Abstract. This paper is concerned with singular convolution operators in $\mathbb{R}^d$, $d \geq 2$, with convolution kernels supported on radial surfaces $y_d = \Gamma(|y'|)$. We show that if $\Gamma(s) = \log s$, then $L^p$ boundedness holds if and only if $p = 2$.

This statement can be reduced to a similar statement about the multiplier $m(\tau, \eta) = |\tau|^{-i\eta}$ in $\mathbb{R}^2$. We also construct smooth $\Gamma$ for which the corresponding operators are bounded for $p_0 < p \leq 2$ but unbounded for $p \leq p_0$, for given $p_0 \in [1, 2)$. Finally we discuss some examples of singular integrals along convex curves in the plane, with odd extensions.

1. Introduction

This paper is primarily concerned with singular integral operators $T$ in dimensions $d \geq 2$ defined for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ by

$$Tf(x', x_d) = \text{p.v.} \int f(x' - y', x_d - \Gamma(|y'|)) \frac{\Omega(y')}{|y'|^{d-1}} dy'$$

where $x' \in \mathbb{R}^{d-1}$. We assume that $\Gamma : (0, \infty) \to \mathbb{R}$ is a smooth function, $\Omega \in L^q(S^{d-2})$ for some $q > 1$ and

$$ \int_{S^{d-2}} \Omega(\theta) d\sigma(\theta) = 0.$$  

We include the case $d = 2$ with the interpretation of $S^0 = \{-1, 1\}$ and the surface measure being counting measure.

It is easy to see using (1.2) that the principal value integral (1.1) exists everywhere for $f \in \mathcal{C}_0^\infty$. The question is for which $p \in (1, \infty)$ the operator $T$ extends to a bounded operator on $L^p(\mathbb{R}^d)$. If we consider the case of convex $\Gamma$ it is known that, then $L^2$ boundedness implies $L^p$ boundedness for $1 < p < \infty$ (see [10], [2] for the case $d = 2$ and [8] for the case $d \geq 3$, at least in the case of smooth $\Omega$). Moreover it was shown in [8] (again assuming that $\Omega$ is smooth and $\Gamma$ is $C^1$ in $(0, \infty)$) that in dimension $d \geq 3$ the operators $T$ are bounded in $L^2(\mathbb{R}^d)$, without any convexity assumption on $\Gamma$. Our primary concern here is whether $T$ extends to a bounded operator on $L^p$ without any further restriction on $\Gamma$. Our first theorem shows that this is not the case, in fact in our example $\Gamma$ is chosen to be concave.
Theorem 1.1. Suppose that $\Omega \in L^q(S^{d-2})$ where $q > 1$ and suppose that the cancellation property (1.2) holds. Suppose $\Gamma(t) = \log t$. Then $T$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ if and only if $p = 2$ or $\Omega = 0$ almost everywhere.

Remark. The analogous maximal operator $M_\gamma$ defined as the pointwise supremum of averages over $\{(x + y', \log(|x + y'|)) : |y'| \leq h\}$, $h > 0$, is unbounded on all $L^p$ spaces, see the argument in [14, p. 1291]. Moreover the $L^2$ estimate may fail if the standard homogeneous Calderón-Zygmund kernels $\Omega(y'/|y'|)|y'|^{1-d}$ are replaced by other (standard) singular kernels, such as the kernel for fractional integration of imaginary order, see Remark 2.3 below.

We shall see that the unboundedness of $T$ for $p \neq 2$ follows from a negative result for a Fourier multiplier on $\mathbb{R}^2$. In what follows $M^p$ denotes the class of Fourier multipliers of $L^p$ and $\|m\|_{M^p}$ is the $L^p$ operator norm of the convolution operator with Fourier multiplier $m$.

Proposition 1.2. Let $\chi$ be a bounded function in $C^1(\mathbb{R})$ and define

$$h(\tau, \eta) = \chi(\eta)\tau^{-i\eta}. \tag{1.3}$$

Then $h \in M^p(\mathbb{R}^2)$ if and only if $p = 2$ or $\chi \equiv 0$.

If $\chi_+$ denotes the characteristic function of $(0, \infty)$, then the same statement holds with $h(\tau, \eta)$ replaced by $h_\pm(\tau, \eta) = h(\tau, \eta)\chi_+(\pm \tau)$.

Remark. This result should be compared with the fact that for every $\eta$ the multiplier $\tau \mapsto |\tau|^{-i\eta}$ is a multiplier in $M^p(\mathbb{R})$ for $1 < p < \infty$ (it is the multiplier corresponding to fractional integration of imaginary order; the $L^p$ boundedness follows from the Marcinkiewicz multiplier theorem).

In our second theorem we exhibit operators $T$ with a prescribed range of $L^p$ boundedness.

Theorem 1.3. Suppose $1 < r \leq 2$. There is a function $\Gamma$ defined on $[0, \infty)$ with $\Gamma(0) = 0$, such that the symmetric extension $\Gamma(|x'|)$ to $\mathbb{R}^{d-1}$ is smooth and such that the following holds.

Let $d \geq 2$ and $T$ be as in (1.1), where $\Omega \in L^q(S^{d-2})$ for some $q > 1$ and the cancellation property (1.2) is assumed. Then $T$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ if and only if $r \leq p \leq r/(r - 1)$ or $\Omega = 0$ almost everywhere.

Remarks. (i) Let $1 \leq r < 2$. A slight modification of our construction yields $\Gamma$ such that $T$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $r < p < r/(r - 1)$ or $\Omega = 0$ a.e.

(ii) Examples where the maximal operator associated to the curve is bounded on some $L^p$ spaces but not on others have been constructed by M. Christ [4], see also Vance, Wright and Wainger [15] and unpublished work by Wierdl. Examples of this kind for singular integral operators seem to be new; however in [3] an example of a convex $\Gamma$ was constructed, so that the Hilbert transform associated to the odd extension was bounded only on $L^2(\mathbb{R}^2)$.

