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BOURGIN–YANG TYPE THEOREM AND ITS APPLICATION TO Z_2-EQUIVARIANT HAMILTONIAN SYSTEMS

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Abstract. We will be concerned with the existence of multiple periodic solutions of asymptotically linear Hamiltonian systems with the presence of Z_2–action. To that purpose we prove a new version of the Bourgin–Yang theorem. Using the notion of the crossing number we also introduce a new definition of the Morse index for indefinite functionals.

1. Introduction

Let E be a separable Hilbert space and $\Phi : E \to R$ a continuously differentiable functional which is of the form $\Phi(z) = \frac{1}{2} \cdot \langle Lz, z\rangle + \psi(z)$, where $L : E \to E$ is a selfadjoint operator with $dim\ kerL < \infty$ and such that 0 is isolated in $\sigma(L)$. Suppose that Φ is even and assume additionally that the gradient $\nabla\Phi : E \to E$ is asymptotically linear, i.e. $\nabla\Phi$ satisfies the following limit conditions:

$$\nabla\Phi(z) = I_0 z + o(\|z\|) \quad \text{as } \|z\| \to 0, \tag{1.1}$$

$$\nabla\Phi(z) = I_\infty z + o(\|z\|) \quad \text{as } \|z\| \to \infty, \tag{1.2}$$

where $I_0, I_\infty : E \to E$ are selfadjoint linear operators which differ from L by compact linear maps. In fact, we have a slightly weaker assumption than (1.2) (see condition c5. in §3).

Our first aim is to obtain multiplicity results concerning the number of pairs of critical points of an even functional Φ. More precisely, we give estimates from below for the number of critical points of Φ in terms of a Morse type index defined for a pair (I_0, I_∞). Clearly, the operator I_0 is the derivative of $\nabla\Phi(z)$ at the origin whereas I_∞ can be regarded as a "derivative" of $\nabla\Phi(z)$ at infinity. Since both operators I_0 and I_∞ are compact perturbations of L, it follows that $kerI_0$ and $kerI_\infty$ are of finite dimension. On the other hand, the eigenspaces of I_0 and I_∞ corresponding to positive and negative parts of the spectra $\sigma(I_0)$ and $\sigma(I_\infty)$ are infinite dimensional. In this case Φ is called a strongly indefinite functional. The existence of critical points for strongly indefinite functionals together with applications to differential equations has been studied by many authors. Results obtained by Amann and Zehnder in [AZ1], [AZ2] seem to be the crucial step in such investigations. Morse theory in Hilbert spaces is thoroughly discussed in Chang [Ch] or Mawhin and Willem [MW]. Based on the Conley index (cf. [C]), a different approach to this theory has been developed by Benci (cf. [B1], [B2]).

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In this paper a new definition of the Morse index is presented. The idea is based on the observation that generically the spectral flow crosses the zero level finitely many times while we are moving along a path of compact and linear perturbations of L. Moreover, this crossing number depends only on the endpoints of the path. We do not require that I_0 be an isomorphism, so the origin can be a degenerate critical point for Φ. It turns out that one can also relax the assumption on I_∞ to be an isomorphism when the symmetry condition is involved. This means that we consider the problem with resonance at infinity.

Although we use standard methods of estimating the number of critical values of Φ based on the notion of Z_2-genus (cf. [R]), the approach presented here demands application of an infinite dimensional version of the Bourgin-Yang theorem (see Thm. 2.2 and cf. [Y]). This is a new topological result which generalizes that obtained by Granas in [G].

Our next purpose is to apply abstract results of Section 5 to problems of estimating of the number of periodic solutions of nonautonomous Hamiltonian systems

$$\dot{z} = J\nabla H(z,t) \tag{1.3}$$

with the presence of symmetry. Our symmetry assumption means that $\nabla H(-z,t) = -\nabla H(z,t)$ for every $z \in E$ and $t \in R$. Problems of that type have been considered by Amann and Zehnder [AZ2], Benci [B3], Clark [C1], Ambrosetti and Rabinowitz [AR] , Long [L1], Chang (cf. [Ch] and references there). We suppose that system (1.3) is asymptotically linear (see Def. 6.1) and, using the notions of the L–index and the L–nullity introduced in Section 4 (see Def. 4.1) we give a definition of positive and negative indices and the nullity for (1.3) (see Def. 6.1).

Such numbers defined for a pair of symmetric (time independent) matrices have been considered by Amann and Zehnder in [AZ1], [AZ2]. Also Ekeland introduced the index of a positive definite matrix (cf. [E]), and later Li and Liu in [LL] defined the index and the nullity for an arbitrary symmetric matix. All those numbers are related to each other which is seen directly from their definitions. One should also mention the Maslov index theory introduced by Conley and Zehnder in [CZ] in order to obtain the existence of more than one nontrivial solution of (1.3). This theory was later extended by Long and Zehnder [LZ] and Long [L2] so that the index is well defined for a "degenerate" periodic solution. Nevertheless, as long as we consider an asymptotically linear Hamiltonian system with $H \in C^1(R^{2N} \times R, R)$ the (generalized) Maslov index is defined at the origin and at infinity only. It follows from results of Long [L2] that this is the case when this invariant carries the same information as the indices mentioned above (if defined) and the one introduced in this paper.

The results we obtain here for Z_2–Hamiltonian systems improve those of Amann and Zehnder [AZ2], Benci [B3] and Long [L1]. All these authors require that the Hamiltonian H be twice continuously differentiable with bounded Hessian. This is because they use generalized Morse theory (cf. [C]) only after reducing the problem to a finite dimensional one. We will always assume that H is of class C^1. In the case considered here the linear part of $\nabla H(z,t)$ at the origin and at infinity depends on time, which (except for [L1]) was not allowed by the authors mentioned above. Moreover, our assumption on the behavior of $\nabla H(z,t)$ at infinity is weaker than usual (see Def. 6.1). In the recent paper by Fei [F1] it is also assumed that $H \in C^1$ and that the linear part of $\nabla H(z,t)$ depends on t. Using the notion of the Maslov

index and an abstract theorem due to Benci (see Thm. 0.1. in [B3]), a multiplicity result for periodic solutions of (1.3) is established. In particular, Thm. 3.1 in [F1] is obtained here as Cor. 6.1, but other theorems are proved under assumptions which are different from ours. Actually, our approach is completely different from all those of the papers mentioned above. We work directly in infinite dimensional spaces and use rather geometrical arguments. Methods used to prove basic results of this paper are classical, finite dimensional ones adopted only in the natural way to our infinite dimensional situation.

It is worth pointing out that our approach can be also applied in the case when the symmetry condition is dropped. We will briefly discuss this problem in Section 7.

2. Bourgin–Yang type theorems

Throughout this paper each map is supposed to be continuous. Let E be a Banach space. Consider a map $T : S \to Aut(E)$ from a unit sphere in E into the space of bounded automorphisms of E which is even and such that its image is relatively compact in $Aut(E)$. For convenience we will denote the image of T at x by T_x. Our first aim is to prove the following.

Theorem 2.1. *Let $f : S \to E^{\infty-1}$ be a map of a unit sphere in a Banach space E into a closed subspace of E of codimension 1 which is of the form*

$$f(x) = T_x(x) + K(x),$$

where $K : S \to E$ is compact. If f is an odd map, then f has a zero.

Proof. The map f can be written in the form

$$f(x) = T_x(x + F(x)),$$

where $F : S \to E$ is defined by $F(x) = T_x^{-1}K(x)$. Clearly, F is compact and for all $x \in S$ one has $F(-x) = -F(x)$. If a map $h(x) = x + F(x)$ has a zero, then also f has one, and our proof is finished. Let us assume that $h(S) \subset E \setminus \{0\}$. Since h is a compact perturbation of the identity, the Leray–Schauder degree of h, denoted by $deg_{LS}h$, is well defined. Moreover the condition $h(-x) = -h(x)$ for all $x \in S$ implies that $deg_{LS}h$ is odd, which means in particular that it is different from zero.

The Banach space E can be represented as a direct sum of the space $E^{\infty-1}$ and a 1–dimensional space E_1, $E = E^{\infty-1} \oplus E_1$. Let us choose an element $w \in E_1$ such that $\|w\| = 1$ and define a new map $Y : S \to E$ by the formula

$$Y(x) = T_x^{-1}(w).$$

One checks immediately that Y is an even and compact map. Now we need the following fact.

Lemma 2.1. *There exist $x_0 \in S$ and $t_0 > 0$ such that*

$$x_0 + F(x_0) = t_0 \cdot Y(x_0).$$

Proof of Lemma. Assume on the contrary that for every $x \in S$ and every positive $t \in R$ one has

$$x + F(x) - t \cdot Y(x) \neq 0.$$

Set $c = \inf\{\|Y(x)\|; x \in S\}$ and notice that c is a positive number. Choosing $t = t_0$ so large that

$$t_0 \cdot c > \sup\{\|F(x)\| + 1; x \in S\},$$

one can define a homotopy $H_1(x,s) = x + (1-s)\cdot F(x) - t_0 \cdot Y(x)$ joining $x + F(x) - t_0 \cdot Y(x)$ with $x - t_0 \cdot Y(x)$ and such that $H_1(x,s) \neq 0$ for all $x \in S$ and $s \in [0,1]$. By our assumptions we have also another homotopy $H_2 : S \times I \to E \setminus \{0\}$, $H_2(x,s) = x + F(x) - s \cdot t_0 \cdot Y(x)$, joining $x + F(x)$ with $x + F(x) - t_0 \cdot Y(x)$, and therefore we see that $x + F(x)$ and $x - t_0 \cdot Y(x)$ are homotopic as maps of the sphere S into $E \setminus \{0\}$. By the homotopy invariance property of the Leray–Schauder degree we conclude that $deg_{LS}(x - t_0 \cdot Y(x)) \neq 0$.

On the other hand, one can extend the map $P(x) = x - t_0 \cdot Y(x)$ to a map $\hat{P}$ from a closed unit ball $B(0,1) \subset E$ centered at the origin into $E \setminus \{0\}$ in the following way.

Since the closure $\overline{Y(S)}$ of $Y(S)$ is compact in $E \setminus \{0\}$, there exists a half line, say $l = \{x \in E; x = t \cdot e_0, t \geq 0, \|e_0\| = 1\}$, such that $l \cap \overline{Y(S)} = \emptyset$. Define a function $\nu : E \setminus \{0\} \to R$ by

$$\nu(x) = \begin{cases} 1 & \text{if } \|x\| \geq t_0 \cdot c, \\ \frac{t_0 \cdot c}{\|x\|} & \text{if } 0 < \|x\| \leq t_0 \cdot c, \end{cases}$$

and a map $Z : B(0,1) \to E \setminus \{0\}$

$$Z(x) = \begin{cases} (1 - \|x\|) \cdot e_0 - t_0 \cdot \|x\| \cdot Y(\frac{x}{\|x\|}) & \text{if } x \neq 0, \\ e_0 & \text{if } x = 0, \end{cases}$$

which is compact. Then the map $\hat{Z} : B(0,1) \to E \setminus \{0\}$ defined by $\hat{Z}(x) = \nu(Z(x)) \cdot Z(x)$ is compact and such that $\|\hat{Z}(x)\| \geq t_0 \cdot c > 1$.

