OVERGROUPS OF IRREDUCIBLE LINEAR GROUPS, II

BEN FORD

Abstract. Determining the subgroup structure of algebraic groups (over an algebraically closed field $K$ of arbitrary characteristic) often requires an understanding of those instances when a group $Y$ and a closed subgroup $G$ both act irreducibly on some module $V$, which is rational for $G$ and $Y$. In this paper and in Overgroups of irreducible linear groups, I (J. Algebra 181 (1996), 26–69), we give a classification of all such triples $(G, Y, V)$ when $G$ is a non-connected algebraic group with simple identity component $X$, $V$ is an irreducible $G$-module with restricted $X$-high weight(s), and $Y$ is a simple algebraic group of classical type over $K$ sitting strictly between $X$ and $SL(V)$.

1. Introduction

E. B. Dynkin in 1957 [3, 2] classified the maximal closed connected subgroups of simple algebraic groups when the underlying (algebraically closed) field has characteristic 0. Seitz ([9, 10]) and Testerman ([13]) completed the same program in positive characteristic in the 1980’s. Their analyses for the classical group cases were based primarily on a striking result: If $G$ is a simple algebraic group and $\varphi : G \rightarrow SL(V)$ is a tensor indecomposable irreducible rational representation, then with specified exceptions the image of $G$ is maximal among closed connected subgroups of one of the classical groups $SL(V)$, $Sp(V)$, or $SO(V)$. What is most striking is the brevity of the list of exceptions.

From a slightly different perspective, the question these authors answered was: Given a closed, connected subgroup $G$ of $SL(V)$ for some vector space $V$, with $G$ acting irreducibly on $V$, find all possibilities for closed, connected overgroups $Y$ of $G$ in $SL(V)$.

This question of irreducible overgroups appears in other contexts as well, sometimes for non-connected subgroups. Here and in [4] we present results for some such non-connected subgroups, namely, those with simple identity components. The overall program is to classify all possible triples $(G, Y, V)$ with $G$ and $Y$ both closed subgroups of $SL(V)$ acting irreducibly on $V$, $G < Aut(Y)$, $Y \neq SL(V), SO(V), or Sp(V)$, and $Y$ a simple group of classical type (the corresponding question for $Y$ of exceptional type is also open). We give complete results for the case when $G$ is not connected but has simple identity component $X$ and the $T_Y$-high weight and $T_X$-high weights of $V$ are restricted. Specifically, the papers are concerned with the proof of Theorem 1.
Theorem 1. Let $G$ be a non-connected algebraic group, over a field $K$ of arbitrary characteristic $p \geq 0$, with simple identity component $X$. Let $V$ be an irreducible $KG$-module with restricted $X$-high weight(s). Let $Y$ be a simple algebraic group of classical type such that $X < Y < SL(V)$ and $G \leq \text{Aut}(Y)$. Then $V|_Y$ is irreducible with restricted high weight if and only if $Y = \text{SO}(V)$, $Y = \text{Sp}(V)$, or $(X,Y,V)$ appears in Table 1 or Table 2, see §6.

If $G$ has simple identity component $X$, then $G \leq \text{Aut}(X)$ (there is also the possibility of a torus acting as scalars on each of the $X$-composition factors of $V$; we ignore this trivial possibility). Since we require that $G \neq X$, we therefore may restrict our attention to $X$ of type $A_n$, $D_m$, or $E_6$. We assume henceforth that $Y$ is simply connected and that $X$ and $Y$ act on $W$, the natural module for $Y$.

The analysis is different depending on whether $X$ acts reducibly or irreducibly on $W$. We settled the reducible case in [4], and we consider the irreducible case here. Also, we will assume here that the involutory graph automorphism of $X$, if it is in $G$, also acts on $W$ (though it need not be in $Y$), as we deal with the case when it does not act on $W$ in the final section of [4].

If $V|_X$ is irreducible, then we are in the case examined by Seitz in [9], with the additional condition that $X$ has an outer automorphism which acts on $V$. We examine Table 1 of that paper, and find that we have such a situation in the examples there labelled $I_4$, $I_5$, $I_6$ for $n = 3$, $II_1$, $S_1$, $S_8$ (in $S_8$ we could take $G = X(t)$, $G = X(s)$, or $G = X(s,t)$, where $t,s$ are outer automorphisms of $X$ of order 2 and 3 respectively), and $MR_4$. These examples are collected in Table 1, and henceforth we shall assume that $V|_X$ is reducible.

1.1. Notation and Conventions. All structures are assumed to be constructed over the same algebraically closed field $K$, of characteristic $p \geq 0$. Throughout, $X$ will denote a simple algebraic group over $K$ admitting an outer automorphism (so $X$ is of type $A_m$, $D_m$, or $E_6$). A fixed standard graph automorphism of order 2 will be denoted by $t$, and if $X$ has an outer automorphism of order 3 (i.e. if $X = D_4$), we will fix one and denote it by $s$. Thus $G = X(t)$ except possibly when $X = D_4$, in which case we also consider $G = X(s)$ and $G = X(s,t)$.

For any reductive group $H$ we consider, with fixed maximal torus $T$, $\Sigma(H)$ will denote the roots of $H$ relative to $T$. If $\gamma \in \Sigma(H)$, we let $h_\gamma : K^* \to T$ be the one-parameter subgroup of $T$ such that $\alpha(h_\gamma(x)) = x^<a,\gamma>$ for any $\alpha \in \Sigma(H)$ and $x \in K^*$.

We let $B_X$ be a fixed $t$-stable Borel subgroup of $X$, containing a fixed $t$-stable maximal torus $T_X$. Define sets of simple roots $\{\beta_1, \beta_2, ..., \beta_m\} = \Pi(X) \subseteq \Sigma(X)$ and fundamental dominant weights $\{\delta_1, \delta_2, ..., \delta_m\}$ with respect to $T_X$ and $B_X$ but with the opposite of the standard convention: The set of positive roots $\Sigma^+(X)$ is defined by $B_X = U_X T_X$ where $U_X = \prod U_\alpha$ for $\alpha \in \Sigma^+(X)$. Then for $J \subseteq \Pi(X)$, $P_X$ is the opposite of the standard parabolic corresponding to $J$. We assume the $\delta_i$ are numbered so that $\delta_i$ corresponds to $\beta_i$ for every $i$.

The group $Y$ will be a simple algebraic group over $K$ of classical type and rank $n$ ($A_n$, $B_n$, $C_n$ or $D_n$), such that $X < Y$ and $G \leq \text{Aut}(Y)$. Let $\{\alpha_1, \alpha_2, ..., \alpha_n\} = \Pi(Y)$ be a set of simple roots of $Y$ and $\{\lambda_i\}$ be the set of fundamental dominant weights such that $\lambda_i$ corresponds to $\alpha_i$. Notation and conventions similar to those used for $X$ are used for parabolic subgroups of $Y$.

For a group $H$ acting on a module $M$, $[M, H^l]$ will denote the $l$-fold commutator of $H$ with $M$. 


The $K$-vector space $V$ is assumed to be a restricted irreducible $Y$-module with high weight $\lambda = \sum a_i \lambda_i$, such that $V$ is irreducible as a $G$-module but not as an $X$-module (see the comment at the end of the previous subsection). We assume that the $T_X$-high weights of $V$ are restricted as well. So if $G = X(t)$, then $V|_X = V_1 \oplus V_2$, where each of $V_1, V_2$ is a restricted irreducible $X$-module.

The natural module for $Y$ will be denoted by $W$. We assume that $W$ is irreducible as an $X$-module, and $\delta$ will denote its $T_X$-high weight. As in [4], we will always assume that $Y$ is the smallest of $\text{SL}(W), \text{SO}(W), \text{Sp}(W)$ containing $X$.

Finally, we assume that $G$ acts on $W$, as the case when it does not was considered in [4].

We label Dynkin diagrams for the groups we will be dealing with as follows, and we always number fundamental roots and fundamental dominant weights to agree with this labelling:

- $A_l$: \[
\begin{array}{cccccc}
  & 1 & 2 & \cdots & l - 1 & l \\
\end{array}
\]
- $B_l$: \[
\begin{array}{cccccc}
  & 1 & 2 & \cdots & l - 1 & l \\
\end{array}
\]
- $C_l$: \[
\begin{array}{cccccc}
  & 1 & 2 & \cdots & l - 1 & l \\
\end{array}
\]
- $D_l$: \[
\begin{array}{cccccc}
  & 1 & 2 & \cdots & l - 2 & l - 1 \\
\end{array}
\]
- $E_6$: \[
\begin{array}{cccccc}
  & 1 & 3 & 4 & 5 & 6 \\
  & &  & & & 2 \\
\end{array}
\]

We will sometimes use the standard partial order on weights: $\nu \succ \mu$ if and only if $\nu - \mu$ is a sum of positive roots.

2. $Q_X$-LEVELS AND EMBEDDINGS OF PARABOLICS

In this section we introduce important facts about the “commutator series” of a module of a simple algebraic group.

**Lemma 2.1.** If $H$ is a simple algebraic group whose root system has only one root length, then restricted irreducible $H$-modules are tensor indecomposable (in particular, restricted irreducible $X$-modules are tensor indecomposable).

**Proof.** This is part of 1.6 of [9].

**Lemma 2.2.** Let $M$ be an irreducible restricted $H$-module with high weight $\gamma$ for some simple algebraic group $H$. Let $P$ be a proper parabolic subgroup of $H$, with $P = QL$ a Levi decomposition. Then $M/[M, Q]$ is irreducible for $L$ and for $L' = [L, L]$, with $T_{L'}$-high weight $\gamma|_{T_{L'}}$.

**Proof.** This is 1.7 and 2.1 of [9].

Let $H, M, \gamma,$ and $P$ be as in the last lemma. Let $\{\epsilon_i\}$ be the set of fundamental roots of $H$. 

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Definition 2.3. Let $\mu$ be a weight of $M$, say $\mu = \gamma - \sum c_i \epsilon_i$, with each $c_i \geq 0$. The $Q$-level of $\mu$ is $\sum c_j$, where the sum ranges over those $j$ for which $\epsilon_j \in \Pi(H) - \Pi(L')$. The $Q$-level $l$ of $M$ is the sum of weight spaces for weights having $Q$-level $l$ and is denoted $M_l$.

Lemma 2.4. Let $H$, $M$, and $P$ be as above. If $H$ is simply laced or if $p > 2$ ($p > 3$ for $H = G_2$), then
1. $[M, Q^l] = \bigoplus M_\mu$, the sum taken over those weights $\mu$ having $Q$-level at least $l$.
2. $[M, Q^l]/[M, Q^{l+1}] \cong M_l$.
3. $\dim([M, Q^l]/[M, Q^{l+1}]) \leq s \cdot \dim([M, Q^{l-1}]/[M, Q^l])$, where $s$ is the number of positive roots $\beta$ such that $U_{-\beta} \leq Q$ and $\beta = \epsilon_i + \beta'$ for some $\epsilon_i \in \Pi(H) - \Pi(L')$, with $\beta' = 0$ or a sum of roots in $\Pi(L')$.
4. $\dim([M, Q^l]/[M, Q^{l+1}]) \leq \dim(Q) \cdot \dim([M, Q^{l-1}]/[M, Q^l])$.

Proof. This is 2.3 of [9].

We will write $M^l(Q)$ for the quotient $[M, Q^{l-1}]/[M, Q^l]$.

Lemma 2.5. Let $H = A_1$; let $c$ be an integer such that $0 < c < p$; and let $\gamma_1, \ldots, \gamma_l$ be the fundamental dominant weights for $H$. The irreducible module $M$ having high weight $c\gamma_1$ or $c\gamma_l$ has all weight spaces of dimension 1; in particular, $\dim(M) = (l+c)!/l!c!$.

Proof. This is 1.14 of [9].

We will occasionally use the Weyl character formula for dimensions of Weyl modules.

Finally, it was shown in [4] that when $X$ acts irreducibly on $W$, we may assume $W$ is in fact restricted as an $X$-module:

Lemma 2.6. If $X$ acts irreducibly on $W$, then as an $X$-module, $W$ has a restricted high weight.

Now let $P_X$ be a parabolic subgroup of $X$ and $P_X = Q_X L_X$ be a Levi decomposition with $T_X \leq L_X$ (if $P_X$ is $t$-stable, choose $L_X$ to also be $t$-stable). Now $X$ acts irreducibly on $W$ with high weight $\delta$, which is restricted by the Lemma above. We gave in [4] the following construction of a parabolic subgroup $P_Y$ of $Y$ (with $P_Y = Q_Y L_Y$ a Levi decomposition) such that $P_X \leq P_Y$, $Q_X \leq Q_Y$, $L_X \leq L_Y$. Let $Z = Z(L_X)^\circ$.

Lemma 2.7. The stabilizer in $Y$ of the commutator series
$$W > [W, Q_X] > [W, Q_X, Q_X] > \cdots > 0$$
is a parabolic subgroup $P_Y$ of $Y$ satisfying the following:
1. $P_X \leq P_Y$ and $Q_X \leq Q_Y = R_u(P_Y)$.
2. $L_Y = C_Y(Z)$ is a Levi factor of $P_Y$ containing $L_X$.
3. If $T_Y$ is a maximal torus of $Y$ containing $T_X$, then $T_Y \leq L_Y$.

Proof. This is 2.7 of [4].

We give more information about this embedding for particular groups $X$ and parabolic subgroups $P_X$ below and in subsequent sections. For the next Lemma, we assume that $t \in G$ (where $t$ is the fixed outer automorphism of $X$) and $V|_X =
$V_1 \oplus V_2$, with $V_1, V_2$ irreducible $X$-modules. This Lemma was proved in [4, 2.8 and 2.9]:

**Lemma 2.8.** If $P_X$ is a $t$-stable parabolic subgroup of $X$ and $P_X$ is embedded in a parabolic subgroup $P_Y$ of $Y$ as above, then

1. $P_Y$ is likewise $t$-stable;
2. $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$ as $L_X$-modules.

Let $P_Y = L_Y Q_Y$ be a parabolic subgroup of $Y$. For each $\gamma \in \Sigma(Y) - \Sigma(L_Y^t)$, we define a certain normal subgroup $K_{\gamma}$ of $P_Y$, as in [9, page 44]: Let $\Sigma_\gamma(Y)$ denote the set of roots in $\Sigma(Y)$ having $\gamma$-coefficient $-1$ and zero coefficient for other roots in $\Sigma(Y) - \Pi(L_Y^t)$. Then let $K_{\gamma}$ be the product of those $T_Y$-root subgroups $U_{\beta}$ for $\beta \in \Sigma^- (Y) - \Sigma^- (L_Y^t) - \Sigma_\gamma(Y)$. From the commutator relations it follows that $K_{\gamma}$ is normal in $P_Y$ and we let $Q_{\gamma} = Q_Y/K_{\gamma}$. This construction also applies to a parabolic subgroup $P_X$ of $X$. In particular, if $P_X$ is a maximal parabolic subgroup corresponding to $\alpha \in \Pi(X)$, then set $Q_{\alpha} = Q_X/K_{\alpha}$, where $K_{\alpha}$ is the product of those $T_X$-root subgroups corresponding to roots having $\alpha$-coefficient strictly less than $-1$.

The lemma below will be used heavily in dimension arguments.

**Lemma 2.9.** If $P_X = Q_X L_X$ is a maximal parabolic subgroup of $X$ corresponding to $\alpha \in \Pi(X)$ and $P_X$ is embedded in a parabolic subgroup $P_Y$ of $Y$ as in Lemma 2.7, then $\dim(V^2(Q_Y)) \leq \dim(Q_{\alpha}^X) \cdot \dim(V^1(Q_X))$.

**Proof.** This is part of Proposition 2.14 in [9].

**Lemma 2.10.** If $P_X = Q_X L_X$ is a maximal parabolic subgroup corresponding to $\alpha \in \Pi(X)$, then

1. $K_{\alpha} = [Q_X, Q_X]$.
2. $Q_{\alpha}^X$ is an irreducible $L_X^t$-module with $-\alpha$ as its $T_{L_X^t}$-high weight.

**Proof.** See 3.2 in [9] (remembering that $X$ is of type $A_m$, $D_m$, or $E_6$).

Again assume $t \in G$. Let $P_X$ be a parabolic subgroup of $X$ (not necessarily $t$-stable) containing the fixed $t$-stable Borel subgroup $B_X$. Embed $P_X$ in a parabolic subgroup $P_Y$ of $Y$ via the above construction. Write $L_Y^t = L_1 \times \cdots \times L_r$, a direct product of simple groups. By Lemma 2.2, $L_Y^t$ acts irreducibly on $V^1(Q_Y) = V/[V, Q_Y]$. Then $V^1(Q_X) = V^1 \otimes \cdots \otimes V^r$ where for each $i$, $V^i$ is an irreducible module for $L_i$. The embedding $L_X \to L_Y$ gives an embedding of $L_X^t$ into $L_1 \times \cdots \times L_r$, and via the projections $L_X^t \to L_i$, any $L_i$-module, in particular $V^i$, can be regarded as a module for $L_X^t$.

Since $Q_X \leq Q_Y$, we have $[V, Q_X] \leq [V, Q_Y]$ and hence $V/[V, Q_Y]$ is a quotient of $V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$, with each of these summands irreducible $L_X^t$-modules. Since $L_X^t \leq L_Y^t$, this implies that either $V/[V, Q_Y]$ is irreducible for $L_X^t$ or $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$. Lemma 2.8 tells us that the latter happens when $P_X$ is $t$-stable. The following was proved in [4, 2.11]:

**Lemma 2.11.** If $V$, $P_X = L_X Q_X$, $P_Y = L_Y Q_Y$, and $L_i$ are as above with $P_X$ $t$-stable, then only one $L_i$ acts nontrivially on $V/[V, Q_Y]$. 

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3. The case $X = A_m$

As always, let $X < Y$ be simple algebraic groups over an algebraically closed field $K$ of characteristic $p$ (= 0 or a prime), with $X$ admitting an involutory graph automorphism $t$ which also acts on $Y$ and with $Y$ of classical type. Let $\{\beta_i\}$ ({$\alpha_i$} = $\Pi(Y)$) be the set of simple roots for $X$ ($Y$) and let $\{\delta_i\}$ ({$\lambda_i$}) be the corresponding fundamental dominant weights for $X$ ($Y$). The fixed $t$-stable Borel subgroup $B_X$ of $X$ contains a $t$-stable maximal torus $T_X$. Let $V = V(\lambda)$ be a restricted irreducible $t$-stable $Y$-module with high weight $\lambda = \sum a_i \lambda_i$, such that $V|_{X(t)}$ is irreducible, but $V|_Y = V_1 \oplus V_2$, with $V_1, V_2$ restricted irreducible $X$-modules. We denote the $T_X$-high weight of $V_1$ by $b_1 \delta_1 + b_2 \delta_2 + \cdots + b_m \delta_m$, so the $T_X$-high weight of $V_2$ is $b_{m+1} \delta_1 + b_{m+2} \delta_2 + \cdots + b_{2m} \delta_m$. The natural module for $Y$ is $W$.

The main result of the section is

**Theorem 3.1.** If $X$ acts irreducibly on $W$ with high weight $\delta$, $t$ acts on $W$, and $X$ is of type $A_m$, then $p \notin \{2, 3, 5, 7\}$, $X = A_3$, $Y = D_4$, $\delta = 2 \delta_2$, and the high weights of $V|_X, V|_Y$ are as in $U_5$ of Table 2.

Notice that since $t$ acts on $W$, the high weight $\delta = d_1 \delta_1 + \cdots + d_m \delta_m$ of $W|_X$ must be symmetric, i.e. $d_1 = d_m, d_2 = d_{m-1}$, etc. But then by a result of Steinberg ([11, page 226]), $X$ fixes a nondegenerate bilinear form on $W$; the form is orthogonal if $p \neq 2$.

The strategy we use to rule out most possibilities for the high weight $\delta$ is to show that the construction (outlined in Lemma 2.7) of a parabolic subgroup of $Y$ containing the fixed ($t$-stable) Borel subgroup of $X$ gives a contradiction in all but a few cases. After giving the Lemma which we usually use to produce the contradiction, we will treat the $A_2$ and $A_3$ cases first, followed by the general argument.

3.1. Some Facts About $P_Y$. We use the construction given in Lemma 2.7 of a parabolic subgroup $P_Y < Y$ containing the fixed $t$-stable Borel subgroup $B_X$. Namely, $P_Y$ is taken to be the stabilizer in $Y$ of the flag in $W$ given by “$U_X$-levels.”

We want to use Lemma 3.2 below to produce a contradiction in most cases; we will show that $L_Y$ has a factor of type $A_1$ only under strong conditions. Before proceeding with the general proof, we need some facts about the flag in $W$ of which $P_Y$ is the stabilizer.

Recall that for $i \geq 0, W_i = \sum_{e_1 + e_2 + \cdots = i} W_{\delta - e_1 \beta_1 - e_2 \beta_2 - \cdots}$, the sum taken over $e_j \geq 0$. Each space $W_i$ is $T_X(t)$-stable, and if $u \in U_{-\alpha}$, then

$$u W_{\delta - e_1 \beta_1 - e_2 \beta_2 - \cdots} \subseteq \sum_{m \geq 0} W_{\delta - e_1 \beta_1 - e_2 \beta_2 - \cdots - m \alpha}.$$

So $B_X(t)$ stabilizes each factor

$$\left( \sum_{i \geq m} W_i \right) / \left( \sum_{i \geq m+1} W_i \right).$$

By Lemma 2.4, $W_i \cong [W, U_X^m]/[W, U_X^{m+1}]$.

Let $l$ be minimal with respect to $[W, U_X^{m+1}] = 0$, and notice that $l$ is then the level of the low weight $-\delta$. If $Y = \text{Sp}(W)$ or $Y = \text{SO}(W)$, with the form denoted by ( , ), then we noted in [4, proof of 2.7] that $(u, v) = 0$ for $u \in W_i, v \in W_j$ unless
i + j ≤ l. Thus the \( W_i \) for \( i > l/2 \), along with a maximal totally singular subspace of \( W_{l/2} \) (if \( l \) is even), span a maximal totally singular subspace of \( W \).

