

SPECTRAL GAP ESTIMATES ON COMPACT MANIFOLDS

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ABSTRACT. For a compact Riemannian manifold with boundary, its mass gap is the difference between the first and second smallest Dirichlet eigenvalues. In this paper, taking a variational approach, we obtain an explicit lower bound estimate of the mass gap for any compact manifold in terms of geometric quantities.

0. INTRODUCTION

Let (M, g) be a compact Riemannian manifold with non-empty boundary, ∂M . The Laplace-Beltrami operator (Laplacian) acting on functions on M is defined by

$$\Delta(f) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial_i} \left(\sqrt{G} g^{ij} \frac{\partial f}{\partial_j} \right),$$

where the metric on M is given by $ds^2 = g_{ij} dx^i \otimes dx^j$, $(g^{ij}) = (g_{ij})^{-1}$ and $G = \det(g_{ij})$.

Eigenvalues of the Laplacian under Dirichlet boundary conditions are constants l_i which satisfy

$$\begin{cases} \Delta u_i + l_i u_i = 0, & x \in M, \\ u_i(x) = 0, & x \in \partial M, \end{cases}$$

for some nonzero eigenfunctions u_i . It is well known that the spectrum of the Dirichlet problem satisfies

$$\text{Spec}_D(M) = \{0 < l_1 < l_2 \leq l_3 \cdots \rightarrow \infty\},$$

so that the gap $l_2 - l_1$, sometimes referred to as the “mass gap”, is nontrivial.

In [S-W-Y-Y] the mass gap is estimated by analyzing the function $\phi = \frac{u_2}{u_1}$. They show that

$$\Delta \phi + 2 \nabla \log u_1 \cdot \nabla \log \phi + (l_2 - l_1) \phi = 0.$$

Therefore, $l_2 - l_1$ appears as an eigenvalue of a certain partial differential operator. Moreover, since it is easy to see that ϕ is smooth on M and satisfies $\frac{\partial \phi}{\partial \nu} = 0$, $l_2 - l_1$ is a Neumann eigenvalue of a certain partial differential operator. Thus, the gradient estimate techniques of Li and Yau [L-Y] can be employed to show that

$$l_2 - l_1 \geq \frac{\pi}{4D^2}$$

where M is a convex Euclidean domain and D is the diameter of M . The assumption of convexity seems to be crucial, however, to the argument since the Hessian

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of $\log(u_1)$ appears in the gradient estimate calculation. To control this term a log-concavity result of Brascamp and Lieb [B-L] is used. The result fails for non-convex domains and manifolds.

The first author and Cheng [C-O] were able to estimate the mass gap on Euclidean domains satisfying a rolling R-ball condition,

$$l_2 - l_1 \geq C(R, H, V, n),$$

where the second fundamental form of the boundary, Π , satisfies $\Pi \geq -H$ ($H \geq 0$), V is the volume of M and n is the dimension. This was done by introducing a weighted Cheeger's constant

$$h_u = \inf \frac{\int_H u}{\min\{\int_{M_1} u, \int_{M_2} u\}}$$

where the infimum is taken over all hypersurfaces H with $M \setminus H = M_1 \cup M_2$ and $\partial M_1 \cap \partial M_2 = H$. They then show that

$$l_2 - l_1 \geq \frac{1}{4} h_u^2$$

and give a lower bound for h_u in terms of R , H , V and n (see [C-O] for details).

In this paper, we show (Theorem 1.3)

$$l_2 - l_1 \geq C(K, H, R, D),$$

on a compact Riemannian manifold with nonempty boundary satisfying a rolling R-ball condition. In this formulation, $-K$ bounds the Ricci curvature from below, H is a positive, semi-definite matrix which bounds the second fundamental form of ∂M , R is determined by the rolling ball condition and D is the diameter of M .

As in [C-O], the key to the estimation technique is finding a lower bound for a ϕ^2 weighted Rayleigh-Ritz quotient on M , with some appropriately chosen ϕ (see Proposition 1.1, below). For suitable ϕ , for example the first Dirichlet eigenfunction, and bounded Euclidean domains satisfying rolling R-ball conditions, one can control the global behavior of the weighting function in terms of the distance function to the boundary—a Harnack-type inequality. Replacing the weighting function with the distance function allows one to use simple calculus to estimate the Rayleigh-Ritz quotient on M . Though these techniques do not carry over directly to the manifold setting, further study of the argument in [C-O] reveals that the essential elements of the estimation process are a Harnack inequality for the weighting function of the form $\phi(x) \leq C\phi(y)$ for all $x, y \in M$ satisfying $0 < d(x, \partial M) < 2d(y, \partial M)$, where C is constant and the use of a weak Neumann-Poincaré inequality on the manifold (see assumptions 1–4, below). The rolling ball condition leads to the Harnack inequality by controlling the growth of ϕ near the boundary of M in terms of the boundary geometry (H) and the global geometry (K). This naturally leads to an estimate of the constant, C , in terms of a possibly very small R , dependent on the smallest focal length as detailed in the remark following the proof of Theorem 1.3 and derived explicitly in §2.

The organization of this paper is as follows: In §1 we introduce notation and basic definitions and then prove the main theorem. In §2, which can be read independently, we prove a comparison result for the first Dirichlet eigenfunction. This verifies assumption (4), which is necessary for the main theorem.

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1.

Let l_1 and l_2 be the first and second Dirichlet eigenvalues of M . We start with the following proposition which was established by S. Y. Cheng and the first author in [C-O].

Proposition 1.1 (Cheng & Oden).

$$l_2 - l_1 = \inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 u^2 dx}{\inf_{k \in \mathbb{R}} \int_M |f - k|^2 u^2 dx}$$

where u is the positive eigenfunction corresponding to l_1 , i.e. $\Delta u - l_1 u = 0$ and $u > 0$ on M .