Finally, a map $\hat{P} : B(0,1) \to E \setminus \{0\}$ given by the formula $\hat{P}(x) = x + \hat{Z}(x)$ is an extension of the map P of the form "identity – compact", and this implies that $deg_{LS} P = 0$ which is a contradiction. □

By our lemma we have therefore that $x_0 + F(x_0) = t_0 \cdot Y(x_0)$ for some $x_0 \in S$, $t_0 > 0$, and

$$f(x_0) = T_{x_0}(x_0 - F(x_0)) = T_{x_0}(t_0 \cdot Y(x_0)) = t_0 \cdot T_{x_0}(Y(x_0)) = t_0 \cdot w$$

does not belong to $E^{\infty-1}$. This contradicts our assumption. □

Now, let H be a Hilbert space and let $L : H \to H$ be a selfadjoint bounded endomorphism. Suppose that $\nu : S \to R$ is a bounded, continuous function defined on a unit sphere in H. The following fact is a consequence of Theorem 2.1.

Corollary 2.1. *If $f : S \to H^{\infty-1}$ is a map from a unit sphere in H into a closed subspace of H of codimension 1 which is of the form*

$$f(x) = e^{\nu(x)L} x + K(x),$$

where K is compact, then $f(x_0) = f(-x_0)$ for some $x_0 \in S$.

Proof. Let us define an odd map $g : S \to H^{\infty-1}$ by putting

$$g(x) = f(x) - f(-x) = (e^{\nu(x)L} + e^{\nu(-x)L})x + K(x) - K(-x).$$

Since L is a selfadjoint operator, an isomorphism e^{sL}, $s \in R$, is positive definite; therefore the map $T : S \to Aut(H)$, $T(x) = e^{\nu(x)L} + e^{\nu(-x)L}$, is well defined. Clearly, T is an even and compact map. By Theorem 2.1 the map g has a zero, which means that $f(x_0) = f(-x_0)$ for some $x_0 \in S$. □

In particular, if f is an odd map, which does not necessarily mean that ν has to be even and K is odd, we have $f(x_0) = 0$ for some $x_0 \in S$.

Corollary 2.1 is a type of an infinite dimensional version of the Borsuk–Ulam theorem. Let us recall that for any paracompact space X equipped with a free action of Z_2 one can define the Z_2–genus $\gamma(X)$, which is the smallest natural number k such that there is an odd map $e : X \to R^k \setminus \{0\}$. If no such map exists for any k we say that $\gamma(X) = +\infty$. For simplicity we will write the genus instead of the Z_2–genus. Theorem 2.1 can be generalized in the following way.

Theorem 2.2. *Let $f : S \to E^{\infty-k}$ be a map of a unit sphere in a Banach space E into a closed subspace of E of codimension k which is of the form*

$$f(x) = T_x(x) + K(x),$$

where T and K are as in Theorem 2.1. Assume that f is odd and denote by A_f the set of zeroes of f. Then:

1. *A_f is compact and symmetric with respect to the origin in E.*
2. *The genus $\gamma(A_f)$ of A_f is at least k.*

Proof. One can easily show that f is a proper map, which means by definition that the inverse image $f^{-1}(C)$ of any compact set $C \subset E$ is compact. Thus $A_f = f^{-1}(0)$ is compact. It is also symmetric, because f is odd.

Assume that the genus $\gamma(A_f) < k$, which means that there exists an odd map $\tilde{g} : A_f \to R^{k-1} \setminus \{0\}$. One can extend this map to a bounded and odd map $g : S \to R^{k-1}$. Consider a map $h : S \to E^{\infty-k} \oplus R^{k-1} \approx E^{\infty-1}$ defined by the formula

$$h(x) = f(x) + g(x) = T_x(x) + K(x) + g(x)$$

which is odd and satisfies all the assumptions of Theorem 2.1. Thus there exists $x_0 \in S$ such that $h(x_0) = 0$. As a consequence of the last equality we obtain that $f(x_0) = 0$ and $g(x_0) = 0$, which is impossible since $f(x_0) = 0$ implies $x_0 \in A_f$, and this in turn implies that $g(x_0) \neq 0$. Our proof is finished. □

As an immediate consequence of Theorem 2.2 we can formulate an infinite dimensional version of Bourgin–Yang theorem as follows.

Corollary 2.2. *If $f : S \to H^{\infty-k}$ is a map from a unit sphere in H into a closed subspace of H of codimension k which is of the form*

$$f(x) = e^{\nu(x)L}x + K(x),$$

where K is a compact map, then the genus of the set

$$A_f = \{x \in S;\ f(x) = f(-x)\}$$

is greater than or equal to k.

3. Pseudo-gradient vector fields

In this section we will suppose that H is a Hilbert space and $L : H \to H$ is a bounded linear selfadjoint operator such that either 0 is an isolated point in the spectrum $\sigma(L)$ or $0 \notin \sigma(L)$. We also assume that $dim\ kerL < \infty$. Consider a C^1–functional $\Phi : H \to R$ of the form

$$\Phi(z) = \frac{1}{2} \cdot \langle Lz, z\rangle + \psi(z)$$

satisfying the following conditions:

c1. $\Phi(\Theta) = 0$;

c2. $\nabla\psi$ is completely continuous ;

c3. $\nabla\psi(z) = B_0 z + o(\|z\|)$, $z \to 0$;
c4. $\nabla\psi(z) = B_\infty z + o(\|z\|)$, $z \to \infty$.

Here B_0 and B_∞ are selfadjoint compact operators.

Definition 3.1. Given a function $\Psi \in C^1(H, R)$, we say that a vector field $P : H \to H$ is a pseudo–gradient vector field for Ψ provided that:

- P is locally Lipschitz continuous at least on the set $H \setminus \nabla\Psi^{-1}(0)$;
- there are two positive numbers $\nu > \eta$ such that for all $z \in H$ one has

$$\|Pz\| \le \nu \cdot \|\nabla\Psi(z)\| \text{ and } \langle Pz, \nabla\Psi(z)\rangle \ge \eta \cdot \|\nabla\Psi(z)\|^2.$$

In particular we have from the definition that the sets of zeroes of $\nabla\Psi$ and P coincide. Let us denote $A = \{z \in H;\ \nabla\Phi(z) \neq 0\}$. In our considerations we will need the following two lemmas.

Lemma 3.1. *Assume that $L + B_0$ and $L + B_\infty$ are isomorphisms. Then there is a pseudo-gradient vector field for Φ*

$$Pz = Lz + Cz$$

such that

1. *$C : H \to H$ is completely continuous,*
2. *there exists a number $\alpha > 0$ such that $Pz = (L + B_0)z$ if $\|z\| < \alpha$,*
3. *there exists a number $\beta > 0$ such that $Pz = (L + B_\infty)z$ if $\|z\| > \beta$.*

If we suppose additionally that ψ is even, then P can be made odd.

Proof. We apply the method presented in [R] (Appendix A). Put $\rho(z) = \nabla\psi(z) - B_0 z$ and choose $\alpha > 0$ and $\beta > 3\alpha$ such that

$$\|\rho(z)\| < \frac{d_0}{4} \cdot \|z\| \quad \text{if } \|z\| \le 2\alpha \tag{3.1}$$

and

$$\|\nabla\psi(z) - B_\infty z\| < \frac{d_\infty}{4} \cdot \|z\| \quad \text{if } \|z\| \ge \beta - \alpha, \tag{3.2}$$

where $d_0 = dist(0, \sigma(L + B_0))$ and $d_\infty = dist(0, \sigma(L + B_\infty))$.

For every $z \in W = \{z \in A\ ;\ 2\alpha \le \|z\| \le \beta - \alpha\}$ there is an open neighbourhood N_z in H such that the inequalities

$$\|(L + B_0)y + \rho(z)\| < 2 \cdot \|\nabla\Phi(y)\|$$

and

$$\langle (L + B_0)y + \rho(z), \nabla\Phi(y)\rangle \ > \ \frac{1}{2} \cdot \|\nabla\Phi(y)\|^2$$

are satisfied for all $y \in N_z$. With no loss of generality we can assume that $diam(N_z) < \alpha$ for all $z \in W$. The family $\mathcal{N} = \{N_z : z \in W\}$ is an open covering of W. By Stone's theorem $\mathcal{N}$ has a locally finite refinement $\{M_j\}$. Define a new family

$$\mathcal{M} = \{M_j\} \cup \{M_0 = \{z \in H; \|z\| < 2\alpha\}\} \cup \{M_\infty = \{z \in H; \|z\| > \beta - \alpha\}\}$$

which is a locally finite covering of $A \cup \{0\}$. If $\overline{\gamma}_j(z)$ is a distance of z from the complement of M_j, then $\overline{\gamma}_j$ is a Lipschitz function and $\overline{\gamma}_j(z) = 0$ for $z \notin M_j$.

Put $\gamma_j(z) = \overline{\gamma}_j(z)/\sum_i \overline{\gamma}_i(z)$, then $\gamma_j(z)$ is a partition of unity associated with the covering $\mathcal{M}$. Each M_j ($j \neq 0, \infty$) is included in some N_{z_j}. Define $\overline{P} : A \cup \{0\} \to H$ by the formula

$$\overline{P}z = Lz + Cz = Lz + B_0 z + \gamma_\infty(z) \cdot (B_\infty - B_0)z + \sum_{j \neq 0,\infty} \gamma_j(z) \cdot \rho(z_j).$$

It follows from the construction that $\overline{P}$ is locally Lipschitz continuous and that conditions *2* and *3* are satisfied. By Mazur's theorem (cf. [DG]) which says that a convex hull of a compact set in a Banach space is compact, we find that C is completely continuous.

Our next step is to show that there are two constants $0 < \eta < \nu$ such that

$$\|\overline{P}z\| \leq \nu \cdot \|\nabla\Phi(z)\| \tag{3.3}$$

and

$$\langle \overline{P}z, \nabla\Phi(z)\rangle \ \geq \eta \cdot \|\nabla\Phi(z)\|^2 \tag{3.4}$$

for all $z \in A \cup \{0\}$. In order to do this we will consider three cases.

Case 1. Assume that $z \in B_{2\alpha} - \bigcup_{j\neq 0} M_j$. From the construction we have $\overline{P}z = (L + B_0)z$. Note that

$$2 \cdot \|\rho(z)\| \leq \frac{d_0}{2} \cdot \|z\| \leq d_0 \cdot \|z\| \leq \|(L + B_0)z\|;$$

thus the inequality

$$2 \cdot \|\nabla\Phi(z)\| \geq 2 \cdot \|(L + B_0)z\| - 2 \cdot \|\rho(z)\|$$

implies

$$\|\overline{P}z\| = \|(L + B_0)z\| \leq 2 \cdot \|\nabla\Phi(z)\|,$$

and therefore we can put $\nu = 2$. Using the definition of $\nabla\Phi$ we see that

$$\langle (L + B_0)z, \nabla\Phi(z)\rangle = \|\nabla\Phi(z)\|^2 - \langle \rho(z), \nabla\Phi(z)\rangle. \tag{3.5}$$

It follows from the inequality

$$\|\nabla\Phi(z)\| \geq \|(L + B_0)z\| - \|\rho(z)\| \geq d_0 \cdot \|z\| - \frac{d_0}{4} \cdot \|z\| \geq 2 \cdot \|\rho(z)\|$$

that

$$\|\rho(z)\| \cdot \|\nabla\Phi(z)\| \leq \frac{1}{2} \cdot \|\nabla\Phi(z)\|^2.$$

Applying the above and Schwarz's inequality to (3.5), we obtain

$$\langle \overline{P}z, \nabla\Phi(z)\rangle = \langle (L + B_0)z, \nabla\Phi(z)\rangle \geq \frac{1}{2} \cdot \|\nabla\Phi(z)\|^2;$$

thus η can be taken to be $\frac{1}{2}$.