(iii) In an appendix (§5) we include some observations related to the examples in [3] and [4], dealing with singular integrals with convolution kernels supported on curves $\{(t, \gamma(t))\}$ in the plane; here $\gamma$ is the odd extension of a convex function on $(0, \infty)$.
2. $L^2$-estimates

We shall now consider the case
\[ \Gamma(t) = \log t \]
and show that $T$ is bounded on $L^2$ (provided that $\Omega \in L^q$, $q > 1$). This is achieved by showing that
\[
m_R(\xi) = \int_{|x'| \leq R} e^{-i\langle x', \xi \rangle + \xi_d \log |x'|} \frac{\Omega(x'/|x'|)}{|x'|^{d-1}} \, dx'
\]
(2.1)
\[
= \int_0^R e^{-i\xi_d \log r} \int_{S^{d-2}} e^{-i(r\theta, \xi')} \Omega(\theta) d\sigma(\theta) \frac{dr}{r}
\]
is bounded uniformly in $\xi$ and $R$ and converges to a bounded function as $R \to \infty$. By changing variables $r \mapsto r|\xi'|$ and using the cancellation of $\Omega$ we see that
\[
m_R(\xi) = e^{i\xi_d \log |\xi'|} M_R(\xi'/|\xi'|, \xi_d)
\]
with
\[
M_R(\vartheta, \xi_d) = \int_0^R e^{-i\xi_d \log r} \int_{S^{d-2}} (e^{-i(r\vartheta, \theta)} - 1) \Omega(\theta) d\sigma(\theta) \frac{dr}{r}
\]
for $\vartheta \in S^{d-2}$.

We split $M_R = \sum_{i=1}^3 E_i^R$ where
\[
E_1^R(\vartheta, \xi_d) = \int_0^R e^{-i\xi_d \log r} \int_{\vartheta, |r, \vartheta| \leq 1} (e^{-i(r\theta, \vartheta)} - 1) \Omega(\theta) d\sigma(\theta) \frac{dr}{r},
\]
(2.4)
\[
E_2^R(\vartheta, \xi_d) = \int_0^R e^{-i\xi_d \log r} \int_{\vartheta, |r, \vartheta| \geq 1} e^{-i(r\theta, \vartheta)} \Omega(\theta) d\sigma(\theta) \frac{dr}{r},
\]
\[
E_3^R(\vartheta, \xi_d) = -\int_0^R e^{-i\xi_d \log r} \int_{\vartheta, |r, \vartheta| \geq 1} \Omega(\theta) d\sigma(\theta) \frac{dr}{r}.
\]
First observe that
\[
|E_1^R(\vartheta, \xi_d)| \leq \int |\Omega(\theta)| \int_0^{\min\{|(|\vartheta, \theta|)^{-1}, R\}} |e^{-i(r\vartheta, \theta)} - 1| \frac{dr}{r} d\sigma(\theta) \leq C.
\]
To estimate $E_2^R$ interchange the order of the integration and observe that after a change of variables $s = r|\vartheta, \theta|$ in the inner integral we have
\[
E_2^R(\vartheta, \xi_d) = \int_{|\vartheta, \theta| \geq R^{-1}} \Omega(\theta) e^{i\xi_d \log |\vartheta, \theta|} u_+(\xi_d, R|\vartheta, \theta|) d\sigma(\theta)
\]
\[
+ \int_{|\vartheta, \theta| \leq R^{-1}} \Omega(\theta) e^{i\xi_d \log |\vartheta, \theta|} u_-(\xi_d, R|\vartheta, \theta|) d\sigma(\theta)
\]
where
\[
u_{\pm}(\gamma, N) = \int_1^N \exp(-i(\pm s + \gamma \log s)) \frac{ds}{s}.
\]
We show that $u$ is uniformly bounded in $\gamma$ and $N \geq 1$.

Assume first that $|\gamma| > 1/2$. Then we split the integral (2.5) into three parts depending on whether $|\gamma| \geq 5s$ or $s < |\gamma|/5$ or $|\gamma|/5 < s < 5|\gamma|$. The integral over $s \in [|\gamma|/5, 5|\gamma|]$ is trivially bounded.
If $N > 5|\gamma|$, then we integrate by parts to get
\[
\int_{5|\gamma|}^{N} e^{-i(\pm s + \gamma \log s)} \frac{ds}{s} = \int_{5|\gamma|}^{N} \frac{d(e^{i(\pm s + \gamma \log s)})}{\mp is - i\gamma}
\]
\[
= i \left( \frac{e^{-i(\pm N + \gamma \log N)}}{\gamma \mp N} - \frac{e^{-i(5\gamma \log 5\gamma)}}{\gamma \mp 5|\gamma|} \right)
\]
\[
\mp i \int_{5|\gamma|}^{N} e^{-i(\pm s + \gamma \log s)} \frac{ds}{(\gamma \pm s)^2}
\]
and this is bounded (since $|\gamma| \geq 1/2$).

We treat the integral \( \int_{1}^{\gamma/5} e^{-i(\pm s + \gamma \log s)} \frac{ds}{s} \) similarly. If $|\gamma| < 1/2$ and $N \geq 1$, then
\[
(2.6) \int_{1}^{N} e^{-i(\pm s + \gamma \log s)} \frac{ds}{s} = \pm i(e^{\mp iN} \gamma^{-1} - e^{\mp 1}) \pm (i\gamma + 1) \int_{1}^{N} e^{\mp is} s^{-i\gamma - 2} ds
\]
which is bounded. This shows that $|\mathcal{E}^R_2(\vartheta, \xi_d)| = O(1)$, uniformly in $R$.

Finally to estimate $\mathcal{E}^R_3(\vartheta, \xi_d)$ we observe that
\[
\mathcal{E}^R_3(\vartheta, \xi_d) = -\int_{|\vartheta| \geq 1/R} \Omega(\theta) \int_{r=|\vartheta|}^{R} e^{-i\xi_d \log r} \frac{dr}{r} d\sigma(\theta)
\]
\[
= -\mathcal{E}^R_{3,1}(\vartheta, \xi_d) + \mathcal{E}^R_{3,2}(\vartheta, \xi_d)
\]
where
\[
\mathcal{E}^R_{3,1}(\vartheta, \xi_d) = \int_{S^{d-2}} \Omega(\theta) \int_{r=|\vartheta|}^{R} e^{-i\xi_d \log r} \frac{dr}{r} d\sigma(\theta),
\]
\[
\mathcal{E}^R_{3,2}(\vartheta, \xi_d) = \int_{|\vartheta| \leq 1/R} \Omega(\theta) \int_{r=|\vartheta|}^{R} e^{-i\xi_d \log r} \frac{dr}{r} d\sigma(\theta).
\]