Proof.

$$\begin{aligned} & \inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 u^2}{\inf_{k \in \mathbb{R}} \int_M |f - k|^2 u^2} \\ &= \inf_{\substack{f \in C^1(\bar{M}) \\ \int_M f u^2 = 0}} \frac{\int_M |\nabla f|^2 u^2}{\int_M f^2 u^2} \\ &= \inf_{\substack{f \in C^1(\bar{M}) \\ \int_M f u^2 = 0}} \frac{\int_M (|\nabla(fu)|^2 - f^2 |\nabla u|^2 - 2fu \langle \nabla f, \nabla u \rangle)}{\int_M (fu)^2} \\ &= \inf_{\substack{f \in C^1(\bar{M}) \\ \int_M f u^2 = 0}} \frac{\int_M |\nabla(fu)|^2 - \int_M f^2 |\nabla u|^2 + \frac{1}{2} \int_M f^2 \Delta(u^2)}{\int_M (fu)^2} \\ &= \inf_{\substack{f \in C^1(\bar{M}) \\ \int_M f u^2 = 0}} \frac{\int_M |\nabla(fu)|^2 - l_1 \int_M f^2 u^2}{\int_M (fu)^2} \\ &= \inf_{\substack{f \in C^1(\bar{M}) \\ \int_M f u^2 = 0}} \frac{\int_M |\nabla(fu)|^2}{\int_M (fu)^2} - l_1 \\ &\geq \inf_{\substack{g \in C_0^1(\bar{M}) \\ \int_M g u = 0}} \frac{\int_M |\nabla g|^2}{\int_M g^2} - l_1 \geq l_2 - l_1. \end{aligned}$$

On the other hand, let v be an eigenfunction of l_2 . Then $f_0 = \frac{v}{u} \in C^\infty(\bar{M})$ as shown in [S-W-Y-Y] and

$$l_2 - l_1 = \frac{\int_M |\nabla f_0|^2 u^2}{\inf_{k \in \mathbb{R}} \int_M |f_0 - k|^2 u^2} \geq \inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 u^2}{\inf_{k \in \mathbb{R}} \int_M |f - k|^2 u^2}.$$

Thus, the proposition is proved.

In the following, we are going to use the variational characterization of $l_2 - l_1$ to give a lower bound estimate. We consider more generally the problem of estimating

$$\inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 \varphi^2 dx}{\inf_{k \in \mathbb{R}} \int_M |f - k|^2 \varphi^2 dx},$$

where $\varphi(x)$ is a given function on M and $\varphi > 0$ on M . We assume

1. M satisfies volume doubling property, i.e. $\exists C_1 > 0$ s.t. for any ball $B_x(2r) \subset M$ we have $\frac{|B_x(2r)|}{|B_x(r)|} \leq C_1$.
2. M satisfies interior rolling R-ball condition. i.e. for all $x \in \partial M, \exists B_p(R) \subset M$ s.t. $B_p(R) \cap \partial M = \{x\}$.
3. M satisfies a weak Neumann-Poincaré inequality on balls, i.e. for any ball $B_x(r) \subset M, B_x(r) \cap \partial M = \emptyset$, we have

$$\inf_{k \in \mathbb{R}} \int_{B_x(\frac{r}{2})} |f - k|^2 dx \leq C_2 r^2 \int_{B_x(r)} |\nabla f|^2 dx$$

for all $f \in C^1(\bar{M})$, where C_2 is a constant independent of x and r .

4. $\exists C_3 > 0$ s.t. $\varphi(x) \leq C_3 \phi(y)$ for all $x, y \in M$ with $0 < d(x, \partial M) \leq 2d(y, \partial M)$.

Theorem 1.2. *Under assumptions 1, 2, 3, and 4, we have*

$$\inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 \varphi^2 dx}{\inf_{k \in \mathbb{R}} \int_M |f - k|^2 \varphi^2 dx} \geq C(C_1, C_2, C_3, R, \mu, D),$$

where μ is the first nonzero Neumann eigenvalue of $M_{\frac{R}{2}} = \{x \in M : d(x, \partial M) \geq \frac{R}{2}\}$ and $D = \text{diam}(M)$, the diameter of M .

Our proof of the theorem is adapted from [J] and [SC-S].

Proof. We begin with the following claim.

Claim 1. There exists a collection of balls $\mathcal{F} = \{B_{x_i}(r_i) : i \in I\}$ s.t.

- (i) $\bigcup_{i \in I} B_{x_i}(2r_i) = M$ and $B_{x_i}(r_i) \cap B_{x_j}(r_j) = \emptyset$ for any $i \neq j, i, j \in I$,
- (ii) $d(B_{x_i}(r_i), \partial M) = 10^3 r_i$,
- (iii) $\#\{B_{x_i}(r_i) \in \mathcal{F} : \bigcap B_{x_i}(10r_i) \neq \emptyset\} \leq C(C_1)$.

In fact, let B_1 be a ball in M of largest radius r_1 satisfying $d(B_1, \partial M) = 10^3 r_1$. Let B_2 be a ball in M of largest radius r_2 satisfying $d(B_2, \partial M) = 10^3 r_2$ and $B_2 \cap B_1 = \emptyset$. Keep choosing balls in this way to obtain \mathcal{F} . We check that \mathcal{F} satisfies (i), (ii) and (iii). Let $x \in M$. Then $\exists r > 0$ s.t. $d(B_x(r), \partial M) = 10^3 r$. We may assume $B_x(r) \notin \mathcal{F}$. It is clear then $\exists B \in \mathcal{F}$ s.t. the radius of B satisfies $r(B) \geq r$ and $B \cap B_x(r) \neq \emptyset$.

Write $B = B_{x_i}(r_i)$. Then for $y \in B_{x_i}(r_i) \cap B_x(r)$,

$$d(x, x_i) \leq d(x, y) + d(y, x_i) \leq r + r_i \leq 2r_i.$$

Thus $x \in B_{x_i}(2r_i)$ and $M = \bigcup_{i \in I} B_{x_i}(2r_i)$. So (i) follows. (ii) is trivially true by the construction. To check (iii), suppose $\bigcap_{i \in J} B_{x_i}(10r_i) \neq \emptyset$. Let $y \in \bigcap_{i \in J} B_{x_i}(10r_i)$. Then $d(x_i, y) \leq 10r_i$. Thus for any $x \in B_{x_i}(r_i), d(x, y) \leq d(x, x_i) + d(x_i, y) \leq 11r_i$. Hence $\bigcup_{i \in J} B_{x_i}(r_i) \subset B_y(20r)$, where $r = \max\{r_i\} = r_{i_0}$. On the other hand,

$$d(x_{i_0}, x_j) \leq d(x_i, y) + d(y, x_j) \leq 10r_{i_0} + 10r_j \leq 20r.$$

So

$$\begin{aligned} d(x_j, \partial M) &\geq d(x_{i_0}, \partial M) - d(x_{i_0}, x_j) \\ &\geq d(B_{x_{i_0}}(r_{i_0}), \partial M) - 20r \\ &\geq 10^3 r_{i_0} - 20r > 101r. \end{aligned}$$

Hence $d(B_{x_j}(r_j), \partial M) > 100r$ and $r_j > 10^{-1}r$. In conclusion,

$$\bigcup_{i \in J} B_{x_i}(r_i) \subset B_y(20r) \subset B_{x_j}(30r) \subset B_{x_j}(300r_j)$$

Therefore

$$\sum_{i \in J} |B_{x_i}(r_i)| \leq |B_y(20r)| \leq \frac{\sum_{j \in J} |B_{x_j}(300r_j)|}{\#\{B_{x_i}(r_i) : \bigcap B_{x_i}(10r_i) \neq \emptyset\}}.$$

Hence

$$\#\{B_{x_i}(r_i) : \bigcap B_{x_i}(10r_i) \neq \emptyset\} \leq \frac{\sum_{j \in J} |B_{x_j}(300r_j)|}{\sum_{j \in J} |B_{x_j}(r_j)|} \leq C(C_1)$$

and (iii) follows.