Case 2. Assume that $z \in \bigcup_{j\neq 0,\infty} M_j \cap \{z \in H; \|z\| < \beta - \alpha\}$. Since the covering $\mathcal{M}$ is locally finite, $z \in M_j$ only for a finite members of $\mathcal{M}$, say $M_{j_1}, \ldots, M_{j_t}$. By our construction we have

$$\|(L + B_0)z + \rho(z_{j_i})\| < 2 \cdot \|\nabla\Phi(z)\|$$

and

$$\langle (L + B_0)z + \rho(z_{j_i}), \nabla\Phi(z)\rangle > \frac{1}{2} \cdot \|\nabla\Phi(z)\|^2$$

for $i = 1, \ldots, t$.

Therefore

$$\begin{aligned}\|\overline{P}z\| &= \|(L+B_0)z+\sum_{i=1}^{t}\gamma_{j_i}(z)\rho(z_{j_i})\| \\ &\le \|\sum_{i=1}^{t}\gamma_{j_i}(z)((L+B_0)z+\rho(z_{j_i}))\|+\|\gamma_0(z)\cdot(L+B_0)z\| \\ &\le \sum_{i=1}^{t}\gamma_{j_i}(z)\|(L+B_0)z+\rho(z_{j_i})\|+\gamma_0(z)\cdot\|(L+B_0)z\| \\ &\le \sum_{i=0}^{t}\gamma_{j_i}(z)\cdot 2\cdot\|\nabla\Phi(z)\| = 2\cdot\|\nabla\Phi(z)\|.\end{aligned}$$

Thus we can put $\nu = 2$. Analogously

$$\langle\overline{P}z,\nabla\Phi(z)\rangle = \langle(L+B_0)z+\sum_{i=1}^{t}\gamma_{j_i}(z)\cdot\rho(z_{j_i}),\nabla\Phi(z)\rangle$$

$$= \sum_{i=1}^{t}\gamma_{j_i}(z)\cdot\langle(L+B_0)z+\rho(z_{j_i}),\nabla\Phi(z)\rangle+\gamma_0(z)\cdot\langle(L+B_0)z,\nabla\Phi(z)\rangle$$

$$> \frac{1}{2}\cdot\sum_{i=0}^{t}\gamma_{j_i}(z)\cdot\|\nabla\Phi(z)\|^2 = \frac{1}{2}\cdot\|\nabla\Phi(z)\|^2,$$

and η can be chosen to be $\frac{1}{2}$.

Case 3. If $\|z\| > \beta-\alpha$, then using inequality (3.2) one can proceed the same way as in Case 1 to obtain appropriate inequalities.

Now, $\overline{P}$ can be uniquely extended by zero to the map $P : H \to H$, and obviously inequalities (3.3) and (3.4) hold for each $z \in H$ when $\overline{P}$ is replaced by P.

If $\psi(x)$ is an even map, then the gradient vector field $\nabla\Phi(x)$ is odd. Thus, choosing a symmetric partition of unity we obtain a pseudo-gradient vector field which is odd. □

It may happen that both operators $L+B_0$ and $L+B_\infty$ are not isomorphisms. In this case we have $ker(L+B_0) < \infty$ and $ker(L+B_\infty) < \infty$. We also know that $im(L+B_0)$ and $im(L+B_\infty)$ are invariant subspaces of $L+B_0$ and $L+B_\infty$ respectively, since both operators are selfadjoint and moreover $I_0 = (L+B_0)_{|imL+B_0}$ as well as $I_\infty = (L+B_\infty)_{|imL+B_\infty}$ are linear isomorphisms. Let us replace the condition **c4.** by a weaker one:

c5.: $\nabla\psi(z) = B_\infty z+\tau(z)$, where B_∞ is a selfadjoint compact operator, $\|\tau(z)\| \le \varepsilon\cdot\|z\|+c$ for all $z \in H$ and ε is sufficiently small.

The above lemma can be slightly generalized in the following way.

Lemma 3.2. *Suppose that $\varepsilon < \frac{1}{2}\cdot dist(0,\sigma(I_\infty))$. Then there is a pseudo–gradient vector field for Φ*

$$Pz = Lz+Cz$$

such that

1. *$C : H \to H$ is completely continuous,*
2. *there is a number $\alpha > 0$ such that $Pz = (L+B_0)z$ if $z \in im(L+B_0)$ and $\|z\| < \alpha$,*

3. *there is a number* $\beta > 0$ *such that* $Pz = (L + B_\infty)z$ *if* $z \in im(L + B_\infty)$ *and* $\|z\| > \beta$.

If we suppose additionally that ψ *is an even map, then* P *can be made odd.*

Although there are some parts of proof which demand more delicate constructions as well as more careful estimations, the idea is basically the same as in the proof of Lemma 3.1, and therefore we will not go into detail.

Remark 3.1. One can actually prove that there are a number $\beta > 0$, a pseudo–gradient vector field P for Φ and a neighborhood U of the set $\{z \in im(L + B_\infty); \|z\| \geq \beta\}$ such that $Pz = (L + B_\infty)z$ for all $z \in U$ and U is "cone–shaped", i.e.

$$z \in U \Longrightarrow t \cdot z \in U \quad \text{for all } t \geq 0. \tag{3.6}$$

We make use of this remark in Section 5.

4. The L–index of a compact selfadjoint operator

Throughout this section we will assume that $H = \bigoplus_{k=0}^{\infty} H_k$ is a separable Hilbert space. Subspaces H_k are supposed to be mutually orthogonal and of finite dimension. Let $L : H \to H$ be a bounded linear selfadjoint operator with a pointed spectrum $\sigma(L)$ such that $kerL = H_0$ and $L(H_k) = H_k$ for $k > 0$. Let $\{L_t\} = \{(L + B_t) : H \to H;\ t \in [0, 1]\}$ be a continuous family of linear maps, where $B_t : H \to H$ is compact and selfadjoint for all $t \in [0, 1]$. Roughly speaking, we would like to know how many eigenvalues change their sign while we are "moving" along the path $\{L_t\}$. The precise meaning of this statement is given in Definition (4.1). We begin our considerations with the following.

Proposition 4.1. *There are a finite dimensional subspace* $\widetilde{V}$ *of* H *and a number* $\overline{h} > 0$ *such that the inequality*

$$\|L_t(z)\| \geq \overline{h} \cdot \|z\| \tag{4.1}$$

is satisfied for all $z \in \widetilde{V}^\perp$ *and* $t \in [0, 1]$.

Proof. Let V_t denote the kernel of the operator L_t. If $c_t = dist(0, \sigma(L_t) \setminus \{0\})$, then $\|L_t(z)\| \geq c_t \cdot \|z\|$ whenever $z \in V_t^\perp$. Since $\{L_t\}$ is a continuous family of operators, there is an open neighbourhood O_t of t in $[0, 1]$ such that $\|L_s(z)\| \geq \frac{1}{2} c_t \cdot \|z\|$ for all $z \in V_t^\perp$ and $s \in O_t$. Since the family $\{O_t\}_{t \in [0,1]}$ is an open covering of the unit interval, one can choose a finite subcovering of $[0, 1]$, say $\{O_{t_1}, ..., O_{t_r}\}$. Let us define $\widetilde{V} = span\{v \in V_{t_i}\ ;\ i = 1, ..., r\}$ and $\overline{h} = min\{\frac{1}{2} \cdot c_{t_i}\ ;\ i = 1, ..., r\}$. It is then easily seen that the inequality (4.1) holds for all $z \in \widetilde{V}^\perp$ and $t \in [0, 1]$. □

Put

$$V = V_0 \oplus V_1 \oplus \widetilde{V}, \quad h = min\{\overline{h}, 1\}$$

and

$$b = max\{\|L_t\| + 1; 0 \leq t \leq 1\}.$$

Let $P_n : H \to H$ be an orthogonal projection onto a subspace $W = \bigoplus_{k=0}^{n} H_k$. Choose $n \in N$ so large that

$$\|B_t - P_n B_t P_n\| \leq \frac{h}{12} \quad \text{for all } t \in [0, 1] \tag{4.2}$$

and

$$\|P_n(z) - z\| \leq \frac{h}{12 \cdot b} \cdot \|z\| \text{ for } z \in V. \tag{4.3}$$

Consider a family of selfadjoint operators $\{L_t^W\} = \{(L + P_n B_t P_n)_{|W} : W \to W;\, t \in [0,1]\}$. We are interested in a change of spectra of operators $\{L_t^W\}$ when t runs from 0 to 1.

Lemma 4.1. *If $j = 0$ or 1 and λ is an eigenvalue of L_j^W, then either $\mid \lambda \mid \leq \frac{1}{3} \cdot h$ or $\mid \lambda \mid \geq \frac{2}{3} \cdot h$. Moreover L_j^W has exactly $dim\ kerL_j$ eigenvalues λ (counted with multiplicity) satisfying $\mid \lambda \mid \leq \frac{h}{3}$.*

Proof. We prove our lemma for $j = 0$. Consider a restriction of P_n to the space V, which by inequality (4.3) is a monomorphism. In particular, $dim\ kerL_0 = dimP_n(kerL_0)$. Let $z \in kerL_0$ and $y = P_n(z)$; then by (4.2) we obtain

$$\|(L + P_n B_0 P_n)z\| \ \leq\ \frac{h}{12} \cdot \|z\|$$

and by (4.3)

$$\begin{aligned}\|(L + P_n B_0 P_n)z - (L + P_n B_0 P_n)y\| \ &\leq\ \|L + P_n B_0 P_n\| \cdot \|z - y\| \\ &\leq \frac{h}{12 \cdot b}\|L + P_n B_0 P_n\| \cdot \|z\| \ \leq\ \frac{h}{12} \cdot \|z\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|L_0^W(y)\| \ &\leq\ \|(L + P_n B_0 P_n)y - (L + P_n B_0 P_n)z\| + \|(L + P_n B_0 P_n)z\| \\ &\leq\ \frac{h}{6} \cdot \|z\|\end{aligned}$$