Now
\[
\mathcal{E}^R_{3,1}(\vartheta, \xi_d) = -\int_{S^{d-2}} \Omega(\theta) \frac{R^{-i\xi_d} - |\vartheta, \theta|^{i\xi_d}}{-i\xi_d} d\sigma(\theta) = -\int_{S^{d-2}} \Omega(\theta) \frac{1 - |\vartheta, \theta|^{-i\xi_d}}{-i\xi_d} d\sigma(\theta)
\]
where we have used the cancellation of $\Omega$ again. We see that
\[
|\mathcal{E}^R_{3,1}(\vartheta, \xi_d)| \leq \int_{S^{d-2}} |\Omega(\theta)| \frac{|e^{-i\xi_d \log |\vartheta, \theta|} - 1|}{|\xi_d|} d\sigma(\theta)
\]
\[
\leq \int_{S^{d-2}} |\Omega(\theta)| \log |\vartheta, \theta|^{-1} d\sigma(\theta)
\]
and the last integral is bounded uniformly in $\vartheta$ because of our assumption $\Omega \in L^q$. Moreover by a straightforward estimate
\[
\mathcal{E}^R_{3,2}(\vartheta, \xi_d) \leq \int_{|\vartheta| \leq 1/R} |\Omega(\theta)| \left[ \log R + \log |\vartheta, \theta|^{-1} \right] d\sigma(\theta)
\]
\[
\leq 2 \int_{S^{d-2}} |\Omega(\theta)| \log |\vartheta, \theta|^{-1} d\sigma(\theta).
\]
We have shown that $M_R$ is bounded uniformly in $(\vartheta, \xi d)$. An examination of the above argument also shows that if $|\xi d| \leq J$ and $J \geq 1$, then for $J \leq R \leq R'$

$$|M_R(\vartheta, \xi d) - M_R(\vartheta, \xi d)| \leq C_J \int_{|\langle \theta, \vartheta \rangle| \leq 10JR^{-1}} |\Omega(\theta)|(1 + \log |\langle \theta, \vartheta \rangle|^{-1})d\sigma(\theta)$$

$$+ \int_{|\langle \theta, \vartheta \rangle| \geq R^{-1}} |\Omega(\theta)|(R|\langle \theta, \vartheta \rangle|)^{-1}d\sigma(\theta)$$

which is $O(R^{-1+1/3}(1 + \log R))$. Therefore $\lim_{R \to \infty} M_R(\xi d)(\xi'/\xi^\alpha, \xi d)$ exists and the convergence is uniform with respect to $(\xi', \xi d)$ in compact subsets of $(\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}$.

Since each $M_R$ is easily seen to be a smooth function on $\mathbb{S}^{d-1} \times \mathbb{R}$ we have proved

**Proposition 2.1.** Suppose that $\Gamma(t) = \log t$, $\Omega \in L^q(\mathbb{S}^{d-2})$, $q > 1$, and that (1.2) holds. Then $T$ is bounded on $L^2(\mathbb{R}^d)$ and the Fourier transform of its convolution kernel is given by

$$m(\xi) = e^{i\xi d \log(|\Omega'|)} M(\xi'/\xi^\alpha, \xi d)f$$

where $M$ is a bounded continuous function on $\mathbb{S}^{d-2} \times \mathbb{R}$.

**Remark 2.2.** If $\Omega$ is odd, then $T$ is $L^2$ bounded if (1.2) holds and $\Omega$ is merely in $L^1(\mathbb{S}^{d-2})$. To see this one uses the method of rotations (see [1]). Define

$$H_\vartheta f(x) = \text{p.v.} \int f(x' - t\vartheta, x_d - \log |t|)\frac{dt}{t};$$

then one can see by transferring our result in two dimensions to $d$ dimensions that $H_\vartheta$ is bounded on $L^2(\mathbb{R}^d)$ with operator norm independent of $\vartheta$. If $\Omega$ is odd, then $T = C \int_{\mathbb{S}^{d-2}} \Omega(\vartheta)H_\vartheta d\sigma(\vartheta)$ and the $L^2$ boundedness of $T$ follows. For general $\Omega$ satisfying (1.2) the assumption $\Omega \in L \log L(\mathbb{S}^{d-2})$ yields $L^2$ boundedness of $T$.

**Remark 2.3.** For $\alpha \neq 0$ let $m_\alpha(\tau) = |\tau|^\alpha$ and $k_\alpha = \mathcal{F}^{-1}[m_\alpha]$, then $k_\alpha$ is a standard singular integral kernel on $\mathbb{R}^{d-1}$ (although not homogeneous of degree $1 - d$). For $f \in C_0^\infty(\mathbb{R}^d)$ define

$$\mathcal{H}_\alpha f(x) = \int f(x' - t, x_d - \log |t|)k_\alpha(t)dt.$$ 

Then $\mathcal{H}_\alpha$ is unbounded on $L^2(\mathbb{R}^d)$. To see this observe that the associated multiplier

$$c_\alpha \int_{\mathbb{R}^{d-1}} e^{-i(\langle \xi', x' \rangle + (\xi_d + \alpha\log |\xi'|)|x'|^{1-d}}dx'$$

is unbounded as $\xi_d \to -\alpha$.

For later use we shall now show that for $\xi_d \neq 0$ the function $M$ is actually differentiable as a function of $\xi d$; in particular we shall show that

$$|\xi d \frac{\partial M(\vartheta, \xi d)}{\partial \xi d}| \leq C \text{ if } 0 < |\xi d| \leq 1/2.$$ 

The proof of (2.7) follows the lines above. Differentiation with respect to $\xi d$ gives another factor of $-i \log r$ in the formulas (2.4). In the estimation of $\xi_d^1(\vartheta, \xi d)$ this yields an additional factor of $\log |\langle \theta, \vartheta \rangle|^{-1}$ which is harmless in view of our assumption $\Omega \in L^q(S^{d-2})$. In the estimation of $\xi_d^2(\vartheta, \xi d)$ we shall only need to consider the term corresponding to (2.6) since we assume that $|\xi d| \leq 1/2$, and we get boundedness of the derivative (again the calculation yields an additional factor.
of \( \log |\langle \theta, \vartheta \rangle|^{-1} \). The term corresponding to \( E_3^R(\theta, \xi_d) \) has to be handled with some care; it is a difference of \( E_{3,1}^R(\theta, \xi_d) \) and \( E_{3,2}^R(\theta, \xi_d) \) given by

\[
E_{3,1}^R(\theta, \xi_d) = -i \int_{S^{d-2}} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{\log r}{r} dr d\sigma(\theta),
\]

\[
E_{3,2}^R(\theta, \xi_d) = -i \int_{|\langle \theta, \vartheta \rangle| \leq 1/R} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{\log r}{r} dr d\sigma(\theta).
\]

