TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 4, April 1999, Pages 1445–1460
S 0002-9947(99)02389-2

C^*-ALGEBRAS GENERATED BY A SUBNORMAL OPERATOR

KIT C. CHAN AND ŽELJKO ČUČKOVIĆ

ABSTRACT. Using the functional calculus for a normal operator, we provide a result for generalized Toeplitz operators, analogous to the theorem of Axler and Shields on harmonic extensions of the disc algebra. Besides that result, we prove that if T is an injective subnormal weighted shift, then any two nontrivial subspaces invariant under T cannot be orthogonal to each other. Then we show that the C^*-algebra generated by T and the identity operator contains all the compact operators as its commutator ideal, and we give a characterization of that C^*-algebra in terms of generalized Toeplitz operators. Motivated by these results, we further obtain their several-variable analogues, which generalize and unify Coburn's theorems for the Hardy space and the Bergman space of the unit ball.

1. INTRODUCTION

On a separable Hilbert space H, a bounded linear operator $S : H \to H$ is said to be a *subnormal operator* if there exists a Hilbert space L containing H and a normal operator $N : L \to L$ such that H is invariant under N and the restriction $N|_H$ is S. If H reduces N, then S is actually normal. If there is no proper subspace of L that contains H and reduces N, then N is called a minimal normal extension of S. In this paper we discuss the C^*-algebra, denoted by $C^*(S)$, generated by S and the identity operator I on H, by using the functional calculus for a minimal normal extension N of S. In particular, we make connection with the theorem of Axler and Shields [1] which says that the algebra generated by the disc algebra and a function that is harmonic but not analytic on the disc is the algebra of all continuous functions on the closed disc. When S is an injective subnormal weighted shift operator, we completely characterize the C^*-algebra $C^*(S)$ in terms of the continuous functions on the spectrum of N. Using this characterization we can determine when an operator in $C^*(S)$ is compact.

C. A. Berger has shown that every subnormal weighted shift T is unitarily equivalent to the operator M_z on certain Hilbert space of analytic functions. Based on this unitary equivalence R. E. Frankfurt has studied a lot of properties of T in his papers [10], [11], [12]. In Sections 2 and 3, we further develop the theory of a subnormal weighted shift based on Berger's result. Then in Section 4, we discuss the C^*-algebra $C^*(T)$. Most of the techniques that we use in that section call for

Received by the editors December 4, 1995 and, in revised form, March 4, 1998.
1991 *Mathematics Subject Classification.* Primary 47B20, 32A37, 46L05; Secondary 46E20, 47A13, 47B37.
Key words and phrases. C^*-algebras, subnormal operators, weighted shifts, functional calculus, analytic functions, joint spectrum, Toeplitz operators.

the theory of a single subnormal operator, and a minimal normal extension of it. A good reference for this subject is found in Conway's book [6].

In Section 5 we generalize our results to their n-variable analogues. In particular, we show that if U denotes the C^*-algebra generated by the identity and the Toeplitz operators whose symbols are the coordinate functions, and if K denotes the ideal of compact operators, then U contains K and U/K is isometrically $*$-isomorphic to the algebra of continuous functions on the unit sphere. To show that we can employ only some of the techniques that appear in Sections 2, 3, and 4, and we need more techniques from the theory of C^*-algebras and operator theory. The major difference is that when we study the n-variable analogues, we have to deal with n commuting subnormal operators. However, it was shown by A. Lubin [19] that there exist commuting subnormal n-variable shift operators that do not have commuting normal extensions.

2. Subnormal unilateral shifts

One of the most studied classes of linear operators on a separable Hilbert space is the weighted shifts; see the survey article on this topic by Allen Shields [24]. These operators are important because they provide numerous interesting examples in operator theory, and have a strong connection with analytic function theory. The connection is even stronger when the shifts are subnormal, and in this section we make use of the connection to prove that any two nontrivial invariant subspaces of an injective subnormal unilateral weighted shift cannot be orthogonal to each other.

A linear operator T on an infinite dimensional separable complex Hilbert space H is called a *unilateral weighted shift* with *weight sequence* $\{w_k : k \geq 0\}$ if there is an orthonormal basis $\{e_k : k \geq 0\}$ of H such that

$$Te_k = w_k e_{k+1},$$

for all integers $k \geq 0$. It is known that, from [24, Page 52], T is unitarily equivalent to the unilateral weighted shift with nonnegative weight sequence $\{|w_k| : k \geq 0\}$. In addition, one can easily check that T is injective if and only if every weight w_n is nonzero. In this paper we consider only those weighted shifts that are injective, and so in the rest of this paper, when we say T is a *unilateral shift* we always mean that T is an injective unilateral weighted shift. Equivalently, we always make the following assumption:

Assumption. $w_k > 0$ for all integers $k \geq 0$.

When the unilateral shift T is subnormal, T has a nice connection with function theory, due to an unpublished result of C. A. Berger, quoted as Theorem 1 below. To explain that connection we require the following notation. For a compactly supported Borel measure μ on the complex plane $\mathbf{C}$, we use $P^2(\mu)$ to denote the closure of the polynomials in $L^2(\mu)$. On $P^2(\mu)$ we define the linear operator of multiplication by the position function z as $M_z : P^2(\mu) \to P^2(\mu)$ such that $M_z f = zf$.

Theorem 1 (Berger)**.** *If T is an injective unilateral shift with $\|T\| = 1$, then the following are equivalent:*
(a) T is subnormal.

(b) There is a positive Borel measure ν on the closed interval $[0,1]$, with $\nu[0,1]=1$ and with 1 in the support of ν, such that if $d\mu = d\theta d\nu/2\pi$, then T is unitarily equivalent to M_z on $P^2(\mu)$.

An interested reader can find Theorem 1 in [6, Theorem 8.16, Page 159], or a more descriptive version in [10, Theorem 8]. With the measures ν and μ as stated in the theorem, we point out that for $m \neq k$,

$$\int_{[0,1]} \int_0^{2\pi} z^m \overline{z^k} \, \frac{d\theta d\nu}{2\pi} = 0.$$

Thus the sequence $\{z^k\}_0^\infty$ is an orthogonal basis of $P^2(\mu)$. If we let $\|\cdot\|$ denote the norm of $P^2(\mu)$, and let $\beta_k = \|z^k\|$, then the sequence $\{z^k/\beta_k\}$ is an orthonormal basis of $P^2(\mu)$. It follows that, by a Hilbert space argument, a function is in $P^2(\mu)$ if and only if it has a power series representation of the form

$$\sum_{k=0}^\infty a_k \frac{z^k}{\beta_k}$$

where the sequence $\{a_k\}$ satisfies $\sum |a_k|^2 < \infty$. Every such power series representation is analytic in the open unit disc Δ, as indicated in [6, Proposition 8.19, Page 163]. We now want to show a slightly stronger statement: *The largest open set in which all power series in $P^2(\mu)$ converge is exactly Δ.* This statement is now deduced by using [24, Theorem 10, Page 73]: *If we denote $R = \liminf \beta_k^{1/k}$, then the largest open disc in which all the power series in $P^2(\mu)$ converge is $\{z : |z| < R\}$.* To show that $R=1$, it suffices to show that $R \le 1$, which is obvious.

From now on we identify a subnormal unilateral shift with the multiplication operator M_z without further comment. Note that the class of spaces $P^2(\mu)$ includes the Hardy space H^2 and the Bergman space A^2.