By our assumptions it is clear that $\|z\| \ \leq\ 2 \cdot \|y\|$. Thus

$$\|L_0^W(y)\| \ \leq\ \frac{h}{3} \cdot \|y\|.$$

Suppose now that $y \in P_n(kerL_0)^{\perp} \cap W$. Let $P = Id - \overline{P}$, where $\overline{P} : H \to H$ is the orthogonal projection onto $kerL_0$. Since for every $v \in kerL_0$ one has

$$\mid \langle v, y \rangle \mid = \mid \langle v - P_n(v), y \rangle \mid \leq\ \|v - P_n(v)\| \cdot \|y\|,$$

it follows by (4.3) that $\|\overline{P}y\| \ \leq\ \frac{1}{12} \cdot \|y\|$. The last inequality implies that $\|Py\| \geq \frac{11}{12} \cdot \|y\|$, and therefore, by our assumptions on h, we obtain $\|L_0(Py)\| \ \geq\ h \cdot \|P(y)\| \ \geq \frac{11}{12} h \cdot \|y\|$. Since $\|L_0(y)\| = \|L_0(Py)\| \ \geq \frac{11}{12} h \cdot \|y\|$, we conclude by (4.2) that

$$\begin{aligned}\|L_0^W(y)\| \ &\geq\ \|L_0(y)\| - \|(L_0 - L_0^W)y\| \\ &\geq \frac{11}{12} h \cdot \|y\| - \|L_0 - L_0^W\| \cdot \|y\| \ \geq \frac{10}{12} h \cdot \|y\|.\end{aligned}$$

We have proved that the space W can be presented as a direct sum of two eigenspaces of L_0^W, $W = W_1 \oplus W_2$, such that $dimW_1 = dim\ kerL_0$ and $\|L_0^W(z)\| \leq \frac{1}{3} h \cdot \|z\|$ for $z \in W_1$ and $\|L_0^W(z)\| \geq \frac{2}{3} h \cdot \|z\|$ for $z \in W_2$. (We do not need them to be explicitly determined.) This completes our proof. □

Let e_j^-, e_j^0 and e_j^+ be the numbers of eigenvalues of L_j^W which are not greater than $-\frac{2}{3} \cdot h$, which have modulus not greater than $\frac{1}{3} \cdot h$ and which are not less than $\frac{2}{3} \cdot h$, where $j = 0, 1$.

Definition 4.1. The number $i_L^-(B_0, B_1) = e_0^- - e_1^-$ (resp. $i_L^+(B_0, B_1) = e_0^+ - e_1^+$) is called the negative (resp. positive) L–index of a pair (B_0, B_1). The number $i_L^0(B_0, B_1) = e_0^0 + e_1^0$ is called the L–nullity of a pair (B_0, B_1).

If B_0 is trivial then we simply write $i_L^-(B_1)$ instead of $i_L^-(0, B_1)$ and call this number the L–index of B_1. Similarily we define $i_L^+(B_1)$. The number $i_L^0(B_1)$ is by definition equal to $dim\ ker(L + B_1)$.

One can show that the definition of L–indices does not depend on the choice of P_n as long as the inequalities (4.2) and (4.3) are satisfied. Let $\{L'_t\} = \{(L + B'_t) : H \to H\ ;\ t \in [0, 1]\}$ be another family of operators such that $B_0 = B'_0$ and $B_1 = B'_1$. If P_m is an appropriate projection chosen for $\{L'_t\}$, then $P_k : H \to H$ with $k = max\{m, n\}$ works for both families $\{L_t\}$ and $\{L'_t\}$. From this we conclude that in fact our definition is independent of the choice of the family $\{L_t\}$ and depends only on its endpoints L_0 and L_1. The basic properties of the L–index are listed in the following.

Proposition 4.2. *If $B_0, B_1, B_2 : H \to H$ are selfadjoint and compact operators, then:*

a.

$$i_L^-(B_0, B_1) + i_L^-(B_1, B_2) = i_L^-(B_0, B_2); \tag{4.4}$$

b. $i_L^-(B_0, B_1) = -i_L^-(B_1, B_0)$, *and in particular* $i_L^-(B_0, B_1) = i_L^-(B_1) - i_L^-(B_0)$*;*

c. $i_L^-(B_0, B_1) + i_L^0(B_0, B_1) + i_L^+(B_0, B_1) = 2 \cdot dim\ ker L_1$, *and in particular* $i_L^-(B_0, B_1) + i_L^0(B_0, B_1) + i_L^+(B_0, B_1) = 0$ *if L_1 is an isomorphism;*

d. *if $L + B_0$ is an isomorphism, then $i_L^-(B_0, B_1) = i_L^-(B'_0, B_1)$ for any selfadjoint and compact operator B'_0 which is sufficiently close to B_0;*

e. $i_{L+B_0}^-(B_1, B_2) = i_L^-(B_0 + B_1, B_0 + B_2)$.

All the properties follow immediately from Def. 4.1, and therefore we skip the proof.

5. Some abstract theorems

In this section we consider a C^1–functional $\Phi : H \to R$ of the form

$$\Phi(z) = \frac{1}{2} \cdot \langle Lz, z\rangle + \psi(z)$$

satisfying conditions **c1.**, **c2.**, **c3.** and **c5.** from Section 3. We will also keep all the assumptions on the Hilbert space H and the operator $L : H \to H$ as in the previous section. Put $d_0 = dist(0, \sigma(L + B_0) \setminus \{0\})$ and $d_\infty = dist(0, \sigma(L + B_\infty) \setminus \{0\})$.

Suppose, first, that the eigenspace F^+ of $L + B_\infty = L_\infty$ corresponding to the positive part of the spectrum $\sigma(L+B_\infty)$ is a closed subspace of the space H^+ which in turn is the eigenspace of $L + B_0 = L_0$ corresponding to the positive part of the spectrum $\sigma(L + B_0)$.

Our first theorem is as follows.

Theorem 5.1. *Let Φ be an even functional satisfying the above conditions. Assume additionally that $L + B_\infty$ is an isomorphism and that the codimension of F^+ in H^+ is equal to $f < +\infty$. Then, Φ has at least f pairs of nonzero critical points.*

Proof. By Lemma 3.2 there is an odd pseudo–gradient vector field P for Φ which is linear in the space imL_0 if $\|z\| < \alpha$ and in $imL_\infty = H$ if $\|z\| > \beta$. Let F^- be the eigenspace of L_∞ corresponding to the negative part of the spectrum $\sigma(L_\infty)$,

and let F be the orthogonal complement of F^+ in H^+. The subspaces F^+ and F^- are mutually orthogonal in H, and $H = F^+ \oplus F^-$. Denote by $D_s(V)$ (resp. $S_s(V)$) a closed disc (resp. sphere) centered at the origin with radius s in a normed space V and consider the Cartesian product of two discs

$$M = D_{\beta_1}(F^+) \times D_{\beta_2}(F^-),$$

where $\beta_1 > \beta$ and the radius β_2 $(> \beta)$ is chosen so large that the quadratic form $\langle (L + B_\infty)z, z\rangle$ is negative for all $z \in D_{\beta_1}(F^+) \times S_{\beta_2}(F^-)$.

Put

$$D = \{z \in S_\alpha(H^+)\ ;\ \exists t > 0\ ,\ \eta(t, z) \in S_{\beta_1}(F^+) \times D_{\beta_2}(F^-)\},$$

where η is a flow defined by Cauchy's problem

$$\begin{cases} \frac{d\eta}{dt} & = & P(\eta), \\ \eta(0, z) & = & z. \end{cases}$$

By continuity of η we see that D is an open subset of $S_\alpha(H^+)$, and therefore its complement $K = S_\alpha(H^+) \setminus D$ is closed.

Lemma 5.1. *The Z_2–genus of K, $\gamma(K)$, is greater than or equal to f.*

Proof of Lemma. Obviously, $K \subset H^+$ is symmetric with respect to the origin in H. Assume, on the contrary, that there exists an odd map $\delta : K \to S(R^{f-1})$. Then δ can be extended to an odd and bounded map $l : S_\alpha(H^+) \to R^{f-1}$.

On the other hand we have a Z_2–map $h_1 : D \to S_{\beta_1}(F^+) \times D_{\beta_2}(F^-)$ given by the formula

$$h_1(z) = e^{\nu(z)L}z + U_1(\nu(z), z)$$

such that:

- $\nu : D \to R$ is "the first stricking time" function, and thus it is even ;
- $U_1 : D \to H$ is odd and $U_1(a, \cdot)$ is compact for all $a \in R$

(cf. e.g. [S]). Observe that if $\pi_{H^+} : H \to H$ is an orthogonal projection onto H^+ then a map $h_2 : D \to S_{\beta_1}(F^+) \times F$, $h_2 = \pi_{H^+} \circ h_1$, is still of the form

$$e^{\nu(z)L}z + U_2(\nu(z), z)$$

where $U_2 : D \to H^+$ has the same properties as listed above for U_1.

The set $C = l^{-1}(0)$ is a subset of D that is nonempty, closed and symmetric with respect to the origin. We show that the map $\nu_{|C} : C \to R$ is bounded. Assume, on the contrary, that there is a sequence $(z_n)_{n\in N} \subset C$ such that $(\nu(z_n))_{n\in N}$ tends monotonically to infinity. If (z_n) includes a converging subsequence $(z_{n_k}) \overset{k\to\infty}{\longrightarrow} z_0$, then $z_0 \in C$ and $\nu(z_0) < +\infty$. From the continuity of the flow we obtain that $(\nu(z_{n_k}))_{k\in N}$ is bounded. Assume, then, that there is no converging subsequence included in (z_n). With no loss of generality we may suppose that

$$\exists \rho > 0 \quad \forall n, m \in N\ \|z_n - z_m\| > \rho.$$

Let $c = inf\{\|Lz\|\ ;\ z \in H^+, \|z\| = 1\} = min\{x \in R\ ;\ x \in \sigma(L_{|H^+})\}$. Since $L : H^+ \to H^+$ is a selfadjoint and positive isomorphism, we have for all $z \in H^+$ and $a \geq 0$

$$\|e^{aL}z\| \ \geq\ (1 + a^2c^2)^{\frac{1}{2}} \cdot \|z\|.$$

Choose $a \in R$ such that

$$(1 + a^2c^2)^{\frac{1}{2}} \cdot \rho > 2 \cdot (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} + 1.$$

There is $n_0 \in N$ such that for all $n \geq n_0$ we have $\nu(z_n) \geq a$. Since ν is "the first sticking time" function, we see that

$$\forall\, n \geq n_0 \quad \|e^{\nu(z_{n_0})L} z_n - U_2(\nu(z_{n_0}), z_n)\| \ < (\beta_1^2 + \beta_2^2)^{\frac{1}{2}}.$$