Now let $M_R = \{x \in M; d(x, \partial M) \geq R\}$. Let $\mathcal{L} = \{B_i \in \mathcal{F} : x_i \notin M_R\}$. For $B_i \in \mathcal{L}$, let $y_i \in \partial M$ s.t. $d(x_i, y_i) = d(x_i, \partial M)$. By the interior rolling R-ball condition, $\exists B_{q_i}(R)$ s.t. $B_{q_i}(R) \cap \partial M = y_i$. Then clearly $d(q_i, y_i) = d(q_i, \partial M) = R$. Let $\overline{q_i y_i}$ be a minimal geodesic. Then $\overline{q_i y_i} \perp \partial M$. In particular, this implies such a minimal geodesic is unique up to a reparametrization. Also, let $\overline{x_i y_i}$ be a minimal geodesic realizing the distance between x_i and ∂M . Then $\overline{x_i y_i} \perp \partial M$. This forces $x_i \in \overline{q_i y_i}$. We denote by l_i the segment $\overline{q_i x_i}$ of $\overline{q_i y_i}$. We then define, for $B_i \in \mathcal{L}$, $\mathcal{F}(B_i) = \{A \in \mathcal{F} : 2A \cap l_i \neq \emptyset\}$. Let $\mathcal{H} = \{A \in \mathcal{F} : A \in \mathcal{F}(B_i) \text{ for some } B_i \in \mathcal{L}\}$. We have the following claim.

Claim 2. (i) $A \in \mathcal{F}(B)$ implies $r(A) \geq 10^{-1}r(B)$.

(ii) For any $A \in \mathcal{H}$, let

$$A(\mathcal{L}) = \{B \in \mathcal{L} : A \in \mathcal{F}(B)\}.$$

Then

$$r(A)^2 |A|^{-1} \sum_{B \in A(\mathcal{L})} \#\mathcal{F}(B) |B| \leq C(C_1) R^2.$$

We first check (i). For $A \in \mathcal{F}(B)$, $2A \cap l_B \neq \emptyset$. Let $y \in 2A \cap l_B$ and x be the center of the ball B . Then

$$\begin{aligned} d(A, \partial M) &\geq d(y, \partial M) - d(A, y) \\ &\geq d(x, \partial M) - d(A, y) \\ &\geq 10^3 r(B) - d(A, y) \\ &\geq 10^3 r(B) - 2r(A). \end{aligned}$$

Thus $10^3 r(A) \geq 10^3 r(B) - 2r(A)$ and $r(A) \geq 10^{-1}r(B)$.

To check (ii), we show

- (a) $\#\{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\} \leq C(C_1)$ for any $r > 0$, where C is independent of r .
 (b) $\#\mathcal{F}(B) \leq C \ln \frac{CR}{r(B)}$, $C = C(C_1)$.
 (c) $\exists \varepsilon = \varepsilon(C_1) > 0$ s.t.

$$\sum_{B \in A(\mathcal{L}), r \leq r(B) \leq 2r} |B| \leq C |A| \left(\frac{r}{r(A)}\right)^\varepsilon \text{ for } A \in \mathcal{H}$$

where $r > 0$ is arbitrary, $C = C(C_1)$ constant.

For (a), suppose $A_1, A_2 \in \{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\}$ and $y_1 \in 2A_1 \cap l_B$, $y_2 \in 2A_2 \cap l_B$. Let $A_1 = B_{\xi_1}(r_1)$ and $A_2 = B_{\xi_2}(r_2)$. Since for $i = 1, 2$,

$$d(y_i, \partial M) \geq d(\xi_i, \partial M) - d(\xi_i, y_i) \geq 10^3 r_i - 2r_i$$

and

$$d(y_i, \partial M) \leq d(\xi_i, \partial M) + d(\xi_i, y_i) \leq 10^3 r_i + 3r_i,$$

we have

$$\begin{aligned} d(y_1, y_2) &\leq |d(y_1, \partial M) - d(y_2, \partial M)| \\ &\leq (10^3 + 3)\frac{11}{10}r - (10^3 - 2)r \\ &\leq 15r. \end{aligned}$$

Now it is clear that

$$A_2 \subset B_{y_1}(20r) \text{ and } d(y_1, \partial M) \geq 998r.$$

Therefore

$$\bigcup \{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\} \subset B_{y_1}(20r).$$

Using the volume doubling property, we conclude

$$\#\{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\} \leq C(C_1).$$

Now (b) follows from (a). In fact, $A \in \mathcal{F}(B)$ implies that $10^{-1}r(B) \leq r(A) \leq R$. Therefore

$$\begin{aligned} \#\mathcal{F}(B) &\leq \#\{A \in \mathcal{F}(B) : 10^{-1}r(B) \leq r(A) \leq \frac{11}{10}10^{-1}r(B)\} \\ &+ \#\{A \in \mathcal{F}(B) : \frac{11}{10}10^{-1}r(B) \leq r(A) \leq (\frac{11}{10})^2 10^{-1}r(B)\} \\ &+ \dots + \#\{A \in \mathcal{F}(B) : (\frac{11}{10})^k 10^{-1}r(B) \leq r(A) \leq (\frac{11}{10})^{k+1} 10^{-1}r(B)\} \end{aligned}$$

where k satisfies $(\frac{11}{10})^k 10^{-1}r(B) \leq R \leq (\frac{11}{10})^{k+1} 10^{-1}r(B)$. From (a) we know $\#\mathcal{F}(B) \leq Ck$. But $k \leq C \ln \frac{CR}{r(B)}$. Thus $\#\mathcal{F}(B) \leq C \ln \frac{cR}{r(B)}$.