Since every function f in $P^2(\mu)$ is analytic on Δ, we can evaluate f at a point ζ in Δ. It is known, from [6, Proposition 8.19, Page 163], that the linear functional $\lambda_\zeta : f \mapsto f(\zeta)$ is continuous on $P^2(\mu)$, and furthermore by the principle of uniform boundedness we deduce that if E is a compact subset of Δ, then $\sup\{\|\lambda_\zeta\| : \zeta \in E\}$ is finite. It follows that there exists a positive constant C, depending on E and μ, such that for all $\zeta \in E$ and all $f \in P^2(\mu)$,

$$|f(\zeta)| \le C\|f\|.$$

With this inequality, one can easily show the following proposition, using that $\nu[R,1] > 0$.

Proposition 2. *If $0 < R < 1$, then there exists a positive constant C, depending on R and the measure μ, such that for every function f in $P^2(\mu)$,*

$$\|f\|^2 \le C \int_{[R,1]} \int_0^{2\pi} |f|^2 \frac{d\theta}{2\pi} d\nu.$$

One can now see, from Proposition 2, that the norm of $P^2(\mu)$ is equivalent to the norm given by $\int_{[R,1]} \int_0^{2\pi} |f|^2 \frac{d\theta}{2\pi} d\nu$. With this equivalence we now identify the approximate point spectrum $\sigma_{ap}(T)$ of a subnormal unilateral shift T. Without doubt this is well-known, but we present it in the next proposition because the techniques we use in its proof are also important to our work later in the present paper. If $\|T\| = 1$, then it is known, from [6, Page 158], that the spectrum $\sigma(T)$ of T is the closed unit disc $\overline{\Delta}$. Thus $\partial\Delta \subset \sigma_{ap}(T)$.

Proposition 3. *If T is an injective subnormal unilateral shift with $\|T\| = 1$, then $\sigma_{ap}(T) = \partial\Delta$.*

Proof. It suffices to show that if $\zeta \in \Delta$, then $T - \zeta$ is bounded below. For that we let $R > 0$ satisfy $|\zeta| < R < 1$. Using Proposition 2, we can obtain

$$\|(T-\zeta)f\|^2 \geq \frac{1}{C}(R - |\zeta|)^2\|f\|^2. \qquad \square$$

It is known that even when the unilateral shift T is not subnormal, there does not exist a pair of proper invariant subspaces S_1 and S_2 (not necessarily orthogonal complements) such that the underlying Hilbert space is the direct sum of S_1 and S_2; see [24, Corollary 2, Page 63]. As a result, T is not reducible. This means that T does not have a nontrivial reducing subspace. When T is subnormal, we have the following result.

Theorem 4. *If T is an injective subnormal unilateral shift with two nontrivial invariant subspaces S_1 and S_2, then S_1 and S_2 are not orthogonal to each other.*

Proof. Without loss of generality we can assume that $\|T\| = 1$ because T and $T/\|T\|$ have the same invariant subspaces. By Theorem 1, we can assume that T is M_z on $P^2(\mu)$. Suppose that S_1 and S_2 are two orthogonal nontrivial invariant subspaces of M_z. We prove the theorem by picking f from S_1 and g from S_2, and then show that either f is the zero function or g is.

Since S_1 and S_2 are invariant under M_z, for any nonnegative integers k and m, the functions $z^k f$ and $z^m g$ are orthogonal. In other words,

$$\int_{\overline{\Delta}} z^k \overline{z}^m f\overline{g}\, d\mu = 0.$$

By the linearity of integrals, this equation still holds if we replace $z^k\overline{z}^m$ by any polynomial in z and $\overline{z}$. Consequently by the Stone-Weierstrass theorem, for any continuous function h on $\overline{\Delta}$, we have

$$\int_{\overline{\Delta}} h f\overline{g}\, d\mu = 0.$$

Since the measure $f\overline{g}d\mu$ is necessarily regular, by [21, Theorem 2.18, Page 48], an application of the Riesz Representation theorem shows that $f\overline{g}d\mu$ is the zero measure on $\overline{\Delta}$ and thus

$$\int_{\overline{\Delta}} |fg| d\mu = 0.$$

We now focus on the case when the measure ν has a point mass at 1, say $\nu\{1\} = a$ where $0 < a \leq 1$. In this case one can show that if p is a polynomial, then

$$\int_0^{2\pi} |p(e^{i\theta})|^2\, a\frac{d\theta}{2\pi} \;\leq\; \int_0^{2\pi}\int_0^1 |p(re^{i\theta})|^2\,\frac{d\theta}{2\pi} d\nu(r) \;<\; \infty.$$

Thus a sequence of polynomials that is Cauchy in $P^2(\mu)$ is also Cauchy in $P^2(\theta)$. It follows that $P^2(\mu)$ is contained in the Hardy space H^2 and so the functions f and g are in H^2. Hence we have $\int |fg| d\theta = 0$. This in turn implies that f is the zero function or g is.

The second case is when the measure ν does not have a point mass at 1. If there exists a point $r \in (0,1)$ such that $|fg| = 0$ on the circle $\{|z| = r\}$, then by Cauchy's integral formula we see that $fg = 0$ in $r\Delta$, and thus either f is the zero function in Δ or g is. If there does not exist such a point r, then for every

$a \in (0,1)$ there exists a point z with $|z| = a$ such that $|f(z)g(z)| > 0$. By our assumption that $\nu\{1\} = 0$, there exists $\rho \in [0,1)$ such that ρ is in the support of ν. Since the set $\{z : |f(z)g(z)| > 0\}$ is open, it contains an open disc E centered at a point z_0 with $|z_0| = \rho$. It follows that $\mu(E) > 0$. Hence $\int_E |fg| d\mu > 0$, which is a contradiction. □

The conclusion of Theorem 4 is not surprising when T is the operator $M_z : H^2 \to H^2$. This is simply because of the fact that any two nontrivial invariant subspaces of H^2 must have nonzero intersection, due to Beurling's characterization of all the invariant subspaces; see [8]. As a result the operator $M_z : H^2 \to H^2$ is irreducible. There are other proofs for its irreducibility, as given in [14, Page 40] and also in [20, Page 38]. However, none of these proofs can be generalized to work for unilateral weighted shifts, which are also irreducible as indicated before. In the case that the shift is subnormal, Theorem 4 gives another proof for the following result:

Corollary 5. *Every injective subnormal unilateral shift is irreducible.*

In the case of the Bergman space A^2, its invariant subspaces are very different from those of the Hardy space. H. Bercovici, C. Foias and C. M. Pearcy [2] have proved that there exists a family $\{S_\alpha : \alpha \in \mathbf{C}\}$ of closed subspaces in A^2 invariant under M_z such that $S_\alpha \cap S_\beta = \{0\}$ if $\alpha \neq \beta$. Theorem 4 complements this result by proving that S_α cannot be orthogonal to S_β if $\alpha \neq \beta$.

3. C^*-ALGEBRAS

In the preceding section we built up the necessary tools to discuss the C^*-algebra generated by a subnormal unilateral shift and the identity operator. Before we start our discussion, we need to investigate some results on the C^*-algebra generated by a general subnormal operator, not necessarily a subnormal unilateral shift.