Since the map $U_2(\nu(z_{n_0}), \cdot)$ is compact we may suppose that the sequence $(U_2(\nu(z_{n_0}), z_n)$ converges and that $\lim_{n\to\infty} U_2(\nu(z_{n_0}), z_n) = y_{n_0}$. For all $n \geq n_1$ let the following inequality be satisfied:

$$\|U_2(\nu(z_{n_0}), z_n) - y_{n_0}\| \leq \frac{1}{2}.$$

Then, for $m, n \ > \ max\{n_0, n_1\}$ we obtain

$$\begin{aligned} 2 \cdot (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} + 1 < \ & \|e^{\nu(z_{n_0})L} z_n - e^{\nu(z_{n_0})L} z_m\| \\ \leq \ & \|e^{\nu(z_{n_0})L} z_n - U_2(\nu(z_{n_0}), z_n)\| \\ & + \|U_2(\nu(z_{n_0}), z_n) - y_{n_0})\| + \|y_{n_0} - U_2(\nu(z_{n_0}), z_m)\| \\ & + \|U_2(\nu(z_{n_0}), z_m) - e^{\nu(z_{n_0})L} z_m\| \ < \ 2 \cdot (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} + 1, \end{aligned}$$

which is a contradiction. Thus the map $\overline{U_2} : C \to H^+$ defined by $\overline{U_2}(z) = U_2(\nu(z), z)$ is compact. Denote by μ an even and bounded extension of $\nu_{|C}$ to $S_\alpha(H^+)$. Let U_3 be an extension of $\overline{U_2}$ to an odd and compact map from $S_\alpha(H^+)$ to H^+. Such an extension does exist due to results by Dugundji (cf. [G]) and Mazur (cf. [DG]). Now, one can define a map $h_3 : S_\alpha(H^+) \to H^+$,

$$h_3(z) = e^{\mu(z)L} z + U_3(z).$$

If $\pi_F : H^+ \to H^+$ is the orthogonal projection onto F, then one can consider a map $h : S(H^+) \to F^+$,

$$h(z) = h_3(z) - \pi_F \circ h_3(z),$$

and finally define a map $g : S_\alpha(H^+) \to F^+ \oplus R^{f-1} \approx (H^+)^{\infty-1}$ by the formula

$$g(z) = h(z) + l(z)$$

which obviously is odd and satisfies all the assumptions of Corollary 2.1. Thus, there exists $z_0 \in S_\alpha(H^+)$ such that $g(z_0) = 0$. This is equivalent to the fact that $h(z_0) = 0$ and $l(z_0) = 0$. It follows from the last equality that $z_0 \in C$ and therefore $h(z_0) \in S_{\beta_1}(F^+)$ $(h(z_0) \neq 0)$, which is a contradiction. □

It is clear that under our assumptions the functional Φ satisfies the Palais–Smale condition. Let $\mathcal{E}$ be a family of sets in $H \setminus \{0\}$ which are closed in H and symmetric with respect to the origin.

For $1 \leq j \leq f$ define

$$\gamma_j = \{A \in \mathcal{E} \quad \forall t \geq 0 \ \eta(t, A) \subset M \& \gamma(A) \geq j\}.$$

This family of sets possesses the following properties:

1. $\gamma_j \neq \emptyset, \quad 1 \leq j \leq f$;
2. $\gamma_1 \supset \gamma_2 \supset ... \supset \gamma_f$;
3. if $A \in \gamma_s$ and $B \in \bigcup_j \gamma_j$ and $\phi : A \to B$ is an odd and continuous map then $B \in \gamma_s$;
4. if $A \in \gamma_j$ and $B \in \mathcal{E}$ with $\gamma(B) \leq s < j$, then $\overline{A \setminus B} \in \gamma_{j-s}$.

Indeed, 1. is a consequence of the above lemma, 2. is trivial, and 3. and 4. follow immediately from basic properties of the genus. If

$$c_j = \sup_{A\in\gamma_j} \inf_{z\in A} \Phi(z), \quad 1 \le j \le f,$$

then $-\infty < c_f \le ... \le c_2 \le c_1 < +\infty$.

Lemma 5.2. *If $c_j = ... = c_{j+p} = c$ and $K_C = \{z \in M;\ \Phi(z) = c \quad and \quad \nabla\Phi(z) = 0\}$, then $\gamma(K_C) \ge p+1$. In particular, c_j defined as above is indeed a critical value of Φ.*

Proof of Lemma. Suppose that $\gamma(K_C) \le p$. By the Palais–Smale condition $K_C \subset intM$ is compact, and that is why there exists a positive number δ such that the set

$$\widehat{K}_C = \{z \in M; \quad \|z - K_C\| \le \delta\}$$

has the following properties:

- $\widehat{K}_C \in \mathcal{E}$;
- $\gamma(\widehat{K}_C) = \gamma(K_C) \le p$.

Take $\mathcal{O} = int\widehat{K}_C$ and $\overline{\varepsilon} > 0$ such that c is the only critical value of $\Phi_{|M}$ in $[c-\overline{\varepsilon}, c+\overline{\varepsilon}]$. Define $A_s = \{z \in H; \quad \Phi(z) \ge s\}$. There is an $\varepsilon \in (0, \overline{\varepsilon})$ such that

$$\forall z \in A_{c-\varepsilon} \cap M \setminus \mathcal{O} \exists t_z > 0 \quad \text{with} \quad \eta(t_z, z) \in A_{c+\varepsilon} \cup \partial M.$$

Choose t_z to be "the first striking time"; then $\nu : A_{c-\varepsilon} \cap M \setminus \mathcal{O} \to R$, $\nu(z) = t_z$, is continuous, and the map $\phi(\cdot) = \eta(\nu(\cdot), \cdot) : A_{c-\varepsilon} \cap M \setminus \mathcal{O} \to A_{c+\varepsilon} \cup \partial M$ is odd.

Now, take $A \in \gamma_{j+p}$ such that $inf\{\Phi(z)\ ;\ z \in A\} > c-\varepsilon$. By 4., $\gamma(\overline{A \setminus \widehat{K}_C}) \ge j$, and obviously $\overline{A \setminus \widehat{K}_C} \in \gamma_j$. Therefore the image $\phi(\overline{A \setminus \widehat{K}_C}) \in \gamma_j$ and

$$c \ge inf\{\Phi(z); \quad z \in \phi(\overline{A \setminus \widehat{K}_C})\} = c + \varepsilon$$

(notice that $\phi(\overline{A \setminus \widehat{K}_C}) \subset M$), which is a contradiction.

The thesis of Theorem 5.1 is a direct consequence of the above lemma and our proof is complete. □

Remark. Following the lines of the proof of Lemma 5.1, one can actually prove that the genus of $\gamma(K)$ is $+\infty$ provided F is an infinite dimensional subspace of H^+.

Theorem 5.2. *Suppose that Φ is even and that the number ε from condition c5. satisfies $\varepsilon < d_\infty$. Assume that the L–nullity $i_L^0(B_\infty) = 0$ and that*

$$m = max\{i_L^-(B_0, B_\infty),\ i_L^+(B_0, B_\infty)\} > 0.$$

Then Φ possesses at least m pairs of nonzero critical points.

Proof. Since Lemma 3.2 cannot be directly applied to the functional Φ, we have to begin with some modifications. Choose $a_0 > 0$ so large that $\|\tau(z)\| < \|(L + B_\infty)(z)\|$ for $\|z\| \ge a_0$. Then, by Lemma 7.9 in [S] there exist $a_2 > a_1 \ge a_0$ and $\xi \in C^\infty(R, [0,1])$ such that

$$\xi(s) = \begin{cases} 1 & \text{if } s \le a_1, \\ 0 & \text{if } s \ge a_2, \end{cases}$$

and the functional

$$\tilde{\Phi}(z) = \frac{1}{2} \cdot \langle (L + B_\infty)(z), z\rangle + \xi(\|z\|) \cdot (\psi(z) - \frac{1}{2} \cdot \langle B_\infty(z), z\rangle)$$

satisfies

$$\|\nabla\tilde{\Phi}(z)\| \ \geq 1 \quad \forall z \text{ with } a_1 \leq \|z\| \leq a_2$$

and is even. The critical points of Φ and $\tilde{\Phi}$ coincide, and since

$$\|\nabla(\xi(\|z\|)\cdot(\psi(z) - \frac{1}{2}\cdot\langle B_\infty(z), z\rangle))\| \leq c$$

if c is sufficiently large, Lemma 3.2 can be applied to $\tilde{\Phi}$. In fact, $\nabla\tilde{\Phi}(z)$ is linear if $\|z\| \geq a_2$, and obviously one can arrange things so that the inequalities (3.3) and (3.4) are satisfied with constants $\nu = 2$ and $\eta = \frac{1}{2}$. Therefore there is an odd pseudo–gradient vector field for $\tilde{\Phi}$, $P(z) = L(z) + C(z)$, such that

$$P(z) = (L + B_0)(z) \quad \text{if } \ z \in im(L + B_0) \quad \text{and } \|z\| \leq \alpha,$$

$$P(z) = (L + B_\infty)(z) \quad \text{if } \ \|z\| \geq \beta,$$

for some positive numbers $\alpha \ < \ \beta$.

With no loss of generality we may assume that $m = i_L^+(B_0, B_\infty)$. Let $\{e_n\}_{n\in N}$ be an orthogonal basis of H consisting of eigenvectors of $L_0 = L + B_0$. Choose a number $n \in N$ so large that

$$\|(B_\infty - B_0) - Q_n(B_\infty - B_0)Q_n\| \leq \frac{1}{2}\cdot d_\infty, \tag{5.1}$$

where $Q_n : H \to H$ is an orthogonal projection onto $W = span\{e_i \ ; \ 1 \leq i \leq n\}$, and define a real function $\mu : R \to R$

$$\mu(t) = \begin{cases} 1 & \text{if } t \leq a_2, \\ -\frac{1}{a_2}\cdot t + 2 & \text{if } a_2 \leq t \leq 2\cdot a_2, \\ 0 & \text{if } t \geq 2\cdot a_2. \end{cases}$$

Now, we can slightly modify our vector field P, putting

$$\overline{P}(z) = \begin{cases} P(z) & \text{if } \|z\| \leq a_2, \\ L_0(z) + \mu(\|z\|)(B_\infty - B_0)(z) & \\ \qquad + (1 - \mu(\|z\|))Q_n(B_\infty - B_0)Q_n(z) & \text{if } \|z\| \geq a_2. \end{cases}$$

Clearly $\overline{P}$ is also an odd pseudo–gradient vector field for $\tilde{\Phi}$. It follows from (5.1) that

$$\{(L_0)_t\} = \{L_0 + (1-t)(B_\infty - B_0) + t\cdot Q_n(B_\infty - B_0)Q_n) \ ; \ 0 \leq t \leq 1\}$$

is a family of isomorphisms. Thus $i_{L_0}^+(B_\infty - B_0, Q_n(B_\infty - B_0)Q_n) = 0$, and therefore using properties (4.4) a. and e. we obtain

$$\begin{aligned} i_{L_0}^+(Q_n(B_\infty - B_0)Q_n) &= i_{L_0}^+(0, Q_n(B_\infty - B_0)Q_n) \\ &= i_{L_0}^+(0, B_\infty - B_0) = i_L^+(B_0, B_\infty) = m. \end{aligned}$$

Let G^+ be the eigenspace of $L_0 + Q_n(B_\infty - B_0)Q_n$ corresponding to the positive part of the spectrum $\sigma(L_0 + Q_n(B_\infty - B_0)Q_n)$. It is easily seen that there is an automorphism of the form "identity + compact", $Id + K : H \to H$ such that the image of G^+ by $Id + K$ is a closed subspace of H^+ of codimension m. In fact, there exists a Lipschitz continuous deformation of the identity map $A : H \times [a_2, a_2 + 1] \to H$ such that $A(\cdot, s)$ is a linear automorphism of the form "identity + compact" for $a_2 \leq s \leq a_2 + 1$ and $A(\cdot, a_2 + 1)$ maps G^+ onto a closed subspace of H^+ of codimension m.