Now for \( \xi_d \neq 0 \)

\[
\int_{r=a}^R e^{-i\xi_d \log r} \frac{\log r}{r} dr = i\xi_d^{-1} R^{-i\xi_d}(\log R - i\xi_d^{-1}) - i\xi_d^{-1} a^{-i\xi_d}(\log a - i\xi_d^{-1}).
\]

Using this for \( a = |\langle \theta, \vartheta \rangle|^{-1} \) we may copy the argument for \( E_{3,1}^R(\theta, \xi_d) \), \( E_{3,2}^R(\theta, \xi_d) \) above, producing an additional factor of \( \xi_d^{-1} \). Moreover the limiting argument above can be carried over as long as we stay away from \( \xi_d = 0 \). This yields (2.7).

### 3.1. The model multiplier in two dimensions.

We now give a proof of Proposition 1.2. Clearly \( h \in M_2 \) since \( h \) is bounded. Let \( 1 < p < 2 \) and assume that \( \chi \) is not identically zero. We argue by contradiction and assume that \( h \in M^p \). Our proof is related to an argument by Littman, McCarthy and Rivièr\[9\].

We may choose an interval \( I = (\alpha_0, \alpha_1) \) so that \( \chi(\eta) \neq 0 \) if \( \eta \) belongs to the closure of \( I \). Let \( \Phi \in \mathcal{S}(\mathbb{R}) \) so that the Fourier transform \( \hat{\Phi} \) is compactly supported in \( I \) but does not identically vanish. Let \( \beta \) be a \( C_\infty \) function so that \( \beta \) is supported in \( \{ \tau : |\tau| \leq 1 \} \), \( \beta(\tau) = 1 \) if \( |\tau| \leq 1/2 \).

Let

\[
g_N(\tau, \eta) = \sum_{k=10}^{N} \frac{\hat{\Phi}(\eta)}{\chi(\eta)} \beta(\tau - e^{2k}) e^{-i\eta(2^k - \log \tau)}.
\]

Then it is easy to see by the sharp form of the Marcinkiewicz multiplier theorem ([13, p. 109]) that

\[
\|g_N\|_{M^p} \leq C_p \text{ for } 1 < p < \infty.
\]

Let

\[
h_N(\tau, \eta) = \sum_{k=10}^{N} \hat{\Phi}(\eta) \beta(\tau - e^{2k}) e^{-i\eta 2^k},
\]

then \( h_N = g_N h \) and therefore

\[
\|h_N\|_{M^p} \leq C_p \|h\|_{M^p}.
\]

However we shall show that

\[
(3.1) \quad \|h_N\|_{M^p} \geq cN^{1/p-1/2}
\]

so \( h \) cannot be in \( M^p \).

Define \( f_N \) by

\[
\hat{f}_N(\tau, \eta) = \sum_{k=10}^{N} \beta(\tau - e^{2k}) \hat{\Psi}(\eta)
\]

where \( \hat{\Psi} \) is compactly supported but equals 1 on the support of \( \hat{\Phi} \), so \( \Phi = \Phi \ast \Psi \).
Then by Littlewood-Paley theory
\[ \|f_N\|_p \approx \left\| \left( \sum_{k=10}^{N} |\mathcal{F}^{-1}[\beta]|^2 \right)^{1/2} \right\|_p \approx N^{1/2}. \]

But
\[ \mathcal{F}^{-1}[h_Nf_N](x) = \sum_{k=10}^{N} \mathcal{F}^{-1}[\beta^2](x_1)e^{ix_1e^{2k}} \Phi(x_2 - 2^k) \]
and since \( \Phi \neq 0 \) is a Schwartz function it is easy to see that
\[ \|\mathcal{F}^{-1}[h_Nf_N]\|_p \geq cN^{1/p}. \]

This yields (3.1) and therefore the desired contradiction. The above argument also proves the corresponding statement for the multiplier \( h_+ \) and then also for \( h_- \).

3.2. Failure of \( L^p \)-boundedness in Theorem 1.1. We now show that if \( \Gamma(t) = \log t \) and if \( T \) is bounded on \( L^p(\mathbb{R}^d) \), then \( p = 2 \), assuming that \( \Omega \) is not identically 0. By the Riesz-Thorin theorem we may assume that \( 1 < p < \infty \). Let \( \chi_+ \) be the characteristic function of \((0, \infty)\). If \( m \) is the corresponding multiplier, then we know by de Leeuw’s theorem [7] that for almost all \( \vartheta \in S^{d-2} \) the function \((\tau, \eta) \to \chi_+(\tau)m(\vartheta, \eta)\) is a Fourier multiplier on \( L^p(\mathbb{R}^2) \).

Now \( m(\tau \vartheta, \eta) = |\tau|^{\eta}M(\vartheta, \eta) \) for \( \tau > 0 \), by Proposition 2.1. Let \( K_\Omega \) be the kernel \( \Omega(x'/|x'|)|x'|^{1-d} \) on \( \mathbb{R}^{d-1} \). Then its Fourier transform in \( \mathbb{R}^{d-1} \) is homogeneous of degree zero and equals \( M(\xi'/|\xi'|, 0) \). The latter cannot be zero almost everywhere by uniqueness of Fourier transforms. Therefore there is \( \vartheta \in S^{d-2} \) such that \( m(\tau \vartheta, \eta) \) is a Fourier multiplier on \( L^p(\mathbb{R}^2) \) and such that \( M(\vartheta, 0) \neq 0 \). Since \( M \) is continuous in \( \eta \) there is \( 0 < \epsilon < 1/2 \) and \( c > 0 \) such that \( |M(\vartheta, \eta)| \geq c \) for \( \epsilon/2 \leq \eta \leq \epsilon \). Let \( \chi \) be a \( C^\infty \) function supported in \( (\epsilon/2, \epsilon) \), not identically zero.