Now we come to (c). First, note that for $B \in A(\mathcal{L})$, $r(B) < 10r(A)$ by (i) of Claim 2. So $\frac{r}{r(A)} < 10$. Choose $\eta_0 \in \partial M$ s.t. $d(\eta_0, A) = d(A, \partial M) = 10^3 r(A)$. Let $s = 10^{10} r(A)$. For $\delta > 0$, denote

$$R(\delta) = \{\eta \in B(\eta_0, s + 2\delta s) : d(\eta, \partial M) < \delta s\}.$$

Then for $B \in A(\mathcal{L})$ and $r(B) \leq 2r$, we have $B \subset R(\frac{r}{r(A)})$. In fact, let $B = B_x(r(B))$. Then

$$d(x, \partial M) \leq d(B, \partial M) + r(B) \leq 10^3 r(B) + 2r \leq 10^4 r.$$

So $B \subset \{\eta : d(\eta, \partial M) < 10^{10}r\}$. To show $B \subset B(\eta_0, s + 2\delta s)$, $\delta = \frac{r}{r(A)}$, note that $A \in \mathcal{F}(B)$. Therefore, $\exists y \in \overline{xq} \cap 2A$. Hence

$$d(x, y) \leq d(y, \partial M) \leq 3r(A) + d(A, \partial M) \leq (10^3 + 3)r(A).$$

So

$$d(x, \eta_0) \leq d(x, y) + d(y, \eta_0) \leq 2(10^3 + 3)r(A)$$

and for $z \in B_x(r(B))$,

$$d(z, \eta_0) \leq d(z, x) + d(x, \eta_0) < 10^{10}r(A) + 2r.$$

Therefore $B_x(r(B)) \subseteq R(\frac{r}{r(A)})$. On the other hand, by the volume doubling property, we have $|A| \approx |B(\eta_0, 2s)|$. Thus, (c) follows if we can show

$$|R(\delta)| \leq C\delta^\varepsilon |B(\eta), 2s|, \quad 0 < \delta < 10.$$

We first show that $\exists \sigma(C_1) > 0$ s.t.

$$(1 + \sigma)|R(\frac{1}{4}\delta)| \leq |R(\delta)| \quad \text{for } 0 < \delta < 10^{10}.$$

Choose a maximal set of points $\xi_1, \dots, \xi_N \in \partial M$ s.t. $B(\xi_i, \delta s) \cap B(\eta_0, s) \neq \emptyset$ and $\{B(\xi_i, \delta s); i = 1, \dots, N\}$ are pairwise disjoint. It is then easy to check that

$$R(\frac{1}{4}\delta) \subset \bigcup_{j=1}^N B(\xi_j, 10\delta s).$$

Also, $\exists B(\eta_j, \frac{\delta s}{2}) \subset \bar{M} \setminus \partial M$ s.t. $d(\eta_j, \xi_j) \leq \frac{\delta s}{2}$. In fact, since $A \in \mathcal{H}$, $A \in \mathcal{F}(B)$ for some $B \in \mathcal{L}$. Thus, $d(A, \partial M) \leq r(A) + R$ and $r(A) \leq 10^{-2}R$. By the interior rolling R-ball condition, $\exists q_j$ s.t. $B_{q_j}(R) \cap \partial M = \{\xi_j\}$. Now choose $\eta_j \in \overline{q_j \xi_j}$ s.t. $d(\eta_j, \xi_j) = \frac{\delta s}{2} \leq \frac{r(A)}{2} < 10^{-2}R$. Then

$$B(\eta_j, \frac{\delta s}{2}) \subset B_{q_j}(R) \subset \bar{M} \setminus \partial M \quad \text{and} \quad d(\eta_j, \xi_j) = \frac{\delta s}{2}.$$

Now we have

$$B(\eta_j, \frac{\delta s}{4}) \subset R(\delta) \quad \text{and} \quad B(\eta_j, \frac{\delta s}{4}) \cap R(\frac{1}{4}\delta) = \emptyset.$$

Moreover, $\exists \sigma > 0$ s.t. $|B(\eta_j, \frac{\delta s}{4})| > \sigma |B(\xi_j, 10\delta s)|$. Therefore,

$$\sum_{j=1}^N |B(\eta_j, \frac{\delta s}{4})| > \sigma \sum_{j=1}^N |B(\xi_j, 10\delta s)| \geq \sigma |R(\frac{1}{4}\delta)|.$$

Consequently,

$$|R(\delta)| \geq |R(\frac{1}{4}\delta)| + |\bigcup_{j=1}^N B(\eta_j, \frac{\delta s}{4})| > (1 + \sigma)|R(\frac{1}{4}\delta)|$$

as the balls $B(\eta_j, \frac{\delta s}{4})$ are mutually disjoint.

For any $\delta > 0$, write $\delta \cong (\frac{1}{4})^k \delta_0$, where δ_0 fixed. Then

$$|R(\delta)| \leq \left(\frac{1}{1+\sigma}\right)^k |R(\delta_0)| \leq C\delta^\varepsilon |B(\eta_0, 2s)|.$$

Thus (c) follows.

Now (ii) of Claim 2 can be checked. In fact,

$$\begin{aligned} & r(A)^2 |A|^{-1} \sum_{B \in \mathcal{A}(\mathcal{L})} \#\mathcal{F}(B) |B| \\ & \leq r(A)^2 |A|^{-1} \sum_{B \in \mathcal{A}(\mathcal{L})} C \ln \frac{CR}{r(B)} |B| \\ & \leq r(A)^2 |A|^{-1} \sum_{k=-\infty}^0 \sum_{\substack{B \in \mathcal{A}(\mathcal{L}) \\ 2^k r(A) \leq r(B) \leq 2^{k+1} r(A)}} C \ln \frac{CR}{r(B)} |B| \\ & \leq r(A)^2 |A|^{-1} \sum_{k=-\infty}^0 C \ln \frac{CR}{2^k r(A)} |A| \left(\frac{2^k r(A)}{r(A)}\right)^\varepsilon \\ & \leq r(A)^2 \sum_{k=-\infty}^0 \left[\left(C \ln \frac{CR}{r(A)}\right) 2^{k\varepsilon} - (C \ln 2) k 2^{k\varepsilon} \right] \\ & \leq r(A)^2 \left[C \ln \frac{CR}{r(A)} + C \right] \\ & \leq CR^2 \text{ as } r(A) \leq CR. \end{aligned}$$

With those two claims, we can now finish the proof of the theorem.

For $B \in \mathcal{L}$, let A_1, \dots, A_l be the elements in $\mathcal{F}(B)$ s.t. $A_1 = B$, $q \in 2A_l$. Let

$$f'_B = \int_B \frac{f\varphi^2}{\int_B \varphi^2}, f'_0 = \frac{\int_{M_{R/2}} f\varphi^2}{\int_{M_{R/2}} \varphi^2}. \text{ Then}$$

$$\begin{aligned} & \int_{4B} |f - f'_0|^2 \varphi^2 \\ & \leq 2 \int_{4B} |f - f'_{4B} + \sum_{i=1}^{l-2} (f'_{4A_i} - f'_{4A_{i+1}})|^2 \varphi^2 + 2 \int_{4B} \varphi^2 |f'_{l-1} - f'_0|^2 \\ & \leq 2l \left[\int_{4B} |f - f'_{4B}|^2 \varphi^2 + \int_{4B} \varphi^2 \sum_{i=1}^{l-2} |f'_{4A_i} - f'_{4A_{i+1}}|^2 \right] + 2 \int_{4B} \varphi^2 |f'_{l-1} - f'_0|^2. \end{aligned}$$