Obviously, the sets of zeroes of $\overline{P}$ and $\tilde{P} : H \to H$ defined by the formula $\tilde{P}(z) = A(\overline{P}(z), \|z\|)$ coincide. Consequently, by Thm. 5.1 we see that $\tilde{P}$ possesses at least m nonzero pairs of zeroes, which implies that the number of pairs of nonzero critical points of Φ is greater than or equal to m. This completes our proof. □

Now, we assume that $F^+ \oplus F^0$ is a closed subset of H^+, where $F^0 = kerL_\infty$. Theorem 5.1 can be generalized in the following way.

Theorem 5.3. *Let Φ be an even functional satisfying the same conditions as in Thm. 5.1 except that $L + B_\infty = L_\infty$ is an isomorphism. Suppose that the codimension of $F^+ \oplus F^0$ in H^+ is equal to $f < +\infty$. Then Φ has at least f pairs of nonzero critical points.*

Proof. First of all we apply Lemma 3.2 together with Remark 3.1 in order to deal with a pseudo–gradient vector field P for Φ given in a more convenient form. Consider a Cartesian product of two discs

$$M = D_{\beta_1}(F^+ \oplus F^0) \times D_{\beta_2}(F^-),$$

where $\beta_1 > \beta$ and $\beta_2 > \beta$ are chosen so large that the the set $M_1 = D_{\beta_1}(F^+ \oplus F^0) \times S_{\beta_2}(F^-)$ is included in U (see Rem. 3.1) and the quadratic form $\langle (L + B_\infty)(z), z\rangle$ is negative on M_1. We may assume that the set of critical points of Φ is included in $intM$.

If η is a flow defined by Cauchy's problem

$$\begin{cases} \frac{d\eta}{dt} & = & P(\eta), \\ \eta(0, z) & = & z, \end{cases}$$

then, defining the set

$$D = \{z \in S_\alpha(H^+) \; ; \; \exists t > 0 \; , \; \eta(t, z) \in U \cap \partial M\},$$

we easily see that D is open in $S_\alpha(H^+)$, since η is continuous. Following the lines of the proof of Lemma 5.1, one can show that the genus of the set $K = S_\alpha(H^+) \setminus D$ satisfies $\gamma(K) \geq f + w$, where $w = dimF^0$. It is also clear that as long as $dim\ kerL < \infty$ the function $\Phi_{|M} : M \to R$ satisfies the Palais–Smale condition.

Let $\mathcal{E}$ be a family of sets in $H \setminus \{0\}$ which are closed in H and symmetric with respect to the origin.

For $1 \leq j \leq f$ define

$$\gamma_j = \{A \in \mathcal{E} \; ; A = \{\eta(\nu(z), z) \; ; \; z \in B \subset K \;\text{ and }\; \gamma(B) \geq j + w\}\},$$

where ν runs through all bounded functions from K into R_+ satisfying $\nu(z) = \nu(-z)$.

This family of sets possesses all the properties 1,..,4. listed just before Lemma 5.2. Put

$$c_j = \sup_{A \in \gamma_j} \inf_{z \in A} \Phi(z), \quad 1 \leq j \leq f.$$

If A is any member of γ_j , $j = 1, ..., f$, then obviously $\inf_{z \in A} \Phi(z) > 0$; moreover, $A \cap D_\beta(H) \neq \emptyset$, a direct consequence of the classical Borsuk–Ulam antipodal theorem. Thus

$$0 < c_f \leq ... \leq c_2 \leq c_1 < +\infty.$$

Proceeding as in the proof of Lemma 5.2, we deduce our result. □

As a consequence of the above theorem we obtain

Theorem 5.4. *Suppose that Φ is even and that the number ε from condition c5. satisfies $\varepsilon < d_\infty$. Assume that*

$$m = max\{i_L^-(B_0, B_\infty) - i_L^0(B_\infty), i_L^+(B_0, B_\infty) - i_L^0(B_\infty)\} > 0.$$

Then Φ possess at least m pairs of nonzero critical points.

Proof. Using the same arguments as in the proof of Thm. 5.2, one can modify Φ to a functional $\tilde{\Phi}$ in such a way that:

- the critical points of Φ and $\tilde{\Phi}$ coincide,
- $\|\nabla\tilde{\Phi}(z) - L_\infty(z)\| \le c$, for sufficiently large c and $z \in D_\beta(H) \cup U_1$,

where U_1 is a neighborhood of the set $\{z \in imL_\infty;\ \|z\| \ge \beta\}$ satisfying condition (3.6). Consequently, there is an odd pseudo–gradient vector field for $\tilde{\Phi}$, $P(z) = L(z) + C(z)$, such that

$$P(z) = (L + B_0)(z) \quad \text{if } z \in im(L + B_0) \text{ and } \|z\| \le \alpha,$$
$$P(z) = (L + B_\infty)(z) \quad \text{if} \quad z \in U_1,$$

where $\alpha < \beta$ are positive numbers.

With no loss of the generality we may assume that the set of zeroes of P is included in $D_\beta(H)$ and that $m = i_L^+(B_0, B_\infty) - i_L^0(B_\infty)$. Let $\{e_n\}_{n\in N}$ be an orthogonal basis of H consisting of eigenvectors of $L_0 = L + B_0$ with $\|e_n\| = \beta$ for all $n \in N$. It is clear that $e_n \in U_1$ if n is sufficiently large, say $n \ge N_0$.

One can fix $n \in N$ so large that the following conditions are satisfied:

1.
$$\|(B_\infty - B_0) - Q_n(B_\infty - B_0)Q_n\| \le \frac{1}{2} \cdot d_\infty \tag{5.2}$$
where $Q_n : H \to H$ is the orthogonal projection onto $W = span\{e_i\ ;\ 1 \le i \le n\}$
2. The eigenspace F_ρ of $L_0 + Q_n(B_\infty - B_0)Q_n$ corresponding to the part of the spectrum $\rho = \{\lambda \in \sigma(L_0 + Q_n(B_\infty - B_0)Q_n)\ ;\ |\lambda| > \frac{2}{3} \cdot d_\infty\}$ is included in $D_\beta(H) \cup U_1$.

(Observe, that $L_0 + Q_n(B_\infty - B_0)Q_n$ is an approximation of $L_\infty = L_0 + (B_\infty - B_0)$.)

Choose an open neighborhood U_2 of the set $\{z \in F_\rho\ ;\ \|z\| \ge \beta + 2\}$ satisfying condition (3.6) and such that $dist(D_\beta(H) \cup (H \setminus U_1), \overline{U_2}) > 0$. Now, define an even C^1–function $\mu : H \to [0, 1]$,

$$\mu(z) = \begin{cases} 1 & \text{iff } z \in D_\beta(H) \cup (H \setminus U_1), \\ 0 & \text{iff } z \in \overline{U_2 \setminus D_{2\beta+2}(H)}, \end{cases}$$

and consider a new vector field $\overline{P} : H \to H$

$$\overline{P}(z) = \begin{cases} P(z) \quad \text{if } z \in D_\beta(H) \cup (H \setminus U_1), \\ L_0(z) + \mu(z)(B_\infty - B_0)(z) \\ \qquad + (1 - \mu(z)) \cdot Q_n(B_\infty - B_0)Q_n(z) \quad \text{if } z \in U_1. \end{cases}$$

Clearly, $\overline{P}$ is an odd pseudo–gradient vector field for $\tilde{\Phi}$, and therefore $\overline{P}^{-1}(0) = P^{-1}(0) \subset D_\beta(H)$.

It follows from (5.2) that

$$\begin{aligned} i_{L_0}^+(B_\infty - B_0) &\ge i_{L_0}^+(Q_n(B_\infty - B_0)Q_n) - i_{L_0}^0(Q_n(B_\infty - B_0)Q_n) \\ &\ge i_{L_0}^+(B_\infty - B_0) - i_{L_0}^0(B_\infty - B_0) \end{aligned}$$

and by (4.4) d. and e. we obtain

$$i^+_{L_0}(Q_n(B_\infty - B_0)Q_n) - i^0_{L_0}(Q_n(B_\infty - B_0)Q_n) \geq i^+_L(B_0, B_\infty) - i^0_L(B_\infty) = m.$$

Let G^+ be the eigenspace of $L_0 + Q_n(B_\infty - B_0)Q_n$ corresponding to the positive part of the spectrum $\sigma(L_0 + Q_n(B_\infty - B_0)Q_n)$, and let

$$G^0 = ker(L_0 + Q_n(B_\infty - B_0)Q_n).$$

We see from the above inequality that there is an automorphism of the form "identity + compact", $Id + K : H \to H$, such that the image of $G^+ \oplus G^0$ by $Id + K$ is a closed subspace of $H^+ \oplus H^0$ of codimension greater than or equal to m. In fact, there exists a Lipschitz continuous deformation of the identity map $A : H \times [\beta + 2, \beta + 3] \to H$ such that $A(\cdot, s)$ is a linear automorphism of the form "identity + compact" for $\beta + 2 \leq s \leq \beta + 3$, and $A(\cdot, \beta + 3)$ maps $G^+ \oplus G^0$ onto a closed subspace of $H^+ \oplus H^0$ of codimension not less than m.

Obviously, the sets of zeroes of $\overline{P}$ and $\tilde{P} : H \to H$ defined by $\tilde{P}(z) = A(\overline{P}(z), \|z\|)$ coincide. Consequently, by Thm. 5.3 $\tilde{P}$ possess at least m nonzero pairs of zeroes, which implies that the number of pairs of nonzero critical points of Φ is greater than or equal to m. This completes our proof. □

6. Asymptotically linear Hamiltonian systems

For a Hamiltonian $H \in C^1(R^{2N} \times R, R)$ which is 2π–periodic in t, consider the Hamiltonian system of differential equations

$$\dot{z} = J\nabla H(z, t) \tag{6.1}$$

where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ is the standard symplectic matrix.

We will be concerned with the existence of 2π–periodic solutions of (6.1). Let us denote by $E = H^{\frac{1}{2}}(S^1, R^{2N})$ the Hilbert space of 2π–periodic, R^{2N}–valued functions

$$z(t) = a_0 + \sum_{k=1}^{\infty} a_k \cdot cos(kt) + b_k \cdot sin(kt), \text{ where } a_0, a_k, b_k \in R^{2N},$$

with the inner product given by

$$\langle z, z' \rangle = 2\pi \cdot a_0 \cdot a_0^{'} + \pi \cdot \sum_{k=1}^{\infty} k \cdot (a_k \cdot a_k^{'} + b_k \cdot b_k^{'}). \tag{6.2}$$

If $\|\nabla H(z, t)\| \leq c_1 + c_2 \cdot \|z\|^s$ for every $(z, t) \in R^{2N} \times R$ and some positive s, then $z(t)$ is a 2π–periodic solution of (6.1) if and only if it is a critical point of the functional $\Phi \in C^1(E, R)$ defined by

$$\Phi(z) = \frac{1}{2} \cdot \langle Dz, z \rangle + \phi(z), \tag{6.3}$$

where

$$\langle Dz, z \rangle = \int_0^{2\pi} \langle -J \cdot \dot{z}, z \rangle \, dt \quad \text{and} \quad \phi(z) = -\int_0^{2\pi} H(z, t) dt \tag{6.4}$$

(cf. [R]). It is also shown in [R] that the mapping $\nabla\phi$ is compact.