We let H be an infinite dimensional separable Hilbert space, and let $S : H \to H$ be a subnormal operator with $\|S\| = 1$. Since any two minimal normal extensions of S are unitarily equivalent by [6, Corollary 2.7, Page 129], we let the minimal normal extension of S be denoted by $N : L \to L$, where L is a Hilbert space containing H. With a scalar-valued spectral measure μ_N on the spectrum $\sigma(N)$ of N, the functional calculus for a normal operator defines an isometrical $*$-isomorphism ρ from $L^\infty(\mu_N)$ to the abelian von Neumann algebra generated by N. If g is a function in $L^\infty(\mu_N)$, then we denote the operator $\rho(g)$ by $g(N)$. Using this functional calculus and the orthogonal projection P from L onto H, we can define the *Toeplitz operator* $T_g : H \to H$ by

$$T_g f = P g(N) f \quad \text{for all } f \in H.$$

The function g is called the *symbol* of the operator T_g.

The Toeplitz operators that we are interested in this paper are those having continuous symbols. This is because of a result from [6, Lemma 13.4, Page 211]: *If g is a function in $C(\sigma(N))$, then T_g is an operator in the C^*-algebra, denoted by $C^*(S)$, generated by the subnormal operator S and the identity operator I.*

The operator N and the identity operator I on L generate a commutative C^*-algebra, denoted by $C^*(N)$, and there is an isometrical $*$-isomorphism ρ_N from $C^*(N)$ to the algebra $C(\sigma(N))$ of all continuous functions on $\sigma(N)$ with $\rho_N(N) = z$; this follows from [6, Theorem 1.8, Page 56]. This isomorphism ρ_N is the restriction of the isomorphism ρ^{-1} on $C^*(N)$. Of course the C^*-algebra $C^*(S)$ is not commutative, and so we need to consider the commutator ideal J of $C^*(S)$. By definition J is the norm closed ideal generated by $\{AB - BA : A, B \in C^*(S)\}$. If we use $C(\sigma_{ap}(S))$ to denote the algebra of all continuous functions on the approximate point spectrum $\sigma_{ap}(S)$ of S, then from [6, Page 212] we know that the following diagram commutes:

$$\begin{tikzcd}
C^*(N) \arrow[rr, "\rho_N"] \arrow[d, "\theta"] & & C(\sigma(N)) \arrow[d, "\chi"] \\
C^*(S) \arrow[r, "\pi"] & C^*(S)/J \arrow[r, "\rho_S"] & C(\sigma_{ap}(S))
\end{tikzcd}$$

The above diagram gives us a method to relate $C^*(S)$ to the commutative algebra $C^*(N)$, and it is useful to discuss Toeplitz operators with continuous symbols. We now describe how the maps in the diagram work. Since ρ_N is an isometrical $*$-isomorphism, every operator in $C^*(N)$ must be of the form $\rho_N^{-1}(g) = \rho(g) = g(N)$ for some $g \in C(\sigma(N))$. The mapping θ is linear and is defined by $\theta(g(N)) = T_g$. The mapping π is the quotient map. Note that $\sigma_{ap}(S) \subset \sigma(N)$ (see [6, Page 132]), and χ is the restriction map of the continuous functions on $\sigma(N)$ to $\sigma_{ap}(S)$. John Bunce [3] has shown that the map ρ_S is an isometrical $*$-isomorphism, with $\rho_S(T_{\overline{z}^k z^m} + J) = \overline{z}^k z^m$ for all nonnegative integers k and m; one may also refer this result to Pages 210 and 212 of [6]. It follows that if $g \in C(\sigma(N))$, then by using the Stone-Weierstrass theorem, we obtain $\rho_S(T_g + J) = g|_{\sigma_{ap}(S)} = \chi(g)$. From the properties of these mappings, we deduce the following lemma.

Lemma 6. *For every function g in $C(\sigma(N))$,*

$$T_{g^k} + J = T_g^k + J.$$

Proof. Since the diagram commutes, we see that for any function g in $C(\sigma(N))$,

$$\rho_S^{-1} \circ \chi(g^k) = \pi \circ \theta \circ \rho_N^{-1}(g^k).$$

The left-hand side of the above equation is

$$\rho_S^{-1}(\chi(g)^k) = (\rho_S^{-1}(\chi(g)))^k = (T_g + J)^k = T_g^k + J.$$

On the other hand, we have

$$\pi \circ \theta \circ \rho_N^{-1}(g^k) = \pi(T_{g^k}) = T_{g^k} + J. \qquad \square$$

Since $\|S\| = 1$, the spectrum $\sigma(S)$ of S is included in the closed unit disc $\overline{\Delta}$. From [6, Theorem 2.11, Page 131], we know that $\sigma(N) \subset \sigma(S)$, and so $\sigma(N)$ is also included in $\overline{\Delta}$. Although in general g is defined on $\sigma(N)$, we are now interested in those g that are continuous in $\overline{\Delta}$.

We use $C(\overline{\Delta})$ to denote the algebra of continuous function on $\overline{\Delta}$, and give $C(\overline{\Delta})$ the supremum norm over $\overline{\Delta}$. We use $A(\overline{\Delta})$ to denote the disc algebra, the subalgebra of all functions in $C(\overline{\Delta})$ that are analytic in Δ.

If f is a function in $C(\overline{\Delta})$, then $A(\overline{\Delta})[f]$ denotes the norm closed algebra generated by f and $A(\overline{\Delta})$. It is interesting to see when $A(\overline{\Delta})[f]$ is the whole algebra $C(\overline{\Delta})$. One interesting answer is provided as follows:

Theorem 7 (Axler and Shields [1, Theorem 4])**.** *Let f be a function in $C(\overline{\Delta})$ that is harmonic but not analytic on Δ. Then $A(\overline{\Delta})[f]$ equals $C(\overline{\Delta})$.*

This theorem motivates us to discuss its analogue in the setting of Toeplitz operators as defined before, corresponding to the subnormal operator S with $\|S\| = 1$. The major problem is that $C(\overline{\Delta})$ is a commutative algebra while Toeplitz operators, in general, do not commute.

We use $\tau(\overline{\Delta})$ to denote the C^*-algebra generated by all Toeplitz operators T_g whose symbol g is in $C(\overline{\Delta})$. Since ρ_N is an isometrical $*$-isomorphism, we see that the adjoint T_g^* is the same as $T_{\overline{g}}$, and so the C^*-algebra $\tau(\overline{\Delta})$ is also the closed algebra generated by all Toeplitz operators T_g whose symbol g is in $C(\overline{\Delta})$.

We use $[A(\overline{\Delta})]$ to denote the algebra generated by all Toeplitz operators T_g whose symbol g is in $A(\overline{\Delta})$. By $[A(\overline{\Delta})]/J$, we denote the subalgebra generated in $C^*(S)/J$ by the equivalence classes $T_g + J$ with $g \in A(\overline{\Delta})$. In addition, we use $\tau(\overline{\Delta})/J$ to denote the C^*-subalgebra generated by $T_g + J$ with $g \in C(\overline{\Delta})$.

Theorem 8. *If f is a function in $C(\overline{\Delta})$, and f is harmonic but not analytic on Δ, then the closed algebra generated by $[A(\overline{\Delta})]/J$ and $T_f + J$ equals $\tau(\overline{\Delta})/J$.*

Proof. By Theorem 7, for any function g in $C(\overline{\Delta})$ and any positive ϵ, there exist functions $h_0, \cdots, h_k$ in $A(\overline{\Delta})$ such that if $h = h_0 + h_1 f + h_2 f^2 + \cdots + h_k f^k$, then $|h(z) - g(z)| < \epsilon$ for all z in $\overline{\Delta}$. Thus if P is the orthogonal projection from L onto H, then

$$\|T_g - T_h\| = \|P\left(g(N) - h(N)\right)\| \le \|g(N) - h(N)\|.$$

It follows from the functional calculus for a normal operator (see [6, Page 93]) that $\|g(N) - h(N)\|$ is equal to the supremum norm of $g - h$ on $\sigma(N)$, and so we have shown that $\|T_g - T_h\| < \epsilon$.