Since

$$\begin{aligned} & \int_{4A_i \cap 4A_{i+1}} \varphi^2 |f'_{4A_i} - f'_{4A_{i+1}}|^2 \\ & \leq 2 \left[\int_{4A_i} |f - f'_{4A_i}|^2 \varphi^2 + \int_{4A_{i+1}} |f - f'_{4A_{i+1}}|^2 \varphi^2 \right], \end{aligned}$$

and

$$\begin{aligned}
& \int_{4B} \varphi^2 |f'_{l-1} - f'_0|^2 \\
& \leq \frac{\int_{4B} \varphi^2}{\int_{4A_{l-1} \cap M_{R/2}} \varphi^2} \left(\int_{4A_{l-1} \cap M_{R/2}} \varphi^2 |f'_{l-1} - f'_0|^2 \right) \\
& \leq 2 \frac{\int_{4B} \varphi^2}{\int_{4A_{l-1} \cap M_{R/2}} \varphi^2} \left(\int_{4A_{l-1} \cap M_{R/2}} (|f - f'_{l-1}|^2 + |f - f'_0|^2) \varphi^2 \right) \\
& \leq 2 \frac{\int_{4B} \varphi^2}{\int_{4A_{l-1} \cap M_{R/2}} \varphi^2} \left(\int_{4A_{l-1}} |f - f'_{l-1}|^2 \varphi^2 + \int_{M_{R/2}} |f - f'_0|^2 \varphi^2 \right),
\end{aligned}$$

we conclude

$$\begin{aligned}
& \int_{4B} |f - f'_0|^2 \varphi^2 \\
& \leq 2l \left[\int_{4B} |f - f'_{4B}|^2 \varphi^2 + \sum_{i=1}^{l-2} \frac{\int_{4B} \varphi^2}{\int_{4A_i \cap 4A_{i+1}} \varphi^2} \int_{4A_i \cap 4A_{i+1}} \varphi^2 |f'_{4A_i} - f'_{4A_{i+1}}|^2 \right] \\
& \quad + 4 \frac{\int_{4B} \varphi^2}{\int_{4A_{l-1} \cap M_{R/2}} \varphi^2} \left(\int_{4A_{l-1}} |f - f'_{l-1}|^2 \varphi^2 + \int_{M_{R/2}} |f - f'_0|^2 \varphi^2 \right) \\
& \leq 4l \left[\int_{4B} |f - f'_{4B}|^2 \varphi^2 \right. \\
& \quad \left. + \sum_{i=1}^{l-2} \frac{\int_{4B} \varphi^2}{\int_{4A_i \cap 4A_{i+1}} \varphi^2} \left(\int_{4A_i} |f - f'_{4A_i}|^2 \varphi^2 + \int_{4A_{i+1}} |f - f'_{4A_{i+1}}|^2 \varphi^2 \right) \right] \\
& \quad + 4 \frac{\int_{4B} \varphi^2}{\int_{4A_{l-1} \cap M_{R/2}} \varphi^2} \left(\int_{4A_{l-1}} |f - f'_{l-1}|^2 \varphi^2 + \int_{M_{R/2}} |f - f'_0|^2 \varphi^2 \right).
\end{aligned}$$

From assumption (4) on φ , it is easy to see that

$$\frac{\int_{4B} \varphi^2}{\int_{4A_i \cap 4A_{i+1}} \varphi^2} \leq C(C_3) \frac{|B|}{|4A_i \cap 4A_{i+1}|} \leq C(C_1, C_3) \frac{|B|}{|A_i|}$$

and

$$\frac{\int_{4B} \varphi^2}{\int_{4A_{l-1} \cap M_{R/2}} \varphi^2} \leq C(C_1, C_3) \frac{|B|}{|A_{l-1}|}.$$

On each A_i , we also have $\varphi(x) \leq C_2 \varphi(y)$ for all $x, y \in A_i$. Therefore by assumption (3),

$$\int_{4A_i} |f - f'_i|^2 \varphi^2 \leq C_2 r^2(A_i) \int_{4A_i} |\nabla f|^2 \varphi^2.$$

On $M_{R/2}$, $\varphi(x) \leq C_4(C_3, R, D) \varphi(y)$ for $x, y \in M_{R/2}$, where D is the diameter of M . Thus

$$\int_{M_{R/2}} |f - f'_0|^2 \varphi^2 \leq \frac{C_4}{\mu} \int_{M_{R/2}} |\nabla f|^2 \varphi^2.$$

In conclusion, we have

$$\begin{aligned} \int_{4B} |f - f'_0|^2 \varphi^2 &\leq C \#\mathcal{F}(B)|B| \sum_{A_i \in \mathcal{F}(B)} r(A_i)^2 |A_i|^{-1} \int_{4A_i} |\nabla f|^2 \varphi^2 \\ &\quad + \frac{C_4|B|}{\mu|A_l|} \int_{M_{R/2}} |\nabla f|^2 \varphi^2. \end{aligned}$$

Using assumption (1), it is easy to see $|A_l| \geq C(C_1, D, R)|M|$. Thus

$$\begin{aligned} \int_{4B} |f - f'_0|^2 \varphi^2 &\leq C \#\mathcal{F}(B)|B| \sum_{A_i \in \mathcal{F}(B)} r(A_i)^2 |A_i|^{-1} \int_{4A_i} |\nabla f|^2 \varphi^2 \\ &\quad + \frac{C_5|B|}{\mu|M|} \int_{M_{R/2}} |\nabla f|^2 \varphi^2. \end{aligned}$$

Summing over all $B \in \mathcal{L}$, we have

$$\begin{aligned} &\int_{M \setminus M_R} |f - f'_0|^2 \varphi^2 \\ &\leq \sum_{B \in \mathcal{L}} \int_{4B} |f - f'_0|^2 \varphi^2 \\ &\leq C \sum_{B \in \mathcal{L}} \#\mathcal{F}(B)|B| \sum_{A \in \mathcal{F}(B)} r(A)^2 |A|^{-1} \int_{4A} |\nabla f|^2 \varphi^2 + C_5 \frac{|M \setminus M_R|}{\mu|M|} \int_{M_{R/2}} |\nabla f|^2 \varphi^2 \\ &\leq C \sum_{A \in \mathcal{H}} r(A)^2 |A|^{-1} \sum_{B \in \mathcal{A}(\mathcal{L})} |B| \#\mathcal{F}(B) \int_{4A} |\nabla f|^2 \varphi^2 + \frac{C_5}{\mu} \int_{M_{R/2}} |\nabla f|^2 \varphi^2 \\ &\leq CR^2 \int_M |\nabla f|^2 \varphi^2 + \frac{C_5}{\mu} \int_{M_{R/2}} |\nabla f|^2 \varphi^2, \end{aligned}$$

where we have used (ii) of Claim 2. Thus

$$\begin{aligned} \int_M |f - f'_0|^2 \varphi^2 &\leq \int_{M \setminus M_R} |f - f'_0|^2 \varphi^2 + \int_{M_{R/2}} |f - f'_0|^2 \varphi^2 \\ &\leq (CR^2 + \frac{C_5 + 1}{\mu}) \int_M |\nabla f|^2 \varphi^2. \end{aligned}$$

Therefore

$$\frac{\int_M |\nabla f|^2 \varphi^2}{\inf_{k \in \mathbb{R}} \int_M |f - k|^2 \varphi^2} \geq \frac{1}{CR^2 + \frac{C_6}{\mu}},$$

where $C_6 = C(C_1, C_2, C_3, R, D)$ and $C = C(C_1, C_2, C_3)$. The theorem is proved.