From (2.7) we see that \( \eta \to \chi(\eta)[M(\vartheta, \eta)]^{-1} \) is a Fourier multiplier on \( L^p \), with bounds uniform in \( \vartheta \). Therefore \( \chi(\eta)\chi_+(\tau)|\tau|^{\eta} \) is a Fourier multiplier on \( L^p(\mathbb{R}^2) \) and by Proposition 1.2 this implies that \( p = 2 \).

4. Examples for specific \( L^p \) spaces

In this section we give a proof of Theorem 1.3. For each \( p_0 \), with \( 1 < p_0 \leq 2 \), we construct an even function \( \Gamma \in C^\infty(\mathbb{R}) \) such that \( \Gamma(0) = 0 \) and \( \Gamma(t) = 0 \) for \( t \geq 1 \), and such that the operator \( T \) as in (1.1) is bounded on \( L^p(\mathbb{R}^d) \) if and only if \( p_0 \leq p \leq p_0 \) or \( \Omega = 0 \) a.e.

Let \( \zeta \in C^\infty(\mathbb{R}) \) so that \( \zeta(t) = 1 \) if \( t > 1/4 \) and \( \zeta(t) = 0 \) if \( t < -1/4 \). Let \( \delta = \{\delta_n\} \) be a sequence of positive numbers, so that \( |\delta_n| \leq 1 \) and \( \lim_{n \to \infty} \delta_n = 0 \).

Let \( \{\gamma_n\}_{n=1}^\infty \) be a sequence of positive numbers such that \( \gamma_{n+1} \leq \gamma_n/10 \) for all \( n \geq 1 \). Our function \( \Gamma \) is then defined by
\[ \Gamma(t) = \sum_{n=1}^{\infty} \gamma_n \zeta(2^{n^2+n} \delta_n^{-1}(|t| - 2^{-n^2}(1 - \delta_n))) \zeta(2^{n^2+n} \delta_n^{-1}(2^{-n^2}(1 + \delta_n) - |t|)). \]

Then for \( n \geq 1 \)
\[ \Gamma(t) = \begin{cases} \gamma_n & \text{if } 2^{-n^2}(1 - \delta_n + \delta_n 2^{-n^2}) \leq |t| \leq 2^{-n^2}(1 + \delta_n - \delta_n 2^{-n^2}), \\ 0 & \text{if } 2^{-(n+1)^2}(1 + \delta_{n+1} + \delta_{n+1} 2^{-n-3}) \leq |t| \leq 2^{-n^2}(1 - \delta_n - \delta_n 2^{-n^2}) \end{cases} \]
and \( \Gamma(t) = 0 \) for \( |t| \geq 2 \).
Theorem 4.1. Let $\Gamma$ be as in (4.1), $T$ and $\Omega$ as in §1, $1 < p < \infty$ and let $s(p) = |1/p - 1/2|^{-1}$. Then $T$ is bounded on $L^p$ if and only if $\delta \in \ell^{s(p)}$ or $\Omega = 0$ almost everywhere.

Theorem 1.3 is an immediate consequence, except for the fact that the even function $\Gamma$ may not be smooth at the origin. This however can be achieved by an appropriate choice of $\gamma_n$, for example, $\gamma_n \leq \gamma_{n-1} \exp(-2^n\delta_n^{-1})$ for all $n \geq 2$.

Proof of Theorem 4.1. Let $I_n = [2^{-n^2}(1 - \delta_n), 2^{-n^2}(1 + \delta_n)]$ and

$$T_n f(x) = \int_{|y'| \in I_n} f(x' - y', x_d - \gamma_n) \frac{\Omega(y')}{|y'|^{d-1}} dy'.$$

It is easy to see that $T = \sum_{n=1}^\infty T_n + \mathcal{H} + \sum_{n=1}^\infty K_n$ where the $L^p$ operator norm of $K_n$ is $O(2^{-n})$, for $1 \leq p \leq \infty$ and where $\mathcal{H}$ is the extension to $L^p(\mathbb{R}^d)$ of a variant of a Calderón-Zygmund operator acting in the $x'$ variables; the $L^p$ boundedness for $1 < p < \infty$ follows from [1]. It therefore suffices to examine the operator $\sum_n T_n$.

Let $L_k$ denote the standard Littlewood-Paley operator on $\mathbb{R}^{d-1}$, i.e.,

$$\widehat{L_k} f(\xi) = \phi(2^{-k} |\xi|) \hat{f}(\xi)$$

where $\phi$ is a $C_0^\infty$ function supported on $\frac{1}{2} \leq t \leq 2$ such that $\sum_{k=-\infty}^\infty \phi(2^{-k} |t|) = 1$ for $t \neq 0$.

Then for some $\epsilon > 0$, depending on $p > 1$ and $q > 1$

$$(4.2) \quad \|L_{k+n}T_n\|_{L^p} \leq A \min \{2^{-\epsilon k}, \delta_n\};$$

see e.g. [6].

Define $\Delta_n = \sum_{j=n^2-n+1}^{n^2+n} L_j$, $\Delta_n = \sum_{j=n^2-n+2}^{n^2+n+2} L_j$, so that $\Delta_n \Delta_n = \Delta_n$. Observe by (4.2) that

$$\sum_{n=1}^\infty \|T_n - T_n\Delta_n\|_{L^p \to L^p} < \infty$$

for all $p \in (1, \infty)$. The $L^p$ boundedness of $T$, under the assumption $\delta \in \ell^s$, follows by a well known argument using Littlewood-Paley theory (see [12] and [5]). For convenience we include the short proof. Without loss of generality assume $1 \leq p \leq 2$. By Littlewood-Paley theory (or Calderón-Zygmund theory for vector-valued singular integrals [13, ch. II]) the inequality $\|\{\Delta_n f\}\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{p}$ holds for all $p \in (1, \infty)$, similarly the corresponding inequality involving $\Delta_n$. Since the $L^p$ operator norm of $T_n$ is $O(\delta_n)$ we see that

$$\left\| \sum_n \Delta_n T_n \Delta_n f \right\|_p \leq C_p \left\{ \{T_n \Delta_n f\}\right\}_{L^p(\mathbb{R}^d)} \leq C_p \left\{T_n \Delta_n f\right\}_{L^p(\mathbb{R}^d)}$$