Since each function h_i, for $0 \le i \le k$, is a uniform limit of polynomials on $\overline{\Delta}$ and H is invariant under N, it follows that H is invariant under $h_i(N)$. Thus if x is a vector in H, then

$$T_{h_i f^i}(x) = P(f^i(N)h_i(N))x = P(f^i(N))h_i(N)x = T_{f^i}T_{h_i}(x).$$

Hence by Lemma 6, there exists an operator Y_i in J with $0 \le i \le k$, such that

$$T_{h_i f^i} = (T_f^i + Y_i)T_{h_i}.$$

This implies, along with our conclusion in the last paragraph, that if we let $Y = Y_1 T_{h_1} + \cdots + Y_k T_{h_k}$, then

$$\|T_g - T_h\| = \|T_g - T_{h_0} - T_f^1 T_{h_1} - \cdots - T_f^k T_{h_k} - Y\| < \epsilon.$$

Since J is an ideal and $Y \in J$, we have

$$\|\pi(T_g) - \pi\left(T_{h_0} - T_f^1 T_{h_1} - \cdots - T_f^k T_{h_k}\right)\| < \epsilon.$$

This completes the whole proof.

We now want to focus our discussion on a Toeplitz operator associated with a subnormal unilateral shift T with $\|T\| = 1$. By Theorem 1 we can view T as M_z from $P^2(\mu)$ to $P^2(\mu)$. Note that a scalar-valued spectral measure μ_N for the minimal normal extension N of T is different from μ. In fact μ_N is a measure on $\sigma(N)$ while μ is a measure on $\overline{\Delta}$. When M_z is on the Bergman space A^2, then $\sigma(N)$ is $\overline{\Delta}$, and when M_z is on the Hardy space H^2, then $\sigma(N)$ is $\partial\Delta$.

We now fix a scalar-valued spectral measure μ_N and define the Toeplitz operators correspondingly. Let $\tau(\sigma(N))$ be the C^*-algebra generated by all Toeplitz operators $T_\psi : P^2(\mu) \to P^2(\mu)$ whose symbol ψ is in $C(\sigma(N))$.

L. A. Coburn [5, Theorem 1] has proved the several-variable version of the following result: *If T is M_z on the Hardy space, or on the Bergman space, then $\tau(\sigma(N))$ contains the whole ideal of compact operators, and furthermore $\tau(\sigma(N)) = \{T_\psi + A : \psi \in C(\sigma(N))$ and A compact$\}$.* In addition, Coburn also proved that the algebra $\tau(\sigma(N))$ corresponding to the Hardy space is $*$-isomorphic to that algebra corresponding to the Bergman space. We now combine the Hardy space and Bergman space versions of Coburn's result into one setting, by generalizing some main ideas in Coburn's proofs. We use K to denote the algebra of all compact operators on $P^2(\mu)$.

Lemma 9. *If T is an injective subnormal unilateral shift, then the algebra $\tau(\sigma(N))$ contains K.*

Proof. The idea of this proof is to use a theorem of R. F. Olin and J. E. Thomson (see [6, Theorem 13.9, Page 213]): *If S is an irreducible subnormal operator such that $S^*S - SS^*$ is compact, then for all u and v in $C(\sigma(N))$, the operator $T_{uv} - T_uT_v$ is compact.*

We first note that Corollary 5 shows that T is irreducible, and in order to use the theorem of Olin and Thomson, we must show that $T^*T - TT^*$ is compact. Note that T is an injective subnormal shift, and by definition there exists a weight sequence $\{w_n\}_0^\infty$ and an orthonormal basis $\{e_n\}_0^\infty$ such that $Te_n = w_n e_{n+1}$ for all integers $n \geq 0$. One can easily check that if T^* denotes the adjoint of T, then $T^*e_n = w_{n-1}e_{n-1}$ for every integer $n \geq 1$, and $T^*e_0 = 0$.

Any vector f in the Hilbert space can be written as $f = \sum a_n e_n$ with $\sum |a_n|^2 < \infty$. A direct computation shows that

$$(T^*T - TT^*)f = a_0 w_0^2 e_0 + \sum_{n=1}^{\infty} a_n (w_n^2 - w_{n-1}^2) e_n.$$

Thus $T^*T - TT^*$ is a diagonal operator with its diagonal given by $\{w_0^2, w_1^2 - w_0^2, \cdots\}$. Since T is subnormal, it is also hyponormal which by definition means that $T^*T - TT^*$ is a positive operator. It follows that $w_{n+1} \geq w_n$ for all integers $n \geq 0$. Without loss of generality, we assume that $\|T\| = 1$ and so $\sup w_n = 1$. Thus $\{w_n\}$ is a sequence increasing to 1, and hence $w_{n+1} - w_n \to 0$ as $n \to \infty$. This in turn implies that the diagonal operator $T^*T - TT^*$ is compact; see [15, Problem 171]. Hence the conclusion of the theorem of Olin and Thomson holds.

To finish the proof, we use the following result [9, Corollary 4.1.10, Page 97] in C^*-algebra theory: *If W is an irreducible C^*-algebra of bounded linear operators on a Hilbert space and W contains a nontrivial compact operator, then W contains the whole ideal of compact operators.* Here W be irreducible means that the only closed subspaces of the Hilbert space invariant under every operator in W be the zero subspace and the whole space.

For the C^*-algebra $\tau(\sigma(N))$, Corollary 5 tells us that it is irreducible. We now want to show that $\tau(\sigma(N))$ contains a nontrivial compact operator. We observe that T_z and $T_{\overline{z}}$ are in $\tau(\sigma(N))$. Note that $\overline{z}$ is orthogonal to $P^2(\mu)$ since for all integers $k \geq 0$,

$$\int_{\overline{\Delta}} z^k \overline{\overline{z}} d\mu \;=\; \int_{\overline{\Delta}} z^{k+1} d\mu \;=\; 0.$$

Furthermore $P(z\overline{z})$ is a nonzero constant because if k is a positive integer, then

$$\int_{\overline{\Delta}} z\overline{z}\overline{z^k}\, d\mu = 0,$$

and for $k = 0$, we have

$$\int_{\overline{\Delta}} z\overline{z} d\mu \;=\; \int_{[0,1]} r^2 d\nu \;\neq\; 0.$$

Hence $0 = T_z P(\overline{z}) = T_z T_{\overline{z}}(1) \neq T_{\overline{z}} T_z(1) = P(\overline{z}z)$, and so the theorem of Olin and Thomson implies that $\tau(\sigma(N))$ contains a nonzero compact operator. $\square$

We remark here that in the proof of Lemma 9, we show that if T is a subnormal unilateral shift, then T has the following two properties: T is irreducible and $T^*T - TT^*$ is compact. We know, from [6, Lemma 13.8, Page 213], that if A is an operator with these two properties, then the commutator ideal of $C^*(A)$, the C^*-algebra generated by A and the identity I, is the ideal of all compact operators. Hence we have $J = K$. As indicated in the beginning of this section, $\tau(\sigma(N))$ is contained in $C^*(T)$. Thus Lemma 9 implies that the quotient algebra $\tau(\sigma(N))/K$ is commutative.

