$

for all $x \neq y$, then T is bounded from $\mathrm{bmo}_z(\overline{\Omega})$ to $\mathrm{bmo}_z(\overline{\Omega})$ and from $\mathrm{bmo}_r(\Omega)$ to $\mathrm{bmo}_r(\Omega)$.

Proof. For $q > n$, if $\frac{1}{q} + \frac{1}{q'} = 1$, then $q' < \frac{n}{n-1}$ and

$$\int_\Omega K(x,y)^{q'} dy < \infty.$$

Hence T is bounded from $L^q(\Omega)$ to L^∞. Thus for $g \in \mathrm{bmo}_z(\overline{\Omega})$,

$$\|T(g)\|_{\mathrm{bmo}_z} \le \|T(g)\|_\infty \le C\|g\|_{L^q} \le C\|g\|_{\mathrm{bmo}_z},$$

where the last inequality follows from the John-Nirenberg inequality (see the remark preceding Theorem 5.8). Similarly, we get the boundedness on bmo_r. □

Lemma 5.11. *The spaces $\mathrm{bmo}_z(\overline{\Omega})$ and $\mathrm{bmo}_r(\Omega)$ are closed under multiplication by smooth functions, i.e. if $g \in \mathrm{bmo}_z(\overline{\Omega})$ and $\varphi \in \mathcal{C}^1(\overline{\Omega})$, then $\varphi g \in \mathrm{bmo}_z(\overline{\Omega})$ and*

$$\|\varphi g\|_{\mathrm{bmo}_z} \le C\|\varphi\|_{\mathcal{C}^1}\|g\|_{\mathrm{bmo}_z},$$

and similarly for $\mathrm{bmo}_r(\Omega)$.

Proof. To prove this for bmo_z, it suffices to prove it for $\mathrm{bmo}(\mathbf{R}^n)$. We begin by noting that the bmo norm only changes by a factor of 2 if in equation 5.5, for each cube Q, we replace the mean value g_Q by some constant c_Q. Take $\varphi \in C^1$ with compact support. Now for every cube Q with $|Q| < 1$, letting $c_Q = \varphi(x_Q) g_Q$, where x_Q is the center of Q, we get

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |\varphi(x)g(x) - \varphi(x_Q)g_Q|dx &\le \frac{1}{|Q|}\int_Q |\varphi(x)g(x) - \varphi(x)g_Q|dx \\
&\quad + \frac{1}{|Q|}\int_Q |\varphi(x)g_Q - \varphi(x_Q)g_Q|dx \\
&\le \|\varphi\|_\infty \|g\|_{\mathrm{bmo}} + C|g_Q|\|\varphi\|_{C^1}|Q|^{1/n} \\
&\le \|\varphi\|_\infty \|g\|_{\mathrm{bmo}} + C\|g\|_{L^n}\|\varphi\|_{C^1} \\
&\le C\|\varphi\|_{C^1}\|g\|_{\mathrm{bmo}}.
\end{aligned}$$

Clearly if $|Q| \ge 1$,

$$\frac{1}{|Q|}\int_Q |\varphi(x)g(x)|dx \le \|\varphi\|_\infty \|g\|_{\mathrm{bmo}}.$$

Thus $\varphi g \in \mathrm{bmo}$ with norm bounded by $\|\varphi\|_{C^1}\|g\|_{\mathrm{bmo}}$.

The proof for bmo_r is obtained from the proof above by considering only cubes $Q \subset \Omega$. □

We are now ready to prove the theorems.

Proof of Theorem 5.8. We will give the proof for bmo_r, and indicate where changes need to be made for bmo_z.

Set $T_{j,l} = \frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_l}$, defined on $L^2(\Omega)$. Take $g \in \mathrm{bmo}_r(\Omega)$. We want to show $T_{j,l}(g) \in \mathrm{bmo}_r(\Omega)$ with $\|T_{j,l}(g)\|_{\mathrm{bmo}_r} \le C\|g\|_{\mathrm{bmo}_r}$.

As usual, we introduce a partition of unity $\{\eta_\mu\}$, $\mu = 0, \dots, k$, with η_0 supported inside Ω, and η_μ, $\mu \ge 1$, supported in an open set U_μ, with $\bigcup U_\mu \supset \partial\Omega$. In each U_μ

we assume the existence of a coordinate system $(t_1, \dots, t_n)$, where $(t_1, \dots, t_{n-1})$ are tangential coordinates, and t_n is the normal variable.

We want to show $\eta_\mu T_{j,l}(g) \in \mathrm{bmo}_r(\Omega)$ for every μ. We begin with $\mu = 0$.

Case 0: Interior derivatives. We can write

$$\eta_0 T_{j,l} = XY\mathbf{G},$$

where X and Y are vector fields with compact support in Ω (i.e. take $X = \eta_0 \frac{\partial}{\partial x_j}$, $Y = \tilde{\eta}\frac{\partial}{\partial x_l}$ with $\tilde{\eta} = 1$ in the support of η_0).

Note that the operator $\mathbf{G}XY$ is bounded on $\mathrm{bmo}_r(\Omega)$, since for $g \in \mathrm{bmo}_r(\Omega)$ and a an h^1_z atom, we have

$$\begin{aligned} |\langle \mathbf{G}XYg, a\rangle| &= |\langle g, Y^*X^*\mathbf{G}a\rangle| \\ &\leq \|g\|_{\mathrm{bmo}_r} \|Y^*X^*\mathbf{G}a\|_{h^1_z} \\ &\leq C\|g\|_{\mathrm{bmo}_r} \end{aligned}$$

by Theorem 5.1. The same proof applies to bmo_z by using Lemma 3.5.

Furthermore, by the local smoothing properties of $\mathbf{G}$, the operator $XY\mathbf{G} - \mathbf{G}XY$ is smoothing of order 1, hence satisfies the hypothesis of Lemma 5.10, and thus is bounded on $\mathrm{bmo}_r(\Omega)$. This shows $XY\mathbf{G} = \eta_0 T_{j,l}$ is bounded on $\mathrm{bmo}_r(\Omega)$.

We now look at the cases where $\mu \geq 1$. Let $U = U_\mu$. Then we have that

$$\eta_\mu T_{j,l} = XY\mathbf{G},$$

where now X and Y are vector fields with supports in U. Recalling the coordinates $(t_1, \dots, t_n)$ defined in U, we can rewrite X and Y in terms of the vector fields $\frac{\partial}{\partial t_i}$, with smooth coefficients. By Lemma 5.11, it suffices to consider separately the following three cases.

Case 1: Tangential derivatives. We assume that X and Y are tangential vector fields supported in U. Then again we have that the operator $\mathbf{G}XY$ is bounded on $\mathrm{bmo}_r(\Omega)$, since we can integrate by parts as in Case 0. Thus it remains to show that the operator $XY\mathbf{G} - \mathbf{G}XY$ satisfies the hypothesis of Lemma 5.10. We will show that it is a smoothing operator of order 1.