Remark. Notice that the assumptions (1), (2) and (3) are stable under quasi-isometry on (M, g) .

In order to apply Theorem 1.1 to estimate $l_2 - l_1$, we need to verify assumptions (1), (2) and (3) for (M, g) and (4) for u . (4) is verified in §2. To verify assumptions (1), (2) and (3) we assume now the following for (M, g) .

- (5) $Ric_M \geq -K$ on M for some constant $K \geq 0$.

- (6) The second fundamental form of ∂M w.r.t. the outward unit normal satisfies $\Pi \geq -H$, where $H \geq 0$.

Theorem 1.3. *Suppose (M, g) satisfies assumptions (2), (5) and (6). Then $l_2 - l_1 \geq C(K, H, R, D)$, where R is chosen to be “small”. (See Remark.)*

Proof. It has been shown in [W] that M satisfies (1) with $C_1 = C(K, R)$ by arranging R “small”. M satisfies (3) by fact $Ric_M \geq -K$ and $C_2 = C(K, D)$ (see [SC-S]). To estimate μ from below, we use the result in [C].

Note first that $M_{R/2}$ satisfies interior r-rolling ball condition with $r \geq \frac{R}{4}$. In fact, $\forall x \in \partial M_{R/2}$, $\exists y \in \partial M$ s.t. $d(x, y) = d(x, \partial M)$. Since M satisfies interior R-rolling ball condition, $\exists B_p(R) \subset M$ s.t. $B_p(R) \cap \partial M = \{y\}$. Now let \overline{py} be the minimal geodesic. Let $q \in \overline{py}$ s.t. $d(p, q) = \overline{pq} = \frac{1}{4}R$. Then it can be easily checked that $B_q(\frac{1}{4}R) \subset M_{R/2}$ and $B_q(\frac{1}{4}R) \cap \partial M_{R/2} = \{x\}$.

Next, we want to estimate the second fundamental form of $\partial M_{R/2}$ from below. Let $f(x) = d(x, \partial M)$. Then $M_{R/2} = \{f(x) \geq \frac{R}{2}\}$ and $\partial M_{R/2} = \{f(x) = \frac{R}{2}\}$. Since $|\nabla f| = 1$ on $M \setminus M_R$, it is easy to see that $\Pi_{\partial M_{R/2}} = \text{Hessian}(f)$ on M . By the index comparison theorem in [H-K] or [Wr], we have, by choosing R small,

$$\text{Hessian}(f) \geq -C(R, H).$$

Thus, $\Pi_{\partial M_{R/2}} \geq -C(R, H)$. By [C], we conclude that

$$\mu \geq C(R, H, K, D)$$

In conclusion, we have

$$l_2 - l_1 \geq C(R, D, K, H)$$

where R is chosen to be “small”.

Remark. The number R is chosen to satisfy the following:

- (a) $\sqrt{K_R} \tan(R\sqrt{K_R}) \leq \frac{H}{2} + \frac{1}{2}$,
- (b) $H \tan(R\sqrt{K_R}) \leq \frac{1}{2}\sqrt{K_R}$, where K_R is the upper bound of the sectional curvature of M on the set $M \setminus M_R$. Clearly, if M is a Euclidean domain, then $K_R = 0$ and R can be taken as in (2).

2.

In this section we will verify property (4) for the first eigenfunction given Dirichlet conditions on a manifold M with $\partial M \neq \emptyset$.

In order to do this we will need bounds on $\Delta d(x, \partial M)$ for x “close” to ∂M . To this end we recall the notation from §1 and set

$$M_\delta = \{x \in M \mid d(x, \partial M) > \delta\}.$$

We define a map $\Psi : \partial M \times [0, \infty) \rightarrow M$ by setting $\Psi(y, \rho) = \exp(\rho N_y)$ where N_y is the inward pointing unit normal at $y \in \partial M$ and recall the following standard facts about Ψ :

1. If ∂M satisfies an interior rolling R-ball condition, then there exists $\delta > 0$ such that $\Psi : \partial M \times [0, \delta) \rightarrow M$ is C^2 .
2. For the δ of (1), $d(x, \partial M) = d(x, y) = \rho$, and $\rho(x)$ is C^2 for $x \in M \setminus M_\delta$.

3. If dV is the volume element on M and Φ is the volume element on ∂M , then $\Psi^*(dV) = \mathcal{H}(y, \rho)\Phi \wedge d\rho$ and $\Delta\rho(x) = \Delta d(x, \partial M) = \frac{\mathcal{H}'}{\mathcal{H}}$ on $M \setminus M_\delta$.

Many proofs of properties (1), (2) and (3) can be found in the literature. A proof of property (3) can be found in [G-H-L]. Proofs of (1),(2) and (3) can also be found in [O]. In the next lemma we shall bound $\Delta d(x, \partial M)$ as well as get an estimate on δ .