Choose $e_1, \ldots, e_{2N}$ the standard basis in R^{2N} and denote

$$E(0) = span\{e_1, \ldots, e_{2N}\},$$

$$E(k) = span\{(cos(kt)) \cdot e_m + (sin(kt)) \cdot J \cdot e_m : m = 1, \dots, 2N\}, \quad k \in Z - \{0\}.$$

It is seen from (6.2), (6.3) that D is a differential operator in E which is explicitly given by

$$Dz = \sum_{k=1}^{\infty} -J \cdot b_k \cdot cos(kt) + J \cdot a_k \cdot sin(kt).$$

So, $Dz = 0$ if z is a constant function and $Dz = \pm z$ for every $z \in E(\pm k)$, $k > 0$. Put $E^0 = E(0)$, $E^+ = \bigoplus_{k=1}^{\infty} E(k)$, $E^- = \bigoplus_{k=1}^{\infty} E(-k)$. Obviously, $E = E^+ \oplus E^0 \oplus E^-$, the spaces E^+, E^0, E^- are mutually orthogonal, and if $z = z^+ + z^0 + z^- \in E^+ \oplus E^0 \oplus E^-$, then

$$\langle Dz, z\rangle = \|z^+\|^2 - \|z^-\|^2 \text{ and } Dz = z^+ - z^-.$$

Assume that

$$\begin{aligned} J\nabla H(z,t) &= JA_\infty(t)z + \psi(z,t), \\ J\nabla H(z,t) &= JA_0(t)z + o(\|z\|) \text{ as } \|z\| \to 0, \end{aligned} \tag{6.5}$$

uniformly in t, for continuous loops of symmetric $2N \times 2N$–matrices $A_0(t + 2\pi) = A_0(t)$ and $A_\infty(t + 2\pi) = A_\infty(t)$. The mapping $\psi : R^{2N} \times R \to R^{2N}$ satisfies $\|\psi(z,t)\| \le \varepsilon\|z\| + c$ for some positive numbers ε , c and for all $(z,t) \in R^{2N} \times R$. Consider two operators $B_0, B_\infty : E \to E$,

$$\langle B_0 z, z\rangle = -\int_0^{2\pi} A_0(t)z \cdot z \, dt$$

and

$$\langle B_\infty z, z\rangle = -\int_0^{2\pi} A_\infty(t)z \cdot z \, dt.$$

Since both operators B_0,B_∞ are compact, one can define the D–indices $i_D^{\pm}(B_0, B_\infty)$ and the D–nullity $i_D^0(B_0, B_\infty)$.

Definition 6.1. If $J\nabla H(z,t)$ satisfies (6.5) with

$$\varepsilon < min\{|\lambda| ; \lambda \in \sigma(D + B_\infty) \setminus \{0\}\},$$

then we say that the Hamiltonian system (6.1) is the asymptotically linear. The numbers $i_D^{\pm}(B_0, B_\infty)$ (resp. $i_D^0(B_0, B_\infty)$) are called the indices (resp. the nullity) of asymptotically linear Hamiltonian system (6.1).

For our convenience we will also write $i^{\pm}(A_0(t), A_\infty(t))$ (resp. $i^0(A_0(t), A_\infty(t))$) and $i^{\pm}(A(t))$ (resp. $i^0(A(t))$) instead of $i_D^{\pm}(B_0, B_\infty)$ (resp. $i_D^0(B_0, B_\infty)$) and $i^{\pm}(B)$ (resp. $i^0(B)$).

Theorem 6.1. *Suppose that the Hamiltonian system* (6.1) *is asymptotically linear. Assume that $H(z,t)$ is even with respect to $z \in R^{2N}$, $i^0(A_\infty(t)) = 0$ and*

$$m = max\{i^-(A_0(t), A_\infty(t)), i^+(A_0(t), A_\infty(t))\} > 0.$$

Then, the system (6.1) *has at least m pairs of nontrivial 2π–periodic solutions (in addition to the trivial one $z = 0$).*

Proof. Since (6.1) is asymptotically linear we have

$$\dot{z} = J\nabla H(z,t) = J(A_0(t)z + \nabla\Psi_0(z,t))$$

with $\Psi_0 \in C^1(R^{2N} \times R, R)$, which is 2π–periodic in $t \in R$, even with respect to $z \in R^{2N}$, and $\|\nabla\Psi_0(z,t)\| = o(\|z\|)$ uniformly in t as $\|z\| \to 0$.

On the other hand,

$$\dot{z} = J\nabla H(z,t) = J(A_\infty(t)z + \nabla\Psi_\infty(z,t)),$$

where $\Psi_\infty \in C^1(R^{2N} \times R, R)$ is 2π–periodic in $t \in R$, even with respect to $z \in R^{2N}$, and $\|\nabla\Psi_\infty(z,t)\| \leq \varepsilon \cdot \|z\| + c$ for all $(z,t) \in R^{2N} \times R$ with

$$\varepsilon < dist(\{0\}, \sigma(D + B_\infty)).$$

Following the general construction given at the beginning of this section, we find that the functional $\Phi : E \to R$ corresponding to system (6.1) is defined by the formula

$$\Phi(z) = \frac{1}{2} \cdot \langle L_0 z, z\rangle + \phi_0(z),$$

where $L_0 = D + B_0$ is uniquely determined by the equations

$$\langle L_0 z, z\rangle = \langle (D + B_0)z, z\rangle = \int_0^{2\pi} \langle -J \cdot \dot{z} - A_0(t)z, z\rangle dt$$

and

$$\phi_0(z) = -\int_0^{2\pi} \Psi_0(z,t)dt.$$

The gradient $\nabla\phi_0$ is a completely continuous odd map, and by our assumptions $\|\nabla\phi_0(z)\| = o(\|z\|)$, $\|z\| \to 0$. Obviously, Φ can be also represented in the form

$$\Phi(z) = \frac{1}{2} \cdot \langle L_\infty z, z\rangle + \phi_\infty(z),$$

where $L_\infty = D + B_\infty$ is defined by

$$\langle L_\infty z, z\rangle = \langle (D + B_\infty)z, z\rangle = \int_0^{2\pi} \langle -J \cdot \dot{z} - A_\infty(t)z, z\rangle \, dt$$

and

$$\phi_\infty(z) = -\int_0^{2\pi} \Psi_\infty(z,t)dt.$$

It is known that the inequality $\|\nabla\Psi_\infty(z,t)\| \leq \varepsilon \cdot \|z\| + c$ for all $(z,t) \in R^{2N} \times R$ implies

$$\|\nabla\phi_\infty(z)\| \leq \varepsilon \cdot \|z\| + c_1$$

for all $z \in E$ (cf. [S], p.413).

Therefore $\phi_0(z) = (B_\infty - B_0)(z) + \nabla\phi_\infty(z)$, and the functional $\Phi(z) = \frac{1}{2} \cdot \langle L_0(z), z\rangle + \phi_0(z)$ satisfies all the assumptions of Theorem 5.2. Thus Φ possesses at least m pairs of nonzero critical points which correspond to nontrivial 2π–periodic solutions of the system (6.1). □

Let us assume for a moment that $i^0(A_0(t), A_\infty(t)) = 0$. Then, by Proposition 4.2 we see that $i^-(A_0(t), A_\infty(t)) = -i^+(A_0(t), A_\infty(t))$, and thus we come to a result which has been recently obtained by Fei (see Thm. 1.3. in [F1]).

Corollary 6.1. *Suppose that the Hamiltonian system* (6.1) *is asymptotically linear. Assume that $H(z,t)$ is even with respect to $z \in R^{2N}$, $i^0(A_0(t), A_\infty(t)) = 0$ and $m = \mid i^-(A_0(t), A_\infty(t)) \mid > 0$. Then, the system* (6.1) *has at least m pairs of nontrivial 2π–periodic solutions (in addition to the trivial one $z = 0$).* □

Our next theorem concerns a Hamiltonian system with resonance at infinity. It turns out that the presence of the group action allows us to drop the assumption that $D + B_\infty$ is an isomorphism.

Theorem 6.2. *Suppose that the Hamiltonian system* (6.1) *is asymptotically linear. Assume additionally that $H(z,t)$ is even with respect to $z \in R^{2N}$ and that*

$$m = max\{i^-(A_0(t), A(t)) - i^0(A_\infty(t)), i^+(A_0(t), A(t)) - i^0(A_\infty(t))\} > 0.$$

Then the system (6.1) *has at least m pairs of nontrivial 2π–periodic solutions (in addition to the trivial one $z = 0$).*

The proof of this theorem is practically the same as the proof of Theorem 6.1. We only need to apply Theorem 5.4 instead of Theorem 5.2.

7. Remarks on the existence theorems without symmetry

In this section sufficient conditions for the existence of a nontrivial periodic solution of a Hamiltonian system (6.1) satisfying conditions (6.5) is given. We drop the assumption that the Hamiltonian $H(z,t)$ is even with respect to $z \in R^{2N}$. Consider a C^1–functional $\Phi : H \to R$ of the form

$$\Phi(z) = \frac{1}{2} \cdot \langle L(z), z\rangle + \psi(z)$$

satisfying conditions **c1., c2., c3.** and **c5.** as in Section 3. Let the space H and the operator $L : H \to H$ be as at the beginning of Section 4., so that the L–index and the L–nullity are defined. We begin with the following.

Theorem 7.1. *Let Φ be as above and assume that $i_L^0(B_0, B_\infty) = 0$.*

If $i_L^-(B_0, B_\infty) \neq 0$ (or equivalently $i_L^+(B_0, B_\infty) \neq 0$), then Φ has at least one nonzero critical point.

Proof. Assume on the contrary that the origin is the only critical point of Φ. With no loss of generality we may assume that the operator B_0 is identically equal to zero, so that L is an isomorphism. Similarily to the proof of Theorem 5.2 we modify Φ to a functional $\tilde{\Phi} : H \to R$ of the form $\tilde{\Phi}(z) = \frac{1}{2} \cdot \langle L(z), z\rangle + \psi_1(z)$ such that:

- the critical points of Φ and $\tilde{\Phi}$ coincide;
- $\nabla\psi_1$ is completely continuous and $\|\nabla\psi_1(z)\| = o(\|z\|)$ as $\|z\| \to 0$;
- $\nabla\psi_1(z) = B_\infty(z) + \tau(z)$ and $\tau(z)$ is bounded.