As in the proof of estimate 3.1 in Section 3, write

$$\mathbf{G} = \mathbf{E} + \mathbf{H},$$

where the operator $\mathbf{E}$ is convolution with the Newtonian potential E, and $-\mathbf{H} = \mathbf{PRE}$, the Poisson integral of the restriction of $\mathbf{E}$ to the boundary. Now $\mathbf{E}$ is smoothing of order 2, so its commutator with a differential operator of order 2 is smoothing of order 1. As for $\mathbf{P}$, since X and Y are tangential, we have that

$$XY\mathbf{P} - \mathbf{P}XY = \mathbf{P}'X + \mathbf{P}''Y + \mathbf{P}''', \tag{5.7}$$

where $\mathbf{P}'$, $\mathbf{P}''$ and $\mathbf{P}'''$ are Poisson type operators of order 0 (on $\partial\Omega$)—see [GS], pp. 167-168. Composing this with $\mathbf{RE}$, and noting that $\mathbf{R}$ commutes with X and Y, while again the commutator with $\mathbf{E}$ is smoothing of order 1, we get that

$$\begin{aligned} XY\mathbf{H} - \mathbf{H}XY &= -XY\mathbf{PRE} + \mathbf{PRE}XY \\ &= -(XY\mathbf{P} - \mathbf{P}XY)\mathbf{RE} - \mathbf{P}(XY\mathbf{RE} - \mathbf{RE}XY) \\ &= -(\mathbf{P}'X + \mathbf{P}''Y + \mathbf{P}''')\mathbf{RE} - \mathbf{PR}(XY\mathbf{E} - \mathbf{E}XY), \end{aligned}$$

which is smoothing of order 1.

Case 2: One normal derivative. We assume X is a tangential vector field and $Y = \frac{\partial}{\partial t_n}$. Denote $Y\mathbf{G}$ by T_n and the normal derivative by $\frac{\partial}{\partial \rho}$. Then the kernel of T_n is $\frac{\partial}{\partial \rho_x} G(x,y)$, so we can define the dual operator T_n^* by

$$T_n^*(f) = \int_\Omega \frac{\partial}{\partial \rho_y} G(y,x) f(y) dy.$$

Consider the operator $X^* T_n^*$. Using the same techniques as in Case 2 of the proof of Theorem 5.1, namely the cancellation obtained from the tangential derivative, and the size estimates on the kernel and its derivatives (which are the same regardless of whether we differentiate $G(x,y)$ in the x or the y variable), we get that $X^* T_n^*$ is bounded on h_z^1. This shows that the dual operator, $(X^* T_n^*)^* = T_n X$, is bounded on bmo_r. The same techniques apply for the case of bmo_z. In fact, when we have one tangential derivative, as in $X^* T_n^*$, we can even show boundedness from h_d^1 to h_z^1, since we gain some cancellation.

Thus it remains to show $XT_n - T_n X$ is bounded on bmo_r. In light of Lemma 5.10, it suffice to prove that $XT_n - T_n X$ is smoothing of order 1.

Again, this follows from the calculus of pseudo-differential and Poisson-type operators. If we write

$$\mathbf{G} = \mathbf{E} - \mathbf{PRE}$$

as above, then

$$T_n = \frac{\partial}{\partial \rho}\mathbf{E} - \frac{\partial}{\partial \rho}\mathbf{PRE}.$$

Again the first term is smoothing of order 1, and therefore so is its commutator with X.

As for the second term, setting $\mathbf{P}_n = \frac{\partial}{\partial \rho}\mathbf{P}$, we get a Poisson-type operator of order 1. Thus its commutator with the tangential vector field X also gives a Poisson-type operator of order 1. Composing with $\mathbf{RE}$, and recalling that $\mathbf{R}$ commutes with X, give

$$\begin{aligned} X\mathbf{P}_n\mathbf{RE} - \mathbf{P}_n\mathbf{RE}X &= (X\mathbf{P}_n - \mathbf{P}_n X)\mathbf{RE} + \mathbf{P}_n(X\mathbf{RE} - \mathbf{RE}X) \\ &= (X\mathbf{P}_n - \mathbf{P}_n X)\mathbf{RE} + \mathbf{P}_n\mathbf{R}(X\mathbf{E} - \mathbf{E}X), \end{aligned}$$

which is smoothing of order 1.

Case 3: Two normal derivatives. The boundedness of $\frac{\partial^2}{\partial \rho^2}\mathbf{G}$ on $\mathrm{bmo}_r(\Omega)$ follows from the previous two cases by writing $\frac{\partial^2}{\partial \rho^2}$ in terms of the Laplacian, purely tangential derivatives, mixed tangential and normal derivatives, and first order terms. The Laplacian composed with $\mathbf{G}$ gives the identity, while the first order derivatives of $\mathbf{G}$ are smoothing of order 1, and are therefore bounded on $\mathrm{bmo}_r(\Omega)$ by Lemma 5.10. Finally, Lemma 5.11 guarantees that multiplication by smooth coefficients does not affect the boundedness on $\mathrm{bmo}_r(\Omega)$.

Having proved these four cases, we have concluded the proof of Theorem 5.8. □

Proof of Theorem 5.9. The proof follows the same lines as for the Dirichlet problem. After localizing, we reduce to the problem of showing that the operator $XY\widetilde{\mathbf{G}}$ is bounded on $\mathrm{bmo}_r(\Omega)$, where $\widetilde{\mathbf{G}}$ is the solution operator for the Neumann problem, and X and Y are vector fields in $\overline{\Omega}$. Case 0, where X and Y are supported in the interior of Ω, is exactly the same as for the Dirichlet problem, using the interior regularity for the Neumann problem. Thus we can proceed to the cases where X and Y are supported in a coordinate neighborhood of a point on the boundary.

In the case where both X and Y are tangential, we use the h^1_z regularity of the Neumann problem to show that the operator $\widetilde{\mathbf{G}}XY$ is bounded on $\mathrm{bmo}_r(\Omega)$. Thus it remains to show the boundedness of the operator $XY\widetilde{\mathbf{G}} - \widetilde{\mathbf{G}}XY$, which we again do using Lemma 5.10, i.e. by showing it is a smoothing operator of order 1.

As in the proof of Claim 4.4 in Section 4, we write

$$\widetilde{\mathbf{G}} = \mathbf{E} + \widetilde{\mathbf{H}} + S,$$

where $\widetilde{\mathbf{H}} = -\mathbf{P}\left(Q\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E}\right)$, S is a smoothing error, and Q is smoothing of order 1 on the boundary. We want to look at $XY\widetilde{\mathbf{G}} - \widetilde{\mathbf{G}}XY$, where X and Y are tangential vector fields. Again, the commutator with $\mathbf{E}$ gives us an operator of one degree of smoothing, and similarly for the smoothing error S. Thus we need only concern ourselves with $\widetilde{\mathbf{H}}$.