For a subnormal operator S and its the minimal normal extension N, Keough [17] has the following result: *If S is irreducible and $S^*S - SS^*$ is compact, then $C^*(S) = \{T_u + A : u \in C(\sigma(N))$ and A compact$\}$.* For a reference of this result, one may also see [6, Page 216]. The conclusion of this result certainly holds when S is a subnormal unilateral shift. Using Keough's result we now prove the next theorem.

Theorem 10. *If T is an injective subnormal unilateral shift, then $\tau(\sigma(N)) = C^*(T)$.*

Proof. Let ψ be a function continuous on $\sigma(N)$ and let $[T_\psi]$ be the equivalence class in $\tau(\sigma(N))/K$ that contains T_ψ. One can show that the map $\psi \to [T_\psi]$ is a $*$-homomorphism from $C(\sigma(N))$ into $\tau(\sigma(N))/K$. The range of this map is a closed $*$-subalgebra in $\tau(\sigma(N))/K$, by [9, Corollary 1.8.3, Page 21]. Since the range contains $[T_\psi]$ for all ψ in $C(\sigma(N))$, the range is $\tau(\sigma(N))/K$. Thus $\tau(\sigma(N)) = \{T_u + A : u \in C(\sigma(N))$ and A compact$\}$, which is $C^*(T)$. $\square$

One question arises naturally: If T is a subnormal unilateral shift with unit norm and ψ is a continuous function on $\sigma(N)$, when is T_ψ compact? The answer is that for T_ψ to be compact, it is necessary and sufficient that ψ vanishes on $\partial\Delta$. This answer can be deduced by using Proposition 3 and the result from [6, Corollary 13.6, Page 212]: *If S is subnormal, N is its minimal normal extension, and u is a function in $C(\sigma(N))$, then $T_u \in J$ if and only if $u(\lambda) = 0$ for each $\lambda \in \sigma_{ap}(S)$.* Along with Theorem 10, this answer tells us that the map that takes the equivalence class $[T_\psi]$ in $\tau(\sigma(N))/K$ to the restriction of ψ on $\partial\Delta$ is an isometrical $*$-isomorphism onto the algebra $C(\partial\Delta)$ of continuous functions on $\partial\Delta$. Thus we have proved the following.

Corollary 11. *$\tau(\sigma(N))/K$ is isometrically $*$-isomorphic to $C(\partial\Delta)$.*

4. Subnormal bilateral shifts

In the preceding two sections we obtained a few results for an injective subnormal unilateral shift, and in this section we discussed those results for an injective subnormal bilateral shift. A linear operator T on an infinite dimensional separable complex Hilbert space is called a *bilateral weighted shift* with *weight sequence* $\{w_k : -\infty < k < \infty\}$ if there is an orthonormal basis $\{e_k : -\infty < k < \infty\}$ of H such that

$$Te_k = w_k e_{k+1}$$

for all integers k. Similar to our discussions on unilateral shifts, we assume that every bilateral shift is injective and further assume, by unitary equivalence, that $w_k > 0$ for all integers k. In the case when the bilateral shift T is subnormal, the situation is similar but different from the setting in the previous two sections because T cannot be viewed as M_z acting on $P^2(\mu)$. An appropriate modification is required. If μ is a regular Borel measure on the closed unit disc $\overline{\Delta}$, then we use $R^2(\mu)$ to denote the closed linear span of $\{z^k : -\infty < k < \infty\}$ in $L^2(\mu)$. With this notation, an analogue of Theorem 1 can be stated for a bilateral shift [6, Theorem 8.17, Page 161]: *If T is an injective bilateral shift with $\|T\| = 1$, then T is subnormal if and only if there is a probability Borel measure ν on $[0, 1]$ with 1 in the support of ν such that $\int_0^1 r^k d\nu(r) < \infty$ for every integer k, and T is unitarily equivalent to M_z on $R^2(d\nu d\theta/2\pi)$.*

In the rest of the present section, T denotes an injective subnormal bilateral shift with $\|T\| = 1$. It has been shown in [6, Page 163] that if we denote $m(T) = \inf\{\|Tf\| : \|f\| = 1\}$, then the support of the measure ν is contained in the interval $[m(T), 1]$, with both end points 1 and $m(T)$ lying in the support of ν, and furthermore every function in $R^2(\mu)$ is analytic in the annulus $\{z : m(T) < |z| < 1\}$. As an analogue of Theorem 4, we offer the following statement: *If T has two nontrivial invariant subspaces S_1 and S_2, then S_1 and S_2 are not orthogonal.*

If N is the minimal normal extension of T and $\sigma(N)$ is the spectrum of N, then we use $\tau(\sigma(N))$ to denote the C^*-algebra generated by the Toeplitz operators $T_g : R^2(\mu) \to R^2(\mu)$ with $g \in C(\sigma(N))$. We now offer the following analogue of Lemma 9 for a bilateral shift: *If T is an injective subnormal bilateral shift, then the algebra $\tau(\sigma(N))$ contains the ideal K of all compact operators.* One can prove this statement by modifying the proof of Lemma 9. For instance, the corresponding argument in that lemma shows that if T is bilateral, then $T^*T - TT^*$ is compact. The only difference is when we want to show that $T_z T_{\overline{z}}(1) \neq T_{\overline{z}} T_z(1)$ due to the fact that $\overline{z}$ is no longer orthogonal to $R^2(\mu)$ in the bilateral case. Nevertheless, one can check that $T_z T_{\overline{z}}(1)$ is a constant function with value $1/\|z^{-1}\|^2$, while $T_{\overline{z}} T_z(1)$ is a constant function with value $\|z\|^2$. We remark that $\|z\| \neq 1/\|z^{-1}\|$, because if they were equal, then it would follow that

$$1 = \int_{[0,1]} 1 d\nu = \int_{[0,1]} r \frac{1}{r} d\nu = \left(\int_{[0,1]} r^2 d\nu \right)^{\frac{1}{2}} \left(\int_{[0,1]} \frac{1}{r^2} d\nu \right)^{\frac{1}{2}}.$$

The last equality in the previous line is the equality in Cauchy's Inequality and it happens only when r is a scalar multiple of $1/r$. However that is not possible since ν is supported at both 1 and $m(T)$.

A review of the proof of Theorem 10 shows that the theorem holds for a bilateral shift, and so we have the following statement: *If T is an injective subnormal bilateral* *shift, then* $\tau(\sigma(N)) = C^*(T)$. In addition, by modifying the argument for the proof of Corollary 11, we have the following analogous result: $\tau(\sigma(N))/K$ *is isometrically* $*$*-isomorphic to* $C(\sigma_{ap}(T))$.

5. Several variables

It is natural to ask to what extent the results in the preceding sections hold in the setting of n variables, where $n \geq 2$. In Sections 2, 3 and 4, we derived a few properties of a subnormal shift using its related measure defined on the closed unit disc $\overline{\Delta}$. In this section, we use B to denote the open unit ball of $\mathbf{C}^n$. As a generalization of the measure $d\nu d\theta/2\pi$ defined in Section 2, throughout the present section we consider the measure $d\mu = d\nu d\sigma$ on the closed unit ball $\overline{B}$, where $d\nu$ is a Borel probability measure on $[0, 1]$ with the point 1 in the support of ν, and $d\sigma$ is the normalized surface area measure on ∂B, the unit sphere of $\mathbf{C}^n$. We let $P^2(\mu)$ denote the closure of the n-variable polynomials in $L^2(\mu)$. It is well-known that the monomials form an orthogonal basis in $P^2(\mu)$.