Let \( w^+ \) be an \( TX \)-high weight vector of \( W \). Then \( W_0 = \langle w^+ \rangle, W_l = \langle w_0 w^+ \rangle, \) and \( B_X \) is contained in the full stabilizer \( P_Y \) of the flag

\[
W = \sum_{i \geq 0} W_i \geq \sum_{i \geq 1} W_i \geq \cdots \geq \sum_{i \geq l} W_i = \langle w_0 w^+ \rangle \geq 0.
\]

Let \( P_Y = L_Y Q_Y \) be a Levi decomposition of \( P_Y \); then if \( u \in U_X, w \in W_m, \) we have \( uw - w \in \sum_{i \geq m+1} W_i, \) so \( U_X \leq Q_Y \). We have \( T_X \leq L_Y = C_Y(Z) \) for \( Z = Z(L_X)^{\circ} \).

Choose a basis for each \( W_i \) (with the basis for \( W_{l/2} \) chosen maximally hyperbolic — note that \( W_{l/2} \cong \left( \sum_{i \geq l/2} W_i \right) / \left( \sum_{i > l/2} W_i \right) \) is the only possible non-singular quotient in the flag); the union of these bases is a basis for \( W \). With respect to this basis, \( L_Y' \) consists of block matrices, each block corresponding to \( W_i \) for some \( i \). On the other hand, each \( W_i \) for \( 0 < i \leq l/2 \) corresponds to a connected component of the Dynkin diagram for \( L_Y \). So the only possibilities for an \( A_1 \) to appear as one of the simple factors of \( L_Y' \) are when \( \dim(W_i) = 2 \) for some \( i < l/2 \) or \( \dim(W_{l/2}) \leq 4 \).

To show \( \dim(W_i) \geq m \), it suffices to find \( m \) \( TX \)-weights of \( W \) which occur in \( W_i \). By the result in [12], weights which appear in characteristic 0 also appear in characteristic \( p \). This is the approach we use to obtain contradictions for most embeddings \( X \hookrightarrow Y \).

For a \( TX \)-weight \( \omega \) of \( W, \omega + \omega^t = \omega - w_0 \omega \) is a sum of roots, and we let \( l_\omega \) be the height of \( \omega + \omega^t \) in the root lattice (the number of summands when we express \( \omega + \omega^t \) as a sum of fundamental roots). So for \( l \) as above, \( l = l_\delta \) where \( \delta \) is the \( TX \)-high weight of \( W \).

The constructions in this section also apply to the embedding of an arbitrary parabolic subgroup \( P_X = Q_X L_X \) of \( X \) in a parabolic subgroup \( P_Y \) of \( Y \), using \( Q_X \)-levels in place of the \( U_X \)-levels. The weights appearing in \( W_i \) then are those of the form \( \delta - e_1 \beta_1 - \cdots - e_m \beta_m \) where the sum of the \( e_j \) for \( \beta_j \in \Pi(X) - \Pi(L_X') \) is \( i \). Again by Lemma 2.4, \( W_i \cong [W, Q_X^i]/[W, Q_X^{i+1}] \).

One more fact: If \( P_X \) is a \( t \)-stable parabolic subgroup of \( X \) (including \( P_X = B_X \), then each \( W_i \) is clearly \( t \)-stable (since then \( W_{\delta - e_1 \beta_1 - e_2 \beta_2 - \cdots - e_m \beta_m} \) is sent by \( t \) to \( W_{\delta - e_m \beta_1 - e_{m-1} \beta_2 - \cdots - e_1 \beta_m} \)). So \( P_Y \) is \( t \)-stable. But then \([V, Q_Y]\) is \( TX(t) \)-stable, hence \([V, Q_X] = [V, Q_X] \leq [V, Q_Y] \) since \( Q_X \leq Q_Y \), and we get equality because \( V/[V, Q_X] \) is an irreducible \( TX(t) \)-module and \( TX \leq L_Y \). So if \( P_X \) is \( t \)-stable, then \([V, Q_Y] \) is a sum of two irreducible \( L_X' \)-modules.

The following lemma provides the basis for the proof of the section’s main result; it will also be used throughout the paper.

**Lemma 3.2.** If \( P_M \) is a \( t \)-stable parabolic subgroup of \( Y \) such that \( B_X < P_Y, U_X < Q_Y \), \( T_X < L_X' \) (where \( P_Y = Q_Y L_Y, B_X = U_X T_X \) are the Levi decompositions), then at least one of the simple factors of \( L_X' \) has type \( A_1 \); and if this factor corresponds to \( \alpha_j \), then \( \alpha_j = 1 \). In addition, \( \alpha_i = 0 \) for \( \alpha_i \in \Pi(L_X'), i \neq j \).

**Proof.** This is Lemma 5.2 of [4].

3.2. **The Cases** \( X = A_2 \) and \( X = A_3 \). We use Lemma 3.2 heavily. As always, \( \delta = d_1 \delta_1 + d_2 \delta_2 + \cdots \) is the \( TX \)-high weight of \( W \). Let \( \beta \) be a nonnegative sum of fundamental roots of \( X \), of height \( j \) in the root lattice. Note that if \( \delta - \beta \) is a dominant weight such that the \( X \)-module with high weight \( \delta - \beta \) has \( \geq m \) weights at level \( i \), then \( W \) has \( \geq m \) weights at level \( j + i \). So in our attempt to prove that
there cannot be a $U_X$-level of dimension 2 in $W$, we will proceed by induction on the high weight $\delta$.

$X = A_2$. Since $\delta$ is symmetric, $\delta = a\delta_1 + a\delta_2$ for some $a > 0$. Here we will always have $\dim(W_1) = 2$, since the only two weight spaces in level 1 are $\delta - \beta_1$ and $\delta - \beta_2$, both of dimension 1; we will deal with level 1 after we discuss levels 2 and higher. In evaluating the numbers of weights at these levels, we will first use an induction to deal with the case $a \geq 4$, and then deal with $a = 3$ and $a = 2$.

Assume $a = 4$. Then $l = l_5 = 16$, so we must check to level 8 (we must list three weights at every level 2-7, and 5 at level 8). We have the weights in the table below:

<table>
<thead>
<tr>
<th>Level</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\delta - 2\beta_1, \delta - \beta_1 - \beta_2, \delta - 2\beta_2$</td>
</tr>
<tr>
<td>3</td>
<td>$\delta - 3\beta_1, \delta - 2\beta_1 - \beta_2, \delta - 3\beta_2$</td>
</tr>
<tr>
<td>4</td>
<td>$\delta - 4\beta_1, \delta - 3\beta_1 - \beta_2, \delta - 4\beta_2$</td>
</tr>
<tr>
<td>5</td>
<td>$\delta - 4\beta_1 - \beta_2, \delta - 3\beta_1 - 2\beta_2, \delta - 2\beta_1 - 3\beta_2$</td>
</tr>
<tr>
<td>6</td>
<td>$\delta - 4\beta_1 - 2\beta_2, \delta - 3\beta_1 - 3\beta_2, \delta - 2\beta_2 - 4\beta_2$</td>
</tr>
<tr>
<td>7</td>
<td>$\delta - 5\beta_1 - \beta_2, \delta - 4\beta_1 - 3\beta_2, \delta - \beta_1 - 3\beta_2$</td>
</tr>
<tr>
<td>8</td>
<td>$\delta - 6\beta_1 - 2\beta_2, \delta - 5\beta_1 - 3\beta_2, \delta - 4\beta_1 - 4\beta_2, \delta - 3\beta_1 - 5\beta_2, \delta - 2\beta_1 - 6\beta_2$.</td>
</tr>
</tbody>
</table>

So $L_Y' \leq 1$ has no factors of type $A_2$ except possibly the factor $L_1$ corresponding to $W_1$.

Assume $a > 4$. Then $\delta - \beta_1 - \beta_2 = (a - 1)\delta_1 + (a - 1)\delta_2$ is dominant, and by induction $\delta$ has enough weights at all levels except possibly at levels 1, 2 and 3. At level 2, we have $\delta - 2\beta_1, \delta - \beta_1 - \beta_2$, and $\delta - 2\beta_2$; at level 3, $\delta - 3\beta_1, \delta - 2\beta_1 - \beta_2$, and $\delta - 3\beta_2$. So again, $L_1$ is the only possible $A_2$-factor of $L_Y'$.

Assume $a = 3$. Then $l = 12$; we must check dimensions to level 6. In levels 2-5, we have enough weights as above. So we must show that $W_6$ has dimension at least 5. The weights at level 6 are $\delta - 4\beta_1 - 2\beta_2, \delta - 3\beta_1 - 3\beta_2$, and $\delta - 2\beta_1 - 4\beta_2$. If $p \neq 7$, then $\dim(W_{\delta - 3\beta_1 - 3\beta_2}) \geq 3$, so $\dim(W_6) \geq 5$. So unless $p = 7$, here again we have only the $L_1 = A_1$ possibility.

If $a = 2$, then $l = 8$ and we must check dimensions to level 4. For level 2, we have enough weights as above. At level 3, we have $\delta - 2\beta_1 - \beta_2$ and $\delta - \beta_1 - 2\beta_2$; if $p \neq 5$, each has dimension 2, so $\dim(W_3) > 3$. At level 4, the weights are $\delta - 3\beta_1 - \beta_2, \delta - 2\beta_1 - 2\beta_2$, and $\delta - \beta_1 - 3\beta_2$. If $p \neq 5$, then $\delta - \beta_1 - 2\beta_2$ has dimension $\geq 3$, so $\dim(W_4) \geq 5$. As above, unless $p = 5$, we have only the $L_1 = A_1$ possibility.

From the construction of $P_Y$ we can see that in any case covered above (including $(a,p) = (2,5),(3,7)$), there is only one node in the Dynkin diagram between the $L_i$, since there are no $U_X$-levels of dimension 1 other than $\langle w^+ \rangle$. Also notice that from the $a = 2$ and $a = 3$ cases above we know the embeddings in the cases $(a,p) = (2,5),(3,7)$ (we simply compute the dimensions of the levels): If $a = 2, p = 5$, then $\dim(W_3) = 19$, so $Y$ has type $B_3$ and $P_Y$ is the parabolic subgroup of $Y$ corresponding to the indicated nodes

If $a = 3, p = 7$ then $\dim(W) = 37$, $Y$ is of type $B_{18}$, and $P_Y$ corresponds to

So the possibilities are 1) $a_2 = 1$, with the first simple factor of $L_Y'$ corresponding to $\cdots$; 2) $a = 2, p = 5$; 3) $a = 3, p = 7$; 4) $a = 1$. By Lemma 3.2, in the
marking for the high weight of \( V \) on the Dynkin diagram for \( Y \) there is only one non-zero label on the nodes representing \( L_Y \), and this non-zero label must be a 1 on a node corresponding to an \( A_1 \) factor of \( L_Y' \); call this node \( \gamma \). By the comment above, all nodes in the Dynkin diagram are either in or adjacent to \( \Pi(L_Y') \) (except possibly in case 4). Our aim is to show that all nodes except \( \gamma \) have marking 0.

As in the introduction, let \( V^2(Q_Y) = [V, Q_Y]/[V, Q_Y, Q_Y] \); similarly define \( V^2(U_X) \) and \( V^2(U_X) \) for \( i = 1, 2 \). Recall that \( \lambda \) is the \( T_Y \)-high weight of \( V \). By Lemma 2.4, the weights in \( V^2(Q_Y) \) are those of the form \( (\lambda - \beta)|_{T_Y} \) where \( \beta \in \Sigma^+(Y) \) and there exists \( \epsilon \in \Pi(Y) - \Pi(L_Y) \) such that \( \beta - \epsilon \in \Sigma(L_Y) \). Then by Lemma 2.4, we have \( \dim(V^2(Q_Y)) \leq \dim(V^2(U_X)) \leq 4 \).

First consider the case \( a = 1 \). If \( p \neq 3 \), then \( \dim(W) = 8 \) and \( Y = D_4 \); if \( p = 3 \), then \( \dim(W) = 7 \) and \( Y = B_3 \). In both cases, constructing \( P_Y \) as usual, we have \( \alpha_2 \in \Pi(L_Y') \), so the only possibility for an \( A_1 \) factor of \( L_Y \) to occur is \( \Pi(L_Y') = \{\alpha_2\} \) (as all other nodes adjoin \( \alpha_2 \)). For \( p \neq 3 \), if another node has a non-zero label, say \( \alpha_3 \) (all are equivalent by symmetry for this argument), then in \( V^2(Q_Y) \) we have the high weights \( \lambda - \alpha_3|_{T_Y} \), giving a composition factor of dimension 3 and \( \lambda - \alpha_2 - \alpha_1 \) and \( \lambda - \alpha_2 - \alpha_4 \) each giving one of dimension 1. But this contradicts \( \dim(V^2(Q_Y)) \leq 4 \). So \( \lambda = \lambda_2 \), giving an example of 1) above, which we deal with below. If \( p = 3 \) and \( a_1, a_3 \neq 0 \), then \( \lambda - \alpha_1|_{T_Y}, \lambda - \alpha_3|_{T_Y} \) each give a composition factor of dimension 3 in \( V^2(Q_Y) \), again a contradiction. Finally, irreducible \( B_3 \)-modules with high weights \( e\lambda_1 + \lambda_2 \) or \( \lambda_2 + e\lambda_3 \) are all too large to be the sum of two restricted irreducible \( A_2 \)-modules unless \( e = 0 \) (by counting weights that appear in the Weyl module of \( W_{B_3}(e\lambda_1 + \lambda_2) \) or \( W_{B_3}(\lambda_2 + e\lambda_3) \); all these weights appear in \( V \) by [8]). So again \( \lambda = \lambda_2 \), which we deal with below.

For cases 1)–3), assume there is an \( \epsilon \in \Pi(Y) \) which adjoins \( \gamma \) and has non-zero marking \( m \). If \( \epsilon \) is not an end node, then it also adjoins another factor \( L_t \) of \( L_Y' \) by our comment above that there are never two adjacent nodes outside \( \Pi(L_Y') \); then \( \lambda - \epsilon \) is an \( L_Y' \)-high weight in \( V^2(Q_Y) \), and the \( L_Y' \)-high weight module of this high weight has dimension \( \geq 6 \) \((\lambda - \epsilon)|_{T_{L_Y'}} = 2\lambda_j + \lambda_k \) where \( \lambda_j \) is the fundamental dominant weight corresponding to \( \gamma \) and \( \lambda_k \) is the node of \( L_t \) which adjoins \( \epsilon \) — the \( L_Y' \)-module with this high weight has dimension at least \( 2 \cdot 3 = 6 \). This contradicts \( \dim(V^2(Q_Y)) \leq 4 \).

If \( \epsilon \) is an end node, it cannot be the short root in a \( B_n \) (that root is in \( \Pi(L_Y') \) because we saw that \( \dim(W_{l/2}) \geq 3 \) in all cases). Then we have the picture

\[
\begin{array}{cccccc}
\epsilon & \gamma & \mu & \cdots \\
\end{array}
\]

We have high weights \( (\lambda - \epsilon)|_{T_{L_Y'}} \) of dimension 3 and \( (\lambda - \gamma - \mu)|_{T_{L_Y'}} \) of dimension \( \geq 2 \). Again, this contradicts \( \dim(V^2(Q_Y)) \leq 4 \). So all nodes adjoining \( \gamma \) have 0 label.

Assume there is an \( \epsilon \in \Pi(Y) \) which does not adjoin \( \gamma \) and has non-zero marking \( m \). By Lemma 3.2, \( \epsilon \notin \Pi(L_Y') \), and \( \epsilon \) adjoins \( \Pi(L_Y') \) since every fundamental root is either in \( \Pi(L_Y') \) or adjoins it. If \( \gamma \) is not an end node, we have the pictures (different pieces of the Dynkin diagram for \( Y \))

\[
\begin{array}{cccccc}
\alpha & \gamma & \beta & \mu & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
\epsilon & \gamma & \mu & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 1 & 0 & \cdots & m & 0 & \cdots \\
\end{array}
\]
Here $\lambda - \alpha - \gamma|_{T_{L_{\gamma}}} \oplus \lambda - \beta|_{T_{L_{\gamma}}}$ is a high weight in $V^2(Q_\gamma)$, giving dimension $\geq 1$; $\lambda - \gamma - \delta|_{T_{L_{\gamma}}}$ gives dimension $\geq 2$; and $\lambda - \epsilon$ gives dimension $\geq 2$. Again, this is a contradiction.

The node $\gamma$ can be an end node only in the cases $(a, p) = (2, 5), (3, 7)$ (these are the only cases in which an $A_1$ factor of $L^\gamma_{\gamma}$ corresponds to an end node of the Dynkin diagram for $Y$), with $\gamma$ the short root of a $B_n$; then the picture is

$$\begin{array}{cccccccc}
0 & 0 & 1 & \cdots & \mu & \cdots & 0 & 1 \\
\mu & \delta & \gamma & & & & & & \gamma \end{array}$$

Here $\lambda - \gamma - \delta|_{T_{L_{\gamma}}}$ gives dimension $\geq 4$; and $\lambda - \epsilon|_{T_{L_{\gamma}}}$ gives dimension $\geq 1$, again a contradiction. So all nodes other than $\gamma$ must have label 0.

For both of the above non-end node cases, similar arguments hold if there is a double bond in one of the relevant pieces of the Dynkin diagram for $Y$.

So $\gamma$ is the only node in the Dynkin diagram with a non-zero label. We need only to show that the few remaining possibilities do not lead to examples.

Case 1): $V$ has high weight $\lambda_2$. Then $V \cong \bigwedge^2 W$ by [7, II.2.15]. Regard $W$ as an $X$-module. Remember that $\delta = a\delta_1 + a\delta_2$ is the $T_X$-high weight for $W$. Let $v_1 \in W_3$ be a maximal vector in $W$; $0 \neq v_2 \in W_{\delta - \beta_1}, 0 \neq v_3 \in W_{\delta - \beta_2}$. Then $v_1 \wedge v_2$ and $v_1 \wedge v_3$ are $X$-maximal vectors in $V$, so $KX(v_1 \wedge v_2) \oplus KX(v_1 \wedge v_3) = V$.

We now consider the dimension of $KX(v_1 \wedge v_2)$. The vector $v_1 \wedge v_2$ has weight $2\delta - \beta_1 = (2a - 2)\delta_1 + (2a + 1)\delta_2$. So $\dim(KX(v_1 \wedge v_2)) \leq \dim$(Weyl module) = $\frac{1}{2}(2a - 1)(2a + 2)(4a + 1) = 8a^3 + 6a^2 - 3a - 1$. Also, $\dim(KX(v_1 \wedge v_3)) = \dim(KX(v_1 \wedge v_2))$ (since $t$ interchanges them), so $\dim(V) \leq 16a^3 + 12a^2 - 6a - 2$.

On the other hand, $\dim(W) \geq 3a^2 + 3a + 1$ (this is the number of weights that appear in the Weyl module with the same $T_X$-high weight as $W$; all these weights appear in $W$), and $\dim(V) = \dim(\bigwedge^2 W) = \binom{\dim(W)}{2} \geq \frac{(3a^2 + 3a + 1)(3a^2 + 3a + 1)}{2} = (9a^4 + 18a^3 + 12a^2 + 3a)/2$. So $9a^4 + 18a^3 + 12a^2 + 3a \leq 2(16a^3 + 12a^2 - 6a - 2)$. But this has no solutions in positive integers. So this case is ruled out.

The only cases not ruled out by the above now follow.

Case 2): $(a, p) = (2, 5)$ (here remember $Y = B_0$). If $\lambda = \lambda_7$ then the picture is

$$\begin{array}{cccccccc}
0 & 0 & 1 & \cdots & \epsilon & \cdots & 0 & 0 \\
\epsilon & \gamma & \mu & & & & & & \mu \end{array}$$

and $\lambda - \gamma - \epsilon|_{L_{\gamma}}$, $\lambda - \gamma - \mu|_{L_{\epsilon}}$ each give dimension $\geq 3$. So $\lambda \neq \lambda_7$. If $\lambda = \lambda_9$, then $\dim(V) = 2^9 = 512$. But the dimension of any irreducible $A_2$-module with high weight $c\delta_1 + d\delta_2$ $(c, d < 5, c \neq d)$ is at most 90. So $V$ is too large to be the sum of two restricted irreducible modules for $X$.

Case 3): $(a, p) = (3, 7)$ (here $Y = B_{18}$) with $\lambda = \lambda_{18}$. Here $\dim(V) = 2^{18}$. But $c, d < 7, c \neq d$; again, $V$ is too large to be the sum of two restricted irreducible $X$-modules.

So $X$ is not of type $A_2$.

$X = A_3$. We use a similar induction. Let $\delta = a\delta_1 + b\delta_2 + a\delta_3$ be the $T_X$-high weight of $W$. First we eliminate the case $b = 0$ with the $A_1$ factor of $L^\gamma_\gamma$ (referred to in Lemma 3.2) corresponding to $\alpha_2$.

Assume $\delta = a\delta_1 + a\delta_3$ and $\lambda = \lambda_2 + \cdots$. Let $P_X = L_X Q_X$ be the maximal parabolic subgroup of $X$ corresponding to $\beta_3 \in I(X)$, and embed $P_X$ in a parabolic.
then \( \dim(W/1) = 3 \), so \( L_1 \) is trivial; if \( a > 0 \), then \( \dim(W_1) = 3 \), so \( L_1 \) is of type \( A_2 \). Again, there are enough weights at level 2. So there are no possible \( A_1 \) factors of \( L'_Y \) here.

Assume \( V/[V, Q_Y] \) is reducible as an \( L_X \)-module; that is,

\[
V/[V, Q_Y] = L_X \oplus V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X].
\]

Then \( Z \leq Z(L_Y) \) since \( L_Y = C_Y(Z) \). So \( Z \) induces scalars on \( V/[V, Q_Y] \) (an irreducible \( L_Y \)-module):

\[
Z = \left\{ \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^{-3} \end{pmatrix} \mid c \in K^* \right\}
\]

\[
= \{ h_{\beta_1}(c) h_{\beta_2}(c^2) h_{\beta_3}(c^3) \mid c \in K^* \}.
\]

Now \( h_{\beta_1}(c) h_{\beta_2}(c^2) h_{\beta_3}(c^3) \) acts by multiplication by \( e^{\delta_1 + 2\delta_2 + 3\delta_3} \) on a high weight vector \( v_1 \in V_1 \) and as multiplication by \( e^{\delta_1 + 2\delta_2 + 3\delta_1} \) on a high weight vector \( v_2 \in V_2 \).