Lemma 2.1. *Let $\rho(x) = d(x, \partial M)$. Suppose we have constants $b \geq a > 0$, $b \geq 1$ such that $-a^2 \leq \text{Sec}(M) \leq b^2$. Then ρ is C^k on $M \setminus M_\delta$, if ∂M is C^k and*

$$\sum_{i=1}^{n-1} \frac{-b \sin b\rho - k_i(y) \cos b\rho}{\cos b\rho - \frac{k_i(y)}{b} \sin b\rho} \leq \Delta\rho \leq \sum_{i=1}^{n-1} \frac{-a \sinh a\rho - k_i(y) \cosh a\rho}{\cos ha\rho - \frac{k_i(y)}{a} \sinh a\rho}$$

where $k_i(y)$ are the principal curvatures on ∂M at the unique point $y \in \partial M$ such that $\rho(x) = d(x, y)$ and

$$\delta = \frac{1}{4(n-1)(b+H+1/R)}.$$

Proof. Let $x \in M$ with $\rho(x) = d(x, \partial M) = d(x, y)$ with $y \in \partial M$. Suppose k_i are the principal curvatures at y . At least locally, on the space form of constant curvature b^2 we may immerse a hypersurface with curvatures k_i . To derive the first inequality, we consider a hypersurface \bar{H} immersed in $S^n(1/b)$, the n -sphere of radius $1/b$. The second inequality is derived by replacing $S^n(1/b)$ with the hyperbolic space $H^n(1/a)$ and repeating the argument. Let $D \subseteq \bar{H}$ be an open set and define, as above, $\bar{\Psi} : D \times [0, \infty) \rightarrow S^n(1/b)$ by

$$\bar{\Psi}(y, \bar{\rho}) = \exp \bar{\rho} N_y$$

where N_y is the inward pointing unit normal at y . Since geodesics on $S^n(1/b)$ starting at a point y and having initial direction N_y have the form

$$\cos b\bar{\rho}y + \frac{1}{b} \sin b\bar{\rho}N_y$$

we may write

$$\bar{\Psi}(y, \bar{\rho}) = \cos b\bar{\rho}y + \frac{1}{b} \sin b\bar{\rho}N_y.$$

On \bar{H} there exist $n-1$ orthonormal principal vectors E_i . We extend E_i by parallel translation to vector fields $E_i(\bar{\rho})$ along the normal geodesic $\gamma(\bar{\rho}) = \bar{\Psi}(y, \bar{\rho})$. Let $\gamma_i(t)$ be geodesics on \bar{H} such that $\gamma_i(0) = y$ and $\gamma'_i(0) = E_i(0)$. Then

$$\begin{aligned} \frac{\partial \bar{\Psi}}{\partial \gamma_i}(y, \bar{\rho}) &= \cos b\bar{\rho}\gamma'_i(0) + \frac{1}{b} \sin b\bar{\rho} \frac{\partial N_{\gamma_i(t)}}{\partial \gamma_i}(0) \\ &= \cos b\bar{\rho}E_i - \frac{k_i}{b} \sin b\bar{\rho}E_i \\ &= \left(\cos b\bar{\rho} - \frac{k_i}{b} \sin b\bar{\rho} \right) E_i. \end{aligned}$$

Since $(\cos b\bar{\rho} - \frac{k_i}{b} \sin b\bar{\rho}) E_i$ are variation vector fields, they are in fact Jacobi fields and describe the Jacobian, $J(\bar{\Psi})$, of $\bar{\Psi}$ (see for instance [H-K]), and $J(\bar{\Psi})$ satisfies

$$\begin{aligned} J(\bar{\Psi})(y, \bar{\rho}) &= (\cos b\bar{\rho} - \frac{k_1}{b} \sin b\bar{\rho})(\cos b\bar{\rho} - \frac{k_2}{b} \sin b\bar{\rho}) \cdots (\cos b\bar{\rho} - \frac{k_{n-1}}{b} \sin b\bar{\rho}) \\ &> 0. \end{aligned}$$

The last inequality follows from $\bar{\rho} \leq \delta$ so that

$$\cot b\bar{\rho} \geq \frac{2H}{b}.$$

We have used the fact that $\theta \rightarrow 0$, $\tan \theta$ is approximately θ . Therefore, we may apply the inverse function theorem to find an open set $D' \subseteq D$ where y is a C^{k-1} function of $x = \bar{\Psi}(y, \bar{\rho})$. However, the definition of $\bar{\Psi}$ implies $\nabla \bar{\rho} = N_y$ is a C^{k-1} function of y . Therefore, $\bar{\rho}$ is C^k . The index estimates on $\bar{\mathcal{H}}$ and \mathcal{H} in [H-K] imply $J(\bar{\Psi})(y, \bar{\rho}) = \bar{\mathcal{H}}(y, \bar{\rho}) \leq \mathcal{H}(y, \rho)$ where $\bar{\rho} = \rho$ which implies ρ is C^k on $M \setminus M_\delta$. Applying property (2) we have

$$\Delta\rho \geq \Delta\bar{\rho} = \sum_{i=1}^{n-1} \frac{-b \sin b\rho - k_i(y) \cos b\rho}{\cos b\rho - \frac{k_i(y)}{b} \sin b\rho},$$

which completes the proof of the lemma.

Since $\rho \leq \delta$ from Lemma 2.1 we can draw the following conclusions:

$$\begin{aligned} \Delta\rho &\geq \sum_{i=1}^{n-1} \frac{-b \sin b\rho - k_i(y) \cos b\rho}{\cos b\rho - \frac{k_i(y)}{b} \sin b\rho} \\ &= \sum_{i=1}^{n-1} \frac{-b \tan b\rho - k_i(y)}{1 - \frac{k_i(y)}{b} \tan b\rho} \\ &\geq \sum_{i=1}^{n-1} -2(b + k_i) \\ (2.1) \quad &\geq -2(n-1)(b + H + 1/R). \end{aligned}$$

$$\begin{aligned} \Delta\rho^2 &= 2\rho\Delta\rho + 2|\nabla\rho|^2 \\ &\geq 2\rho(-2(n-1)(b + H + 1/R)) + 2 \\ &\geq -1 + 2 \\ (2.2) \quad &\geq 1. \end{aligned}$$

Now we can place $\sup_M u$ on a compact subset of M :

Lemma 2.2. *Suppose u is the first eigenfunction on M and $u(x_0) = \sup_{x \in M} u$. Then on $M \setminus M_\delta$*

$$u(x) \leq \frac{6\mu}{\delta} \rho(x).$$

Furthermore,

$$d(x_0, \partial M) \geq \frac{\delta}{6\mu}$$

where

$$\mu = \frac{(n-1)a^2}{4} + \frac{4\pi^2}{\delta^2} + \frac{(n-1)^2 a^2}{4 \sinh^2(a/2)}.$$

Proof. Without loss of generality, assume $\sup_{x \in M} u = 1$. Set $\gamma_1 = 2/(3\delta)$. Then using Lemma 2.1 and conclusions (2.1) and (2.2) we have

$$\begin{aligned} \Delta(\rho - \gamma_1 \rho^2) &\leq \sum_{i=1}^{n-1} \frac{-a \tanh a \rho - k_i}{1 - \frac{k_i}{a} \tanh a \rho} - \gamma_1 \\ &\leq 2(n-1)(a + H + 1/R) - \frac{1}{2\delta} \\ &\leq \frac{1}{2\delta} - \frac{2}{3\delta} \\ &= -\frac{1}{6\delta}. \end{aligned}$$