Write

$$\begin{aligned} XY\widetilde{\mathbf{H}} - \widetilde{\mathbf{H}}XY &= -XY\mathbf{P}\left(Q\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E}\right) + \mathbf{P}\left(Q\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E}\right)XY \\ &= -(XY\mathbf{P} - \mathbf{P}XY)Q\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E} - \mathbf{P}(XYQ - QXY)\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E} \\ &\quad -\mathbf{P}Q\mathbf{R}\left(XY\frac{\partial}{\partial\vec{n}}\mathbf{E} - \frac{\partial}{\partial\vec{n}}\mathbf{E}XY\right), \end{aligned}$$

where we have again used the fact that $\mathbf{R}$ commutes with X and Y. From equation 5.7 above, we get that the first term on the right-hand side can be written as

$$(\mathbf{P}'X + \mathbf{P}''Y + \mathbf{P}''')Q\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E}.$$

Since $\frac{\partial}{\partial\vec{n}}\mathbf{E}$ is smoothing of order 1, XQ, YQ and Q are operators of order 0 (or better) on $\partial\Omega$, and $\mathbf{P}'$, $\mathbf{P}''$ and $\mathbf{P}'''$ are Poisson-type operators of order 0, we get that the first term is smoothing of order 1. Similarly, since $XYQ - QXY$ is of order 0 on $\partial\Omega$, the second term is of the same form, hence smoothing of order 1. In the third term, the commutator $XY\frac{\partial}{\partial\vec{n}}\mathbf{E} - \frac{\partial}{\partial\vec{n}}\mathbf{E}XY$ is of order 0, but Q is smoothing of order 1 on $\partial\Omega$, so again we get a term which is smoothing of order 1.

In the case of one tangential derivative (X) and one normal derivative (Y), denoting the operator $Y\widetilde{\mathbf{G}}$ by $\widetilde{T}_n$, and following the argument of Case 2 in the proof of Theorem 5.8 above, we have to show that the operator $X\widetilde{T}_n - \widetilde{T}_nX$ is smoothing of order 1. Again we only need to consider the commutator of X with

$$\frac{\partial}{\partial\rho}\widetilde{\mathbf{H}} = -\mathbf{P}_n\left(Q\mathbf{R}\frac{\partial}{\partial\vec{n}}\mathbf{E}\right),$$

where $\mathbf{P}_n$ is the normal derivative of the Poisson operator $\mathbf{P}$. The argument proceeds as for the Dirichlet problem.

Finally, the case of the two normal derivatives is also handled the same way as in the proof of Theorem 5.8, so that we can conclude the proof of Theorem 5.9. $\square$

6. Relations between the spaces

In this section we will discuss the relations between the two spaces $h^p_z(\overline{\Omega})$ and $h^p_d(\overline{\Omega})$, as well as their relations to the spaces $h^p_z(\Omega)$ and $h^p_r(\Omega)$ considered in [CKS].

We begin by comparing $h^p_z(\overline{\Omega})$ and $h^p_d(\overline{\Omega})$. To do so, we must consider elements of $h^p_z(\overline{\Omega})$ as distributions in $\mathcal{C}^{\infty\prime}_d(\overline{\Omega})$. That is, for $f \in h^p_z(\overline{\Omega})$, we can define a linear functional $\tilde{f}$ on $\mathcal{C}_d^\infty(\overline{\Omega})$ by

$$\langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle,$$

where $\varphi \in \mathcal{C}_d^\infty(\overline{\Omega})$, and $\tilde{\varphi} \in \mathcal{S}(\mathbf{R}^n)$ is an extension of φ. This is independent of the extension because f is supported on the compact set $\overline{\Omega}$, which also shows that $\tilde{f}$ must be continuous on $\mathcal{C}_d^\infty(\overline{\Omega})$. Thus $f \to \tilde{f}$ is a mapping from $h_z^p(\overline{\Omega})$ to $\mathcal{C}_d^{\infty\prime}(\Omega)$. The kernel of this map is

$$h_0^p(\partial\Omega) = \{f \in h^p(\mathbf{R}^n) : \ \langle f, \varphi \rangle = 0 \ \ \forall \varphi \in \mathcal{S} \text{ with } \varphi|_{\partial\Omega} = 0\},$$

which is the space of $h^p(\mathbf{R}^n)$ distributions supported on $\partial\Omega$ and having order zero in the normal direction. (This is not to be confused with any h^p space that can be defined on $\partial\Omega$ as a manifold.)

We can identify the quotient space $h_z^p(\overline{\Omega})/h_0^p(\partial\Omega)$ with a subspace of $\mathcal{C}_d^{\infty\prime}(\Omega)$. As such, we have the following characterization:

Proposition 6.1. *For $p < 1$,*

$$h_d^p(\overline{\Omega}) \cong h_z^p(\overline{\Omega})/h_0^p(\partial\Omega)$$

with comparable norms, i.e. there is a linear operator T from $h_z^p(\overline{\Omega})/h_0^p(\partial\Omega)$ onto $h_d^p(\overline{\Omega})$ such that for every equivalence class $[f] \in h_z^p(\overline{\Omega})/h_0^p(\partial\Omega)$,

$$\|T([f])\|_{h_d^p(\overline{\Omega})} \approx \inf\{\|f\|_{h_z^p(\overline{\Omega})} : f \in [f]\}.$$

Proof. As explained above, since $h_0^p(\partial\Omega)$ is the kernel of the map $f \to \tilde{f}$ from $h_z^p(\overline{\Omega})$ into $\mathcal{C}_d^{\infty\prime}(\overline{\Omega})$, we can define T from $h_z^p(\overline{\Omega})$ by $T([f]) = \tilde{f}$, and this is independent of the choice of the representative f. Thus to show that the image of this map is contained $h_d^p(\overline{\Omega})$, it suffices to show that for every $f \in h_z^p(\overline{\Omega})$, $\tilde{f} \in h_d^p(\overline{\Omega})$ and

$$\|\tilde{f}\|_{h_d^p} \leq \|f\|_{h_z^p}.$$

This can be seen in two ways. Using the maximal function definitions (as in Section 1), we see that by extending every normalized $\mathcal{C}_d^\infty(\overline{\Omega})$ bump function to a normalized bump function in $\mathcal{D}(\mathbf{R}^n)$, we have that

$$m_d(\tilde{f})(x) \leq m(f)(x)$$

for all $x \in \Omega$; hence $\tilde{f} \in h_d^p(\overline{\Omega})$ with

$$\|\tilde{f}\|_{h_d^p} = \|m_d(\tilde{f})\|_{L^p(\Omega)} \leq \|m(f)\|_{L^p(\mathbf{R}^n)} = \|f\|_{h_z^p}.$$

Alternatively, using the atomic decompositions, it is enough to note that for an h_z^p atom a, $\tilde{a} = a$ is already an $h_d^p(\overline{\Omega})$ atom.