$$= C_p \left\{T_n \Delta_n f\right\}_{\ell^p(\mathbb{R}^d)} \leq C_p \left(\sum_n \|T_n\|_{L^p \to L^p} \|\Delta_n f\|_p^p\right)^{1/p}$$

$$\leq C_p \|\delta\|_{\ell^\epsilon} \|\{\Delta_n f\}\|_{\ell^p(\mathbb{R}^d)} \leq C_p \|\delta\|_{\ell^\epsilon} \|\{\Delta_n f\}\|_{L^p(\mathbb{R}^d)}$$

$$\leq C_p \|\delta\|_{\ell^\epsilon} \|f\|_p.$$

We now turn to the proof of the converse. We fix $p \in (1, 2)$ and assume that $T$ is bounded on $L^p$ and that $\Omega$ does not vanish on a set of positive measure; we then have to prove that $\delta \in \ell^s$, $s = s(p)$. 

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Let
\[ m_n(\xi') = \int_{|y'| \leq I_n} e^{i\xi' \cdot y'} \Omega(y'/|y'|)|y'|^{1-d} dy'. \]

Since by (4.1) the operator \( \sum_n T_n \) is bounded on \( L^p \),
\[ m(\xi', \xi_d) = \sum_n e^{i\xi_d n} m_n(\xi') \]
is a bounded multiplier on \( L^p(\mathbb{R}^d) \). Since we assume that \( \Omega \) does not vanish on some set of positive measure, it follows that there is an open set \( U \) on which the Fourier transform \( \Omega d\sigma \) does not vanish, in fact we may assume that \( |\Omega d\sigma(\xi)| \geq A > 0 \) for \( \xi \in U \). By de Leeuw's theorem [6] there is \( \Xi \in U \) so that
\[ u(\tau, \eta) = \sum_n e^{in\eta_n} m_n(\tau \Xi) \]
is a multiplier in \( M^p(\mathbb{R}^2) \).

Since we assume that \( \lim_{n \to \infty} \delta_n = 0 \) we can choose a positive integer \( K \) so that the closed ball of radius \( \delta_n \) and center \( \Xi \) is contained in \( U \) for all \( \ell \geq K \). Let \( \beta \in C^\infty(\mathbb{R}) \) with \( \beta \) supported in \([1/2, 2]\) so that \( \beta(t) = 1 \) in a neighborhood of 1. By the Marcinkiewicz multiplier theorem \( \sum_{\ell=K}^N \beta(\tau - 2^\ell) \) is in \( M^r(\mathbb{R}) \) for every \( r, 1 < r < \infty \), uniformly in \( N \) (here and in what follows we assume that \( N \geq K \)). Therefore the norms in \( M^p(\mathbb{R}^2) \) of the multipliers \( \sum_{\ell=K}^N \sum_n e^{i\eta_n} m_n(\tau \Xi) \beta(\tau - 2^\ell) \) are uniformly bounded.

It follows from (4.2) that the \( M^r(\mathbb{R}^2) \) norm of \( m_n(\tau \Xi) \beta(\tau - 2^\ell) \) is \( O(2^{-(\ell^2-n^2)}) \), where \( \epsilon = \epsilon(r, q) > 0 \) if \( r > 1, q > 1 \). Therefore \( \sum_{\ell=K}^N \sum_n e^{i\eta_n} m_n(\tau \Xi) \beta(\tau - 2^\ell) \) is a Fourier multiplier of \( L^r(\mathbb{R}^2) \) for all \( r \in (1, \infty) \) with bound uniformly in \( N \). Consequently, by our assumption
\[ v_N(\tau, \eta) = \sum_{\ell=K}^N e^{i\eta_n \ell} m(\tau \Xi) \beta(\tau - 2^\ell) \]
is a Fourier multiplier of \( L^p(\mathbb{R}^2) \).

Now let
\[ A_\ell = \int_{1-\delta_\ell}^{1+\delta_\ell} \int_{S^{d-2}}^{1} \Omega(\theta)e^{i\tau \cdot \Xi, \theta} d\theta r^{-1} dr, \]
\[ b_\ell(\tau) = \int_{1-\delta_\ell}^{1+\delta_\ell} \int_{S^{d-2}}^{1} \Omega(\theta)\left[e^{i\tau r \cdot \Xi, \theta} - e^{i\tau (\Xi, \theta)}\right] d\theta r^{-1} dr, \]
then
\[ v_N(\tau, \eta) = \sum_{\ell=1}^N e^{i\eta_n \ell} (A_\ell + b_\ell(\tau)). \]

Observe that for \( \ell \geq K \)
\[ |A_\ell| \geq A \log \left( \frac{1+\delta_\ell}{1-\delta_\ell} \right) \geq A \delta_\ell. \]
Moreover \( \beta(-2^\ell) b_\ell \) is a Fourier multiplier of \( L^1(\mathbb{R}) \), with bound independent of \( \ell \). The \( L^\infty \) norm of this function is \( O(2^{-\ell^2}) \) and therefore by interpolation the
multiplier \( \sum_{k=K}^{N} \beta(\cdot - 2^k) b_k \) belongs to \( M_r(\mathbb{R}) \) for \( r \in (1, \infty) \), with norm bounded in \( N \). We conclude that

\[
w_N(\tau, \eta) = \sum_{\ell=K}^{N} e^{i\eta \tau} \beta(\tau - 2^\ell) A_\ell
\]

belongs to \( M^p(\mathbb{R}^2) \) with norm independent of \( N \).

Let \( \psi \) be a nonnegative smooth function not identically zero, with support in \([-1/2, 1/2]\) and let \( \psi_N(y) = \gamma_{N+1}^{-1/p} \psi(\gamma_{N+1}^{-1} y) \).