A result of Gage ([G]) tells us that $\lambda \leq \mu$, so by setting $\gamma_2 = 6\mu/\delta$ we have

$$\begin{aligned} \Delta(u - \gamma_2(\rho - \gamma_1 \rho^2)) &\geq -\lambda u + \gamma_2 \frac{1}{6\delta} \\ &\geq -\lambda u + \mu \\ &\geq 0. \end{aligned}$$

On ∂M_δ we have

$$\begin{aligned} u - \gamma_2(\rho - \gamma_1 \rho^2) &= u - \gamma_2(\delta - \gamma_1 \delta^2) \\ &= u - \gamma_2 \left(\delta - \frac{1}{2} \delta \right) \\ &= u - \frac{6\mu}{\delta} \frac{1}{2} \delta \\ &\leq u - 3\mu \\ &\leq 0. \end{aligned}$$

The last inequality follows since μ is clearly greater than 1. Therefore, by the maximum principle we have

$$d(x, \partial M) = \rho(x) \geq \frac{1}{\gamma_2} u(x).$$

To bound u from below is more involved. Again the maximum principle will be used but it is necessary for us to bound u from below on $\{\rho = \epsilon\}$ for some suitable ϵ . This is done in the following proposition.

Proposition 2.3. *Let u be the positive eigenfunction of M with respect to Dirichlet boundary conditions and normalized so that $\sup_M u = 1$. Then, for $d(x, \partial M) \geq \frac{\delta}{6\mu}$ we have*

$$(2.3) \quad u(x) \geq \exp\left(-\frac{24\mu d_M}{\delta} \log\left(\frac{50(n-1)^2}{\delta} (\mu^2 + a)\right)\right).$$

Proof. Proposition says that $\sup_{M_\epsilon} u = 1$ with $\epsilon = \frac{6\mu}{\delta}$. We will apply the Harnack inequality to bound u from below on M_ϵ . For any $x \in M_\epsilon$, $B_\epsilon(x) \subseteq M$. The gradient estimate in [S-Y] says that

$$\left| \frac{\nabla u}{u} \right| \leq 4 \frac{l_1^2}{\epsilon^2} + \frac{8(n-1)^2}{\epsilon^2} (\epsilon^2 + a)$$

on $B_{\epsilon/2}(x)$. Therefore, on $B_{\epsilon/2}(x)$,

$$\sup_{B_{\epsilon/2}(x)} u \leq \frac{l_1^2}{\epsilon} + \frac{8(n-1)^2}{\epsilon^2} (\epsilon + a) \inf_{B_{\epsilon/2}(x)} u.$$

Since $l_1 \leq \mu$, we have

$$(2.4) \quad \sup_{B_{\epsilon/2}(x)} u \leq \frac{8(n-1)^2}{\epsilon} (\mu^2 + a) \inf_{B_{\epsilon/2}(x)} u.$$

Since $\epsilon = \frac{6\mu}{\delta} \leq \frac{R}{2}$ for $n \geq 2$, we see easily that $d_{M_\epsilon} \leq 2d_M$. Applying (2.4) repeatedly on M_ϵ we derive that

$$1 = \sup_{M_\epsilon} u \leq \left(\frac{8(n-1)^2}{\epsilon} (\mu^2 + a) \right)^{\frac{4d_M}{\epsilon}} \inf_{M_\epsilon} u.$$

Therefore,

$$\inf_{M_\epsilon} u \geq \exp \left(-\frac{24\mu d_M}{\delta} \log \left(\frac{50(n-1)^2}{\delta} (\mu^2 + a) \right) \right).$$

This completes the proof of the proposition.

Now we can bound u from below by some multiple of ρ .

Proposition 2.4. *Let u be the first positive eigenfunction of M with respect to Dirichlet boundary conditions and normalized so that $\sup_M u = 1$. For $x \in M$, $d(x, \partial M) \leq \frac{6\mu}{\delta}$, we have*

$$u(x) \geq \rho(x) \frac{1}{4\delta} \exp \left(-\frac{24\mu d_M}{\delta} \log \left(\frac{50(n-1)^2}{\delta} (\mu^2 + a) \right) \right).$$

Proof. Let $C(\delta)$ denote the right-hand side of (2.3).

Let α_1, α_2 be positive constants to be chosen later. Then on T_δ ,

$$(2.5) \quad \begin{aligned} \Delta(u - \alpha_2(\rho + \alpha_1\rho^2)) &= -l_1u - \alpha_2 \left(\sum \frac{-b \sinh b\rho - k_1 \cosh b\rho d}{\cos b\rho - \frac{k_1}{b} \sin b\rho} + \alpha_1 \Delta\rho^2 \right) \\ &\leq -l_1u - \alpha_2 (-2(n-1)(b + H + 1/R) + \alpha_1). \end{aligned}$$

Therefore, the right-hand side of (2.5) will be less than or equal to zero if $\alpha_1 \geq (n-1)(b + H + 1/R)$.

Set $\alpha_1 = \frac{1}{\delta}$. Then, the infimum of $(u - \alpha_2(\rho + \alpha_1\rho^2))$ on $M \setminus M_\delta$ occurs on the boundary of $M \setminus M_\delta$.

For $x \in M$, $d(x, \partial M) = \delta$, we have

$$\begin{aligned} u(x) - \alpha_2(\delta + \alpha_1\delta^2) &\geq C(\delta) - \alpha_2\delta \left(1 + \frac{1}{\delta}\delta \right) \\ &\geq C(\delta) - 2\alpha_2\delta \\ &\geq 0, \end{aligned}$$

if $\alpha_2 = \frac{C(\delta)}{4\delta}$.

So, with these choices of α_1, α_2 we then have on $M \setminus M_\delta$,

$$u \geq \frac{C(\delta)}{4\delta} \left(\rho + \frac{1}{\delta}\rho^2 \right)$$

and therefore

$$u \geq \frac{C(\delta)}{4\delta} \rho,$$

as stated in the proposition.

Lemmas 2.1, 2.2 and Proposition 2.3 immediately give us property (4) of §1. That is,

Theorem 2.5. *Let M be a compact Riemannian manifold with $\partial M \neq \emptyset$, satisfying an interior rolling R -ball condition and having sectional curvature $-a^2 \leq \text{Sec}(M) \leq b^2$ with $b \geq a$. Let $\rho(x) = d(x, \partial M)$ and u be the first eigenfunction satisfying Dirichlet conditions. Then on $M \setminus M_\delta$, with δ chosen as in Lemma 2.1, there are constants $C_i(a, b, H, R, D, n)$, $i = 1, 2$, such that*

$$C_1 \rho(x) \leq u(x) \leq C_2 \rho(x).$$

Furthermore, there exists $C = C(a, b, H, R, D, n) > 0$ such that $u(x) \leq Cu(y)$ for all $x, y \in M$ with $0 < d(x, \partial M) \leq 2d(y, \partial M)$.

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