But by our assumption, \( v_1 \) and \( v_2 \) both have nonzero images in \( V/[V, Q_Y] \); hence \( b_1 + 2b_2 + 3b_3 \neq b_1 + 2b_2 + 3b_1 \), which contradicts \( b_1 \neq b_3 \) because \( V_1 \neq X V_2 \).

So \( V/[V, Q_Y] \) is an irreducible \( L_X \)-module and thus an irreducible \( L'_X \)-module by Lemma 2.2. Assume \( V/[V, Q_Y] = X V_1/[V_1, Q_X] \).

Now we are in the situation studied in [9]: \( V/[V, Q_Y] \) is an irreducible module for \( L'_X \) (of type \( A_2 \)) and for \( L_1 \). As there are no examples matching this setup in [9, Table 1], we know that either the embedding \( L'_X \hookrightarrow L_1 \) is an isomorphism or \( V/[V, Q_Y] \) is the natural module for \( L_1 \). Both possibilities are excluded if \( a > 1 \), as then \( L_1 \) is of type \( A_1 \) with \( l \geq 5 \) (so \( L_1 \neq L'_X \)), and \( a_2 = 1 \) (so \( V/[V, Q_Y] \) is not the natural module for \( L_1 \)). On the other hand, if \( a = 1 \), then necessarily \( L'_X \cong L_1 \) (as \( W/[W, Q_X] \) then has \( L'_X \)-high weight \( \delta_1 \)).

So \( V/[V, Q_Y] \) must have \( L'_X \)-high weight \( a_1 \delta_1 + \delta_2 \), since it has \( L_1 \)-high weight \( a_1 \lambda_1 + \lambda_2 \) and the embedding is an isomorphism. But \( V_1/[V_1, Q_X] \) also has high weight \( b_1 \delta_1 + b_2 \delta_2 \). So \( a_1 = b_1 \) and \( b_2 = 1 \).

Now the argument above can be repeated with the maximal parabolic subgroup of \( X \) corresponding to \( \beta_1 \) instead of \( \beta_3 \), with the conclusion that \( b_3 = a_1 \). But then \( b_1 = b_3 \), a contradiction.

So if \( \delta = a \delta_1 + a \delta_3 \), then \( a_2 = 0 \).

Every weight of the form \( a \delta_1 + b \delta_2 + a \delta_3 \) except \( \delta_1 + \delta_3 \), \( \delta_2 + 2 \delta_3 \), \( 2 \delta_1 + 2 \delta_3 \) or \( \delta_1 + \delta_2 + \delta_3 \) as a subdominant weight. It is easy to check, as in the \( A_2 \) case, that the modules with these latter two high weights have enough weights at every level, so we can proceed by induction: If \( b < 2 \) and \( a > 2 \), then by induction \( \delta - \beta_1 - \beta_2 - \beta_3 = (a - 1) \delta_1 + b \delta_2 + (a - 1) \delta_3 \) has enough weights at all levels; we need to check \( \delta \)-levels 2-5. As before, there are enough weights in each of these levels, so by induction, \( L_1 \) is the only possible factor of \( L'_Y \) of type \( A_1 \).

If \( b \geq 2 \), then by induction \( \delta - \beta_2 \) has enough weights at all levels, and we need to check only \( \delta \)-levels 1 and 2. If \( a = 0 \), then \( \dim(W_1) = 1 \), so \( L_1 \) is trivial; if \( a > 0 \), then \( \dim(W_1) = 3 \), so \( L_1 \) is of type \( A_2 \). Again, there are enough weights at level 2. So there are no possible \( A_1 \) factors of \( L'_Y \) here.
So the possibilities which have not been ruled out are $\delta = \delta_1 + \delta_3$, $\delta = \delta_2$, and $\delta = 2\delta_2$. If $\delta = \delta_2$, then $X = SO(W) = Y$, so there are no examples here. If $\delta = \delta_1 + \delta_3$, we can check the dimensions of the weight spaces and find that $\dim(W) = 15$ if $p \neq 2$, and $P_Y$ is the parabolic subgroup of $Y$ corresponding to the subset $\{\alpha_2, \alpha_4, \alpha_5, \alpha_7\}$ of $\Pi(Y)$, as indicated on this picture of the Dynkin diagram

for $Y = B_7$.

If $\delta = \delta_2$, then $X = SO(W) = Y$, so there are no examples here.

If $\delta = \delta_1 + \delta_3$, we can check the dimensions of the weight spaces and find that $\dim(W) = 15$ if $p \neq 2$, and $P_Y$ is the parabolic subgroup of $Y$ corresponding to the subset $\{\alpha_2, \alpha_4, \alpha_5, \alpha_7\}$ of $\Pi(Y)$, as indicated on this picture of the Dynkin diagram

we see that the parabolic subgroup $P_Y$ corresponds to

(stabilizer of the 2-dimensional level 3 is $SO_2$, which is a torus). So Lemma 3.5 rules out this case.

If $\delta = 2\delta_2$, again we can check dimensions of all the weight spaces to find $\dim(W) = 20$ ($p \neq 2, 3$), $\dim(W) = 19$ ($p = 3$), and $P_Y$ is indicated by the circled nodes of the Dynkin diagram for $Y$.

\[ \text{(p \neq 2, 3)} \]

or

\[ \text{(p = 3)} \]

Notice we need not consider $p = 2$ by Lemma 2.6.

So the cases we must deal with are

\[ W|_X \quad Y \quad \text{and embedding of } B_X < P_Y \]

1) $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$

\[ \text{(p \neq 2)} \]

2) $\begin{bmatrix} 0 & 2 & 0 \end{bmatrix}$

\[ \text{(p \neq 2, 3)} \]

\[ \text{(p = 3)} \]

In the marking for $V$ on $\Pi(Y)$, there is a 1 on one of the nodes $\alpha_j$ corresponding to an $A_1$ factor of $L_Y'$. There can be no other nonzero marking on any of the indicated nodes, since $\dim(V/[V,Q_Y]) = 2$. Recall that $j \neq 2$ by Lemma 3.5; thus, as we see from the pictures above, $j = n$ is the only possibility (or $j = n - 1$ in case 2, but we may assume $j = n$ by symmetry). We claim that $\alpha_n$ is the only node with a nonzero label.

**Claim 3.3.** *In any of the above cases $\lambda = \lambda_n$.***

**Proof.** Since $U_X \leq Q_Y$, we have $[V,Q_Y,Q_Y] \geq [V,U_X,U_X]$. Then because $[V,Q_Y] = [V,U_X]$, we have $\dim([V,Q_Y]/[V,Q_Y,Q_Y]) \leq \dim([V,U_X]/[V,U_X,U_X]) \leq 6$ by Lemma 2.4. The weights that appear in $V^2(Q_Y) = [V,Q_Y]/[V,Q_Y,Q_Y]$ are those of the form $\lambda - \beta$, where if $\beta = \sum e_i \alpha_i$, then the sum of the $e_i$ for those
\( \alpha_i \in \Pi(Y) - \Pi(L'_Y) \) is 1. Let us consider the above cases. Remember \( \lambda = \sum a_i \lambda_i \) is the \( T_Y \)-high weight for \( V \).

1) The node \( \alpha_7 \) has label 1 (\( a_7 = 1 \)). Consider the possibilities for another nonzero label. If \( a_1 \neq 0 \), then \( \lambda - \alpha_1 \) is a high weight in \( V^2(Q_Y) \), giving a composition factor of dimension 4. Another high weight is \( \lambda - \alpha_6 - \alpha_7 \), giving dimension 6 since \( p \neq 2 \). But above we noted that \( \dim(V^2(Q_Y)) \leq 6 \). So \( a_1 = 0 \). If \( a_3 \neq 0 \), then \( \lambda - \alpha_3 \) gives \( \dim(V^2(Q_Y)) \geq 12; \) so \( a_3 = 0 \). If \( a_6 \neq 0 \), then \( \lambda - \alpha_6 \) gives dimension 12 since \( p \neq 2 \). So \( a_6 = 0 \).

2) If \( p \neq 3 \), we are in the case \( Y = D_{10} \), with \( a_{10} = 1 \). If \( a_8 \neq 0 \), then \( \lambda - \alpha_8 \) gives dimension 18 in \( V^2(Q_Y) \). If \( a_5 \neq 0 \), then \( \lambda - \alpha_5 \) gives dimension 18. If \( a_2 \neq 0 \), then \( \lambda - \alpha_2 \) gives dimension 6 and \( \lambda - \alpha_8 - \alpha_9 \) gives dimension 6. If \( a_1 \neq 0 \), then \( \lambda - \alpha_8 - \alpha_9 \) gives dimension 6 and \( \lambda - \alpha_1 \) dimension 2. So in fact \( a_i = 0 \) for all \( i \neq 9 \).

If \( p = 3 \), then \( Y = B_9 \) and \( a_9 = 1 \). If \( a_8 \neq 0 \), then \( \lambda - \alpha_8 \) gives dimension 6 and \( \lambda - \alpha_8 - \alpha_9 \) gives dimension \( \geq 1; \) so \( a_8 = 0 \). If \( a_5 \neq 0 \), then \( \lambda - \alpha_5 \) gives dimension 18; if \( a_2 \neq 0 \), \( \lambda - \alpha_2 \) gives dimension 6 and \( \lambda - \alpha_8 - \alpha_9 \) gives dimension 6. If \( a_1 \neq 0 \), then \( \lambda - \alpha_8 - \alpha_9 \) gives dimension 6 and \( \lambda - \alpha_1 \) dimension 2. So \( a_i = 0 \) for all \( i \neq 9 \). \( \square \)

**Lemma 3.4.** If \( \lambda = \lambda_n \) (i.e. if we are in one of the remaining cases), then as an \( X \)-module, \( W \) has high weight \( 2\delta_2 \) and \( V \) is as in the statement of Theorem 3.1.

**Proof.** We still have to check cases 1) and 2) with \( \lambda = \lambda_n \).

1) Assume \( W \) has \( T_X \)-high weight \( \delta_1 + \delta_3 \) and \( p \neq 2 \). Consider the embedding in a parabolic subgroup \( P_Y \leq Y \) of the parabolic subgroup \( P_X \leq X \) corresponding to \( \{ \beta_1, \beta_2 \} \subseteq \Pi(X) \). Checking the dimensions at different \( Q_X \)-levels as before, we see that for any characteristic, \( P_Y \) corresponds to \( \{ \alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \} \). Since \( \lambda = \lambda_n, \dim(V/[V,Q_Y]) = 16 \) if \( \{ \alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \} \) is isomorphic to a spin module for the simple factor of \( L'_Y \) of type \( B_4 \) corresponding to \( \{ \alpha_1, \alpha_2, \alpha_4, \alpha_5 \} \). The quotient \( V/[V,Q_Y] \) is also an irreducible \( L'_X \)-module (again by considering the action of \( Z = Z(L_X) \) on the two \( T_X \)-high weight vectors of \( V \), but \( L'_X = A_2 \), which has no irreducible representations of dimension 16 in any characteristic. So we have no examples here.

2) Assume \( W|_X \) has high weight \( 2\delta_2 \). If \( p = 3 \), then \( \dim(W) = 19 \) and \( Y = B_9 \). Using \( P_X \) corresponding to \( \{ \beta_1, \beta_3 \} \) as above, we get an embedding of \( P_X \) in the parabolic subgroup \( P_Y \) corresponding to \( \Pi(Y) - \{ \alpha_6 \} \). Since \( \lambda = \lambda_n, \dim(V/[V,Q_Y]) = \dim(\text{spin}(B_3)) = 8 \). But \( V/[V,Q_Y] \) is an irreducible \( L'_X \)-module (by the action of \( Z \) again), and \( L'_X = A_2 \), which has no 8-dimensional irreducible representations in characteristic 3.

If \( p \neq 3 \), then take \( P_X \) as above; again \( P_Y \) corresponds to \( \Pi(Y) - \{ \alpha_6 \} \) and \( \dim(V/[V,Q_Y]) = \dim(\text{spin}(D_4)) = 8 = \dim(V_i/[V_i,Q_X]) \) for \( i = 1 \) or 2, say \( i = 1 \). So \( b_1 = 1, b_2 = 1 \).

Now let \( P_X \) correspond to \( \{ \beta_1, \beta_3 \} \). This \( P_X \) is \( t \)-stable, so

\[
V/[V,Q_Y] = V/[V_i,Q_X] = V_i/[V_i,Q_X] \oplus V_2/[V_2,Q_X].
\]

The embedding gives \( P_Y \) corresponding to \( \Pi(Y) - \{ \alpha_1, \alpha_3 \} \). So here,

\[
\dim(V/[V,Q_Y]) = 16 = \dim(V/[V,Q_X]) = 2 \dim(V_i/[V_i,Q_X]) = 2(b_1 + 1)(b_3 + 1) = 4(b_3 + 1).
\]
So \( b_3 = 3 \). Now \( \dim(V_{D_{10}}(\lambda_n)) = 2^9 \) in any characteristic; \( \dim(V(3\delta_1 + \delta_2 + \delta_3)) = 256 = 2^8 \) when \( p > 7 \) or \( p = 0 \). So \( V|_X = \begin{array}{cccc} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ \end{array} \), \( V|_Y = \text{spin}(D_{10}) \) is a possibility for \( p \neq 2, 3, 5, 7 \).

In this case, consider again the embedding of \( B_X \) in the parabolic subgroup

\[ P_Y \]

We define some temporary notation: Let \( \lambda_{1,j} = (\lambda_1 - \alpha_1 - \cdots - \alpha_j)|_{T_X} \). By the construction of the embedding, we know that

\[
\begin{align*}
\lambda|_{T_X} & = \delta (= 2\delta_2), \\
\lambda_{1,1} = (\lambda_1 - \alpha_1)|_{T_X} & = \delta - \beta_2, \\
\{\lambda_{1,2}, \lambda_{1,3}, \lambda_{1,4}\} & = \{-\delta - \beta_1 - \beta_2, \delta - \beta_2 - \beta_3, \delta - 2\beta_2\}, \\
\{\lambda_{1,5}, \lambda_{1,6}, \lambda_{1,7}\} & = \{-\delta - \beta_1 - 2\beta_2, \delta - 2\beta_2 - \beta_3, \delta - \beta_1 - 2\beta_2 - \beta_3\}, \\
\{\lambda_{1,8}, \lambda_{1,9}, \lambda_{1,10}\} & = \{-\delta - 2\beta_2 - 2\beta_3, \delta - 2\beta_1 - 2\beta_2, \delta - 2\beta_1 - 2\beta_2 - \beta_3\},
\end{align*}
\]

with \( (\lambda_1 - \alpha_1 - \cdots - \alpha_8)|_{T_X} = -((\lambda_1 - \alpha_1 - \cdots - \alpha_{10})|_{T_X}) \) (since \( \lambda_1 - \alpha_1 - \cdots - \alpha_8 = -\lambda_9 + \lambda_10 = -(\lambda_1 - \alpha_1 - \cdots - \alpha_{10}) \)).

This gives enough information to determine the possibilities for \( \alpha_i|_{T_X} \) for \( i = 1, \ldots, 10 \). We can write the \( T_Y \)-high weight for \( V, \lambda_{10} \), in terms of the \( \alpha_i \), and we find that with any of the possible choices made above, \( \{\lambda_{10}|_{T_X}, (\lambda_{10} - \alpha_{10})|_{T_X}\} = \{3\delta_1 + \delta_2 + \delta_3, \delta_1 + \delta_2 + 3\delta_3\} \). So \( V \) contains \( A_3 \)-submodules of these two high weights; since their dimensions add to \( \dim(V) \), we have the case stated in the theorem. \( \square \)

This completes the proof for \( X = A_3 \).

3.3. When Lemma 3.2 Doesn’t Help. Using our standard construction of \( P_Y \) (Lemma 2.7), the obvious situation in which the Lemma 3.2 is of no help is when \( \delta = a\delta_1 + b\delta_j \), i.e. when \( U_X \)-level 0 has dimension 2. In this case \( L_1 \leq L'_Y \) is of type \( A_1 \), corresponding to \( \alpha_2 \in \Pi(Y) \). Remember that \( \delta \) must be symmetric, so that in fact the following is what we will need.

**Lemma 3.5.** The situation \( \delta = a\delta_1 + a\delta_{m-i+1} \) (\( i \leq m/2 \)), \( a_2 = 1 \) does not give any examples in the Main Theorem if \( m > 3 \).

**Proof.** With the given \( \delta \) and with \( P_Y \) as in Lemma 3.2, we have \( \alpha_2 \in \Pi(L'_Y) \) since level 1 in the construction of \( P_Y \) has dimension 2. So Lemma 3.2 tells us that \( \alpha_2 \) is the only nonzero coefficient on \( \Pi(L'_Y) \).

Assume \( i > 1 \). Consider the parabolic subgroup \( P_X \) of \( X \) corresponding to \( \Pi(X) - \{\beta_1, \beta_m\} \), as in the following picture:

\[
\begin{array}{cccc}
\bullet & \bullet & \cdots & \bullet \\
\end{array}
\]

So \( L'_X \) is of type \( A_{m-2} \), and if we embed \( P_X \hookrightarrow P_Y \) by the usual construction, then level 0 (that is, \( W/[W,Q_X]\) of the flag which results is totally singular and has dimension \( \geq 7 \) (the smallest \( W/[W,Q_X]\) could be is when \( L'_X \) is of type \( A_2 \) with a label of 1 on each node; this has dimension 7 in characteristic 3 and dimension 8 in other characteristics); thus \( L_1 \) (the simple factor of \( L'_Y \) corresponding to the quotient \( W/[W,Q_X] \) in the flag) is of type \( A_l \) for some \( l \geq 6 \).
By Lemma 2.11, only one $L_j$ acts nontrivially on $V/[V, Q_Y]$. So $V/[V, Q_Y]$ is a nontrivial (there is at least one nonzero label on $L_1$ in the marking for the high weight of $V/[V, Q_Y]$, namely, on the second node), irreducible $L_1$-module, and as an $L'_X$-module, $V/[V, Q_Y]$ is the sum of two irreducibles, $V_1/[V_1, Q_X]$ and $V_2/[V_2, Q_X]$. The natural module for $L_1$ is isomorphic to $W/[W, Q_Y]$ and is irreducible as an $L'_X$-module. So unless $V_1/[V_1, Q_X] \cong V_2/[V_2, Q_X]$, we are in the situation we consider in this section and by induction no examples arise (the single case which arises below for $X = A_3$ does not arise inductively because $\delta$ does not have the form we are considering here).

Let the $T_X$-high weight of $V_1$ be $b_1 \delta_1 + b_2 \delta_2 + \cdots + b_m \delta_m$ (so the high weight of $V_2$ is $b_m \delta_1 + b_{m-1} \delta_2 + \cdots$). Assume $b_2 = b_{m-1}, b_3 = b_{m-2}, \ldots$ (i.e. $V_1/[V_1, Q_X] \cong V_2/[V_2, Q_X]$). This implies $b_1 \neq b_m$ since $V_1 \neq V_2$. Now take $P_X$ to be another parabolic subgroup of $X$, corresponding to $\Pi(X) - \{\beta_m\}$, and embed $P_X \hookrightarrow P_Y$ via the same construction. Again we have $W/[W, Q_Y] = W/[W, Q_X]$ irreducible for $L'_X$ and for $L_1$.

We show that in this case $V/[V, Q_Y]$ is irreducible as an $L_X$-module (in contrast to the situation when the parabolic subgroup of $X$ is $t$-stable, which forces $V/[V, Q_Y]$ to be the sum of two irreducibles for $L_X$). Let $Z = Z(L_X)^c$. Since $L_Y = C_Y(Z)$, we have $Z \leq Z(L_Y)$, so $Z$ induces scalars on $V/[V, Q_Y]$ (since $L_Y$ acts irreducibly on this module). But if $V/[V, Q_Y]$ is reducible as an $L_X$-module, then $V/[V, Q_Y] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$, and we show that $Z$ acts differently on these two $L_X$-modules:

$$Z = \{ \text{diag}(a, \ldots, a, a^{-m}) | a \in K^* \}$$

so $h_{\beta_1}(a)h_{\beta_2}(a^2)\cdots h_{\beta_m}(a^m)$ acts by multiplication by $a^{b_1+2b_2+\cdots+mb_m}$ on a high weight vector $v_1 \in V_1$ and as multiplication by $a^{b_m+2b_{m-1}+\cdots+mb_1}$ on a high weight vector $v_2 \in V_2$; these two exponents are not equal. Since $v_j$ has a nonzero image in $V_j/[V_j, Q_X]$, this shows that only one of the $V_j/[V_j, Q_Y]$ can have a nonzero image in $V/[V, Q_Y]$. So $V/[V, Q_Y]$ is irreducible as an $L_X$-module.

Assume $V_1$ is the summand which projects nontrivially to $V/[V, Q_Y]$ (so $V_2 \subseteq [V, Q_Y]$). As no irreducible restricted representations of $A_{m-1}$ are tensor decomposable by Lemma 2.1, $L_1$ is the only simple factor of $L'_Y$ to act nontrivially on $V/[V, Q_Y]$. Note that the rank of $L_1$ is $\geq 15$, since this rank is one less than the dimension of $W/[W, Q_Y]$; With $\delta$ of the form we are assuming, the $A_{m-1}$-high weight of $W/[W, Q_Y]$ is of the form $(a\delta_i + a\delta_{m-i+1})|_{T_{X'}}$. The Weyl module for $A_{m-1}$ with this high weight has at least 16 weights, and these weights all appear in the irreducible module $W/[W, Q_Y]$ by the result in [12].

So now we are inductively in the situation examined in [9]: $V/[V, Q_Y]$ is an irreducible module for both $L'_X$ and $L_1$, and it is not the natural module for $L_1$ (since in the labelling for the $L_1$-high weight of $V/[V, Q_Y]$ there is a 1 on the second node of $\Pi(L_1)$). Also, $L'_X \not\cong L_1$ because the natural module for $L_1$, $W/[W, Q_Y]$, has high weight $\delta|_{T_{X'}}$ and thus has dimension larger than $m$, which is the dimension of the natural module for $L'_X$. So any examples here would appear in Table 1 of [9];
examining that table, we see that in fact there are no examples. This completes the case $i > 1$.