Conversely, to show that T is onto and the bounds on the norms can be reversed, we have to show that if $g \in h_d^p(\overline{\Omega})$, we can write $g = \tilde{f}$ for some $f \in h_z^p(\overline{\Omega})$, and

$$\|g\|_{h_d^p} \geq C \inf_{\tilde{f}=g} \|f\|_{h_z^p}.$$

As in Theorem 2.6, write $g = \sum \lambda_j a_j$, where a_j are h_d^p atoms. Suppose we can show that every $a_j = \tilde{f}_j$ for some $f_j \in h_z^p(\overline{\Omega})$ with $\|f_j\|_{h_z^p} \leq C$. Then setting $f = \sum \lambda_j f_j$ in $\mathcal{S}'(\mathbf{R}^n)$, we get that $f \in h_z^p(\overline{\Omega})$ and $\tilde{f} = \sum \lambda_j \tilde{f}_j = g$, since the map $f \to \tilde{f}$ is continuous on $\mathcal{S}'(\mathbf{R}^n)$. Furthermore,

$$\|f\|_{h_z^p} \leq C \left(\sum |\lambda_j|^p\right)^{1/p}$$

and each atomic decomposition of g gives rises to such an f, so

$$\inf_{\tilde{f}=g} \|f\|_{h^p_z} \le C \inf \left(\sum |\lambda_j|^p\right)^{1/p} = C'\|g\|_{h^p_d}.$$

Now if a_j is a type (a) h^p_d atom, or an atom supported in a large cube, then it is also an h^p_z atom, so we can take $f_j = a_j$. Thus it remains to prove the following:

Lemma 6.2. *Suppose $p < 1$. If a is a type* (b) *h^p_d atom with a small supporting cube (i.e. $|Q| < C_\Omega$), then there exists $f \in h^p_z(\overline{\Omega})$ such that $f - a \in h^p_0(\partial\Omega)$ and*

$$\|f\|_{h^p_z(\overline{\Omega})} \le C,$$

with C independent of a.

In particular, this shows that $a = \tilde{f}$, since

$$\langle \tilde{f} - a, \varphi\rangle = \langle f - a, \tilde{\varphi}\rangle = 0$$

for all $\varphi \in \mathcal{C}^\infty_d(\overline{\Omega})$.

Proof of lemma. Since a is supported in $Q \cap \overline{\Omega}$, where the cube Q is small and near the boundary, we can find a neighborhood U of $\partial\Omega$ containing Q, in which there is defined a smooth projection $\pi : U \to \partial\Omega$.

Define a linear functional Λ on $\phi \in \mathcal{S}(\mathbf{R}^n)$ by

$$\langle \Lambda, \phi\rangle = \int a(y)\phi(\pi(y))dy.$$

Since π is smooth, Λ is continuous, hence a tempered distribution. Furthermore, $\langle \Lambda, \varphi\rangle = 0$ whenever $\varphi|_{\partial\Omega} = 0$.

We claim that $\Lambda \in h^p(\mathbf{R}^n)$. To see this, we look at the grand maximal function $m(\Lambda)$, as in Definition 1.1. Take $x \in \mathbf{R}^n$, and let ϕ^x_t be a normalized bump function supported in a ball of radius $R \le 1$ containing x. Then

$$\begin{aligned} |\langle \Lambda, \phi^x_t\rangle| &= \left|\int_Q a(y)\phi^x_t(\pi(y))dy\right| \\ &\le |Q|^{-1/p}\|\phi^x_t\|_\infty |\{y \in \operatorname{supp}(a) : \pi(y) \in \operatorname{supp}(\phi^x_t)\}| \\ &\le C|Q|^{-1/p}\|\phi^x_t\|_\infty \operatorname{diam}(Q) t^{n-1} \\ &\le C'|Q|^{-1/p+1/n} t^{-1} \\ &\le C''|Q|^{-1/p+1/n}(\operatorname{dist}(x, \partial\Omega))^{-1} \end{aligned}$$

since we must have $2t \ge \operatorname{dist}(x, \partial\Omega)$ in order for $\phi^x_t(\pi(y)) \ne 0$ for some y.

If we now vary ϕ^x_t, we see that the maximal function $m(\Lambda)$ is supported in a tubular neighborhood of "radius" 2 around $\partial\Omega$. Thus

$$\begin{aligned} \int_{\mathbf{R}^n} m(\Lambda)(x)^p dx &\le C|Q|^{-1+p/n} \int_{\{x:\operatorname{dist}(x,\partial\Omega)\le 2\}} (\operatorname{dist}(x, \partial\Omega))^{-p} dx \\ &\le C_{Q,\Omega} \end{aligned}$$

since $p < 1$ and Ω is bounded. Thus $\Lambda \in h^p(\mathbf{R}^n)$, and since it is supported on $\partial\Omega$ and has order zero, $\Lambda \in h^p_0(\partial\Omega)$. Note that the h^p norm of Λ depends on a, and may blow up as the supporting cube of a shrinks.

Now let $f = a - \Lambda$ in $\mathcal{S}'(\mathbf{R}^n)$. Since $a \in h^p(\mathbf{R}^n)$ and $\Lambda \in h^p(\mathbf{R}^n)$, we have $f \in h^p(\mathbf{R}^n)$. Also, $f = 0$ on $\mathbf{R}^n \setminus \overline{\Omega}$, so $f \in h^p_z(\overline{\Omega})$. We want to show its norm is bounded by a constant independent of a.

Let us compute the local grand maximal function $m(f)$. For $x \in 2Q$, write

$$m(f)(x) \leq m(a)(x) + m(\Lambda)(x).$$

Now we know from the size condition on a and the L^2 boundedness of the maximal function that

$$\begin{aligned} \int_{2Q} m(a)^p &\leq C\left(\int_{2Q} |a|^2\right)^{p/2} |Q|^{1-p/2} \\ &\leq C. \end{aligned}$$

As for Λ, we have from above that

$$\begin{aligned} \int_{2Q} m(\Lambda)^p(x)dx &\leq C|Q|^{-1+p/n} \int_{2Q} (\operatorname{dist}(x, \partial\Omega))^{-p} dx \\ &\leq C'|Q|^{-1+p/n} \operatorname{diam}(Q)^{1-p} \operatorname{diam}(Q)^{n-1} \\ &\leq C''. \end{aligned}$$

When $x \notin 2Q$, take ϕ_t^x as above and write

$$\begin{aligned} \langle f, \phi_t^x \rangle &= \int_Q a(y)\phi_t^x(y)dy - \int_Q a(y)\phi_t^x(\pi(y))dy \\ &= \int_Q a(y)[\phi_t^x(y) - \phi_t^x(\pi(y))]dy. \end{aligned}$$

But the function $\varphi_t^x(y) = \phi_t^x(y) - \phi_t^x(\pi(y))$ is in $\mathcal{C}_d^\infty(\overline{\Omega})$ (where we might have to use a cutoff function supported in U in order to define φ for all y). Thus by the moment conditions on a,

$$\begin{aligned} |\langle f, \phi_t^x \rangle| &\leq \|\varphi_t^x\|_{\mathcal{C}^{N_p+1}(Q)} |Q|^{\nu_p} \\ &\leq C_\pi \|\phi_t^x\|_{\mathcal{C}^{N_p+1}(Q)} |Q|^{\nu_p} \\ &\leq C t^{-n-(N_p+1)} |Q|^{\nu_p} \\ &\leq C'(\operatorname{dist}(x, Q))^{-n-(N_p+1)} |Q|^{(N_p+1)/n+1-1/p}. \end{aligned}$$

Note that for $\langle f, \varphi_t^x \rangle \neq 0$ we must have either $2t \geq \operatorname{dist}(x, Q)$ or $2t \geq \operatorname{dist}(x, \pi(Q))$, and since $x \notin 2Q$ and Q is a type (b) cube, this gives $t \geq C \operatorname{dist}(x, Q)$ for some constant C.