Similar to the one-variable case, examples include the Hardy and Bergman spaces on B. We remark that whenever $d\nu$ is a Borel probability measure with 1 in the support of ν, the Hilbert space $P^2(d\nu d\sigma)$ consists of functions analytic in B.

It is known from [18, Page 55] that if ζ is a point in B, then the mapping $f \longmapsto f(\zeta)$ is a continuous linear functional on $H^2(B)$. This functional is called the evaluation functional of the point ζ. According to the Riesz representation theorem, there exists a function g in $H^2(B)$ such that if $\langle , \rangle_H$ denotes the inner product in $H^2(B)$, then $f(\zeta) = \langle f, g\rangle_H$ for all f in $H^2(B)$.

Lemma 12. *For every point ζ in B, there exists a function k_ζ in $P^2(\mu)$ such that $\langle f, k_\zeta\rangle = f(\zeta)$ for all functions f in $P^2(\mu)$.*

Proof. One can easily check that k_ζ is given by the formula:

$$k_\zeta(z) = \sum_\alpha \frac{\overline{\zeta^\alpha}}{\|z^\alpha\|} \frac{z^\alpha}{\|z^\alpha\|}. \qquad \square$$

The result of the preceding lemma, along with the idea in the proof of Proposition 2, enables one to prove the following several-variable analogue of that proposition, and we omit the proof.

Proposition 13. *If $0 < R < 1$, then there exists a positive constant C, depending on R and μ, such that for every function f in $P^2(\mu)$,*

$$\|f\|^2 \;\leq\; C \int_{[R,1]} \int_{\partial B} |f|^2 \, d\sigma d\nu.$$

Since $P^2(\mu)$ consists of only analytic functions, we are now ready to prove the several-variable result analogous to Theorem 4. Before we state and prove the result, we need a few definitions. For each integer k with $1 \leq k \leq n$, we let $T_k : P^2(\mu) \to P^2(\mu)$ be defined by $T_k f = z_k f$ for all functions f in $P^2(\mu)$. A closed subspace S of $P^2(\mu)$ is said to be *invariant* if $T_k S \subset S$ for each k.

Theorem 14. *If S_1 and S_2 are two nontrivial invariant subspaces of $P^2(\mu)$, then S_1 and S_2 are not orthogonal to each other.*

Proof. Our proof for Theorem 4 can be modified to work for this theorem, and we only point out the modifications needed. If S_1 and S_2 are invariant and f, g

are functions in S_1 and S_2 respectively, then for all multi-indices α and β, the functions $z^\alpha f$ and $z^\beta g$ are orthogonal. This allows us to conclude that for all continuous functions h on $\overline{B}$,

$$\int_{\partial B}\int_{[0,1]} hf\overline{g}d\nu d\sigma = 0.$$

When the measure ν has a point mass at 1, it is clear that $fg = 0$. When the measure ν does not have a point mass at 1, then one may use Cauchy's integral formula [22, 3.2.4, Page 39] for the unit ball to complete the whole argument. $\square$

Theorem 14 tells us that there does not exist a nontrivial closed subspace of $P^2(\mu)$ that is reducing for all the operators T_k for $1 \leq k \leq n$. Hence the C^*-algebra generated by $T_1, \cdots, T_n$ and the identity operator I does not have a nontrivial closed subspace that reduces every operator in the algebra. This result is important for us to generalize Theorem 10 to the case of several variables, which is one of the goals in this section. Besides that result, we need to generalize the concept of the approximate point spectrum of an operator also. For that purpose we digress to function theory in several variables.

Let $\langle , \rangle$ denote the inner product in $\mathbf{C}^n$; that is, for $z = (z_1, \cdots, z_n)$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ in $\mathbf{C}^n$, we define $\langle z, \zeta\rangle = z_1\overline{\zeta_1} + \cdots + z_n\overline{\zeta_n}$. Let us fix a point $\zeta \in \partial B$ and define a polynomial $G : \overline{B} \to \mathbf{C}$ by

$$G(z) = \frac{1 + \langle z, \zeta\rangle}{2}.$$

For every positive integer k, define

$$c_k = \int_{\overline{B}} |G(z)|^k \, d\mu(z).$$

We now want to obtain a lower estimate on c_k. Let $0 < s < 1$ and $E_s = \{z \in \overline{B} : |G(z)| > s\}$. Thus

$$c_k \geq \int_{E_s} |G(z)|^k \, d\mu(z) \geq s^k \mu(E_s).$$

Since $s < 1$, the set E_s contains a neighborhood of ζ in $\overline{B}$ and so $\mu(E_s) > 0$ and $c_k > 0$. This allows us to define $G_k(z) = |G(z)|^k / c_k$. By our definition of c_k, we have $\int_{\overline{B}} G_k(z)\, d\mu(z) = 1$. In fact the sequence $\{G_k\}$ behaves like an approximate identity as indicated as follows.

Lemma 15. *Suppose that ζ is a point in ∂B. Corresponding to ζ, we define the polynomial $G(z)$ and a sequence of positive numbers $\{c_k\}$ as in the above. If g is a continuous function on $\overline{B}$, then*

$$g(\zeta) = \lim_{k\to\infty} \int_{\overline{B}} g(z) \frac{|G(z)|^k}{c_k}\, d\mu(z).$$

Proof. Since g is uniformly continuous on $\overline{B}$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $z, \eta \in \overline{B}$ and $|z - \eta| < \delta$, then $|g(z) - g(\eta)| < \epsilon$. We now define

$\Omega = \{z \in \overline{B} : |z - \zeta| < \delta\}$, and perform the following calculations:

$$\begin{aligned} & |g(\zeta) - \int_{\overline{B}} g(z) G_k(z)\, d\mu(z)| \\ = \quad & \left(\int_{\overline{B}\setminus\Omega} + \int_{\Omega} \right) |g(\zeta) - g(z)| G_k(z)\, d\mu(z) \\ \leq \quad & 2 \max\{|g(z)| : z \in \overline{B}\} \frac{1}{c_k} \int_{\overline{B}\setminus\Omega} |G(z)|^k \, d\mu(z) + \epsilon. \end{aligned}$$

Now one can use the fact that $|G(z)| < 1$ when $z \in \overline{B}\setminus\{\zeta\}$ and $G(\zeta) = 1$ to complete the whole proof. □

With Lemma 15, we now generalize Proposition 3, by using the following definition given by J. Bunce [4, Definition 1]: If $B(H)$ is the algebra of all bounded linear operators on a Hilbert space H, then for commuting operators $A_1, A_2, \cdots, A_n$ in $B(H)$, their *joint approximate spectrum* is defined by

$$\{(\zeta_1, \cdots, \zeta_n) \in \mathbf{C}^n : B(H)(A_1 - \zeta_1) + \cdots + B(H)(A_n - \zeta_n) \neq B(H)\}.$$

We use $a(A_1, \cdots, A_n)$ to denote this joint approximate spectrum. As a natural generalization of the approximate point spectrum of a single operator, A. T. Dash [7, Proposition 3.2] has the following formulation: *$(\zeta_1, \cdots, \zeta_n)$ is in $a(A_1, \cdots, A_n)$ if and only if there exists a sequence of unit vectors $\{f_m\}$ in H such that for each fixed integer k, the sequence $(A_k - \zeta_k) f_m \to 0$ in norm, as $m \to \infty$.*

Back to our setting, the operators $T_1, \cdots, T_n$ on $P^2(\mu)$ commute, and we can use Dash's proposition to prove the following.

Proposition 16. $a(T_1, \cdots, T_n) = \partial B$.