So we need to consider only the case $i = 1$, i.e. $\delta = a\delta_1 + a\delta_m$, with $\lambda = \lambda_2 + \cdots$. Let $P_X = L_X Q_X$ be the maximal parabolic subgroup of $X$ corresponding to $\beta_j \in \Pi(X)$ and embed $P_X$ in a parabolic subgroup $P_Y = L_Y Q_Y$ of $Y$ via the usual construction. Notice that $L_1$ (the simple factor of $L_Y$ corresponding to the $Q_X$-level 0 of $W$) is of type $A_l$, with $l > 3$ unless $m = 4$ (we have taken care of the cases $m = 2, 3$ in §3.2). We wish to show that for at least one choice of $j$, $V/[V, Q_Y]$ is irreducible as an $L_X$-module. We will again use the action of $Z = Z(L_X)^\circ$ on $T_X$-high weight vectors in $V$:

$$Z = \{ \text{diag}(a^{(m-j+1)}, a^{(m-j+1)}, \ldots, a^{(m-j+1)}, b, \ldots, b) \mid a \in K^+ \}$$

$$= \{ h_{\beta_1}(a^{(m-j+1)}) h_{\beta_2}(a^{2(m-j+1)}) \ldots h_{\beta_j}(a^{j(m-j+1)}) h_{\beta_{j+1}}(a^{j(m-j-1)}) \ldots h_{\beta_m}(a^l) \mid a \in K^+ \}. $$

If $V/[V, Q_Y]$ is reducible as an $L_X$-module, then as above, $Z$ must act as multiplication by the same scalar on a high weight vector $v_1 \in V_1$ as on a high weight vector $v_2 \in V_2$, and we get the equation

$$ (m-j+1)b_1 + 2(m-j+1)b_2 + \cdots + j(m-j+1)b_j + j(m-j)b_{j+1} + \cdots + j \cdot b_m = (m-j+1)b_m + 2(m-j+1)b_{m-1} + \cdots + j(m-j+1)b_{m-j+1}$$

$$+ j(m-j)b_{m-j} + \cdots + j(m-j-1)b_{m-j-1} + \cdots + j \cdot b_1. $$

If we assume that $V/[V, Q_Y]$ is irreducible as an $L_X$-module for every $j$, then we have a system of equations which together imply $b_1 = b_m, b_2 = b_{m-1}, \ldots$. For example, the equations for $j = m$ and $j = m-1$ are

$$b_1 + 2b_2 + \cdots + (m-1)b_{m-1} + mb_m = b_m + 2b_{m-1} + \cdots + mb_1, $$

$$2b_1 + 4b_2 + \cdots + 2(m-1)b_{m-1} + (m-1)b_m = 2b_m + 4b_{m-1} + \cdots + (m-1)b_1. $$

Twice the first equation minus the second gives $(m+1)b_m = (m+1)b_1$. Knowing $b_1 = b_m$, the equations for $j = m$ and $j = m-2$ give $b_2 = b_{m-1}$; continuing in this manner we obtain $b_l = b_{m-l+1}$ for every $l$. But this is impossible, as it would imply that $V$ is reducible for $X(l)$. So for some $j, 1 \leq j \leq m$, $V/[V, Q_Y]$ is irreducible as an $L_X$-module, where $L_X, Q_Y$ are as above.

But then again we are in the situation examined in [9]: $V/[V, Q_Y]$ is an irreducible module for $L_X = A_{j-1} \times A_{m-j}$ and for $L_1$. So one of the following occurs:

1. The embedding $L_X \hookrightarrow L_1$ is an isomorphism.
2. $V/[V, Q_Y]$ is the natural module for $L_1$ (which happens only for $m = 3, 4$, since $L_1$ has rank $\geq 4$ in other cases we consider and the $L_1$-high weight of $V/[V, Q_Y]$ has a nonzero label on the node $\alpha_2$).
3. $V/[V, Q_Y]$ appears in Table 1 of [9].

We deal with 3 first. Of the appearances of the inclusion $A_{j-1} \times A_{m-j} \leq A_l$ in Table 1 of [9], only one (case $I_7$ there) gives the correct restriction of the natural module for $A_l$ to the subgroup. So the possible picture here is $(L_X' = A_{m-1};
\[ L_1 = A_{(m^2 + m - 2)/2}; \text{the high weight of } W|_X = 2\delta_1 + 2\delta_m: \]

\[ V_1/[V_1, Q_X] = \begin{array}{cccccc}
2 & 1 & 0 & \cdots & \bullet & b_m \\
\bullet & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \quad \text{and} \quad V/[V, Q_Y] = \begin{array}{cccc}
0 & 1 & 0 & 0 & \cdots
\end{array}. \]

Now we look at another parabolic subgroup of \( X \): Let \( P_X \) be generated by \( B_X \) and the root subgroups for \( \beta_1 \) and \( \beta_m \); embed \( P_X \) in a parabolic subgroup \( P_Y \) of \( Y \) as usual. By Lemma 2.11, \( L_1 \) is the only simple factor of \( L'_Y \) to act nontrivially on \( V/[V, Q_Y] \). Notice \( \dim(W/[W, Q_X]) = 9 \) (since \( W|_X \) has \( T_X \)-high weight \( 2\delta_1 + 2\delta_m \)), so \( L_1 \) has rank 8. Thus \( 2 \cdot 3(b_m + 1) = \dim(V_1/[V_1, Q_X]) + \dim(V_2/[V_2, Q_X]) = \dim(V/[V, Q_X]) = \dim(V/[V, Q_Y]) = \binom{9}{2} = 36 \), which tells us \( b_m = 5 \). This tells us that the only two \( T_{L_X} \)-high weights of \( V/[V, Q_Y] \) are \((2\delta_1, 5\delta_m)\) and \((5\delta_1, 2\delta_m)\). But in fact \( V/[V, Q_Y] \cong \bigwedge^2(W/[W, Q_Y]) \) \( (p \neq 2 \text{ because of the } 2 \text{ appearing in the picture above and our assumption that the } T_X \)-high weights of \( V \) are restricted), and two \( T_{L_X} \)-high weights of \( \bigwedge^2(W/[W, Q_Y]) \) are \((4\delta_1, 2\delta_m)\) and \((2\delta_1, 4\delta_m)\). Again we have a contradiction.

Next consider item 2. This can only occur if \( \dim(W/[W, Q_X]) \leq 4 \) (since there is a 1 on the second node in the marking for the high weight of \( V/[V, Q_Y] \) on \( L_1 \)) and this occurs only in the cases we excluded \((X = A_2, A_3)\) and the case \( X = A_4 \), with \( j = 4 \) (or, equivalently, \( j = 1 \)), \( \delta = \delta_1 + \delta_4 \). But this in fact gives an instance of item 1 (it implies that \( W/[W, Q_Y] \) has \( T_{L_X} \)-high weight \( \delta_1 \)).

So we are left with item 1. Note that the equations (1) hold for all \( j \neq 1, m \), since in the consideration of item 3 above we obtained a contradiction to \( V/[V, Q_Y] \) being irreducible for \( L'_X \) when \( j \neq 1, m \). We have the pictures

\[
V_1/[V_1, Q_X] = \begin{array}{cccccc}
1 & a_3 & a_m & \cdots & \bullet & b_m \\
\bullet & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}, \\
V/[V, Q_Y] = \begin{array}{cccc}
1 & a_3 & a_m & \cdots
\end{array}.
\]

So \( a_1 = b_1, 1 = a_2 = b_2, \ldots, a_{m-1} = b_{m-1} \). Now let \( P_X \) correspond to the indicated nodes, \( \begin{array}{cccccc}
\bullet & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \), and embed it in a parabolic subgroup of \( Y \) as usual. Then \( V/[V, Q_Y] \) is the sum of two irreducible \( L'_X \)-modules, and the simple factor \( L_2 \) of \( L'_Y \) corresponds to the indicated nodes at the beginning of the Dynkin diagram for \( Y \), \( \begin{array}{cccccc}
\bullet & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \), with \( L_2 \) of rank \( 2m - 3 \). But by the construction of \( P_Y \), we know the embedding \( L'_X \hookrightarrow L_2 \) here (the natural module for \( L_2 \) is \( Q_X \)-level 1 of \( W \) and restricts to \( L'_X \) as the sum of the two irreducible modules with high weights \( \delta_2|_{T_{L_X}} \) and \( \delta_{m-1}|_{T_{L_X}} \)). As this situation gave no examples in [4], we know by induction that either 1) \( a_3 = a_4 = \cdots = a_{2m-2} = 0 \) (i.e. \( V/[V, Q_Y] \cong \bigwedge^2(W/[W, Q_Y]) \)) or 2) \( 1 = a_2 = a_m, a_3 = a_m, \ldots \) (i.e. \( V/[V, Q_Y] \) is irreducible for \( L'_X(t) \)).

We noted that all the equations (1) on the previous page hold for \( j \neq 1, m \). The equation for \( j = m - 1 \) is

\[ 2b_1 + 4b_2 + \cdots + 2(m - 1)b_{m-1} + (m - 1)b_m = 2b_m + 4b_{m-1} + \cdots + (m - 1)b_1. \]
Now if 2) above holds, we know that $b_2 = b_{m-1}, b_3 = b_{m-2}, \ldots$ (remember that $a_i = b_i$ for $1 < i < m$), which together with the above equation gives $b_1 = b_m$, which is a contradiction. So 2) does not occur.

If $a_3 = \cdots = a_{2m-2} = 0$, i.e. 1) holds, then once again we examine the equation above and see that now it reduces to $2b_1 + 4 + (m-1)b_m = 2b_m + 2(m-1) + (m-1)b_1$ (remembering $a_i = b_i$ for $1 < i < m$ and $a_2 = b_2 = 1$) or $b_m - b_1 = 2$. Let $P_X$ be the parabolic subgroup of $X$ corresponding to $\{\beta_1, \beta_m\}$. Then $L_1$ is of type $A_3$, $L_1$ is the only factor of $L'_Y$ acting nontrivially on $V/[V,Q_Y]$ by Lemma 2.11, and the picture is

$$V_1/[V_1,Q_X] = b_1 \otimes b_1 + 2 \quad \text{and} \quad V/[V,Q_Y] = \begin{array}{cccc}
 b_1 & 1 & 0 & \ldots \\
 \end{array}.$$ 

By the Andersen-Jantzen sum formula ([1]), the $A_3$-module with the high weight pictured above is in fact the Weyl module except when $b_1 = p = 2$. But $p > b_m = b_1 + 2$ here, so the dimension of $V/[V,Q_Y]$ is the dimension of the Weyl module, which is $(b_1 + 1)(b_1 + 3)(b_1 + 4)/2$. On the other hand, $\dim(V/[V,Q_Y]) = \dim(V/[V,Q_X]) = 2 \dim(V_1/[V_1,Q_X]) = 2(b_1 + 1)(b_1 + 3)$. These two equations together imply $b_1 = 0$.

So now all the $b_i$ are known: $b_2 = 1, b_m = 2$, and all others are 0. Since one of the coefficients is greater than 1, we know $p \neq 2$. And since the $T_X$-high weight of $W$ is $\delta_1 + \delta_m$, we know $\dim(W) = m(m + 2)$ or $m(m + 2) - 1$. We claim that $\dim(V) \geq \dim(\wedge^2 W)$ (this is clear if $\lambda = \lambda_2$, as then $V = \wedge^2 W$ since $Y = D_n$ and $p \neq 2$).

The $T_Y$-high weight $\lambda$ of $V$ has a $\lambda_2$-coefficient of 1, and $W$ is orthogonal. We claim that any $B_n$ or $D_n$ weight of this form has one of $\lambda_2, \lambda_3,$ or $\lambda_m$ as a subdominant weight. For $B_n$, every fundamental weight except $\lambda_\alpha$ is a sum of roots, and $2\lambda_\alpha$ is a sum of roots, so any $\lambda$ with a $\lambda_2$-coefficient of 1 has $\lambda_2$ or $\lambda_\alpha$ as a subdominant weight. For $D_n$, any fundamental weight $\lambda_k$ for even $k \leq n - 2$ is a sum of roots; for odd $k$ with $1 < k < n - 2$, $\lambda_k$ differs from a sum of roots by $\lambda_3$.

The weight $2\lambda_\alpha$ or $2\lambda_{n-1}$ is either a sum of roots or differs from one by $\lambda_3$; finally, $\lambda_1 + \lambda_2$ has $\lambda_3$ as a subdominant weight. So the claim holds.

Say $\lambda \succ \lambda_1$. Then by the result in [8], every weight which appears in the Weyl module with $T_Y$-high weight $\lambda$ appears in $V$. So

$$\dim(V) \geq \card(\{\omega \text{ is a weight occurring in } V_\lambda(\lambda_i)\})$$

$$+ \card(\text{Weyl group-orbit of } \lambda)$$

$$\geq \left(\frac{\dim(W)}{2}\right) - \dim(W) + \dim(W) = \dim(\wedge^2 W).$$

Now we have a chain of inequalities (the second line is a computation of the dimension of the Weyl module with the specified high weight, using the Weyl dimension formula):

$$\dim(V) = 2 \dim(V_{\lambda_m}(\lambda_2 + 2\lambda_m))$$

$$\leq 2((m-1)m(m+2)(m+3)/4)$$

$$= (m^4 + 4m^3 + m^2 - 6m)/2$$

$$< (m^4 + 4m^3 + m^2 - 6m + 2)/2$$

$$= (m(m+2)^2 - 1)/2$$

$$\leq \dim(V),$$
which is a contradiction. So we have ruled out all possible configurations, and the proof of the lemma is complete.

3.4. The General $X = A_m$ Case. We must prove that there are no triples $(X, Y, V)$ with $X$ acting irreducibly on $W$ and $t$ acting on $W$, for $X$ of type $A_m$ with $m > 3$. We use the same argument as in the $A_2$ and $A_3$ cases to limit the possibilities for the embeddings $X \hookrightarrow Y$, relying on Lemma 3.2. Lemma 3.5 tells us we need not worry about level 1 in the computation of dimensions of $U_X$-levels.

As in the small cases, the method we use to generate weights that appear in a representation is simple: If $\mu$ is a weight in the $X$-module $M$ and $\langle \mu, \beta_i \rangle \geq a > 0$, then $\mu - a\beta_1$ is another weight appearing in $M$.

Every symmetric weight for $X = A_m$ except $\delta_1 + \delta_m$ has either $\delta_2 + \delta_{m-1}$ or $\delta_{(m+1)/2}$ as a subdominant weight ($\delta_i + \delta_{m-i+1}$ is a sum of roots for any $1 \leq i \leq m$). It is relatively simple to find enough weights in each level for $\delta = \delta_2 + \delta_{m-1}$: The level $l_2$ of the low weight $-\delta$ in this case is $2(m + (m - 2)) = 4(m - 1)$, so we must show there are at least three weights at level $j$ for $2 \leq j < 2(m - 1)$ and at least 5 at level $2(m - 1)$; this is easy. Unless otherwise stated, the levels we discuss are $U_X$-levels.

The case $\delta = \delta_{(m+1)/2}$ ($m$ odd) is considerably more difficult. At $U_X$-level 2 there are only the two weights $\delta - \beta_{(m+1)/2} - \beta_{(m-1)/2}$ and $\delta - \beta_{(m+1)/2} - \beta_{(m+3)/2}$; so Lemma 3.2 is of no help here. We have $l_2 = (m + 1)^2/4$, so if we can show that there are at least three weights at each level $3 \leq j < (m + 1)^2/8$ and at least 5 at level $(m + 1)^2/8$ (when this is integral), we will only have level 2 to worry about.

Notice that for any weight $\delta$, the dimensions and numbers of weights of $U_X$-levels of the $X$-module with high weight $\delta$ are symmetric about level $l_\delta/2$. In other words, $\dim(W_i) = \dim(W_{l_\delta-i})$ and the same numbers of weights appear in these two spaces, since $v_0$ interchanges them. So, for instance, if $V_{A_4}(\delta)$ has at least 3 weights at all levels $j$ for $i \leq j \leq l_\delta/2$, then it has at least 3 at all levels $j$ for $i \leq j \leq l_\delta - i$.

For $m = 5, 7$, it is easy to see that there are enough weights at levels 3 through $l_2/2$ (three at every level except level 8 for $m = 7$, in which case there are at least 5 weights). We proceed by induction on $m$ (considering the subsystem group of $X$ of type $A_{m-2}$, corresponding to $\Pi(X) - \{\beta_1, \beta_m\}$). Assume $\delta = \delta_{(m+1)/2}$ for $m \geq 9$.

Notice that $\delta_1 - \delta_{(m+1)/2} + \delta_m$ is a weight at level $l_{\delta-(m+1)/2} = (m - 1)^2/4$ and that if $\delta_{(m+1)/2} - \sum_{i=2}^{m-1} c_i \beta_i$ is a weight at level $j$ in the $A_{m-2}$-module with high weight $\delta_{(m+1)/2}$, then $\delta_{(m+1)/2} - \sum_{i=2}^{m-1} c_i \beta_i$ is a weight at level $j$ in $W$ for $\delta = \delta_{(m+1)/2}$. So, using the comment in the last paragraph and by induction from the $A_{m-2}$ case, $W$ has at least three weights at levels 3 through $((m - 1)^2/4) - 2$. But $((m - 1)^2/4) - 2 > (m + 1)^2/8$ for $m > 7$, which shows that the only possibilities for a simple factor of $L'_4$ of type $A_1$ are the above-mentioned factor corresponding to $U_X$-level 2 and level $(m + 1)^2/8$ when this number is integral.

We need to show that there are at least 5 weights at level $(m + 1)^2/8$ when $4|m + 1$ and $\delta = \delta_{(m+1)/2}$. It is easy to write down 5 such weights for $m = 7$, as noted above; in particular in this case there are two at this level which are symmetric (with respect to the graph automorphism). So assume $m \geq 11$ and consider the $A_{m-4}$-subsystem subgroup of $X$, corresponding to $\Pi(X) - \{\beta_1, \beta_2, \beta_{m-1}, \beta_m\}$. Assume
there are at least two symmetric weights at level \((m - 3)^2/8\) for the \(A_{m-1}\)-module with high weight \(\delta_{(m+1)/2}\): as in the last paragraph each corresponds to a weight of \(W\) at level \((m - 3)^2/8\). Let \(\gamma\) be one of these weights of \(W\). Write \(\gamma = \delta - \sum_{i=3}^{m-2} c_i \beta_i\). For any symmetric weight of \(W\) expressed in this form, either

1) \(c_{(m+1)/2} = c_{(m-1)/2} + 1 = c_{(m+3)/2} + 1\) or

2) \(c_{(m+1)/2} = c_{(m-1)/2} = c_{(m+3)/2}\).

If \(\gamma\) satisfies 1), then \(\gamma = \beta_1 = \beta_2 - \beta_3 - \cdots - \beta_{(m-1)/2} - 2\beta_{(m+1)/2} = \beta_{(m+3)/2} - \cdots - \beta_{m-1}\) is a symmetric weight of \(W\) at level \((m - 3)^2/8 + m - 1 = (m + 1)^2/8\) which satisfies 2); if \(\gamma\) satisfies 2), then \(\gamma = \beta_1 - \beta_2 - \cdots - \beta_{(m-1)/2} - \beta_{(m+3)/2} - \cdots - \beta_{m}\) is a symmetric weight which satisfies 1). Since of the two symmetric weights at level 8 for \(m = 7\), one \(\delta_1 - \beta_1 - \beta_2 - \beta_3 - 2\beta_4 - \beta_5 - \beta_6 - \beta_7\) satisfies 1) and the other \(\delta_4 - \beta_2 - 2\beta_3 - 2\beta_4 - 2\beta_5 - \beta_6\) satisfies 2), we conclude that the two symmetric weights at level \((m + 1)^2/8\) we obtain are in fact distinct. At the same level we have the weights \(\gamma = \beta_1 - \cdots - \beta_{m-1}\) and \(\gamma = \beta_2 - \cdots - \beta_{m}\). So from each of the two symmetric weights at level \((m - 3)^2/8\), we obtain 3 weights at level \((m + 1)^2/8\); the six weights occurring are all distinct. So there are no possible \(A_1\) factors of \(L'_{Y}\) here.

Note that \(\delta = \delta_{(m+1)/2}\) is the only possible high weight of \(W\) which does not have at least 3 weights at level 2. Now the induction is much the same as for \(X = A_3\). Recall that \(\delta = \sum d_i \delta_i\). If \(d_{(m+1)/2} \geq 1\), then \(\delta = \beta_{(m+1)/2}\) is a subdominant weight, so by induction \(\delta\) has enough weights at levels 4 through \(l_{(2)} - 1\), and we can easily check levels 1, 2, and 3.

If \(d_{(m+1)/2} \leq 1\) or \(m\) is even, then let \(k\) be such that \(d_k \neq 0\) but \(d_i = 0\) for all \(k < i < (m+1)/2\). Then \(\delta = \beta_k - \beta_{k+1} - \cdots - \beta_{m-1}\) (at level \(m - 2k + 2\)) is lower in the partial order and still has one of \(\delta_2 + \delta_{m-1}, \delta_{(m+1)/2}\) as a subdominant weight, so by induction \(\delta\) has enough weights at levels \(m - 2k + 4\) and higher. Once again, the missing levels are easy to check.

So the cases not ruled out by the induction are \(\delta = \delta_1 + \delta_m\) and \(\delta = \delta_{(m+1)/2}\) at level 2. For the first case, we can easily write down enough weights in levels 2 through \(m - 1 = (l_{(2)} - 1\). At level \(m\) the only weight is 0, but the dimension of its weight space is \(m\) or \((p|m + 1) = m - 1\). So the only possibilities for a \(U_Y\)-level of dimension 4 or less are \(m = 4\), or \(m = 5\) with \(p = 2\) or 3.