Thus

$$\begin{aligned} \int_{x \notin 2Q} m(f)(x)^p dx &\leq C|Q|^{p(N_p+1)/n+p-1} \int_{x \notin 2Q} (\operatorname{dist}(x, Q))^{-np-(N_p+1)p} \\ &\leq C'|Q|^{p(N_p+1)/n+p-1} \operatorname{diam}(Q)^{n-np-(N_p+1)p} \\ &\leq C'' \end{aligned}$$

since $np + (N_p + 1)p > np + n(1/p - 1)p = n$.

We have now shown that

$$\|f\|_{h_z^p} = \|m(f)\|_{L^p(\mathbf{R}^n)} \leq C,$$

which concludes the proof of Lemma 6.2. □

By the atomic decomposition, this also completes the proof of Proposition 6.1. □

Note that Proposition 6.1 applies only to $p < 1$. For the case $p = 1$, the situation is different. Since elements of $h^1(\mathbf{R}^n)$ are functions, the space $h^1_0(\partial\Omega)$ consists only of the zero function. Thus the quotient space $h^1_z(\overline{\Omega})/h^1_0(\partial\Omega)$ is just $h^1_z(\overline{\Omega})$, the h^1 functions supported on $\overline{\Omega}$. We claim that this space is different from $h^1_d(\overline{\Omega})$.

By the atomic decomposition (Theorem 2.6), we know that the elements of $h^1_d(\overline{\Omega})$ are also functions (since the sums converge in L^1). We want to characterize these functions.

In order to do this, we recall the space $h^p_r(\Omega)$, defined in [M] and [CKS]. Let $\mathcal{D}(\Omega)$ denote the space of $\mathcal{C}^\infty$ functions with compact support in Ω, and let $\mathcal{D}'(\Omega)$ denote its dual, the space of distributions on Ω.

Definition 6.3. The space $h^p_r(\Omega)$ consists of elements of $\mathcal{D}'(\Omega)$ which are the restrictions to Ω of elements of $h^p(\mathbf{R}^n)$, i.e.

$$h^p_r(\Omega) = h^p(\mathbf{R}^n)/\{f \in h^p(\mathbf{R}^n) : f = 0 \text{ on } \Omega\},$$

equipped with the quotient norm

$$\|[f]\|_{h^p_r(\Omega)} = \inf\{\|f\|_{h^p(\mathbf{R}^n)} : f \in [f]\}.$$

Again it is clear that when $p = 1$, this is a space of functions, since elements of $h^1_r(\Omega)$ are just restrictions to Ω of functions in $h^1(\mathbf{R}^n)$. Thus we can state:

Proposition 6.4. *For* $\frac{n}{n+1} < p \leq 1$,

$$h^p_d(\overline{\Omega}) \cong h^p_r(\Omega).$$

Furthermore, considered as subspaces of $L^1(\Omega)$, *we have*

$$h^1_z(\overline{\Omega}) \subsetneq h^1_d(\overline{\Omega}) = h^1_r(\Omega).$$

Proof. The equivalence of $h^p_d(\overline{\Omega})$ and $h^p_r(\Omega)$ follows immediately from the atomic decompositions. From [M] (Theorem 1) and [CKS] (Theorem 2.7), we know that every $f \in h^p_r(\Omega)$ has a decomposition

$$f = \sum \lambda_j a_j$$

in $\mathcal{D}'(\Omega)$, where the a_j are either $h^p(\mathbf{R}^n)$ atoms supported in type (a) cubes, or atoms supported in type (b) cubes and satisfying a size condition, but no moment condition. Note that these are essentially type (a) and type (b) h^p_d atoms, since when $\frac{n}{n+1} < p \leq 1$, the moment conditions 2.3 in Definition 2.3 are null (they follow from the size condition by expanding the $\mathcal{C}^\infty_d$ test function around a point on the boundary). The only difference is that in both [M] and [CKS], type (b) cubes are assumed to be contained in Ω, and in fact their distance from the boundary is assumed to be proportional to their diameter. However, because of the lack of moment conditions, it can be shown that a type (b) h^p_d atom can be decomposed into type (b) h^p_r atoms, as in [M] and [CKS]—see [CKS], pp. 295-296, where this is done for $\mathbf{R}^n_+$.

Note that for this range of p, $h^p_r(\Omega)$ distributions can be applied to test functions with only one order of vanishing at the boundary, and therefore can be considered as distributions in ${\mathcal{C}^\infty_d}'(\overline{\Omega})$. Also for this range of p, the convergence of the atomic decompositions in $\mathcal{D}'(\Omega)$ and in ${\mathcal{C}^\infty_d}'(\overline{\Omega})$ is equivalent. Thus we have the same atomic decomposition for $h^p_r(\Omega)$ and $h^p_d(\overline{\Omega})$, considered as subspaces of ${\mathcal{C}^\infty_d}'(\overline{\Omega})$, which means that these two spaces are the same, with equivalent norms.

The fact that $h^1_z(\overline{\Omega})$ is a subset $h^1_r(\Omega)$ is obvious, since every function $f \in h^1_z(\overline{\Omega})$ can be extended by 0 to a function in $h^1(\mathbf{R}^n)$, and the norm is the same. Thus $f \in h^1_r(\Omega)$ and

$$\|f\|_{h^1_r} \leq \|f\|_{h^1_z},$$

since the h^1_r norm is the infimum of the $h^1(\mathbf{R}^n)$ norms of all possible extensions of f.

To see that $h^1_z(\overline{\Omega})$ is a strict subset of $h^1_r(\Omega)$, we will construct a function $f \in h^1_r(\Omega) \setminus h^1_z(\overline{\Omega})$. By translating and rescaling, we may assume $0 \in \partial\Omega$ and the cone

$$\Gamma = \{x = (x', x_n) : 2|x'| \leq x_n, |x_n| \leq 1\} \subset \Omega.$$

For $j \geq 1$, set

$$Q_j = [-2^{-j-1}, 2^{-j-1}]^{n-1} \times [2^{-j}, 2^{-j+1}],$$

and

$$a_j = |Q_j|^{-1} \chi_{Q_j}.$$

Then a_j are type (b) h^1_r atoms. Let

$$f = \sum_{j=1}^{\infty} \frac{1}{j^2} a_j.$$

Then $f \in h^1_r(\Omega)$.

It is also easy to see that $m(f)(x) \geq \frac{c}{|x|^n \log(1/|x|)}$, for $x \in \Gamma$, $|x| < 1/2$, and hence, $f \notin h^1_z(\Omega)$. This concludes the proof of Proposition 6.4. $\square$

We may now ask what are the relations between the spaces $h^p_z(\overline{\Omega})$ and $h^p_d(\overline{\Omega})$ and the spaces $h^p_r(\Omega)$ when p is small. Recall that in [CKS], in order to relate h^p_z to h^p_r, the quotient space

$$h^p_z(\Omega) = h^p_z(\overline{\Omega}) / \{f \in h^p_z(\overline{\Omega}) : f = 0 \text{ on } \Omega\}$$

was introduced (see Remark 2 following Definition 1.2.) Note that in this case, the null space

$$h^p(\partial\Omega) = \{f \in h^p_z(\overline{\Omega}) : f = 0 \text{ on } \Omega\}$$

is the whole space of h^p distributions supported on $\partial\Omega$, which is strictly larger than $h^p_0(\partial\Omega)$ when $p < \frac{n}{n+1}$.