Proof. In this proof we use z to denote the variable $(z_1, \cdots, z_n) \in \mathbf{C}^n$. If $\zeta = (\zeta_1, \cdots, \zeta_n)$ is a point in ∂B, then Lemma 15 provides us with a polynomial $G(z)$ so that for each integer m with $1 \leq m \leq n$,

$$\lim_{k\to\infty} \int_{\overline{B}} |z_m - \zeta_m|^2 \left| \frac{G(z)^k}{\sqrt{c_{2k}}} \right|^2 d\mu(z) = 0.$$

If we define $f_k(z) = G(z)^k / \sqrt{c_{2k}}$, then the above limit can be rewritten as $\|(T_m - \zeta_m) f_k\| \to 0$. Since f_k is a unit vector in $P^2(\mu)$, Dash's proposition implies that $\partial B \subset a(T_1, \cdots, T_n)$.

On the other hand, if $\zeta = (\zeta_1, \cdots, \zeta_n)$ is a point in B and f is a function in $P^2(\mu)$, then for any R with $0 < R < 1$,

$$\begin{aligned} & \|(T_1 - \zeta_1) f\|^2 + \cdots + \|(T_n - \zeta_n) f\|^2 \\ \geq \quad & \int_{[R,1]} \int_{\partial B} (|z_1 - \zeta_1|^2 + \cdots + |z_n - \zeta_n|^2) |f|^2 \, d\mu(z). \end{aligned}$$

We can choose R so that ζ is in the open set RB, and thus there exists a positive δ such that $(|z_1 - \zeta_1|^2 + \cdots + |z_n - \zeta_n|^2) \geq \delta$ whenever $(z_1, \cdots, z_n) \in B\setminus RB$. This observation, along with Proposition 13, implies that there exists a positive constant C such that

$$\|(T_1 - \zeta_1) f\|^2 + \cdots + \|(T_n - \zeta_n) f\|^2 \geq \frac{\delta}{C} \|f\|^2.$$

Our result follows from another application of Dash's proposition. □

We now digress to proving the compactness of the operator $T_m^* T_m - T_m T_m^*$ for every m with $1 \leq m \leq n$. Without loss of generality, it suffices to prove the case when $m = 1$. For that we can just show that $T_1^* T_1 - T_1 T_1^* (\frac{z^\alpha}{\|z^\alpha\|}) \to 0$, as $|\alpha| \to +\infty$, because it follows that $T_1^* T_1 - T_1 T_1^*$ is a norm limit of finite-rank operators.

To show that limit is zero, we now proceed as follows. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ are two multi-indices, then we use $\alpha \pm \beta$ to denote the multi-index $(\alpha_1 \pm \beta_1, \cdots, \alpha_n \pm \beta_n)$. In addition, we use e to denote the multi-index $(1, 0, \cdots, 0)$, and use w_α to denote the weight $w_\alpha = \|z^{\alpha+e}\| / \|z^\alpha\|$. One can check that

$$T_1 \frac{z^\alpha}{\|z^\alpha\|} = w_\alpha \frac{z^{\alpha+e}}{\|z^{\alpha+e}\|}.$$

By using this identity, along with the definition of the inner product, we have

$$T_1^* \left(\frac{z^\alpha}{\|z^\alpha\|} \right) = \begin{cases} w_{\alpha-e} \frac{z^{\alpha-e}}{\|z^{\alpha-e}\|} & \text{if } \alpha_1 \neq 0, \\ 0 & \text{if } \alpha_1 = 0. \end{cases}$$

From this equation we deduce that

$$T_1^* T_1 - T_1 T_1^* \left(\frac{z^\alpha}{\|z^\alpha\|} \right) = \begin{cases} (w_\alpha^2 - w_{\alpha-e}^2) \frac{z^\alpha}{\|z^\alpha\|} & \text{if } \alpha_1 \neq 0, \\ w_\alpha^2 \frac{z^\alpha}{\|z^\alpha\|} & \text{if } \alpha_1 = 0. \end{cases}$$

To continue the proof we need an estimate on w_α, by using the following identity [22, Proposition 1.4.9, Page 16]:

$$\int_{\partial B} |z^\alpha|^2 \, d\sigma(z) = \frac{(n-1)! \alpha!}{(n-1+|\alpha|)!}.$$

With this formula we can compute w_α:

$$w_\alpha^2 = \frac{\int r^{2|\alpha|+2} d\nu(r) \int |z^{\alpha+e}|^2 d\sigma(z)}{\int r^{2|\alpha|} d\nu(r) \int |z^\alpha|^2 d\sigma(z)} = \frac{(\alpha_1 + 1) \int r^{2|\alpha|+2} d\nu(r)}{(n + |\alpha|) \int r^{2|\alpha|} d\nu(r).}$$

Then we observe that

$$\left(\int r^{2|\alpha|} d\nu \right)^2 = \left(\int r^{|\alpha|+1} r^{|\alpha|-1} d\nu \right)^2 \leq \int r^{2|\alpha|+2} \, d\nu \int r^{2|\alpha|-2} \, d\nu.$$

From this observation, we deduce that the function ρ defined by

$$\rho(|\alpha|) = \frac{\int r^{2|\alpha|+2} \, d\nu}{\int r^{2|\alpha|} d\nu}$$

is an increasing function of $|\alpha|$. Since $\rho(|\alpha|) \leq 1$ for all α, the limit of $\rho(|\alpha|)$ exists as $|\alpha| \to \infty$. With this limit, one can now show that

$$w_\alpha^2 - w_{\alpha-e}^2 = \frac{\alpha_1 + 1}{n + |\alpha|} \rho(|\alpha|) - \frac{\alpha_1}{n + |\alpha| - 1} \rho(|\alpha| - 1)$$

goes to zero, as $|\alpha| \to \infty$. This completes the whole proof.

As in the one-variable case that we discussed in Section 3, the compactness of $T_1^* T_1 - T_1 T_1^*$ has an important corollary in the theory of C^*-algebras. To explain that, we let U denote the C^*-algebra generated by the operators $T_1, \cdots, T_n$ and the identity operator I on $P^2(\mu)$. We also let K denote the ideal of compact operators on $P^2(\mu)$. Note that U contains the nonzero compact operator $T_1^* T_1 - T_1 T_1^*$, and moreover by Theorem 14, U is irreducible. This allows us to use [9, Corollary 4.1.10, Page 97], as in the proof of Lemma 9, to conclude that U contains the whole ideal K. It is then natural to ask what the quotient algebra U/K is like. We identify this quotient algebra in the next theorem. The idea of its proof comes from N. P. Jewell [16, Corollary 4]who has shown that the theorem holds for the case when

$P^2(\mu)$ is the Hardy space. We now generalize his result and use $C(\partial B)$ to denote the algebra of continuous functions on the ∂B.

Theorem 17. *U/K is isometrically $*$-isomorphic to the algebra $C(\partial B)$.*

Proof. Clearly the C^*-algebra U/K is generated by commuting elements $T_1 + K, \cdots, T_n + K$ and $I + K$. Since each $T_m^* T_m - T_m T_m^*$ is compact, $T_m + K$ is normal in U/K, it follows that U/K is a commutative C^*-algebra. The maximal ideal space of U/K is homeomorphic to the joint spectrum σ_j of $T_1 + K, \ \cdots, \ T_n + K$; see, for example, [13, Theorem 1.4, Page 68]. If M denotes the maximal ideal space of U/K, then that *joint spectrum* σ_j is $\{(\psi(T_1+K), \cdots, \psi(T_n+K)) \in \mathbf{C}^n : \psi \in M\}$, by definition.