Consider the case \(\delta = \delta_{(m+1)/2}\) \((m \geq 5\). In the labelling for the \(Y\)-high weight of \(V\), there is a 1 on the third node of the Dynkin diagram. Let \(P_X\) correspond to \(\Pi(X) - \{\beta_1, \beta_m\}\). Then \(\text{dim}(W/W, Q_X) \geq 6\), so the corresponding simple factor \(L_1\) of \(L'_Y\) is of type \(A_4\) for some \(l \geq 5\), and \(V/V, Q_Y\) is an irreducible \(L_1\)-module whose high weight has a \(\lambda_3\)-coefficient of 1 (\(L_1\) is the only simple factor of \(L'_Y\) to act on \(V/V, Q_Y\) by Lemma 2.11). The natural module for \(L_1\) is isomorphic to \(W/W, Q_Y\) and is irreducible as an \(L'_X\)-module, while \(V/V, Q_Y\) is the sum of two irreducible modules for \(L'_X\), interchanged by \(t\). Since \(V/V, Q_Y\) is the natural module for \(L_1\) and no cases with this configuration appeared for \(X = A_3\), by induction we know that \(V/V, Q_Y\) must not be irreducible for \(L'_X(t)\). So \(b_2 = b_{m-1}, b_3 = b_{m-2}\), etc. \((V_{i}(X) = V_{A_{m}}(b_1 \delta_1 + \cdots + b_m \delta_m))\).

Now let \(P_X\) correspond to \(\Pi(X) - \{\beta_m\}\). By the same construction (using the action of \(Z\) on high weight vectors) \(V/V, Q_Y\) is irreducible as an \(L'_X\)-module. The rank of \(L_1\) is at least 9 (since \(W/W, Q_Y \simeq A_{m-1}\)-module with high weight \(\lambda_{(m+1)/2}\), and \(V/V, Q_Y\) is irreducible as an \(L_1\)-module, with a 1 on the third
node of the Dynkin diagram in its high weight labelling ($L_1$ is the only factor of $L'_X$ acting nontrivially on $V/[V,Q_Y]$ since no restricted irreducible $A_{m-1}$-modules are tensor decomposable). So we are again in the case examined in [9]. Examining Table 1 there, we see that there are no examples of the configuration we obtain. So \( \delta = \delta_{(m+1)/2} \) does not occur.

The two cases that remain are 1) \( m = 4, \delta = \delta_1 + \delta_4 \), with the $T_Y$-high weight \( \lambda \) of $V$ having a \( \lambda_n \)-coefficient of 1 (\( Y \) has type $D_{12}$ (\( p \neq 2,5 \)), $B_{11}$ (\( p = 5 \)), or possibly $C_{12}$ (\( p = 2 \)); and 2) \( m = 5, p = 5, \delta = \delta_1 + \delta_5, \lambda \) with a \( \lambda_1 \)-coefficient of 1 (\( Y \) has type $D_{17}$). In both these cases, we can use the fact that \( \dim(V^2(Q_Y)) \leq m \dim(V^4(Q_Y)) \) (Lemma 2.9) to conclude that in fact \( \lambda = \lambda_n \); i.e. $V$ is a spin module for $Y$.

For 1), we first let $P_X$ correspond to \( \{ \beta_2, \beta_3 \} \subseteq \Pi(X) \) and conclude that $b_2 = b_3$ as the resulting configuration does not appear in [4]. Next let $P_X$ correspond to $\Pi(X) - \{ \beta_3 \}$. Then again the construction using the action of $Z$ tells us that $V/[V,Q_Y]$ is irreducible as an $L'_X$-module, and we are back in the situation examined in [9]. Examining Table 1 of [9], we see that there are no examples of modules irreducible for both $A_3 \leq D_8$.

For 2), we use the same constructions and conclude that to have an example, there must be an example of the form $A_4 \leq D_{12}$ in Table 1 of [9]; there is none. So this final possibility is ruled out.

This completes the proof of the theorem.

\[ \square \]

4. THE CASE $X = D_m$

In this section we establish the main result for the case $X = D_m$, $G = X(t)$. Section 3 included the base case of $X = D_3 = A_3$. So throughout this section $X = D_m$ for $m > 3$ and $G = X(t)$. We assume that $t$ acts on $W$, the natural module for $Y$.

All notation ($X \leq Y$, $V$, $V_1$, $V_2$, \( \lambda \), \( \delta \), etc.) is as in previous sections. Recall that $\Pi(X) = \{ \beta_i \}$ is the set of simple roots of $X$; $\Pi(Y) = \{ \alpha_i \}$ is the set of simple roots of $Y$, and $n$ is the rank of $Y$. The main theorem of this section is

**Theorem 4.1.** If $X$ acts irreducibly on the natural module $W$ for $Y$ and $X$ is of type $D_m$ for $m > 3$, then $p = 2, Y = C_m$ or $B_m$, $W$ is the natural module for $X$ and $Y$; and $V$ is the spin module for $Y$, the sum of two spin modules for $X$. This is $U_7$ in Table 2.

**Proof.** Assume $X$ is of type $D_m$. Let \( \delta \) be the $X$-high weight of $W$. Since we are assuming $t$ acts on $W$, we have that \( \delta \) is symmetric with respect to $t$, so by [9, 1.8], $X$ stabilizes a non-degenerate bilinear form on $W$. So $Y$ is of type $B_n, C_n$, or $D_n$.

Let $P_X$ be the maximal parabolic subgroup of $X$ corresponding to $\beta_1$. Then $L'_X$ is of type $D_{m-1}$, and we embed $P_X$ in $P_Y$, a parabolic subgroup of $Y$, via the construction detailed in Lemma 2.7. Then $Q_X \subseteq Q_Y$ and $L'_X \subseteq L'_Y$ for $L_Y = C_Y(Z(L_X)^+)$. By Lemma 2.8, $L_Y$ is $t$-stable. Write $L'_Y = L_1 \times \cdots \times L_m$ where each $L_i$ is simple. By Lemma 2.2, $V/[V,Q_Y]$ is irreducible as an $L'_Y$-module with high weight \( \lambda_{T_Y \cap L'_Y} \); also, Lemma 2.11 tells us that only one $L_i$ acts nontrivially on $V/[V,Q_Y]$. Note that since $P_X$ is $t$-stable, Lemma 2.8 tells us that $V/[V,Q_Y] = V/[V,Q_X] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X]$ is a sum of two irreducible $L'_X$-modules; neither of these $L'_X$ summands is trivial, since that would imply that $V_1 \cong V_2$. 

Knowing this, we can list the possibilities for \( L_i, (V/[V,Q_Y])|_{L_i}, (V/[V,Q_Y])|_{L'_X} \) based on our inductive knowledge about \( L'_X \) (which is of type \( D_{m-1} \)).

1. \( V/[V,Q_Y] \cong \) the natural module for \( L_i \).
2. (U3 in Table 2) The natural module \( W_i \) for \( L_i \) is reducible as an \( L'_X \)-module and different from \( V^1(Q_Y) = V/[V,Q_Y] \), with \( L_i \) of type \( D_m \), \( (V/[V,Q_Y])|_{L_i} = \text{spin}(L_i) \), and \( (V/[V,Q_Y])|_{L'_X} = \text{spin}(L'_X) \oplus \text{spin}(L'_X) \) (one of the two cases in which \( X = D_m \) occurs in the situation examined in [4]).
3. (U2 in Table 2) Same as 2 above, except here \( L_i \) is of type \( B_{m-1} \) and \( (V/[V,Q_Y])|_{L_i}, (V/[V,Q_Y])|_{L'_X} \) are as in the statement of [4, Theorem 3.3].
4. \( W_i \) is irreducible for \( L'_X \).

Here is a lemma to simplify things considerably:

**Lemma 4.2.** In all of the four cases above, if \( \alpha \in \Pi(Y) \) and \( \langle \alpha, \Pi(L_i) \rangle = 0 \), then \( \langle \lambda, \alpha \rangle = 0 \).

**Proof.** Nodes in \( L_j \) for \( j \neq i \) must have 0 label since \( L_j \) acts trivially on \( V^1(Q_Y) \). Assume \( \langle \lambda, \alpha \rangle \neq 0 \) for some \( \alpha \in \Pi(Y) \) with \( \langle \alpha, \Pi(L_i) \rangle = 0 \) (i.e. \( \alpha \) does not adjoin \( \Pi(\alpha) \)). Define \( K_\alpha \) as in the discussion following Lemma 2.9. Then \( Q_Y/K_\alpha \) has dimension 1 by [9, 3.1] (indeed, \( Q_Y/K_\alpha \cong U_\alpha \)). It has an \( L_X \)-submodule \( Q_XL_\alpha/K_\alpha \cong Q_X/(Q_X \cap K_\alpha) \) by [9, 3.3]. By Lemma 2.10(i), \( Q_X = K_\beta = \langle U_\beta | \beta \in \Sigma^-(X), \beta_1 \text{-coefficient of } \beta < -1 \rangle = 1 \) (since \( D_m \) has no roots with a \( \beta_1 \)-coefficient less than \(-1\)). Then by part (ii) of the same Lemma, \( Q_X = Q_X/K_\beta = Q_\beta \) is an irreducible \( L_X \)-module of high weight \( -\beta_1|T_{L_X} \) (and thus dimension \( 2(m-1) \)). But \( Q_X \cap K_\alpha \) is a submodule since \( K_\alpha \) normal in \( P_Y \geq P_X \geq L_X \) since \( Z = Z(L_X)^0 \) induces the full group of scalars on \( Q_X \). So either \( Q_X \leq K_\alpha \) or \( Q_XL_\alpha/K_\alpha \cong Q_X/Q_X/Q_X/Q_X = Q_X \) has dimension \( 2(m-1) \); the latter is impossible (it has dimension 0 or 1). So \( [V,Q_Y] = [V,Q_X] \leq [V,K_\alpha] \). But the weight space \( V_{X,\alpha} \) appears in \([V,Q_Y]\) and not in \([V,K_\alpha]\). This is a contradiction, so \( \langle \lambda, \alpha \rangle = 0 \) for \( \alpha \) not adjoining \( \Pi(L_Y) \).

Now assume \( \langle \lambda, \alpha \rangle \neq 0 \) for some \( \alpha \in \Pi(Y) \) such that \( \langle \alpha, \Pi(L_i) \rangle = 0 \). By the above, there is some \( j \neq i \) such that \( \langle \alpha, \Pi(L_j) \rangle \neq 0 \). The weight \( \lambda - \alpha \) is a weight in \([V,Q_Y] = [V,Q_X]\) but not in \([V,K_\alpha]\), so \( Q_X \leq K_\alpha \) gives a contradiction as above. So \( Q_X \not\leq K_\alpha \). Then \( Q_XL_\alpha/K_\alpha \neq 0 \) and as above \( Q_XL_\alpha/K_\alpha \cong Q_X/Q_X \cong Q_X/K_\beta \) is an irreducible \( L'_X \)-module of dimension \( 2(m-1) \). So \( Q_XL_\alpha \neq V_{L'_X}(-\alpha) \) is an irreducible \( L'_X \)-module of dimension \( \leq 2(m-1) \).

Let \( \mu = \lambda - \alpha \). Then \( \mu \mid_{T_i} = \lambda \mid_{T_i} \) (where \( T_i \) is a maximal torus of \( L_i \)) and \( V_{L'_X}(\mu) = V_{L_i}(\lambda) \otimes V_{L'_X}(-\alpha) \), of dimension \( \geq \dim(V^1(Q_Y))(2m-2) \). By Lemma 2.9, \( \dim(V^2(Q_Y)) \leq \dim(V^1(Q_Y))(2m-2) \), and \( V_{L'_X}(\mu) \leq V^2(Q_Y) \). This forces \( V_{\alpha}(Q_Y) = V_{L'_X}(\mu) = V^2(Q_Y) \), but there is some \( \epsilon \in \Pi(Y) - \Pi(L_i) \) such that \( \langle \epsilon, \Pi(L_i) \rangle \neq 0 \) and \( V_{\epsilon}(Q_Y) \neq 0 \). From this contradiction, we have that in fact \( \langle \lambda, \alpha \rangle = 0 \) for all \( \alpha \) not in or adjoining \( \Pi(L_i) \).

We now look at each of the cases 1–4 in turn.

**4.1. Case 1.**

**Claim 4.3.** \( V^1(Q_Y) \not\cong \) the natural module for \( L_i \), so case 1 on the previous page does not arise.

**Proof.** Assume \( V^1(Q_Y) \) is isomorphic to the natural module \( W_i \) for \( L_i \).
There are several possibilities for the type of $L_i$. Call the node in $L_i$ with a non-zero label $\alpha_l$. If $L_i$ is of type $B_k, C_k, \text{ or } D_k$, then it corresponds to a subdiagram of the Dynkin diagram for $Y$ at the “end” of that diagram; the picture is

\[
\cdots \bullet \bullet \bullet \alpha_l \cdots L_i \cdots .
\]

The only node in $\Pi(Y)$ adjoining $L_i$ adjoins $\alpha_l$; call it $\gamma$. Now $W_i \cong V^1(Q_Y) = V^1_1(Q_X) \oplus V^2_1(Q_X)$, and these are the only two $L_X$-submodules of $V^1(Q_Y)$. They are interchanged by $t$. Also, $Q_Y / K^\gamma_Y$ is an $L^\gamma_Y$-module of high weight $-\gamma$; $Q_X K^\gamma_Y / K^\gamma_Y$ is a non-zero $t$-stable $L_X$-submodule ($Q_X \leq K^\gamma_Y$ gives a contradiction as in the proof of the last lemma). But we just said that the natural module for $L_i$ has no $t$-stable $L_X$-submodules; this forces there to be an $L_j$ adjoining $\gamma$ such that $L^\gamma_X$ projects non-trivially to $L_j$. Thus the natural module for $L_j$ has dimension $\geq 2(m-1)$. But then if $\gamma$ has non-zero label, we have a composition factor of high weight $(\lambda - \gamma)\mid T_{L^\gamma_X}$ in $V^2(Q_Y)$, of dimension $> 2(m-1) \dim(V^1(Q_Y))$. And if $\gamma$ has label 0, we have $\lambda - \gamma - \alpha_l$, also giving dimension $> 2(m-1) \dim(V^1(Q_Y))$. By Lemma 2.9, this is a contradiction. Note that this same argument holds whenever there is a node in $\Pi(Y) - \Pi(L_i)$ adjoining $\alpha_l$ via a single bond. So $L_i$ must be of type $A_k$, with no node outside $\Pi(L_i)$ adjoining $\alpha_l$ via a single bond.

By Lemma 4.3, all nodes in $\Pi(Y)$ not adjoining $\Pi(L_i)$ have marking 0. There is no node outside $\Pi(L_i)$ adjoining $\alpha_l$ via a single bond. The other possibilities for nodes adjoining $L_i$ are the following, which we will consider in turn:

\begin{itemize}
  \item[a)] $\cdots \bullet \alpha_l \gamma ;$
  \item[b)] $\cdots \bullet \alpha_l \gamma_{l-1} ;$
  \item[c)] $\cdots \bullet \alpha_l \gamma_{l-1} ;$
  \item[d)] $\cdots \bullet \alpha_l \gamma_{l-1} ;$
  \item[e)] $\cdots \bullet \alpha_l \gamma_{l-1} ;$
  \item[f)] $\cdots \bullet \alpha_l \gamma_{l-1} ;$
  \item[g)] $\cdots \bullet \alpha_l \gamma_{l-1} ;$
\end{itemize}

a) $Q_Y / K^\gamma_Y \cong V^1(Q_Y)$; as above, this forces there to be an $L_j$ adjoining $\gamma$, which is absurd.

b) The rank of $L_i$ is $k$, so $\dim(V^1(Q_Y)) = k + 1$. We then have $k + 1 = \dim(V^1(Q_Y)) \geq 2^{m-1} > \dim(Q^n)$. (The dimension of $V^1(Q_B)$ is at least $2^{m-1}$ since the high weight of $V^1_1(Q_X)$ is not symmetric with respect to the graph automorphism of $D_{m-1}$; thus the Weyl group orbit of the $T_{L^\gamma_X}$-high weight of $V^1_1(Q_X)$ contains at least $2^{m-2}$ weights.) In particular, $k + 1 \geq 8$ since $m \geq 4$. Note also that $\dim(V^1(Q_Y))$ is even (so $k$ is odd) because it is the sum of two $L^\gamma_X$-modules of equal dimension. If $\gamma$ has a non-zero label in the marking for $\lambda$ on $Y$, then $V^2(Q_Y)$ has a composition factor with high weight $(\lambda - \gamma)\mid T_{L^\gamma_X}$, giving $\dim(V^2(Q_Y)) \geq$
\[ \dim(V_{L^i}((\lambda - \gamma)|_{T_Y})) = \frac{1}{6}(\dim(V^1(Q_Y)) + 2)(\dim(V^1(Q_Y)) + 1)(\dim(V^1(Q_Y))) > 2(m - 1)(\dim(V^1(Q_Y))), \] which is a contradiction to Lemma 2.9. If \( \gamma \) has label 0, then the high weight \((\lambda - \alpha_l - \gamma)|_{T_{Y_{L^i}}} = (\lambda_{l-1} + \lambda_l)|_{T_{Y_{L^i}}} \) appears. This weight has \( k(k + 1) \) conjugates, and the subdominant weight \((\lambda_{l-2})\) has \( \frac{(k+1)!}{6(k-2)!} \) conjugates. But for \( k > 6 \), \( k(k + 1) + \frac{(k+1)!}{6(k-2)!} > (k + 1)^2 > 2(m - 1)(\dim(V^1(Q_Y))) \), again a contradiction. So b) does not arise.

c) Take \( k \) as in b). If \( \gamma \) has non-zero marking, then \( V^2(Q_Y) \) has a composition factor of high weight \((\lambda - \gamma)|_{T_{Y_{L^i}}} = (\lambda_{l-1} + \lambda_l)|_{T_{Y_{L^i}}} \), and we have a contradiction as above. If \( \gamma \) has marking 0, then we have the high weight \((\lambda - \alpha_l - \alpha_{l-1} - \gamma)|_{T_{Y_{L^i}}} = \lambda_{l-2}|_{T_{Y_{L^i}}} \), which has, as above, \( \frac{(k+1)(k-1)}{6} \) conjugates. For \( k \geq 8 \), \( \frac{(k+1)(k-1)}{6} > (k + 1)^2 > 2(m - 1)(\dim(V^1(Q_Y))) \); this is a contradiction to the same result. If \( k = 7 \), then \( \dim(V_{L^i}(\lambda_{l-2}) = (\frac{5}{6}) = 56 > 6 \cdot 8 = \dim(Q^{\beta_1}) \dim(V^1(Q_Y)) \), and again we have a contradiction.

d) Let \( k + 1 = \dim(V^1(Q_Y)) \) as above. In any of these cases, if \( \gamma \) has a non-zero label, then \( V^2(Q_Y) \) has a composition factor (given by the high weight \((\lambda - \gamma)|_{T_{Y_{L^i}}} \) of dimension greater than \( \dim(Q^{\beta_1}) \dim(V^1(Q_Y)) \), giving a contradiction as above. If \( \gamma \) has a zero label, then \( V \cong \) the natural module \( W \) for \( Y \), which is impossible, since \( X \) acts irreducibly on \( W \) but not on \( V \).

\[ \square \]

4.2. Case 2.

**Claim 4.4.** Case 2 (stated above Lemma 4.2) does not arise.

**Proof.** We have the picture below, with the boxed diagrams on the left of type \( D_{m-1} \) and on the right of type \( D_m \):

\[
\begin{array}{c}
\begin{array}{c}
\text{e} \\
\cdots \\
\text{1}
\end{array} \\
\oplus \\
\begin{array}{c}
\text{e} \\
\cdots \\
\text{1}
\end{array} \\
\begin{array}{c}
\text{1} \\
\cdots \\
\alpha_j \\
\cdots \\
\alpha_{n-1}
\end{array}
\end{array}
\]

with \( j = n - m + 1 \) (\( n \) is the rank of \( Y \), as always). If \( \gamma \) has a non-zero label, then in the characteristic 0 case,

\[
\dim(V^2(Q_Y)) \geq \dim(V_{\gamma}(Q_Y)) \geq \dim(V_{D_m}(\lambda_j + \lambda_{n-1})) = (2m - 1)(\dim(V^1(Q_Y)) > \dim(Q^{\beta_1})\dim(V^1(Q_Y))
\]

again a contradiction to Lemma 2.9. The only problem here is when the irreducible module \( V_{D_m}(\lambda_j + \lambda_{n-1}) \) is not the Weyl module.

We can use the Andersen-Jantzen sum formula to check that in those characteristics for which the Weyl module does reduce, it reduces only by \( 2^{m-1} \), making the bound sharp: In these cases, \( \dim(V_{D_m}(\lambda_j + \lambda_{n-1})) = (2m - 2)(\dim(V^1(Q_Y)) \). So if there is anything else in \( V_{\gamma}(Q_Y) \), \( V^2(Q_Y) \) will again be too big. So assume the \( D_m \)-module \( V^1 \) with high weight marking \( \text{1} \to \ldots \to \text{1} \) (of high weight \( \lambda_j + \lambda_{n-1} = \lambda' \)) does reduce. Then the dimension of the weight space \( V^1_{\lambda' - \alpha_j - \cdots - \alpha_{n-1}} \) in \( V(\lambda') \) is \( m - 2 \) (this dimension is \( m - 1 \) in the characteristic 0
case). But in $V$, which has marking

```
  ...  a  ...  γ  ...
```

the dimension of the weight space for the weight $\lambda - \gamma - \alpha_j - \cdots - \alpha_{n-1}$ is $m$ or $m-1$ and this weight space is in $V_\gamma(Q_Y)$. So if $V(N)$ reduces, then there is something else in $V_\gamma(Q_Y)$, and again we get that $V^2(Q_Y)$ is too large. So in fact $\gamma$ has label 0 (i.e. $(\lambda, \gamma) = 0$), and, since we have already shown that nodes not adjoining $L_i$ have 0 label, $V$ is the spin module for $Y$.