We can identify $h^p_z(\Omega)$ with a set of distributions in $\mathcal{D}'(\Omega)$, and equipped with the quotient norm, it is clearly a subspace of $h^p_r(\Omega)$.

By analogy, we can define

$$h^p_d(\Omega) = h^p_d(\overline{\Omega}) / \{f \in h^p_d(\overline{\Omega}) : f = 0 \text{ on } \Omega\},$$

where $f = 0$ on Ω for $f \in \mathcal{C}^{\infty\prime}_d$ means $\langle f, \varphi\rangle = 0$ whenever $\varphi \in \mathcal{C}^\infty_d(\overline{\Omega})$ has compact support in Ω. In other words, the null space is the space of distributions in $h^p_d(\overline{\Omega})$ which are supported on $\partial\Omega$.

Again it is easy to see that elements of $h^p_d(\Omega)$ can be identified with distributions on Ω, since every $f \in \mathcal{C}^{\infty\prime}_d(\overline{\Omega})$ immediately induces a distribution in $\mathcal{D}'(\Omega)$, which is zero precisely when $\langle f, \varphi\rangle = 0$ for every smooth φ with compact support in Ω.

Proposition 6.5. *Considered as subspaces of $\mathcal{D}'(\Omega)$, when $p < 1$,*

$$h^p_z(\Omega) = h^p_d(\Omega) \subset h^p_r(\Omega).$$

Moreover, for $N = 0, 1, 2, \ldots$ and $\frac{n}{n+N+1} < p < \frac{n}{n+N}$,

$$h^p_z(\Omega) = h^p_r(\Omega).$$

Proof. The fact that $h^p_z(\Omega) = h^p_d(\Omega)$ as spaces of distributions on Ω follows almost immediately from Proposition 6.1 and its proof. For consider an equivalence class $[f] \in h^p_z(\Omega)$. Taking a representative $f \in [f]$, $f \in h^p_z(\overline{\Omega})$, we can map f to $\tilde{f} \in h^p_d(\overline{\Omega})$, with

$$\|\tilde{f}\|_{h^p_d(\overline{\Omega})} \leq \|f\|_{h^p_z(\overline{\Omega})}.$$

But if we also have $g \in [f]$, then $f = g$ in $\mathcal{D}'(\Omega)$; hence $\tilde{f} = \tilde{g}$ in $\mathcal{D}'(\Omega)$. Thus we can map $[f]$ to the equivalence class $[\tilde{f}]$ in $h^p_d(\Omega)$, and

$$\|[\tilde{f}]\|_{h^p_d(\Omega)} = \inf_{h \in \tilde{f}} \|h\|_{h^p_d(\overline{\Omega})} \leq \inf_{g \in [f]} \|g\|_{h^p_z(\overline{\Omega})} = \|[f]\|_{h^p_z(\Omega)}.$$

This shows $h^p_z(\Omega) \subset h^p_d(\Omega)$ with a bound on the norms.

Conversely, we show in the proof of Proposition 6.1 that every $f \in h^p_d(\overline{\Omega})$ can be decomposed in $\mathcal{C}^{\infty}_d{}'(\overline{\Omega})$ as $g + b$, where $g \in h^p_z(\overline{\Omega})$, $b \in h^p_0(\partial\Omega)$, and the norm $\|f\|_{h^p_d}$ is equivalent to the infimum of the norms $\|g\|_{h^p_z}$ over all such decompositions. Since this decomposition holds a fortiori in $\mathcal{D}'(\Omega)$, we get that every equivalence class $[f] \in h^p_d(\Omega)$ is the image of an equivalence class $[g] \in h^p_z(\Omega)$, with

$$\|f\|_{h^p_d(\Omega)} \geq C\|[g]\|_{h^p_z(\Omega)}.$$

Thus we have shown that $h^p_z(\Omega) = h^p_d(\Omega)$ with equivalent norms.

Since we have already pointed out that $h^p_z(\Omega) \subset h^p_r(\Omega)$, we now get that $h^p_d(\Omega) \subset h^p_r(\Omega)$. This can also be seen from the atomic decomposition, since type (b) h^p_d atoms are essentially (or can be decomposed into) type (b) h^p_r atoms—see the proof of Proposition 6.4.

For the case $\frac{n}{n+N+1} < p < \frac{n}{n+N}$, in light of the atomic decomposition for h^p_r, it suffices to prove that every h^p_r atom is in $h^p_z(\Omega)$, with bounded norm. This is obvious for atoms supported in large cubes and for type (a) atoms, since they are already h^p_z atoms. Thus it remains to show the following:

Lemma 6.6. *Suppose $\frac{n}{n+N+1} < p < \frac{n}{n+N}$. Let a be a type* (b) *h^p_r atom supported in a cube Q with $|Q| < 1$. Then in $\mathcal{D}'(\Omega)$,*

$$a = \sum \lambda_j a_j,$$

where a_j are h^p_z atoms, and

$$\sum |\lambda_j|^p \leq C,$$

where the constant C is independent of a.

Proof of lemma. By choosing appropriate coordinates, we can assume $0 \in \partial\Omega$ and a is supported in

$$Q = [-\delta, \delta]^{n-1} \times [(A-1)\delta, (A+1)\delta]$$

with $\delta < 1$ and $A \geq 1$.

Note that $N = N_p$, so that the a_j (when supported in small cubes) need to satisfy all moment conditions up to order N. We proceed by induction.

Let $K \leq N$, and assume a has vanishing moments up to order $K-1$ (no moment condition is assumed when $K = 0$). We will show that we can write $a = \sum \lambda_j a_j$ with a_j having vanishing moments up to order K, and $\sum |\lambda_j|^p \leq C$. The a_j will also satisfy the size condition, and the supporting cubes Q_j will satisfy the conditions:

$$A'Q_j \subset \Omega$$

and

$$2A'Q_j \cap \partial\Omega \neq \emptyset,$$

with $A' = (3A+1)/(A+3) \geq 1$. Thus the new atoms will satisfy the induction hypothesis for the next step.

Let $b_0 = a$, and for $j = 1, 2, 3, \ldots$, define b_j by

$$b_j(x) = 2^{j(K+n)} a(2^j x).$$

Then b_j is supported in

$$\widetilde{Q_j} = [-2^{-j}\delta, 2^{-j}\delta]^{n-1} \times [2^{-j}(A-1)\delta, 2^{-j}(A+1)\delta].$$

Also,

$$\int_{\widetilde{Q_j}} b_j(x) x^\alpha dx = 2^{j(K-|\alpha|)} \int_Q a(x) x^\alpha dx,$$

so b_j satisfies the same moment conditions as a. Finally,

$$\|b_j\|_\infty \leq 2^{j(K+n)} |Q|^{-1/p}.$$

Now if we set

$$f_j = b_j - b_{j+1},$$

we have that f_j is supported in the smallest cube containing $\widetilde{Q_j}$ and $\widetilde{Q_{j+1}}$, namely

$$Q_j = [-2^{-(j+2)}(A+3)\delta, 2^{-(j+2)}(A+3)\delta]^{n-1} \times [2^{-(j+1)}(A-1)\delta, 2^{-j}(A+1)\delta],$$

f_j satisfies all the moment conditions that a satisfies, and in addition

$$\begin{aligned}
\int_{Q_j} f_j(x) x^\alpha dx &= \int_{\widetilde{Q_j}} b_j(x) x^\alpha dx - \int_{\widetilde{Q_{j+1}}} b_{j+1}(x) x^\alpha dx \\
&= 2^{j(K-|\alpha|)} \int_Q a(x) x^\alpha dx - 2^{(j+1)(K-|\alpha|)} \int_Q a(x) x^\alpha dx \\
&= 0
\end{aligned}$$

whenever $|\alpha| = K$. As for size,

$$\begin{aligned}
\|f_j\|_\infty &\leq 2^{j(K+n)} |Q|^{-1/p} \\
&= C_{n,p,A} 2^{j(K+n-n/p)} |Q_j|^{-1/p}.
\end{aligned}$$