Now, by Lemma 3.2 there is a pseudo-gradient vector field for $\tilde{\Phi}$, $P(z) = L(z)+C(z)$, such that:

- $C : H \to H$ is completely continuous;
- $P(z) = L(z)$ if $\|z\| \leq \alpha$;
- $P(z) = (L + B_\infty)(z)$ if $\|z\| \geq \beta$,

for some $\beta > \alpha > 0$. One can arrange things so that $\|P(z)\| \leq 2 \cdot \|\nabla\tilde{\Phi}(z)\|$ and $\langle P(z), \nabla\tilde{\Phi}(z)\rangle \geq \frac{1}{2} \cdot \|\nabla\tilde{\Phi}(z)\|^2$ for all $z \in H$. Keeping our notation as in Section 4, we denote by $P_n : H \to H$ an orthogonal projection onto a subspace $W = \bigoplus_{k=0}^n H_k$. One can easily verify that for every $a > 0$ the inequality

$$\|C(z) - P_nC(z)\| \leq a \cdot \|P(z)\| \tag{7.1}$$

holds for all $z \in H$ provided that n is sufficiently large. Choose $n \in N$ so large that (7.1) is satisfied with $a = \frac{1}{8}$ and

$$\|B_\infty - P_nB_\infty P_n\| < \frac{1}{8} \cdot dist(\{0\}, \sigma(L + B_\infty)).$$

Clearly the vector field $\tilde{P}(z) = L(z) + P_n C(z)$ is locally Lipschitz continuous and satisfies the following inequalities:

$$\|\tilde{P}(z)\| \leq 3 \cdot \|\nabla\tilde{\Phi}(z)\|$$

and

$$\langle \tilde{P}(z), \nabla\tilde{\Phi}(z)\rangle \geq \frac{1}{4} \cdot \|\nabla\tilde{\Phi}(z)\|^2$$

for all $z \in H$. Thus, $\tilde{P}$ is a pseudo–gradient vector field for $\tilde{\Phi}$. Notice that the restriction $\overline{P}$ of $\tilde{P}$ to the space W defines a vector field on W, and by our assumptions:

- $\overline{P}(z) = L(z)$ if $\|z\| \leq \alpha$;
- $\overline{P}(z) = L(z) + P_n B_\infty(z)$ if $\|z\| \geq \beta$.

Moreover, by Prop. 4.2 one has

$$i_L^-(0, B_\infty) = i_L^-(0, P_n B_\infty P_n) = i_{L_{|W}}^-(0, P_n B_\infty) \neq 0.$$

Therefore, by the classical Morse theory (applied to $\tilde{\Phi}_{|W}$) we see that $\overline{P}(z_0) = 0$ for some $z_0 \in W$ with $\alpha < \|z_0\| < \beta$. Obviously, $\tilde{P}(z_0) = \overline{P}(z_0) = 0$, which is a contradiction. □

If the functional Φ is of class C^2 at least in some neighbourhood of zero, then the assumption that $L + B_0$ is an isomorphism can be relaxed to the assumption that $\{0\}$ is isolated in $\sigma(L+B_0)\cup\{0\}$, and thus we are in the case where the origin is a degenerate critical point.

Theorem 7.2. *Assume that a C^1–functional Φ is of class C^2 in some neighbourhood of zero $U \subset H$ satisfying conditions c1.,c2.,c3. and c5. If $i_L^0(B_\infty) = 0$ and $i_L^-(B_0, B_\infty) \neq 0$ (or equivalently $i_L^+(B_0, B_\infty) \neq 0$), then Φ has at least one nonzero critical point.*

Proof. Similarily to the proof of Theorem 7.1, assume that zero is the only critical point of Φ. Without loss we may assume that the operator B_0 is equal to zero. We modify Φ to a functional $\tilde{\Phi} : H \to R$ of the form $\tilde{\Phi}(z) = \frac{1}{2} \cdot \langle L(z), z\rangle + \psi_1(z)$ such that:

- the origin is the unique critical point for $\tilde{\Phi}$;
- $\Phi(z) = \tilde{\Phi}(z)$ in a disc $D_r = \{z; \|z\| < r\} \subset U$;
- $\nabla\psi_1(z) = B_\infty(z) + \tau(z)$ and $\tau(z)$ is bounded.

By Lemma 3.2 there exists a pseudo–gradient vector field for $\tilde{\Phi}$, $P(z) = L(z)+C(z)$, satisfying the following properties:

- $C : H \to H$ is completely continuous;
- $P(z) = (L + B_\infty)(z)$ if $\|z\| \geq \beta$;
- $P(z) = \nabla\tilde{\Phi}(z) = \nabla\Phi(z)$ if $\|z\| < \alpha \leq r$,

for some $\beta > \alpha > 0$. Note that we do not need to modify the vector field $\nabla\tilde{\Phi}$ on D_r, since $\tilde{\Phi}$ is of C^2–class and therefore $\nabla\tilde{\Phi}$ is already locally Lipschitz continuous. One can arrange things so that $\|P(z)\| \leq 2 \cdot \|\nabla\tilde{\Phi}(z)\|$ and $\langle P(z), \nabla\tilde{\Phi}(z)\rangle \geq \frac{1}{2} \cdot \|\nabla\tilde{\Phi}(z)\|^2$ for all $z \in H$.

Let us take $0 < s < \frac{1}{2} \cdot \alpha$ such that

$$\|\tilde{\Phi}''(z) - L\| < \frac{1}{2} \cdot \|L^\#\| \text{ for } \|z\| < 2 \cdot s, \tag{7.2}$$

where $L^{\#} = (L_{|H_1})^{-1} : H_1 \to H_1$. Let $T > 0$ be chosen in such a way that $\|P(z)\| \geq T$ unless $\|z\| < s$, and let $n \in N$ be so large that:

- $\|C(z) - P_n C(z)\| \leq \frac{1}{4} \cdot T$ if $\|z\| \leq \beta$;
- $\|B_\infty - P_n B_\infty P_n\| \leq \frac{1}{8} \cdot dist(\{0\}, \sigma(L + B_\infty))$,

where $P_n : H \to H$ is an orthogonal projection onto a subspace $W = \bigoplus_{k=0}^{n} H_k$. One can easily check that the vector field $\tilde{P} : H \to H$, $\tilde{P}(z) = L(z) + P_n C(z)$, is locally Lipschitz continuous and that

$$\|\tilde{P}(z)\| \leq 3 \cdot \|\nabla \tilde{\Phi}(z)\|$$

and

$$\langle \tilde{P}(z), \nabla \tilde{\Phi}(z) \rangle \geq \frac{1}{4} \cdot \|\nabla \tilde{\Phi}(z)\|$$

for all $z \in H \setminus D_s$. Thus $\tilde{P}$ is a pseudo–gradient vector field for $\tilde{\Phi}$ outside the ball D_s, and in particular we have $\|\tilde{P}(z)\| \neq 0$ if $\|z\| \geq s$. The restriction $\overline{P}$ of $\tilde{P}$ to the space W defines a vector field on W which is pseudo–gradient for $\tilde{\Phi}_{|W} : W \to R$.

If $i_L^-(0, B_\infty) > 0$ then by Prop. 4.2 d. we obtain

$$i_L^-(B_\infty) = i_L^-(0, B_\infty) = i_L^-(0, P_n B_\infty P_n) = i_{L|W}^-(0, P_n B_\infty) > 0.$$

Let us denote by $m^+(.)$, $m^0(.)$ and $m^-(.)$ the positive, the zero and the negative Morse indices of the symmetric matrix respectively. Then

$$i_{L|W}^-(0, P_n B_\infty) = m^-(L_{|W}) - m^-(P_n B_{\infty|W})$$

and therefore

$$m^-(P_n B_{\infty|W}) < m^-(L_{|W})$$

Assume that $i_L^-(0, B_\infty) < 0$. Since $i_L^0(B_\infty) = 0$ we have (by Prop. 4.2 c.) that

$$i_L^-(B_\infty) + i_L^+(B_\infty) = 0,$$

and thus $m^-(P_n B_{\infty|W}) > m^-(L_{|W}) + m^0(L_{|W})$. In addition to the above, it follows from (7.2) that

$$\|\nabla \tilde{\Phi}_{|W}(z) - L_{|W}\| < \frac{1}{2} \cdot \|L_W^{\#}\| \quad \text{if} \quad \|z\| < 2 \cdot s,$$

where $L_W^{\#}$ is the inverse of the restriction of the operator L to the space $\bigoplus_{k=1}^{n} H_k$.

Now, applying Theorem (1.3) from [LL], we see that there exists at least one critical point $z_0 \in W$ of $\tilde{\Phi}_{|W}$ which lies outside the ball D_s, and therefore $\overline{P}(z_0) = 0$. Obviously z_0 is also a zero of $\tilde{P}$ and $\|z_0\| \geq s$, which is a contradiction. □

In what follows we formulate two theorems concerning the existence of nontrivial periodic solution of an asymptotically linear Hamiltonian system. We would like to mention that both theorems generalize results of Li and Liu [LL] and of Szulkin [S]. In particular we do not assume that the matrices A_0 and A_∞ are independent of t.

Theorem 7.3. *Assume that the Hamiltonian system* (6.1) *is asymptotically linear and that* $i^0(A_0(t), A_\infty(t)) = 0$. *If*

$$i^-(A_0(t), A_\infty(t)) \neq 0 \quad (\textit{or equivalently } i^+(A_0(t), A_\infty(t)) \neq 0)$$

then the system (6.1) *has at least one nontrivial* 2π*–periodic solution (in addition to the trivial one* $z = 0$*).* □

Actually, the proof is the same as that of Thm. 6.1 except that this time we do not care about the group action and we apply Thm. 7.1 instead of Thm. 5.2 to obtain the right conclusion.

Remark 7.1. The methods developed here do allow us to generalize appropriate theorems contained in [AZ1], [AZ2], [LL], [CZ], [Ch1] and [S]. However, Theorem 7.3 is very close to a known result by Chang [Ch] (Thm. 1.3, p.186). Our assumption on the behavior of ∇H at infinity (see (6.5)) is slightly weaker than that in [Ch], and that is the difference between the two results.

Observe that $\Phi \in C^2(H, R)$ whenever $H(z,t)$ is of C^2–class, and

$$\|H_{zz}(z,t)\| \leq a_1\|z\|^r + a_2 \tag{7.3}$$

for some $a_1, a_2, r > 0$ and for all $(z,t) \in R^{2N} \times R$.

Our next theorem is

Theorem 7.4. *Suppose that $H \in C^2(R^{2N} \times R, R)$ is 2π–periodic in $t \in R$ and satisfies* (7.3). *Assume that the Hamiltonian system* (6.1) *is asymptotically linear and that $i^0(A_\infty(t)) = 0$. If*

$$i^-(A_0(t), A_\infty(t)) \neq 0 \quad (\textit{(or equivalently } i^+(A_0(t), A_\infty(t)) \neq 0),$$

then the system (6.1) *has at least one nontrivial 2π–periodic solution (in addition to the trivial one $z = 0$).*

Proof. Since Φ is a C^2–function, our theorem is a consequence of Thm. 7.2 in the same way as Thm. 7.3 follows from Thm. 7.1. □

Remark 7.2. A similar result has been proved by Fei in [F, Thm. 1.2] with the additional condition that both matrices $A_0(t), A_\infty(t)$ are finitely degenerate (see Def. 2.5 in [F]). In Theorem 7.4 we have also a weaker assumption on the behavior of ∇H at infinity (see (6.5)).

One can still slightly relax the assumption on C^2-differentiability of H. It is enough to suppose that the Hamiltonian $H \in C^1(R^{2N} \times R, R)$ is C^2-differentiable with respect to the first $2N$ variables in $U \times R$, where U is some open neighborhood of $0 \in R^{2N}$. An assumption of that type has been made by Li and Liu in [LL, Thm. 3.6].

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