Now look at a different parabolic subgroup of $X$: Let $P_X = L_X Q_X$ correspond to $\Pi(X) - \{\beta_{m-1}, \beta_m\}$. Use the standard notation for a basis of the Lie algebra of $X$: For a simple Lie algebra with root system $\Phi$ having basis $\{\alpha_1, \ldots, \alpha_m\}$, use the Chevalley basis $\{e, f, h\}_{\alpha \in \Phi^+}$, satisfying the usual relations — in particular, $[e, f, h] = h$. Then each $V_i$ ($i = 1, 2$) is spanned by vectors of the form $w = f_\alpha \ldots f_\beta v_i^+$, where $v_i^+$ is a maximal vector. Order the roots in $\Sigma(X)$ so that the last $f_\alpha$ applied correspond to roots in $\Sigma(X) - \Sigma(L_X')$. Now $V^2_i(Q_X)$ is spanned by vectors $f_\alpha w$ where $w = f_\beta \ldots f_\epsilon v_i^+$ such that all the roots $\beta, \ldots, \epsilon$ are in $\Sigma(L_X')$ and $\alpha$ has $Q_X$-level 1 (i.e. $\alpha$ has $\beta_{m-1}$ or $\beta_m$-coefficient $-1$ and the other, 0). If we take a maximal linearly independent set of such $w$, we have a basis of $V_i^1(Q_X)$, and there are $2(m - 1)$ roots $\alpha$ of $Q_X$-level 1. So $\dim(V^2_i(Q_X)) \leq 2(m - 1) \dim(V_i^1(Q_X))$.

This then gives $\dim(V^2(Q_Y)) \leq 2 \dim(V^2_i(Q_X)) \leq 2(2(m-1) \dim(V_i^1(Q_X))) = 2(m-1) \dim(V^1(Q_Y))$.

If $e = 0$ (with $e$ the labelling on $\beta_1$ as in the picture at the beginning of the proof), then $V|_X$ is a sum of two spin modules. So $\dim(V) = 2(2^{m-1}) = 2^m$. Since $V$ is a spin module for $Y$, $\dim(V) = 2^{n-1}$. So $X = D_{n-1}$. But the natural module for $Y$ has dimension $2n$, and $D_{n-1}$ for $n \geq 5$ has no irreducible restricted modules of this dimension. So $e \neq 0$.

$V$ is the spin module for $Y$, and the parabolic subgroup $P_Y$ of $Y$ in which we embed this new $P_X$ must contain both of the root groups corresponding to the node with a label of 1 in the marking for the $Y$-high weight of $V$ (since $e \neq 0$). Since this 1 is the only non-zero label in the marking, the $L_i$ which contains it is the only $L_j$ acting non-trivially on $V/V_i(Q_Y)$. Now $L_X'$ is of type $A_{m-2}$, and $V_i/[V_i, Q_X]$ has high weight $e\delta_1$.

There are two possibilities for the type of $L_i$: $D_{l+1}$ for some $l \geq 3$ and $A_l$. If $L_i$ is of type $D_{l+1}$, then $\dim(V/[V, Q_Y]) = 2^l$ is a power of 2. By Lemma 2.5, $\dim(V_i/[V_1, Q_X]) = (m-2^l)$, which is not a power of two unless $e = 1$ or $m = 3$. But $m \geq 4$, so this forces $e = 1$. Then $\dim(V/[V, Q_Y]) = (m-2) + 1 = 2^l$. A group of type $D_{l+1}$ for $l \geq 3$ does not contain a group of type $A_{2l-1}$, however, so $L_i$ is not of type $D_{l+1}$ for $l \geq 3$.

If $L_i$ is of type $A_l$, then by Lemma 2.5 we have $l + 1 = 2 \dim(V_i/[V_1, Q_X]) = 2^{(m-2^l)} > 4$. The root $\alpha_n$ (see the picture at the beginning of the proof) is not contained in the set of roots corresponding to $L_Y$ (since $l > 3$), so $V^2(Q_Y)$ contains a composition factor of high weight $(\lambda - \alpha_{n-1} - \alpha_{n-2} - \alpha_n)/Q_Y$, of dimension $(l+1)$. Then we have $\frac{1}{6}(l+1)(l-1) \leq \dim(V^2(Q_Y)) \leq 2(m-1) \dim(V^1(Q_Y)) = 2(m-1)(l+1)$, with $l$ given in terms of $m$ and $e$ above. This is impossible in all
cases except \( m = 4, e = 1 \) and \( m = 5, e = 1 \). So the only two possibilities left here are \( V_1|_X = \begin{array}{c}
1 \\
1 \\
\end{array} \) and \( V_1|_X = \begin{array}{c}
1 \\
1 \\
\end{array} \).

Remember that \( V \) is the spin module for \( Y \), so has dimension a power of 2. So \( V_1 \) has dimension a power of 2. The only time when one of the above two modules has dimension a power of 2 is the \( D_5 \) module of high weight \( \delta_1 + \delta_4 \) in characteristic 5, of dimension \( 2^7 \). So \( \dim(V) = 2^8 \) and \( Y \) is of type \( D_9 \). But \( D_5 \) has no 18-dimensional irreducible representations in characteristic 5. So \( L_i \) is not of type \( A_1 \).

### 4.3. Case 3

**Claim 4.5.** The situation outlined in case 3 (stated above Lemma 4.2) does not arise.

**Proof.** Here \( V \) has high weight marking \( \begin{array}{c}
a_1 \\
a_2 \\
a_{m-1} \\
\end{array} \), restricting to \( D_{m-1} \) and with \( a_2, \ldots, a_m = 1 \) related as in [4, 3.3], and with labels to the left of \( a_1 \) all 0. Let \( P_X \) be the parabolic subgroup of \( X \) corresponding to \( \Pi(X) - \{ \beta_m \} \). As always, embed \( P_X \) in a parabolic subgroup \( P_Y \) of \( Y \) via the construction given in the introduction, so that \( Q_X \leq Q_Y \) and \( L_X \leq L_Y \).

As in the proof of Lemma 3.5, we show that in this case \( V/[V,Q_Y] \) is irreducible as an \( L_X \)-module. Let \( Z = Z(L_X)^\circ \). By construction, \( Z \leq Z(L_Y) \) (where \( L_Y = C_Y(Z) \)), so \( Z \) induces scalars on \( V/[V,Q_Y] \) (since \( L_Y \) acts irreducibly). But if \( V/[V,Q_Y] \) is not irreducible for \( L_X \), then \( V/[V,Q_Y] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] \), and \( Z \) acts differently on these two \( L_X \)-modules:

\[
Z = \{ \text{diag}(a, \ldots, a, a^{-1}, \ldots, a^{-1}) \mid a \in K^* \} \\
= \{ \text{diag}(a^2, \ldots, a^2, a^{-2}, \ldots, a^{-2}) \mid a \in K^* \} \\
= \{ h_{\beta_1}(a^2)h_{\beta_2}(a^4) \ldots h_{\beta_{m-3}}(a^{2(m-2)})h_{\beta_{m-1}}(a^{m-2})h_{\beta_m}(a^m) \mid a \in K^* \}.
\]

The two \( X \)-modules \( V_1, V_2 \) have high weight labelling

\[
\begin{array}{c}
a_1 \\
a_2 \\
a_{m-1} \\
\end{array} \quad \text{and} \quad \begin{array}{c}
a_1 \\
a_2 \\
a_{m-1} + 1 \\
\end{array}
\]

so \( h_{\beta_1}(a^2)h_{\beta_2}(a^4) \ldots h_{\beta_{m-3}}(a^{2(m-2)})h_{\beta_{m-1}}(a^{m-2})h_{\beta_m}(a^m) \) acts differently on a high weight vector \( v_1 \in V_1 \) than on a high weight vector \( v_2 \in V_2 \). Since \( v_1 \) has a non-zero image in \( V_1/[V_1,Q_X] \), this shows that only one of the \( V_i/[V_i,Q_X] \) can be in \( V/[V,Q_Y] \). So \( V/[V,Q_Y] \) is irreducible as an \( L_X \)-module. Assume \( V_1 \) is the summand which projects non-trivially to \( V/[V,Q_Y] \) (so \( V_2 \subseteq [V,Q_Y] \)).

Irreducible restricted \( A_{m-2} \)-modules are tensor indecomposable (Lemma 2.1), so only one of the simple factors \( L_i \) of \( L_Y \) acts non-trivially on \( V/[V,Q_Y] = V^1(Q_Y) \).

The two possibilities for the type of \( L_i \) are \( B_k \) and \( A_k \). If \( L_i \) is of type \( B_k \), then \( V^1(Q_Y) \cong \text{natural module for } L_i \), since there are no overgroups of \( A_1 \) of type \( B_k \) appearing in [9, Table 1]. But we know \( a_m = 1 \), so the only possibility here is \( L_i \) of type \( B_2 \cong C_2 \), and \( V^1(Q_Y) \cong \text{natural module for } C_2 \), of dimension 4. Then \( m = 4, L_X^1 = A_3, X = D_4, \) and \( V^1_1(Q_X) \) is of \( L_X \)-high weight \( \delta_3 \) or \( \delta_1 \). If \( \delta_3 \), then the picture is

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so $\dim(V) = \dim(V_1) + \dim(V_2) = 16$. Since $a_m = 1$, the $T_Y$-high weight of $V$ has at least $l_i |W|/|W_{A_{n-1}}| = 2^n n!/n! = 2^n$ conjugates, where $W$ is the Weyl group of type $B_n$ and $W_{A_{n-1}}$ is that of type $A_{n-1}$. So $n \leq 4$; since $4 = m \leq n$, we have $Y = B_4$. But then if $p \neq 2$, $\dim(W) = 9$, and $D_4$ has no irreducible representations of dimension 9. If $p = 2$, then we have one of the situations studied already in [4], which does give the example listed in the theorem.

If the $T_{L_\lambda}$-high weight of $V_1/[V_1,Q_X]$ is $\delta_1$, then the picture is

In this case, since $\dim(V) = 2 \dim(V_1)$, we have $\dim(V) = 112$ if $p \neq 2$, $\dim(V) = 96$ if $p = 2$. But $Y$ cannot have type $B_4$ unless $p = 2$ (in which case [4] tells us there are no examples of this type) as above, and no irreducible restricted $B_l$-module, for $l \geq 5$, whose high weight has an $\lambda_l$-coefficient of 1 has dimension 112 or 96. So $L_i$ is not of type $B_k$.

So $L_i$ is of type $A_k$. The arguments in Lemma 4.3 showing that nodes not adjoining $L_i$ have marking 0 fail here, since $[V,Q_Y] \neq [V,Q_X]$. So we need new arguments. Assume $\gamma$ is a node in $\Pi(Y)$ which has non-zero label and does not adjoin $\Pi(L_Y)$. The argument in the proof of Lemma 4.2 that $Q_X \leq K_Y$ is still valid here, so $V_{\lambda,\gamma} \notin [V,Q_X]$. We have $V/[V,Q_Y] = (V_1 \oplus V_2)/([V_1,Q_Y] \oplus [V_2,Q_Y])$, so if we can show that $V_{\lambda,\gamma} \notin [V_1,Q_Y]$ we will have a contradiction, since $V_{\lambda,\gamma} \notin [V_1,Q_2]$.

Since $\langle \gamma, \Pi(L_Y) \rangle = 0$, we know that $\lambda|_{T_{L_\lambda}} = (\lambda - \gamma)|_{T_{L_\lambda}}$, and thus $\lambda|_{T_{L_\lambda}} = (\lambda - \gamma)|_{T_{L_\lambda}}$. If $\gamma_1$ is the $T_X$-high weight of $V_1$, then $\lambda|_{T_X} = \gamma_i$ for $i = 1$ or 2; we have the pictures

\[ V_1 = \begin{array}{cccccc}
  \bullet & a_1 & a_2 & \cdots & a_{m-1} & a_{m-1} + 1 \\
  \bullet & a_1 & a_2 & \cdots & a_{m-1} \\
  \bullet & a_1 & a_2 & \cdots & a_{m-1} \\
  \bullet & a_1 & a_2 & \cdots & a_{m-1} \\
\end{array} \quad ; \quad V_2 = \begin{array}{cccccc}
  \bullet & a_1 & a_2 & \cdots & a_{m-1} + 1 \\
  \bullet & a_1 & a_2 & \cdots & a_{m-1} \\
  \bullet & a_1 & a_2 & \cdots & a_{m-1} \\
  \bullet & a_1 & a_2 & \cdots & a_{m-1} \\
\end{array} \]

Assume $\lambda|_{T_X} = \gamma_1$. Then $\gamma_1|_{T_{L_\lambda}} = (\lambda - \gamma)|_{T_{L_\lambda}}$ is the $T_{L_\lambda}$-high weight of $V_1/[V_1,Q_X]$. Then to have the weight $(\lambda - \gamma)|_{T_X}$ in $V_2$ would imply we could subtract roots from $\gamma_2$ and obtain a weight $\epsilon$ which has the same labelling as $\gamma_1$ on $\Pi(L_X)$. Say we subtract $c\beta_1$. To balance this (the labelling on $\beta_1$ is the same for $\gamma_1$ as for $\gamma_2$), we must subtract $2c\beta_2,3c\beta_3,\ldots,(m-2)c\beta_{m-2}$. Say we subtract $d\beta_{m-1}$ and $e\beta_m$. To balance the $\delta_{m-2}$ coefficient, we need $2(m-2)c - (m-3)c - d - e = 0$. To get $a_{m-1}$ as the $\delta_{m-1}$ coefficient, we need $2d - (m-2)c = 1$. These give $e = (m-1)c - d$ and $d = ((m-2)c + 1)/2$.

Now [9, 3.6 iv] gives $\gamma_1|_Z = \beta_m|_Z$. So since $Z \leq T_X$ and $(\lambda - \gamma)|_{T_X} = (\gamma_1 - \gamma)|_{T_X}$, we have $c|_Z = (\lambda - \gamma)|_Z = (\gamma_1 - \gamma)|_Z = (\gamma_1 - \beta_m)|_Z$. We computed $Z$ above and

\[ (\gamma_1 - \beta_m) (h_{\beta_1}(a^2) h_{\beta_2}(a^4) \cdots h_{\beta_{m-2}}(a^{2(m-2)}) h_{\beta_{m-1}}(a^{m-2}) h_{\beta_m}(a^m)) = a^{m-4} \]
where \( A = 2a_1 + 4a_2 + \cdots + 2a_{m-2}(m-2) + a_{m-1}(m-2) + (a_{m-1}+1)m \). So \( e \) must act on \( Z \) as \( A^{A-4} \). Let the \( \alpha_m \)-coefficient of \( e \) be \( a_m^\alpha \); we know all the other coefficients of \( e \) (they are \( a_1, \ldots, a_{m-1} \)). Since \( e \) must act as \( A^{A-4} \) on the element of \( Z \) given above, we have \( A - 4 = 2a_1 + 4a_2 + \cdots + 2a_{m-2}(m-2) + a_{m-1}(m-2) + a_m^\alpha m \), or \( a_m^\alpha m = (a_{m-1}+1)m - 4 \). But \( a_m^\alpha = a_m - 2e + (m-2)c \) (we subtracted \( m-1 \epsilon \beta_{m-2} \) and \( e \beta_m \)). So \( 1 - (4/m) = -2e + (m-2)c = 2c(m-2) - 2c(m-1) + 1 \). This gives \( 4/m = 2c \), or \( 2/m = c \). But \( m \geq 4 \), so \( 2/m = c \) has no integer solutions. This is a contradiction.

If \( \lambda \vert_{T_X} = \gamma_2 \), similar computations also give \( 2/m = c \) and a contradiction as above. So all nodes with non-zero labels must be in \( \Pi(L_i) \) or adjoin \( \Pi(Y) \).

Now let \( \gamma = \alpha_n \) be the “end” node of the Dynkin diagram for \( Y \), which has type \( B_n \). We know \( \alpha_n \) has a label of 1 (and \( \alpha_n \notin \Pi(L_i) \) because \( L_i \) does not have type \( B_1 \)) so, by the above, \( \alpha_n \) adjoins some \( \Pi(L_j) \). If \( j \neq i \), then we may assume \( L_X \) projects non-trivially to \( L_j \) (else we can take \( P_Y \) such that \( L_X' = L_1 \times \cdots \times \widehat{L_j} \times \cdots \) and still have a parabolic satisfying all the hypotheses we used above; then \( \alpha_n \) would not adjoin \( \Pi(L_Y) \) and the above would give a contradiction). So \( L_j \) has rank at least \( m - 1 \), but then all labels to the left of \( L_j \) are 0 and there is no room for \( L_i \), which acts non-trivially on \( V/\langle V, Q_Y \rangle \). So in fact \( \alpha_n \) adjoins \( \Pi(L_i) \). So the picture

\[
\begin{array}{c}
1 \cdots \\
\end{array} \quad 1
\]

is \( \cdots \).

The possibilities are \( L_X, L_Y, V/\langle V, Q_Y \rangle \) which appear in Table 1 of [9]; \( V/\langle V, Q_Y \rangle \) is the natural module for \( L_i \); or \( L_i \) is of type \( A_{m-1} \), with the embedding \( L_X \rightarrow L_i \) an isomorphism. There are only 2 possibilities arising from Table 1 of [9]; they give easy contradictions to the fact that most of the (possibly) non-zero labels for the \( T_X \)-high weights of \( V \) coincide with the corresponding labels for the \( T_Y \)-high weight of \( V \).

Assume that \( V^1(Q_Y) \) is isomorphic to the natural module for \( L_i \). Let \( k \) be the rank of \( L_i \). If the picture is

\[
\begin{array}{c}
e \\
\end{array} \quad \begin{array}{c}1 \\
\end{array} \quad \begin{array}{c}2 \\
\end{array}
\]

then we know that \( p = 5 \) (as this is the only characteristic in which this configuration arises in [4, 3.3]) and that there are no non-zero labels to the left of \( L_i \) (since at most the end \( m \) labels are non-zero as mentioned at the beginning of the proof, and \( k > m \)). Let \( V_1 \) be the \( L_X \)-module which projects non-trivially to \( V/\langle V, Q_Y \rangle \) (not necessarily the first summand pictured above). Here \( Q_X \leq K_X^{\alpha_n} \) (since otherwise \( Q_Y/K_Y^{\alpha_n} \cong V/\langle V, Q_Y \rangle \) would have an \( L_X \)-submodule \( Q_X K_X^{\alpha_n}/K_Y^{\alpha_n} \cong Q_X/K^{\beta_m} \), which it does not since \( V/\langle V, Q_Y \rangle \) is irreducible for \( L_X \) ), so again we will have a contradiction if we can show that \( V_{\lambda-\alpha_n} \not\subseteq V_2 \), since \( [V, Q_Y] = [V_1, Q_X] + V_2, V_{\lambda-\alpha_n} \not\subseteq [V_1, Q_X], \) and \( V_{\lambda-\alpha_n} \subseteq [V, Q_Y] \).

Call the two \( T_X \)-high weights pictured above \( \mu \) and \( \nu \) respectively. If \( e = 0 \), we know from the main result of [4] that \( \dim(V_1) + \dim(V_2) \) is the dimension of the \( B_m \)-module (where \( m = \) the rank of \( X = D_m \) with high weight \( \lambda_{m-1} + \lambda_m \). So if \( n(= \) the rank of \( Y \) ) is greater than \( m \), the dimension of \( V \) is strictly larger than \( \dim(V_1) + \dim(V_2) \), which is a contradiction. So if \( e = 0 \), then \( n = m \) and we must be in the \( D_n < B_n \) case. Assume \( e > 0 \). We have \( (\lambda-\alpha_n)(\mu, \nu) = \)


The rank of $L_m$ is $\lambda|_{T_{L_m}} + \lambda_{m-1}|_{T_{L_m}} = 2\lambda|_{T_{L_m}}$. Assume $\lambda|_{T_X} = \mu$ (i.e. the first summand pictured above is the one that projects non-trivially to $V/\{V, QY\}$). Then $\mu|_{T_{L_m}} = \lambda|_{T_{L_m}}$ is the $T_{L_m}$-high weight of $V/\{V, QY\}$. To have the weight $(\lambda - \alpha_n)|_{T_X}$ in $V_2$ would mean we could subtract roots from $\nu$ and obtain a weight $\epsilon$ which restricts to $T_{L_m}$ in the same way as $2\lambda$ does; i.e. there must be integers $c, \ldots, f$ such that $2e \delta_1 + 2e \delta_{m-1} + a \delta_m = e \delta_1 + 2e \delta_{m-1} + \delta_m - (c \beta_1 + \cdots + d \beta_{m-1} + f \beta_m)$ for some $a$. If we subtract $c \beta_1$, we must subtract $(2c + e) \beta_2, (3c + 2e) \beta_3, \ldots, ((m - 2)c + (m - 3)e) \beta_{m-2}$. Say we subtract $d \beta_{m-1}$ and $f \beta_m$. Then since the coefficients of $\delta_{m-2}, \delta_{m-1}$ remain unchanged, we have $(m - 3)c + (m - 4)e + d + f = 2((m - 2)c + (m - 3)e)$ and $(m - 2)c + (m - 3)e = 2d$. These together give $f = (m(c + e) - e)/2$.

Lemma 3.6 iv) in [9] gives $\alpha_n|_Z = \beta_m|_Z$. So $(\lambda - \alpha_n)|_Z = (\mu - \beta_m)|_Z$. We computed $Z$ above, and

$$(\mu - \beta_m)(a^2 \in_1(a^2) \in_3(a^2) \cdots \in_{m-2}(a^{2(m-2)}) \in_{m-1}(a^{m-2}) \in_m(a^m))$$

$= a^{2c + (m - 2) + 2m - 4}$. So $\epsilon$ must act on $Z$ as $a^{2c + (m - 2) + 2m - 4}$. But $2\lambda$ acts as

$\alpha^{4c + 2(m - 2) + (1 - 2f + (m - 2)c + (m - 3)e)m}$. So these two exponents must be equal. Using the above expression for $\lambda$, this simplifies to $m(e + c) = e + 2$, which has no solutions in non-negative integers with $m \geq 4$. So $\lambda|_{T_X} \neq \mu$.

If $\lambda|_{T_X} = \nu$, then similar calculations give $2f = m(c + e) - e - 3$ and $m(e + c - 3) = e$; this system also has no solutions in non-negative integers with $m \geq 4, e \geq 1$.

The picture above does not occur.