Thus we can let

$$\lambda_j = C_{n,p,A} 2^{j(K+n-n/p)}$$

and

$$a_j = \lambda_j^{-1} f_j.$$

The a_j satisfy the conditions stated above, and

$$\sum |\lambda_j|^p = C'_{n,p,A} \sum_{j=0}^{\infty} 2^{j(K+n-n/p)p} \leq C$$

since $K \leq N < n/p - n$.

It remains to show $a = \sum \lambda_j a_j$ in $\mathcal{D}'(\Omega)$. However, since $a - \sum_0^J \lambda_j a_j = b_{J+1}$ is supported in $\widetilde{Q_{J+1}}$, which is contained in smaller and smaller neighborhoods of $\partial\Omega$ as J increases, we have

$$\lim_{J\to\infty} \langle a - \sum_0^J \lambda_j a_j, \psi \rangle \to 0$$

for every ψ with compact support in Ω (actually one can show that the vanishing of ψ up to order K at the boundary is sufficient). This completes the proof of the lemma. $\square$

As was pointed out, by the atomic decomposition, this also concludes the proof of Proposition 6.5. $\square$

The question arises as to the situation for $p = n/(n+N)$, $N = 0, 1, 2, \ldots$. We already saw in Proposition 6.4 that when $p = 1$, h^1_z is a strict subspace of $h^1_d = h^1_r$ (here the distinction between the spaces and their quotients is irrelevant). Thus we have the following:

Problem. *Let $p = n/(n+N)$, $N = 1, 2, \ldots$. Then considered as subspaces of $\mathcal{D}'(\Omega)$, is it true that $h^p_z(\Omega) \neq h^p_r(\Omega)$?*

In other words, the question is whether for the "critical" values of p, there exists a distribution f on Ω which is the restriction to Ω of an element of $h^p(\mathbf{R}^n)$, but not the restriction to Ω of any element of $h^p(\mathbf{R}^n)$ which is supported on $\overline{\Omega}$. Intuitively, the idea is that one cannot "correct" the lack of cancellation of f by a distribution supported on $\partial\Omega$. To answer the question, one needs to understand the behavior of the maximal function of distributions supported on $\partial\Omega$, and to be able to determine when such a distribution is in $h^p(\mathbf{R}^n)$. While several sufficient conditions are known in general (see [S2], 5.18), there is one case in which the necessary and sufficient conditions are well understood, which is the case where $\partial\Omega$ consists of a single point, i.e. when

$$\Omega = \mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}.$$

In this case, we denote by $\mathcal{S}(\mathbf{R}_+)$ the space of Schwartz functions with support in $\mathbf{R}_+$, and let $\mathcal{S}'(\mathbf{R}_+)$ be the dual space. The space $h^p_r(\mathbf{R}_+)$ is the subspace of $\mathcal{S}'(\mathbf{R}_+)$ consisting of restrictions of elements of $h^p(\mathbf{R})$ to $\mathbf{R}_+$. As above, $h^p_z(\overline{\mathbf{R}_+})$ is the space of elements of $h^p(\mathbf{R})$ which are supported on $\overline{\mathbf{R}_+}$.

Lemma 6.7. *Let $p = 1/(1+N)$, $N = 0, 1, 2, \ldots$. Then there exists a distribution $f \in h^p_r(\mathbf{R}_+)$ such that for every $g \in h^p_z(\overline{\mathbf{R}_+})$,*

$$g|_{\mathbf{R}_+} \neq f$$

in $\mathcal{S}'(\mathbf{R}_+)$.

Proof of lemma. We will give two examples, which are essentially two versions of the same example.

For the first example, we take

$$f_1(x) = \frac{d^{N+1}}{dx^{N+1}}\left[\left(\log\frac{1}{x}\right)^{-1/p}\eta(x)\right],$$

defined for $x > 0$, where η is a smooth function satisfying $\eta(x) = 1$ for $0 < x \leq 1/2$ and $\eta(x) = 0$ when $x \geq 3/4$. Being the derivative of a smooth function on $\mathbf{R}_+$ with bounded support, f_1 defines a distribution in $\mathcal{S}'(\mathbf{R}_+)$. Note that since $N+1 = 1/p$,

$$f_1(x) \approx \left(\log\frac{1}{x}\right)^{-1/p-1} x^{-1/p}\eta(x)$$

as $x \to 0$, so f_1 is not a distribution in $\mathcal{S}'(\mathbf{R})$. To extend f_1 to such a distribution, we let

$$h(x) = \begin{cases} \left(\log \frac{1}{x}\right)^{-1/p} \eta(x) & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

and define

$$F_1 = \frac{d^{N+1}}{dx^{N+1}} h$$

in the sense of distributions. Then $F_1 \in \mathcal{S}'(\mathbf{R})$ and $F_1|_{\mathbf{R}_+} = f_1$.

We build the second example by means of an atomic decomposition, analogous to the one constructed in the proof of Proposition 6.4. For $j \geq 1$, set

$$I_j = [2^{-j}, 2^{-j+1}],$$

$$\lambda_j = \frac{1}{j^{1+1/p}},$$

and

$$a_j = 2^{j/p} \chi_{I_j}.$$

Let

$$f_2 = \sum_{j=1}^{\infty} \lambda_j a_j.$$

Again f_2 defines a distribution in $\mathcal{S}'(\mathbf{R}_+)$, and for $x \approx 2^{-j}$,

$$f_2(x) \approx \frac{1}{j^{1+1/p}} 2^{j/p} \approx \left(\log \frac{1}{x}\right)^{-1/p-1} x^{-1/p}.$$

To extend f_2 to a distribution in $\mathcal{S}'(\mathbf{R})$, we define distributions τ_j by

$$\langle \tau_j, \varphi \rangle = \lambda_j \int a_j(x) \left[\varphi(x) - \sum_{k=0}^{N-1} \frac{1}{k!} \frac{d^k \varphi}{dx^k}(0) x^k \right] dx$$

for every $\varphi \in \mathcal{S}(\mathbf{R})$. Since

$$\begin{aligned} |\langle \tau_j, \varphi \rangle| &= \lambda_j 2^{j/p} \left| \int_{I_j} \left[\varphi(x) - \sum_{k=0}^{N-1} \frac{1}{k!} \frac{d^k}{dx^k} \varphi(0) x^k \right] dx \right| \\ &\leq \lambda_j 2^{j/p} \|\varphi\|_{\mathcal{C}^N} \int_{I_j} x^N \, dx \\ &\leq C \frac{1}{j^{1+1/p}} 2^{j(1/p-N-1)} \|\varphi\|_{\mathcal{C}^N} \\ &= C \frac{1}{j^{1+1/p}} \|\varphi\|_{\mathcal{C}^N}, \end{aligned}$$

we can define a distribution $F_2 \in \mathcal{S}'(\mathbf{R})$ by

$$F_2 = \sum_{j=1}^{\infty} \tau_j.$$

Again we have that

$$F_2|_{\mathbf{R}_+} = \sum_{j=1}^{\infty} \tau_j|_{\mathbf{R}_+} = \sum_{j=1}^{\infty} \lambda_j a_j = f_2.$$

We shall prove the result for f_2; there are parallel arguments for f_1.