Now let \( \alpha = \{\alpha_\ell\} \) be a sequence in \( \ell^2/\mathbb{R} \), so that \( \|\alpha\|_{\ell^2/\mathbb{R}} \leq 1 \). Note that \( 2/p = (s/p)' \). We test \( w_N \) on \( f_N \) with

\[
\hat{f}_N(\tau, \eta) = \sum_{\ell=K}^{N} |\alpha_\ell|^{1/p} \beta(\tau - 2^\ell) \hat{\psi}_N(\eta);
\]

then by Littlewood-Paley theory

\[
\|f_N\|_{L^p} \leq C \left( \left( \sum_{\ell=K}^{N} |\alpha_\ell|^{2/p} \| \mathcal{F}^{-1} [\beta] \|^2 \right)^{1/2} \right)_{L^p} \leq C'
\]

where \( C' \) is independent of \( N \). On the other hand, for \((x, y) \in \mathbb{R}^2\),

\[
\mathcal{F}^{-1}[w_N \hat{f}_N](x, y) = \sum_{\ell=K}^{N} A_\ell |\alpha_\ell|^{1/p} \mathcal{F}^{-1} [\beta^2](x) e^{i\eta x} \psi_N(y - \gamma_\ell).
\]

Since \( \gamma_{N+1} \leq \gamma_{\ell}/10 \), \( \ell = K, \ldots, N \), the supports of the functions \( \psi_N(y - \gamma_\ell) \) are disjoint. Therefore

\[
\left( \sum_{\ell=K}^{N} |A_\ell|^p |\alpha_\ell| \right)^{1/p} \leq C \| \mathcal{F}^{-1}[w_N \hat{f}_N] \|_p \leq C \|w_N\|_{M^p} \|f_N\|_p \leq C'
\]

uniformly in \( N \). This implies by (4.3) that

\[
\sup_{\|\alpha\|_{\ell^s(p)} \leq 1} \sum_{\ell=K}^{\infty} |\delta_\ell|^p |\alpha_\ell| < \infty.
\]

By the converse of Hölder’s inequality it follows that \( \{\delta_\ell^p\} \in \ell^{s/p} \) and therefore \( \delta \in \ell^s \). \( \square \)

5. Appendix: Odd extensions of convex curves in the plane

Here we include some observations concerning odd curves \((t, \gamma(t))\) where \( \gamma \) is convex in \((0, \infty)\). Our examples are modifications of those in [3] and [4]. For \( r > 0 \), \( \epsilon \geq 0 \), and \( j \geq 1 \) set \( \alpha_{\epsilon,j} = \tau^4 - j^{-1} \) for a small \( \tau \) to be chosen later and

\[
\gamma_{\tau,\epsilon}(t) = (2j)^r (2^r + (2^r + 2)^r + \alpha_{\epsilon,j})(t - 4^r) \quad \text{for } 4^r \leq t \leq 4^r(1 + j^{-\epsilon}).
\]

For \( 4^r(1 + j^{-\epsilon}) \leq t \leq 4^r(j^{-\epsilon}) \), extend \( \gamma_{\tau,\epsilon} \) so \( \gamma_{\tau,\epsilon}'(t) \) is constant in this interval, \( \gamma_{\tau,\epsilon}' \) is continuous at \( 4^r(1 + j^{-\epsilon}) \) and \( \gamma_{\tau,\epsilon}(t) \) is continuous for \( t \geq 4 \). Similarly extend \( \gamma_{\tau,\epsilon} \) to \([0, 4]\) with constant positive curvature so that \( \gamma_{\tau,\epsilon}(0) = 0 \). A calculation shows that \( \gamma_{\tau,\epsilon} \) is convex for \( t > 0 \). Finally extend \( \gamma_{\tau,\epsilon} \) as an odd function. The
perturbation by $\alpha_{\epsilon,j}$ in (5.1) is convenient in order that arguments in [4] to study maximal functions should apply to singular integral operators. We consider

$$H_{r,\epsilon} f(x,y) = \text{p.v.} \int \frac{f(x-t,y - \gamma_{r,\epsilon}(t))}{t} dt.$$ 

**Proposition 5.1.**

(i) For any $\epsilon \geq 0$ and $r > 0$, $\|H_{r,\epsilon} f\|_{L^2} \leq A \|f\|_{L^2}$.

(ii) If $p_0 = \frac{2p+2}{2+1}$, then for any $r > 0$, $\|H_{r,\epsilon} f\|_{L^p} \leq A_p \|f\|_{L^p}$ for $p_0 < p < p'$.

(iii) If $r = 1$ and $\frac{3}{2} \leq p < 2$, $H_{r,\epsilon}$ is unbounded on $L^p$ if $\epsilon < \frac{1}{p} - \frac{1}{2}$.

(iv) If $r = 1$ and $p \leq \frac{4}{3}$, $H_{r,\epsilon}$ is unbounded on $L^p$ if $\epsilon \leq \frac{3}{p} - 2$.

(v) If $r$ is a positive integer, then $H_{r,\epsilon}$ is unbounded on $L^p$ if $p < \frac{r+2}{r+1+\epsilon}$.

**Remarks.** Consider the maximal function $\sup_{h>0} h^{-1} \int_0^h |f(x-t,y - \gamma_{r,\epsilon}(t))| dt$.

Then the operator $M$ is unbounded on $L^p$ if $p < \frac{r+2}{r+1+\epsilon}$. This is a slight improvement over a result in [4]. More generally if $r = \frac{m}{n}$ with $m$ and $n$ positive integers, then one can show that $M$ is unbounded if $p < \frac{m+2}{m+1+n\epsilon}$. One achieves this by restricting the values of $j$ in the argument below to be with powers. Obviously many questions remain open.

**Proof of Proposition 5.1.** Clearly (i) follows from [10] since $h_{r,\epsilon}(t) = t\gamma_{r,\epsilon}(t) - \gamma_{r,\epsilon}(t)$ is doubling (see also [3], [16] for a more geometric proof of this result). In particular note that if $I_{j,\epsilon} = [4^j, 4^{j+\epsilon})$, then $\gamma_{r,\epsilon}(t) = s_j t - h_j$ where $s_j = (2j+2)^r + \alpha_{\epsilon,j}$ and $h_j = 4^j ((2j+2)^r - (2j)^r) + \alpha_{\epsilon,j}$.

Now set $I_{j,\epsilon} = \{ t : |t| \in I_{j,\epsilon} \}$ and let

$$G_j f(x,y) = \int_{I_{j,\epsilon}} f(x-t, y - \gamma_{r,\epsilon}(t)) t^{-1} dt.$$ 

Then $H_{r,\epsilon} = \sum_{j=1}^{\infty} G_j + E$. In view of the curvature properties of $\gamma_{r,\epsilon}$ in the complement of $\bigcup_j I_{j,\epsilon}$ (where $h$ is “infinitesimally doubling”) the method of [3] may be applied to yield the $L^q$ boundedness of $E$ for all $q \in (1, \infty)$.