If the non-zero label is on the other end of $L_i$, then the picture is

\[
\begin{array}{c}
 e \quad \cdots \quad 0 \\
 1 \\
\end{array}
\oplus
\begin{array}{c}
 e \quad \cdots \quad 1 \\
 0 \\
\end{array}
= L_i
\begin{array}{c}
 1 \\
 1 \\
\end{array}
\cdots
\begin{array}{c}
 0 \\
 1 \\
\end{array}
\alpha.
\]

The rank of $L_i$ is dim$(V^1(QY)) - 1$; we know that the node in the Dynkin diagram for $Y$ with a label of 1 must be within $m$ nodes of the end (it must be $\alpha_i$ for $i \geq n - m$), as noted at the beginning of this proof. If the second summand in the picture is the one that projects non-trivially to $V^1(QY)$, this forces $e = 0$ but then $V$ has dimension too large to be the sum of the two spin modules for $D_m$. If the first summand projects non-trivially, $e = 1$ for the same reason and $L_i$ has rank $m - 1$.

If there is a node to the left of $L_i$ (i.e. if $n > m$), let $P_Y$ be the parabolic subgroup of $Y$ corresponding to the end $m$ nodes of the Dynkin diagram. Then $V|_{L_Y}$ has a composition factor (not all of $V$) which is isomorphic to the $B_m$ module with high weight $\lambda_1 + \lambda_m$. This $B_m$-module has $D_m$-high weights $\delta_1 + \delta_{m-1}$ and $\delta_1 + \delta_m$ (for the $D_m \leq L_Y$ in the natural way); thus $V$ is too large to be the sum of two $D_m$-modules of these high weights. So this case does not occur. If there is no node to the left of $L_i$, then $Y = B_m$ and we are in one of the cases studied in [4], which tells us there are no examples.

So we are left with the case when the embedding $L_X \hookrightarrow L_i$ is an isomorphism, so $L_i$ is of type $A_{m-1}$ and the labelling of the high weight of $V/\{V, QY\}$ on $\Pi(L_i)$
is the same as the labelling on $\Pi(L'_X)$. The two possible pictures are

(1)

$$
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a1) at (0,0) [circle] {$a_1$};
\node (a2) at (0.5,0) [circle] {$a_2$};
\node (am1) at (2.5,0) [circle] {$a_{m-1}$};
\node (am2) at (3,0) [circle] {$a_{m-1}+1$};
\end{tikzpicture}}
\end{array}
\oplus
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a1) at (0,0) [circle] {$a_1$};
\node (a2) at (0.5,0) [circle] {$a_2$};
\node (am1) at (2.5,0) [circle] {$a_{m-1}$};
\node (am2) at (3,0) [circle] {$a_{m-1}+1$};
\end{tikzpicture}}
\end{array}
= \begin{array}{c}
\text{\begin{tikzpicture}
\node (a1) at (0,0) [circle] {$L_1$};
\node (a2) at (0.5,0) [circle] {$a_2$};
\node (am1) at (2.5,0) [circle] {$a_{m-1}$};
\node (am2) at (3,0) [circle] {$a_{m-1}+1$};
\node (am3) at (3.5,0) [circle] {$1$};
\end{tikzpicture}}
\end{array}
\delta
$$

with the first summand projecting non-trivially to $V^1(Q_Y)$ and

(2)

$$
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a1) at (0,0) [circle] {$a_1$};
\node (a2) at (0.5,0) [circle] {$a_2$};
\node (am1) at (2.5,0) [circle] {$a_{m-1}$};
\node (am2) at (3,0) [circle] {$a_{m-1}+1$};
\node (am3) at (3.5,0) [circle] {$1$};
\end{tikzpicture}}
\end{array}
\oplus
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a1) at (0,0) [circle] {$a_1$};
\node (a2) at (0.5,0) [circle] {$a_2$};
\node (am1) at (2.5,0) [circle] {$a_{m-1}$};
\node (am2) at (3,0) [circle] {$a_{m-1}+1$};
\node (am3) at (3.5,0) [circle] {$1$};
\end{tikzpicture}}
\end{array}
= \begin{array}{c}
\text{\begin{tikzpicture}
\node (a1) at (0,0) [circle] {$L_1$};
\node (a2) at (0.5,0) [circle] {$a_2$};
\node (am1) at (2.5,0) [circle] {$a_{m-1}$};
\node (am2) at (3,0) [circle] {$a_{m-2}$};
\node (am3) at (3.5,0) [circle] {$a_1$};
\node (am4) at (4,0) [circle] {$1$};
\end{tikzpicture}}
\end{array}
\delta
$$

with $b = a_{m-1}$ if the first summand projects non-trivially to $V/[V,Q_Y]$ and $b = a_{m-1}+1$ if it is the second summand which projects non-trivially (as noted at the beginning of this proof, all labels to the left of the highlighted nodes of the Dynkin diagram for $Y$ in the pictures above are 0).

If we have picture (1) above, then the question is whether or not $\delta$ exists. The $B_m$-module with high weight $a_1\lambda_1 + \cdots + a_{m-1}\lambda_{m-1} + \lambda_m$ (where the $\lambda_i$ are for the moment the fundamental dominant weights of this Levi factor of $Y$ of type $B_m$), when considered as a module for the $D_m$ sitting in $B_m$ in the usual way, has high weights $a_1\delta_1 + \cdots + a_{m-1}\delta_{m-1} + (a_{m-1}+1)\delta_m$ and $a_1\delta_1 + \cdots + (a_{m-1}+1)\delta_{m-1} + a_{m-1}\delta_n$ (this was noted in the proof of [4, 3.3]). So if $\delta$ exists, then $V$ is too large to be the sum of two $D_m$-modules of these high weights. So $Y$ in fact has type $B_m$; but then we are back in the situation of [4], which tells us the only possibility here is the one in the statement of the theorem.

If we have picture (2) above, then the relationship between the $a_i$ which we know from [4] tells us that $a_2 = a_{m-2}, a_3 = a_{m-3}, \ldots$, and therefore $a_1 = a_{m-1}$. If the first summand projects non-trivially to $V^1(Q_Y)$, then we in fact have an instance of picture (2), which we covered above.

So we may assume we have picture (2), with the second summand projecting non-trivially, i.e. the isomorphism $L'_X \to L_1$ is given by the graph isomorphism sending $\beta_1 \to \alpha_{m-1}, \beta_2 \to \alpha_{m-2}, \ldots, \beta_{m-1} \to \alpha_1$ (here and below, the $\alpha_i$ are the fundamental roots of $Y$ corresponding to the “end” $m$ nodes of the Dynkin diagram, with the $\lambda_i$ the corresponding fundamental dominant weights).

We now define a normal subgroup of $P_Y$ which is the “one level down” analogue of $K'_Y$ (see Lemma 2.10 and the paragraph preceding it). For a root $\gamma \in \Pi(Y) - \Pi(L_Y)$, let $\Sigma_Y(\gamma)$ be the set of roots in $\Sigma^-(Y)$ which have $\gamma$-coefficient $-1$ and coefficient 0 for other fundamental roots not in $\Pi(L_Y)$. Then as in the introduction, $K'_Y$ is the product of root groups $U_\beta$ for $\beta \in (\Sigma^-(Y) - \Sigma^-(L_Y) - \Sigma_Y(\gamma))$. Analogously, let $\Sigma_Y(2\gamma)$ be the set of roots in $\Sigma^-(Y)$ having $\gamma$-coefficient $-2$ and coefficient 0 for other roots in $\Pi(Y) - \Pi(L_Y)$. Then let $K''_Y$ be the product of root groups $U_\beta$ for $\beta \in (\Sigma^-(Y) - \Sigma^-(L_Y) - \Sigma_Y(\gamma) - \Sigma_Y(2\gamma))$. $K''_Y$ is normal in $P_Y$ by the commutator relations, and $K''_Y \leq K'_Y$. We have previously considered the quotient $Q_Y/K'_Y$; now we wish to look at $K''_Y/K'_Y$. Let $\gamma = \alpha_m$ be the short fundamental root of $Y$. Then $K''_Y/K'_Y \cong \text{the irreducible } L_i\text{-module with high weight } -(2\alpha_m + \alpha_{m-1})|_{T_{Li}} = \lambda_{m-2}|_{T_{Li}}$. 
Consider \( Q^{\beta_m} = Q_X/K^{\beta_m} \), where \( K^{\beta_m} \) is the product of those \( T_X \)-root groups \( U_\gamma \) for \( \gamma \in \Sigma^-(X) \) with \( \beta_m \)-coefficient less than \(-1\) (which in this case is trivial, there are no such roots in a root system of type \( D_m \)). By Lemma 2.10, \( Q^{\beta_m} \) is an \( L'_X \)-module with high weight \( (-\beta_m)|_{T'_X} = \delta_m-2|_{T'_X} \) and thus dimension \( \binom{m}{2} \).

By [9, 3.1], \( Q_Y/K^{\alpha_m} \) is an irreducible \( L'_Y \)-module with high weight \(-\alpha_m|_{T'_Y} = \lambda_m-1|_{T'_Y} \), which implies that \( Q_Y/K^{\alpha_m} \) has dimension \( m \). It has an \( L_X \)-submodule \( \Delta \). If \( \Delta \leq K^{\alpha_m} \), then we see that \( \Delta \) has an \( L'_X \)-module and \( \Delta \Delta \) has dimension \( \binom{m}{2} \), which is impossible. So \( \Delta \leq K^{\alpha_m} \). Then \( K^{\beta_m} = Q'_Y \leq K^{\alpha_m'} \leq K^{2\alpha_m} \), and we can project \( Q_X/K^{\beta_m} \rightarrow K^{\alpha_m}/K^{2\alpha_m} \). If this map has non-zero image then it is in fact an isomorphism, since \( Q_X/K^{\beta_m} \) is an irreducible \( L'_X \)-module and \( \dim(Q_X/K^{\beta_m}) = \dim(K^{\alpha_m}/K^{2\alpha_m}) \). But then \( K^{\alpha_m}/K^{2\alpha_m} \) has \( L'_X \)-high weight \( \delta_m-2 \), whereas we know, by the fact that the isomorphism \( L'_X \rightarrow L_i \) is given by the graph isomorphism sending \( \beta_1 \rightarrow \alpha_{m-1} \), \( \beta_2 \rightarrow \alpha_{m-2} \), \ldots \( \beta_{m-1} \rightarrow \alpha_1 \), that \( K^{\alpha_m}/K^{2\alpha_m} \) must have \( L'_X \)-high weight \( \delta_2 \) (since it has \( L'_X \)-high weight \( \lambda_m-2 \)). So \( 2 = m-2 \), i.e. \( X \) is of type \( D_4 \).

On the other hand, if \( Q_X \leq K^{2\alpha_m} \) (i.e. if the map above is not an isomorphism), then we note that the roots \( \beta \) whose corresponding root subgroups appear in \( K^{2\alpha_m} \) all involve some fundamental root in \( \Pi(Y) - \Pi(L_Y) \) other than \( \alpha_m \) (since \( 3\alpha_m \) appears in no root). So if we let \( P_X = \langle P_Y, U_{\pm\alpha_m} \rangle \), we have a new parabolic subgroup of \( Y \) satisfying \( Q_Y \geq Q_X, L_Y \geq L_X, P_Y \geq P_X \). If now \( V/[V, Q_Y] \) is irreducible as an \( L'_X \)-module, then the first part of the proof of this lemma gives a contradiction. If \( V/[V, Q_Y] \) is the sum of two irreducibles for \( L'_X \cong L_i \), then we have a contradiction because this case would have to appear in [4], and it does not.

So we are left with the picture

\[
\begin{array}{ccc}
\begin{array}{ccc}
 a & b & a+1 \\
 b & a & a+1
\end{array}
& \oplus \\
\begin{array}{ccc}
 a & b & a \\
 b & a & a+1
\end{array}
& =
\begin{array}{cccc}
 L_i \\
 0 & a+1 & b & 1 \\
 \delta & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4
\end{array}
\end{array}
\]

with the first summand projecting non-trivially to \( V^1(Q_Y) \). In fact there are three possibilities: \( a = (p-3)/2, b = (p+1)/2, a = 0, b = (p-5)/2 \) or \( a = b = 0 \).

If \( a = b = 0 \), then the \( T_Y \)-high weight of \( V \) has at least 32 conjugates, making \( V \) too large to be the sum of two \( D_4 \)-modules of dimension 8. We may assume \( \text{rank}(Y) = n \geq 5 \), since otherwise [4] tells us there are no examples.

Suppose \( a = 0, b = (p-5)/2 \). Then \( V_1 \) has \( D_4 \)-high weight \( \frac{p-3}{2}\delta_2 + \delta_4 \), and \( V \) has dimension at least the dimension of the \( B_3 \)-module with high weight \( \lambda_1 + \frac{p-5}{2}\lambda_2 + \lambda_4 \) (the \( \lambda_i \) refer to the fundamental weights corresponding to the end four nodes of the Dynkin diagram for \( Y \)). Thus \( V \), as a module for the obvious \( B_4 < Y \), has a composition factor \( V' \) of high weight \( \lambda_1 + \frac{p-5}{2}\lambda_2 + \lambda_4 \) and another, \( V'' \), of high weight \( \frac{p-3}{2}\lambda_2 + \lambda_4 \) (if \( P_Y \) is the parabolic subgroup of \( Y \) corresponding to the end four nodes, then \( V^2(Q_Y) \) has an \( L_Y \)-high weight \( \lambda - \delta - \alpha_1 \), with the above restriction to \( T_{L_Y} \)). As a module for the \( D_4 < B_4 \), \( V' \) has high weights \( \delta_1 + \frac{p-5}{2}\delta_2 + \delta_4 \) and \( \delta_1 + \frac{p-5}{2}\delta_2 + \delta_3 \), while \( V'' \) has high weights \( \frac{p-3}{2}\delta_2 + \delta_4 \) and \( \frac{p-3}{2}\delta_2 + \delta_3 \). So \( \dim(V) \geq 2 \dim(V_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) + 2 \dim(V_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4)) \). Using the Andersen-Jantzen
sum formula \((1, 6)\), we see that
\[
\dim(V_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) = \dim(W_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) - \dim(W_{D_4}(\delta_1 + \frac{p-9}{2}\delta_2 + \delta_4)),
\]
and
\[
\dim(V_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4)) = \dim(W_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4)) - \dim(W_{D_4}(\frac{p-9}{2}\delta_2 + \delta_4)),
\]
where \(W_{D_4}(\beta)\) denotes the Weyl module for \(D_4\) with high weight \(\beta\) (the second terms on the right hand sides of the above equalities appear only if \(p - 9 \geq 0\)). Calculating these dimensions with the Weyl character formula, we find that
\[
\dim(V) \geq 2(\dim(V_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) + \dim(V_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4))) - 2 \dim(W_{D_4}(\frac{p-5}{2}\delta_2 + \delta_4)) - 2 \dim(V_{D_4}(\frac{p-5}{2}\delta_2 + \delta_4))
\]
\[
\geq 2 \dim(V_{D_4}(\frac{p-5}{2}\delta_2 + \delta_4)) - 2 \dim(V_{D_4}(\frac{p-5}{2}\delta_2 + \delta_4)) = 2 \dim(V_1) = \dim(V_1) + \dim(V_2).
\]
This is a contradiction, so in fact this case does not occur.

Finally, if \(a \neq 0 \neq b\), then we take yet another parabolic: Let \(P_X\) be the parabolic subgroup of \(X\) corresponding to \(\{\beta_1, \beta_2, \beta_3\} \subseteq \Pi(X)\). Then \(L'_X\) is a product of three \(A_1\)'s. Here \(P_X\) is \(t\)-stable, so when we embed \(P_X\) in a parabolic subgroup \(P_Y\) of \(Y\), as usual, we have \([V, Q_Y] = [V, Q_X]\). We denote \(T_{L'_X}\)-weights by \((a_1, a_3, a_4)\), where \(a_i \in \mathbb{Z}\) is the value of the weight on the torus corresponding to \(\beta_i\). The high weight of \(V_1/[V_1, Q_X]\) is \((a, a+1, a)\); that of \(V_2/[V_2, Q_X]\), \((a, a, a+1)\). The weights which appear in \(V_1/[V_1, Q_X]\) have the form \((a-2b_1, a+1-2b_3, a-2b_4)\), and those in \(V_2/[V_2, Q_X]\) have the form \((a-2c_1, a-2c_3, a+1-2c_4)\). The weight spaces in \(V_1/[V_1, Q_X]\) all have dimension 1, and no weight can appear in both. So the weight spaces of \([V, Q_Y]\) have dimension 1.

Those modules for simple algebraic groups which have all weight spaces of dimension 1 are classified in [9, chapter 6]. We have \(\dim(V_1/[V_1, Q_X]) = (a+1)^2(a+2)\), so \(\dim(V/[V, Q_Y]) = 2(a+1)^2(a+2)\). With the labelling on \(Y\) known, we can compare the various possibilities for the factors of \(L'_Y\) which act non-trivially on \(V^1(Q_Y)\) (remembering that they must appear in \([9, 6.1]\)) with the known dimension of \(V^1(Q_Y)\), and we get a contradiction in every case but one: \(p = 13\), with the high weights and the embedding of \(P_X\) in \(P_Y\) as in the picture

\[
\begin{array}{ccc}
5 & 7 & 5 \\
6 & \oplus & 6 \\
\end{array}
\]

But now the \(L'_Y\)-module \(Q_Y/K_Y\) has dimension 2, while \(Q_X/Q_X\) has dimension 8; this forces \(Q_X \leq K_Y\) (otherwise \(Q_X K_Y/K_Y \cong Q_X/Q_X\) is a submodule of dimension 8 of \(Q_Y/K_Y\), which is impossible). But then \(V_{\lambda-\gamma} \notin [V, Q_X] = [V, Q_Y]\), which is absurd since \(\gamma\) has a non-zero label.

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4.4. Case 4. We are left with case 4 on as stated above Lemma 4.2.

**Lemma 4.6.** If $W_i$ is irreducible for $L'_X$, then $p = 2$, $X = D_m$, and $Y = B_n$ or $C_n$, with $V$ a spin module for $Y$ and a sum of two spin modules for $X$.

**Proof.** We are back to the situation where $P_X$ is the maximal parabolic subgroup corresponding to $\Pi(X) - \{\beta_1\}$. Inductively we need only to check the case where $p = 2$, $L'_X = D_m$, and $L_i = B_{m-1}$ or $C_{m-1}$, with $V/[V,Q_Y]$ a spin module for $L_i$ and a sum of two spin modules for $L'_X$; and the single case which occurred in section 3: $L_i = D_{10}, (V/[V,Q_Y])/L_i = \text{spin}(L_i), L'_X = D_3 = A_3, V_i/[V_1,Q_X]$ of $L_X$-high weight $3\delta_3 + \delta_2 + \delta_1$, $p \neq 2,3,5,7$.

If we are in the first setup, then the arguments of the last subsection carry over and we have only the examples in the statement of the Lemma.

So the picture is

\[
e 1 1 3 \oplus e 1 3 1 = \cdots \gamma \cdots.
\]

Assume $\gamma$ has a non-zero label (in the marking for the $Y$-high weight of $V$).

Then $V_\gamma(Q_Y)$ has an $L'_Y$-high weight given by the labelling on the boxed nodes above with a 1 on the node to the right of $\gamma$ and the 1 on the end node as pictured. The dimension of this $D_{10}$-module is at least the number of conjugates of the high weight, which is $2^9 \cdot 10$. But then $\dim(V^2(Q_Y)) \geq \dim(V\gamma(Q_Y)) \geq 2^9 \cdot 10 > 6 \cdot 2^9 = 6 \dim(V_1(Q_Y)) = \dim(Q^3) \dim(V_1(Q_Y))$, which is a contradiction to Lemma 2.9. So $\gamma$ has label 0. By Lemma 4.3, all the other nodes in the diagram for $Y$ have label 0, so in fact $V$ is a spin representation of $Y$.

Now switch parabolics: Let $P_X$ correspond to $\{\beta_1, \beta_2\} < \Pi(X)$. As always, embed $P_X$ in a parabolic $P_Y$ of $Y$, via the usual construction (given in Lemma 2.7). Since $P_X$ is $t$-stable, we again have in this case $[V,Q_Y] = [V,Q_X]$ by Lemma 2.8. Since $V/[V,Q_X] \neq 0$, the subset of $\Pi(Y)$ to which $P_Y$ corresponds must contain the node $\alpha$ that has a label 1. Let $L_i$ be the simple factor of $L_Y$ that contains $\langle U_\alpha \rangle$.

The irreducible $L'_X$-module $V_1/[V_1,Q_X]$ has high weight $(e\delta_1 + \delta_2)/T_{L_X}$. Its dimension is $(e+1)(e+3)$ if $e \neq p - 2$, and $(e+1)(e+6)/2$ if $e = p - 2$. Recall that $\dim(V/[V,Q_Y]) = 2(\dim(V_1/[V_1,Q_X]))$.

If $L_i$ has type $D_l$, then $\dim(V_1/[V_1,Q_X])$ is a power of 2. The dimension $(e+1)(e+3)$ is a power of 2 only when $e = 1$, and $(e+1)(e+6)/2$ is never a power of 2. So the possibility here is $e = 1$, $p \neq 3$.

If $L_i$ has type $A_l$ and $e > 0$, then $l \geq (e+1)(e+6)/2 - 1$ (which is always $\geq 6$, so $\Pi(L_i)$ does not contain both $\alpha_\mu$ and $\alpha_{\mu-1}$). Then $V_2(Q_Y)$ has a composition factor of high weight $(\lambda - \gamma_l - \gamma_{l-1} - \alpha)/T_{\lambda'\gamma'}$ (where $\gamma_{l-1}, \gamma_l$ are the end nodes of $L_i$ and $\alpha$ is the node at the end of $Y$ which is left out of $L_i$), of dimension $(l+1)/3$. So we have $(l+1)/3 \leq 6 \dim(V_2(Q_Y)) = 6(l+1)$ (using Lemma 2.9 again). But $(l+1)/3 \leq 6(l+1)$ is a contradiction for $l \geq 6$, and $l \leq 5$ only for $e = 0$.

We are left with some cases for $e = 1$ and $e = 0$. But $V$ is a spin module for $Y$, so $\dim(V)$ is a power of 2; the only time when $V_1$ as above has dimension a power of 2 is for $e = 1$ and $p > 11$ or $p = 0$, in which case $\dim(V_1) = 2^{15}$. But this would
imply that \( \dim(V) = 2^{16} \), so \( Y \) has type \( D_{17} \). But \( X = D_4 \) has no irreducible representations of dimension 34 (= \( \dim(W) \)) when \( p > 11 \) or \( p = 0 \).

This completes the proof of the theorem.