Claim 6.8. f_2 belongs to $h^p_r(\mathbf{R}_+)$.

This is via the atomic decomposition which defines f_2.

Claim 6.9. The distribution F_2 does not belong to $h^p(\mathbf{R})$, since the local grand maximal function satisfies

$$m(F_2)(x) \approx \left(\log \frac{1}{x}\right)^{-1/p} x^{-1/p}$$

for $0 < x < 1/2$.

To get the lower bound for $m(F_2)$, take $0 < x < 1/2$. We will use a normalized bump function ϕ_x supported in the interval $(-x, 3x)$, with $\phi_x \geq 0$ on $(x, 3x)$ and

$$\phi_x(y) = \frac{y^N}{N!(2x)^{1+N}} \text{ for } y \in [0, x].$$

Note that this is consistent with the condition

$$|D^\alpha(\phi_x)| \leq (2x)^{-1-|\alpha|}$$

for $|\alpha| \leq N+1$. Furthermore,

$$\frac{d^k \phi_x}{dy^k}(0) = 0$$

for all $k \leq N-1$. Thus $\langle F_2, \phi_x \rangle = \sum \lambda_j \int a_j \phi_x$ and we can proceed as in the proof of Proposition 6.4 to get

$$\begin{aligned}
\langle F_2, \phi_x \rangle &\geq \sum_{I_j \subset [0,x]} \lambda_j \int_{I_j} a_j(y) \phi_x(y) dy \\
&= \frac{1}{N!(2x)^{1+N}} \sum_{I_j \subset [0,x]} \lambda_j \int_{I_j} a_j(y) y^N dy \\
&= \frac{2^{N+1}-1}{(N+1)!(2x)^{1+N}} \sum_{I_j \subset [0,x]} \lambda_j 2^{j(1/p-N-1)} \\
&\geq C_N x^{-1-N} \sum_{j \geq -\log_2 x} j^{-1/p-1} \\
&\approx x^{-1/p} \left(\log \frac{1}{x}\right)^{-1/p}.
\end{aligned}$$

This gives the lower bound on the maximal function of F_2.

Now we will prove the upper bound on the maximal function of F_2. Take $0 < x < 1/2$, and let φ_t^x be a bump function supported in a ball of radius $t \leq 1$ containing x.

Suppose first that $t \geq x/4$. If $2^{-j} > 6t \geq x + 2t$, then $\operatorname{supp}(\varphi_t^x) \cap I_j = \emptyset$, and we have

$$\begin{aligned}
|\langle \tau_j, \varphi_t^x \rangle| &= \left| \lambda_j 2^{j/p} \sum_{k \leq N-1} \frac{1}{k!} d^k \varphi_t^x(0) \int_{I_j} x^k dx \right| \\
&\leq C \lambda_j \sum_{k \leq N-1} \frac{1}{k!} t^{-1-k} 2^{j(1/p-1-k)},
\end{aligned}$$

while otherwise we use the estimate

$$|\langle \tau_j, \varphi_t^x \rangle| \leq C\lambda_j \|\varphi_t^x\|_{\mathcal{C}^N} \leq C\lambda_j t^{-1-N}.$$

Thus

$$\begin{aligned}
|\langle F_2, \varphi_t^x \rangle| &\leq C \sum_{k \leq N-1} \frac{1}{k!} t^{-1-k} \sum_{j < -A \log t} \frac{2^{j(N-k)}}{j^{1+1/p}} \\
&\quad + Ct^{-1-N} \sum_{j \geq -A \log t} \frac{1}{j^{1+1/p}} \\
&\leq C \sum_{k \leq N-1} \frac{1}{k!} t^{-1-k} t^{-(N-k)} (-\log t)^{-1/p} \\
&\quad + Ct^{-1-N} (-\log t)^{-1/p} \\
&\leq Ct^{-1/p} \left(\log \frac{1}{t} \right)^{-1/p} \\
&\leq Cx^{-1/p} \left(\log \frac{1}{x} \right)^{-1/p}
\end{aligned}$$

since $(x/t)^{1/p} (\log x / \log t)^{1/p}$ remains bounded when $x \ll t$.

If $t \leq x/4$, we have that φ_t^x vanishes in a neighborhood of 0, and

$$\begin{aligned}
|\langle F_2, \varphi_t^x \rangle| &\leq C \sum_{j \leq -A \log x} \lambda_j \left| \int a_j \varphi_t^x \right| \\
&\leq C \sum_{j \leq -A \log x} \frac{1}{j^{1+1/p}} 2^{j/p} \|\varphi_t^x\|_{L^1} \\
&\leq Cx^{-1/p} \left(\log \frac{1}{x} \right)^{-1/p}.
\end{aligned}$$

Thus we have shown that for $0 < x < 1/2$,

$$m(F_2)(x) \approx x^{-1/p} \left(\log \frac{1}{x} \right)^{-1/p}.$$

Claim 6.10. Then there is no distribution $g \in h^p(\mathbf{R})$ with $g = 0$ on $\mathbf{R}_-$ and $g = f_2$ on $\mathbf{R}_+$.

Recall that F_2 is supported on $\overline{\mathbf{R}_+}$ and $F_2 = f_2$ on $\mathbf{R}_+$. Thus if g is any distribution in $\mathcal{S}'(\mathbf{R})$ with $g = 0$ on $\mathbf{R}_-$ and $g = f_2$ on $\mathbf{R}_+$, then $g - F_2$ is supported on $\{0\}$. Since tempered distributions are of finite order, we can write (see [R], Theorem 6.25)

$$g - F_2 = \sum_{k=0}^{K} c_k \delta^{(k)}(0),$$

where $\delta^{(k)}(0)$ denotes the k-th derivative of the delta function at 0. We may assume $c_K \neq 0$. Then if $K \geq N$, we have

$$m(g - F_2) \geq C|x|^{-N-1}$$

as $x \to 0$, and so by the upper bounds on $m(F)$,

$$m(g) \geq C|x|^{-N-1} = C|x|^{-1/p}$$

for x small, and $m(g) \notin L^p(\mathbf{R})$.

On the other hand, if $K < N$, then

$$m(g - F_2) \leq C|x|^{-N} = C|x|^{1-1/p}$$

as $x \to 0$, so by the lower bounds on $m(F_2)$,

$$m(g) \geq C|x|^{-1/p} \left(\log \frac{1}{x}\right)^{-1/p},$$

for x small, and again $m(g) \notin L^p(\mathbf{R})$.

This completes the proof of the lemma. $\square$

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