By a *character* on a C^*-algebra with identity I, we mean a multiplicative linear functional ρ such that $\rho(I) = 1$. If ρ is a character on U, then $\ker\rho$ is a closed ideal in U, and the quotient $U/\ker\rho$ is a vector space isomorphic to $\mathbf{C}$. Thus $U/\ker\rho$ is a vector space of dimension one, and it must be spanned by $I + \ker\rho$. From this we deduce that $\ker\rho$ is irreducible because U is. Since $\ker\rho$ itself is a C^*-algebra and it contains the nonzero compact operator $T_1^* T_1 - T_1 T_1^*$, it must contain K, by [9, Corollary 4.1.10, Page 97].

If π denotes the quotient map from U onto U/K, then from our discussion in the last paragraph we arrive at the following conclusion: For every ψ in M, the composition $\psi \circ \pi$ is a character ρ on U, and conversely every character ρ on U can be written as $\psi \circ \pi$ for some ψ in M. It follows that

$$\begin{aligned} \sigma_j &= \{(\psi \circ \pi(T_1), \cdots, \psi \circ \pi(T_n)) \in \mathbf{C}^n : \psi \in M\} \\ &= \{(\rho(T_1), \cdots, \rho(T_n)) \in \mathbf{C}^n : \rho \text{ is a character on } U\}. \end{aligned}$$

Along with Proposition 16, the whole proof is completed by the following result of J. Bunce [4, Corollary 4]: *If $A_1, \cdots, A_n$ are commuting hyponormal operators on a Hilbert space H, then their joint approximate spectrum $a(A_1, \cdots, A_n)$ is exactly the collection of points $(\rho(A_1), \cdots \rho(A_n))$ such that ρ is any character on the C^*-algebra generated by $A_1, \cdots, A_n$ and the identity operator I.* ☐

Theorem 17 gives us a nice description for the C^*-algebra U, and now we want to give another description. Before we do that we need the following definitions. Let P denote the orthogonal projection from $L^2(\mu)$ onto $P^2(\mu)$, and let κ denote the support of the measure μ. We also use $C(\kappa)$ to denote the algebra of all continuous functions on κ, and use $\|\cdot\|_\infty$ to denote the supremum norm over κ. If ϕ is a function in $C(\kappa)$, then we can define the Toeplitz operator $T_\phi : P^2(\mu) \to P^2(\mu)$ by $T_\phi f = P(\phi f)$. It is obvious that the operator norm $\|T_\phi\|$ satisfies $\|T_\phi\| \ \leq \ \|\phi\|_\infty$. The Stone-Weierstrass theorem shows that if $\phi \in C(\kappa)$, then T_ϕ is in U. Thus if we use $\tau(\kappa)$ to denote the C^*-algebra generated by all Toeplitz operators T_ϕ with $\phi \in C(\kappa)$, then we have the following generalization of Theorem 10.

Corollary 18. $U = \tau(\kappa)$.

Furthermore, a review of the proof for Theorem 10 gives us the following result:

Corollary 19. $\tau(\kappa) = \{T_u + A : u \in C(\kappa)$ *and A is compact*$\}$.

Note that if $P^2(\mu)$ is the Hardy space, then $\kappa = \partial B$, and if $P^2(\mu)$ is the Bergman space, then $\kappa = \overline{B}$. Coburn [5, Theorem 1] has shown that in the case when $P^2(\mu)$ is the Hardy space, then $\tau(\partial B)/K$ can be identified with the algebra $C(\partial B)$, and in additionwhen $P^2(\mu)$ is the Bergman space, then $\tau(\overline{B})/K$ can be identified again with the same algebra $C(\partial B)$. Theorem 17 unifies the proofs for both of these results and generalizes them to a subnormal setting.

References

1. S. Axler and A. Shields, *Algebras generated by analytic and harmonic functions,* Indiana Univ. Math. J. **36** (1987), 631 - 638. MR **88h:**46102
2. H. Bercovici, C. Foias, and C. M. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory,* CBMS Regional Conf. Ser. in Math., no. 56, Amer. Math. Soc., Providence, R. I. 1985. MR **87g:**47091
3. J. Bunce, *Characters on singly generated C^*-algebras,* Proc. Amer. Math. Soc. **25** (1970), 297 - 303. MR **41:**4258
4. J. Bunce, *The joint spectrum of commuting nonnormal operators*, Proc. Amer. Math. Soc. **29** (1971), 499 - 505. MR **44:**832
5. L. A. Coburn, *Singular Integral Operators and Toeplitz Operators on Odd Spheres,* Indiana Univ. Math. J. **23** (1973), 433 - 439. MR **48:**957
6. J. B. Conway, *Subnormal Operators,* Research Notes in Mathematics 51, Pitman, London, 1981. MR **83i:**47030
7. A. T. Dash, *Joint Spectra*, Studia Math. **45** (1973), 225 - 237. MR **49:**1156
8. P. L. Duren, *Theory of H^p Spaces,* Academic Press, New York, 1970. MR **42:**3552
9. J. Dixmier, *C*-algebras,* North-Holland, New York, 1977. MR **56:**16388
10. R. E. Frankfurt, *Subnormal weighted shifts and related function spaces,* J. Math. Anal. Appl. **52** (1975), 471-489. MR **58:**2407
11. R. E. Frankfurt, *Subnormal weighted shifts and related function spaces II,* J. Math. Anal. Appl. **55** (1976), 2-17. MR **58:**2408
12. R. E. Frankfurt, *Function spaces associated with radially symmetric measures,* J. Math. Anal. Appl. **60** (1977), 502-541. MR **58:**22569
13. T. W. Gamelin, *Uniform Algebras,* Chelsea, New York, 1984.
14. P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity,* Second Edition, Chelsea, New York, 1957.
15. P. R. Halmos, *A Hilbert Space Problem Book,* Second Edition, Springer-Verlag, New York, 1982. MR **84e:**47001
16. N. P. Jewell, *Multiplication by the coordinate functions on the Hardy space of the unit sphere in* $\mathbf{C}^n$, Duke Math. J. **44** (1977), 839 - 851. MR **57:**3904
17. G. E. Keough, *Subnormal operators, Toeplitz operators, and spectral inclusion,* Trans. Amer. Math. Soc. **263** (1981), 125-135. MR **81k:**47033
18. S. G. Krantz, *Function Theory of Several Complex Variables,* John Wiley and Sons, New York, 1982. MR **84c:**32001
19. A. Lubin, *Weighted shifts and products of subnormal operators,* Indiana Univ. Math. J. **26** (1977), 839 - 845. MR **56:**6448
20. H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer-Verlag, New York, 1973. MR **51:**3924
21. W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, New York, 1987. MR **88k:**00002
22. W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980. MR **82i:**32002
23. D. Sarason, *The H^p spaces of an annulus,* Memoirs of the Amer. Math. Soc., Number 56, 1965. MR **32:**6256
24. A. L. Shields, *Weighted shift operators and analytic function theory,* Topics in Operator Theory (ed. C. Pearcy), Math. Surveys 13, Amer. Math. Soc., Providence, R. I., 1974, pp. 49-128. MR **50:**14341

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio 43403-0221
E-mail address: kchan@bgnet.bgsu.edu

Department of Mathematics, University of Toledo, Toledo, Ohio 43606-3390
E-mail address: zcuckovi@math.utoledo.edu