For the remaining assertions of the proposition it suffices to consider $G = \sum_j G_j$. To prove (ii) we consider the analytic family $G_z = \sum_j j^z G_j$. If $\text{Re}(z) < -1$, $G_z$ is clearly bounded in $L^1$. (ii) follows by analytic interpolation if we can show that $G_z$ is bounded in $L^2$ for $\text{Re}(z) < \epsilon$. This however follows by Fourier transform estimates following [11] or [16]. One derives the estimate

$$|\hat{G}_j(\xi)| \leq C_1 \min \{ j^{-\epsilon}; 4^j |\xi_1 + \xi_2 (s_j - 4^{-j} h_j)| + C_2 |\xi_2| 4^{-j} h_j; 4^{-j} |\xi_1 + \xi_2 s_j| \}.$$ 

The first estimate is obvious, the second estimate uses the oddness of $\gamma$ and the estimate $|\sin a| \leq |a|$ and the third uses an integration by parts. It is now straightforward to bound the sum $\sum_{j=1}^{\infty} |j^z \hat{G}_j(\xi)|$ provided that $\text{Re}(z) < \epsilon$.

To obtain conclusion (v) we follow Christ [4]. We test $G$ on the characteristic function $f_N$ of a union of small rectangles $R_{(a,b)}$ centered at lattice points $(a, b)$ with $0 \leq a \leq 2^N$ and $0 \leq b \leq N^r 2^N$,

$$R_{a,b} = \{ (x,y) : a - N^{-r-1} \sigma \leq x \leq a + N^{-r-1} \sigma, b - N^{-1} \sigma \leq y \leq b + N^{-1} \sigma \}.$$ 

Here $\sigma$ is small (to be chosen). We let for each pair of positive integers $\ell$ and $j$

$$S^{\ell,j} = \{ (x, y) : 0 \leq x \leq 2^N, 0 \leq y \leq N^r 2^N, |y - (2j+2)^r x - \ell| \leq N^{-1} \sigma \}.$$
Then $|S^{\ell,j}| \geq \sigma 2^{N}(2N)^{-1}$ if $j \leq N/4$ and $\ell \leq N'2^{N}/10$, moreover if $j' \neq j$, $|S^{\ell,j} \cap S^{\ell,j'}| \leq A\sigma 2^{N}2^{-r}2^{-2}\|j-r-(j')^{-1}| \leq A'\sigma 2^{N}2^{-r}2^{-2}\|j-r-(j')^{-1}|^{-1}$. Fixing $\ell, j$, and $j'$, the number of strips $S^{\ell,j}$ that intersect $S^{\ell,j'}$ is at most $2^{N}|j-r-(j')^{-1}|$. Since there are at most $N$ values of $j'$, the measure of the union of all strips intersecting a given $S^{\ell,j}$ is at most $A\sigma|S^{\ell,j}|$, with $A$ an absolute constant not depending on $\sigma$. We are going to restrict $j$ to $N/5 \leq j \leq N/4$. We estimate $Gf_N$ for points $(x, y)$ in $S^{\ell,j}$ such that $(x, y)$ is in no $S^{\ell,j'}$ with $j' \neq j$ and such that the vertical distance from $(x, y)$ to the top of $S^{\ell,j}$ is between $10^{-5}\tau/N$ and $10^{-6}\tau/N$. If we first choose $\sigma$ sufficiently small and then $\tau = \sigma/100$, we will be estimating $Gf_N$ on a positive fraction of $S^{\ell,j}$. In evaluating $Gf_N$ at such points $(x, y)$ the contribution to $Gf_N$ from pieces of $\gamma_{r, \epsilon}$ with slopes other than $(2j + 2)^{r}$ is zero. The contribution $Gf_N$ at such points comes from two strips $S^{\ell+(2j)^{r}2^{j},j}$ and $S^{\ell-(2j)^{r}2^{j},j}$.

The contribution from $S^{\ell-(2j)^{r}2^{j},j}$ is at least $10^{-2}j^{-\epsilon}N^{-r-1}$. The absolute value of the contribution from $S^{\ell+(2j)^{r}2^{j},j}$ is at most $10^{-3}j^{-\epsilon}N^{-r-1}$. Thus if $G$ is bounded in $L^p$, $N^{-(r+1)p}j^{-p}p\left|\bigcup_{\ell,j} S^{\ell,j}\right| \leq A|\text{supp}(f_N)|$.

Therefore $N^{-(r+1)p}j^{-p}N^{r'}2^{N}(2N)/N \leq AN^{r}2^{N}N^{-r-2}$ which implies for $N \to \infty$ the necessary condition $p \geq r+2$. Note that (iv) is a special case of (v). Finally (iii) follows along the same lines as in §7 of [3]. Let $b_\eta(k) = \int_{4^{k}}^{4^{(1+k^{-\epsilon})}} \sin\{\eta(\alpha_{\epsilon,k}(t - 4^{k}) - 4^{k}+1)\} \frac{dt}{t}$ $= -(\log 2)k^{-\epsilon} \sin(4^{k+1}\eta) + O(k^{-1})$. It then suffices to show that the sequence $b_\eta$ does not belong to $M^p(\mathbb{Z})$ (the class of Fourier multipliers for Fourier series in $L^p(\mathbb{T})$) uniformly for $\pi \leq \eta \leq 3\pi$. The error $O(k^{-1})$ represents the Fourier coefficients of an $L^2$ function and belongs to $M_1(\mathbb{Z})$ for all $r \in [1, \infty]$. Now the argument in [3] shows $b_\eta \notin M^p(\mathbb{Z})$ if $\{k^{-\epsilon-1}/\log^{-1}k\} \notin \ell^2(\mathbb{Z})$ which is true if $\epsilon < 1/p - 1/2$. 

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: seeger@math.wisc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: wainger@math.wisc.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY 2052, AUSTRALIA
E-mail address: jim@maths.unsw.edu.au

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND
Current address: Department of Mathematics, Dominican University, River Forest, Illinois 60305
E-mail address: ziessara@email.dom.edu

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