5. The case \( X = E_6 \)

Here we establish Theorem 1 for the case where \( X = E_6 \) and \( G = X(t) \). Again we assume that \( t \) acts on \( W \), the natural module for \( Y \). Notation \( (X \leq Y, V, V_1, V_2, \alpha, \lambda) \) is as previously defined; in particular, \( \{\beta_1, ..., \beta_6\} \) is the set of simple roots of \( X \), with \( \{\delta_1, ..., \delta_6\} \) the set of fundamental dominant weights, labelled so that \( \langle \delta_i, \beta_j \rangle = \delta_{ij} \). The main theorem is

**Theorem 5.1.** If \( X \) acts irreducibly on the natural module \( W \) for \( Y \), then \( X \) is not of type \( E_6 \).

**Proof.** Since \( t \) acts on \( W \), the \( T_X \)-high weight \( \delta = d_1\delta_1 + \cdots + d_6\delta_6 \) of \( W \) must be symmetric; that is, \( d_1 = d_6 \) and \( d_3 = d_5 \). We will write \( \delta = a\delta_1 + b\delta_2 + b\delta_3 + c\delta_4 + b\delta_5 + a\delta_6 \), so the picture of the \( X \)-high weight of \( W \) is \( a \quad b \quad c \quad b \quad a \quad d \). We use the same method as in section 3, based on Lemma 3.2. As in the case \( X = A_m \), we will investigate the embedding of the fixed Borel subgroup \( B_X \) of \( X \) in a parabolic subgroup \( P_Y \) of \( Y \), via the construction outlined in the proof of Lemma 2.7. In the first subsection, we will show that there are only a few cases in which a factor of type \( A_1 \) might appear in \( L_4 \); following that, we deal with these few cases by investigating the embeddings of other parabolic subgroups of \( X \).

### 5.1. The Almost-Everywhere Argument.

The argument will again be an induction on the partial order on the weight lattice; for the base case of the induction we need

**Lemma 5.2.** If \( X \) is of type \( E_6 \) and \( \delta \neq \delta_2 \), then \( \delta \geq \delta_1 + \delta_6 \).

**Proof.** This is an easy exercise (using the fact that \( \delta \) is symmetric).

Notice that 0 is a weight of \( V_{E_6}(\delta_1 + \delta_6) \) at level 16 by the expression for \( \delta_1 + \delta_6 \) in terms of roots (see [5], for example). So to begin our induction we must show that in \( V_{E_6}(\delta_1 + \delta_6) = V_{E_6}(\delta) \) there are at least three weights in every \( U_X \)-level \( i \) for \( 2 \leq i < 16 \) and at least five at level 16. This is easy to do; we illustrate by giving three weights in each of levels 2–4. Here we are using the usual rules to determine that a weight appears: If \( \mu \) is a \( T_X \)-weight such that \( W_\mu \neq 0 \), and \( \langle \mu, \beta_j \rangle = l > 0 \), then \( W_{\mu - j\beta} \neq 0 \) for every \( 1 \leq j \leq l \). This depends on the result in [8], which says that weights which appear in characteristic 0 also appear in characteristic \( p \), and the fact that in characteristic 0, the \( \beta_l \)-string through \( \mu \) is connected (the set of weights is saturated—see [5, p. 114]).

- **Level 2:** \( \delta - \beta_1 - \beta_3, \delta - \beta_5 - \beta_3, \delta = \beta_1 - \beta_6; \)
- **Level 3:** \( \delta - \beta_1 - \beta_3 - \beta_4, \delta - \beta_5 - \beta_3 - \beta_4, \delta - \beta_1 - \beta_3 - \beta_5 - \beta_4; \)
- **Level 4:** \( \delta - \beta_1 - \beta_3 - \beta_4 - \beta_5, \delta - \beta_5 - \beta_4 - \beta_3 - \beta_6, \delta - \beta_1 - \beta_3 - \beta_4 - \beta_5; \)
- etc.
If $\delta = \delta_2$, then we check all levels and find that the only possibilities for an $A_1$-factor of $L'_Y$ are level 3 and level 11 (the level of the 0 weight). The weight space for the weight 0 has dimension at least 5 in all characteristics, so level 11 is large enough to preclude a corresponding $a_1$-factor of $L'_Y$. We will deal with level 3 later.

Since $\delta$ is symmetric, it has the form $a(\delta_1 + \delta_5) + b(\delta_3 + \delta_4) + c\delta_4 + d\delta_2$ for some nonnegative integers $a, b, c, d$. Now for the induction. Assume $\delta > \delta_1 + \delta_6$ (and $\delta \neq \delta_1 + \delta_6$). Then at least one of the following must be true: 1) $d \geq 2$; 2) $c \geq 2$; 3) $b > 0$; 4) $a > 0$; or 5) $\delta \in \{\delta_4, \delta_2 + \delta_4\}$. We consider each of these possibilities in turn.

1) If $d \geq 2$, then $\delta - \beta_2$ is a dominant weight, still greater than $\delta_1 + \delta_6$ in the partial order, and by induction $\delta$ has enough weights at all levels 3 and higher. So we need to check levels 1 and 2. Level 1 is 2-dimensional if $a = b = 0, c \neq 0$; otherwise $\dim(W_1) = 1$ or $\dim(W_1) \geq 3$. At level 2, if $a = b = c = 0$, we have just the two weights $\delta - 2\beta_2, \delta - \beta - \beta_4$; so this is a case we must consider below.

2) If $c > 1$, then $\delta - \beta_4$ is a dominant weight, still greater than $\delta_1 + \delta_6$ in the partial order, and by induction $\delta$ has enough weights at all levels 3 and higher. So we need to check levels 1 and 2. As above, level 2 is 2-dimensional if $a = b = 0, d \neq 0$. At level 2 are the three weights $\delta - \beta_4 - \beta_3, \delta - \beta_4 - \beta_2$, and $\delta - \beta_4 - \beta_5$.

3) If $b > 0$, then $\delta - \beta_3 - \beta_4 - \beta_5$ is a dominant weight, still greater than $\delta_1 + \delta_6$, and by induction we must check levels 1-4. Level 1 is 2-dimensional only if $a = c = d = 0$; level 2 has at least 5 weights appearing; level 3, at least 5; and level 4, at least 5. So for $b \neq 0$ we need to consider only level 1 with $a = c = d = 0$.

4) If $a > 0$ and $\delta \neq \delta_1 + \delta_6$, then $\delta - \beta_1 - \beta_3 - \beta_4 - \beta_5 - \beta_6$ is a dominant weight, and we need to check levels 1–6. Level one is 2-dimensional only if $b = c = d = 0$; level 2 has at least 5 weights appearing; level 3, at least 4; level 4, at least 5; level 5, at least 5; and level 6, at least 5. So for $a \neq 0$, we have only to consider level 1 with $b = c = d = 0$.

5) The weights that aren’t covered above are $\delta = \delta_4, \delta = \delta_2 + \delta_4$, and $\delta = \delta_2$. Notice that $\delta_4 - \beta_3 - \beta_4 - 2\beta_4 - \beta_5 = \delta_1 + \delta_5$, so we need to check levels 1-6; in this case, level 1 has dimension 1 and levels 2–6 all have dimension at least 3. So $\delta = \delta_4$ gives no $A_1$-factors of $L'_Y$. If $\delta = \delta_2 + \delta_4$, then $\delta - \beta_4 - \beta_2$ is a dominant weight, so we need to check only levels 1-3. Level 1 has dimension 2, so it must be considered; levels 2 and 3 are both big enough.

So the only embeddings for which an $A_1$ factor of $L'_Y$ might appear are the ones that give an obvious level of dimension 2:

1. Level 1 for $\delta = a\delta_1 + b\delta_4; \delta = b\delta_3 + b\delta_5$; or $\delta = c\delta_4 + d\delta_2$ ($a \neq 0 \neq b, c \neq 0 \neq d$).
2. Level 2 for $\delta = d\delta_2$ with $d > 1$.
3. Level 3 for $\delta = \delta_2$.

5.2. The Remaining Cases. We treat the cases listed above in reverse order.

3. Say $\delta = \delta_2$. Then the picture of the embedding of the Borel subgroup of $X$ is

$$a_1 a_2 a_3 1 a_5 \ldots$$

with all simple factors of $L'_Y$ to the right of the picture having rank at least 2, all separated by only a single node.

By Lemma 2.4, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) = 2 \dim(V^1(Q_X)) \leq 2 \cdot 6 \dim(V^1(Q_X)) = 12$. No node other than $\alpha_4$ of the Dynkin diagram for $Y$
which falls in \( \Pi(L'_Y) \) can have a nonzero label in the labelling for \( \lambda \) on \( Y \), since \( \dim(V^2(Y)) = 2 \). Suppose some node \( \alpha_i \in \Pi(Y) - \Pi(L_Y)' \) for \( i > 7 \) has a nonzero label. Then \((\lambda - \alpha_i)|_{V_{1\gamma}} \) is a high weight in \( V^2(Y) \), giving a composition factor of dimension at least 9 (since the two simple factors of \( L_Y \) which it adjoins have rank \( \geq 2 \)). But \( \lambda - \alpha_4 - \alpha_3 \) (if \( a_3 = 0 \)) or \( \lambda - \alpha_3 \) (if \( a_3 > 0 \)) is another high weight in \( V^2(Y) \), giving a factor of dimension at least 1, and \( \lambda - \alpha_4 - \alpha_5 \) (if \( a_5 = 0 \)) or \( \lambda - \alpha_5 \) (if \( a_5 > 0 \)) is another, giving one of dimension at least 3. Adding these up, we see that \( \dim(V^2(Y)) \geq 13 \), which is a contradiction.

So \( a_i = 0 \) for \( i > 5 \). Now embed the maximal parabolic subgroup \( P_X \) of \( X \) corresponding to \( \beta_2 \) into a parabolic subgroup \( P_Y \) of \( Y \) by the same \( Q_X \)-level construction. Here level 0 has dimension 1 and level 1 has dimension 20. Now \( L'_X \) has type \( A_5 \) and the factor \( L_1 \) of \( L'_Y \), corresponding to level 1, has type \( A_{19} \). Since \( a_i = 0 \) for \( i > 5 \), \( L_1 \) is the only factor of \( L'_Y \) acting nontrivially on \( V/[V, Q_Y] \).

By Lemma 2.8, \( V/[V, Q_Y] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X] \). But then we are in the situation of section 3, with \( V/[V, Q_Y] \) an irreducible \( A_5(t) \)-module. Since there were no examples there of this setup, inductively we have none here.

2. If \( \delta = d\delta_2 \) with \( d > 1 \), then the embedding of the Borel subgroup of \( X \) in a parabolic subgroup of \( Y \) is

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \cdots \\
& & 1 & & \\
& & & a_4 & \\
& & & & \\
\end{array}
\]

with, as above, all simple factors of \( L'_Y \) to the right of the picture of rank at least 2, all separated by only a single node. We continue exactly as in case 3, finding no examples.

1. We have \( \delta = a\delta_1 + a\delta_4, \delta = b\delta_3 + b\delta_5, \) or \( \delta = c\delta_4 + d\delta_2 \). The embedding of \( B_X \) in a parabolic subgroup of \( Y \) is as pictured in

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \cdots & \\
& & 1 & & \\
& & & a_3 & \\
& & & & \\
\end{array}
\]

with the second simple factor of \( L'_Y \) having type \( A_4 \) for \( l \geq 2 \) and all simple factors of \( L'_Y \) to the right of the picture having rank at least 2, all separated by only a single node. By the same arguments as above, \( a_i = 0 \) for \( i > 4 \).

We again embed the maximal parabolic subgroup of \( X \) corresponding to \( \beta_2 \) in a parabolic subgroup of \( Y \); now level 0 is large, with dimension at least 20. Because \( a_i = 0 \) for \( i > 4 \), the factor \( L_1 \) of \( L'_Y \) corresponding to this level 0 is the only simple factor of \( L'_Y \) acting nontrivially on \( V/[V, Q_Y] \). As above, we are back in the situation of section 3, which tells us that once again there are no examples.

So \( G = E_6(t) \), with \( E_6 \) acting irreducibly on \( W \), gives no examples.

6. The case \( X = D_4, [G : X] \geq 3 \)

In this section we treat the cases \( G = D_4(s) \) and \( G = D_4(s,t) \), where \( s \) is a graph automorphism of \( D_4 = X \) of order 3 and \( t \) is one of order 2. As mentioned earlier, since \( G < \text{Aut}(Y) \) we have \( s \in Y \), as no simple algebraic group properly containing \( D_4 \) has an outer automorphism of order 3. We assume that \( s \) acts on \( W \) and that \( W \) is irreducible as an \( X \)-module. The case \( G = D_4(t) \) was covered in section 4. All notation is as before. The main result is

**Theorem 6.1.** If \( X \) acts irreducibly on the natural module \( W \) for \( Y \), then we are not in the situation where \( X = D_4 \) with \( G = X(s) \) or \( G = X(s,t) \).
Notice that the assumption that $s$ acts on $W$ forces the $T_X$-high weight $\delta$ of $W$ to be of the form $a\delta_1 + b\delta_2 + a\delta_3 + a\delta_4$, which implies that $t$ acts on $W$. In addition, it implies that $X$ fixes a nondegenerate bilinear form on $W$, which is orthogonal if $p \neq 2$ ([11, Lemma 79]).

If $V|_X = V_1 \oplus V_2 \oplus V_3$ with each $V_i$ irreducible and restricted as an $X = D_4$-module, then $s$ permutes the $V_i$ and $V$ is irreducible as an $X(s)$-module; in this case we may assume that $G = X(s)$. So if $G = X(s,t)$, we may assume that $V$ has six simple factors as an $X$-module.

We based the arguments for $X = A_m$ and $X = E_6$ on Lemma 3.2. We will need an analogous result for the cases we consider here. Recall that $t_s$ is the $U_X$-level of the $T_X$-low weight of $W$. Let $P_Y$ be the parabolic subgroup of $Y$ containing $B_X = U_XT_X$, constructed via $U_X$-levels as outlined in Lemma 2.7, with $P_Y = Q_YL_Y$ the Levi decomposition given in that construction.

**Lemma 6.2.** If $P_Y$ is as above, then $\dim(V/[V,Q_Y]) = 3$ if $G = X(s)$ and $\dim(V/[V,Q_Y]) = 6$ if $G = X(s,t)$. If $G = X(s)$, either $\dim(W_{i/2}) = 4$ or there is a $U_X$-level in $W$ of dimension 3 or 2. If $G = X(s,t)$, there is an $i$ such that $2 \leq \dim(W_i) \leq 6$.

**Proof.** It suffices to prove the statements in the first sentence, since $A_1$, $A_2$, and $A_1 \times A_1$ are the only groups under consideration which have simple modules of dimension 3, and only those groups whose natural modules have dimension at most 6 have an irreducible module of dimension 6.

The construction via $U_X$-levels of the parabolic subgroup $P_Y$ of $Y$ clearly gives an $s$-stable subgroup, as $s$ acts on $W$ and on each $U_X$-level in $W$. The quotient $V/[V,Q_Y]$ is an irreducible $L_Y$-module by [9, Lemma 2.10]. By an argument similar to that used to prove Lemma 2.8, here we have $(V/[V,Q_Y])_{L_X} = V_1/[V_1,U_X] \oplus V_2/[V_2,U_X] \oplus V_3/[V_3,U_X] \oplus V_4/[V_4,U_X] \oplus V_5/[V_5,U_X] \oplus V_6/[V_6,U_X]$, if $t \in G$. But each of these $L_Y$-modules $V_i/[V_i,U_X]$ has dimension 1 by Lemma 2.4. This proves the first statement of the Lemma.

Recall that $P_Y$ is the stabilizer in $Y$ of the flag $0 \leq W_l \leq W_l \oplus W_{l-1} \leq \cdots$, where $l$ is minimal with respect to $[W,U_X^{+1}] = 0$. Each factor $L_i$ of $L_Y$ corresponds to a $U_X$-level $W_i$. Since $L_Y$ has an irreducible module of dimension $j = 3$ or 6, there must be a simple factor of $L_Y$ of rank less than $j$. For $j = 3$, this can happen if $i = l_{i/2}$ and $\dim(W_{l_{i/2}}) = 4$ (we could have the middle level $l_{i/2}$ giving a product of two groups of type $A_1$ if $Y = D_n$; otherwise, this happens only if $\dim(W_i) \leq j$.

This proves the second assertion of the Lemma.

Now if we can show that for a particular $T_X$-high weight $\delta$ of $W$, all $U_X$-levels of $W$ have dimension bigger than $j$ (with $j$ as above), and in the $G = X(s)$ case $\dim(W_{l_{i/2}}) > 4$, then the Lemma will imply that there are no examples with this embedding of $X$ into $Y$.

**Lemma 6.3.** Assume $X = D_4$, $s \in G$, and $X$ acts irreducibly on the natural module $W$ for $Y$. If $\dim(W_i) \leq 3$ where $W_i$ is the $U_X$-level $i$ of $W$, then one of the following holds:

1. $i = 1$ and either $\delta = b\delta_2$ or $\delta = a\delta_1 + a\delta_3 + a\delta_4$ for some $b \neq 0 \neq a$.
2. $\delta = \delta_2$, $\delta = 2\delta_2$, or $\delta = \delta_1 + \delta_3 + \delta_4$.

   If $\dim(W_i) \leq 6$, then one of the following holds:

3. $i = 1$.
4. $i = 2$ and either $\delta = b\delta_2$ for some $b \neq 0$ or $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$. 

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5. \( \delta = \delta_2, \delta = 2\delta_2, \) or \( \delta = \delta_1 + \delta_3 + \delta_4. \)

Finally, if \( \dim(W_{i/2}) \leq 4, \) then \( \delta = \delta_2. \)

**Proof.** We wish to induct on the height in the weight lattice of \( \delta, \) as in the \( X = A_m \) and \( X = E_6 \) cases. Let

\[
\delta = a\delta_1 + b\delta_2 + a\delta_3 + a\delta_4
\]

be the \( T_X \)-high weight of \( W. \) Since \( \delta_1 + \delta_3 + \delta_4 = 2\beta_1 + 3\beta_2 + 2\beta_3 + 2\beta_4 \) and \( \delta_2 = \beta_1 + 2\beta_2 + \beta_3 + \beta_4 \) are sums of roots, every weight which has the form of \( \delta \) has \( \delta_1 + \delta_2 + \delta_3 + \delta_4 \) as a subdominant weight, except \( \delta_2, 2\delta_2, \) and \( \delta_1 + \delta_3 + \delta_4. \)

So to begin our induction we must investigate the numbers of weights at various \( U_X \)-levels of the \( D_4 \)-module with high weight \( \delta_1 + \delta_2 + \delta_3 + \delta_4. \) The weight 0 appears in this module at level 14, so we must check the numbers of weights at levels \( i \) with \( 1 \leq i \leq 14. \) It is not hard to do this (again using the result in [8] that weights which appear in characteristic 0 appear in characteristic \( p \)); we find that there are 4 weights at level 1, 6 at level 2, and 10 or more at every level 3-14. We will exclude level 1 from the discussion below, since it is clear that \( \dim(W_1) = 1 \) if \( a = 0 \) and \( b > 0; \) \( \dim(W_1) = 3 \) if \( a > 0 \) and \( b = 0; \) and \( \dim(W_1) = 4 \) if \( a \neq 0 \neq b. \)

Assume \( \delta = a\delta_1 + b\delta_2 + a\delta_3 + a\delta_4 \) as above, with \( b > 2. \) Then \( \delta - \beta_2 \) is a dominant weight, less than \( \delta \) in the partial order and still having \( \delta_1 + \delta_2 + \delta_3 + \delta_4 \) as a subdominant weight, so by induction \( \delta - \beta_2 \) has enough weights at all levels 3 and higher. So we must check \( \delta \)-levels 2 and 3. There are 4 weights at level 2 if \( a = 0, \) and 7 otherwise. There are at least 7 weights at level 3. So the only possibility for a level of dimension 6 or less is level 2, with \( a = 0. \)

If \( b = 2 \) and \( a > 1, \) then again \( \delta - \beta_2 \) is dominant and lower in the partial order, still with \( \delta_1 + \delta_2 + \delta_3 + \delta_4 \) as a subdominant weight, so by induction \( \delta - \beta_2 \) has enough weights at all levels 3 and higher. Here, level 2 has 7 weights and level 3 has at least 7. So we have no possibilities arising from this embedding (other than level 1).

Finally, if \( b = 1 \) and \( a > 1, \) then \( \delta - \beta_1 - \beta_3 - \beta_4 \) is dominant and lower in the partial order; so by induction we must check \( \delta \)-levels 2, 3, 4, and 5. At all 4 of these levels there are more than 7 weights. This completes the listing of all possibilities for levels of dimension at most 6, except for the three weights \( \delta_2, 2\delta_2, \) and \( \delta_1 + \delta_2 + \delta_3. \)

The last statement of the lemma is proved by noting that by the same argument as above, every symmetric dominant weight except \( \delta_1 \) has \( \delta_1 + \delta_3 + \delta_4 \) as a subdominant weight, and \( \delta_1 + \delta_3 + \delta_4 \) has more than 4 weights at level 9, which is its middle level.

We see in the proof that the only possibilities for 1-dimensional \( U_X \)-levels are level 1 when \( \delta = b\delta_2 \) and level 0. So as in the arguments for earlier cases, we have that for any \( \delta, \) there are never 2 consecutive weights in the Dynkin diagram for \( Y \) which lie outside of \( \Pi(L'_Y) \), except for \( \alpha_1 \) and \( \alpha_2 \) in the case \( \delta = b\delta_2. \)

From this point, we will consider the two possibilities for \( G \) separately.

6.1. \( G = X(s). \) Here we need to find a level of dimension 3 or 2. By the previous Lemma, we need to consider only level 1 for \( \delta = a\delta_1 + a\delta_3 + a\delta_4 \) and the three possible \( \delta \) that were not covered by the induction there: \( \delta_2, 2\delta_2, \) and \( \delta_1 + \delta_3 + \delta_4. \)

We can check these last three directly and find that the only possibilities for a factor
of $L'_Y$ of type $A_1$ or $A_2$ are those corresponding to level 1 for $\delta = \delta_1 + \delta_3 + \delta_4$ and to levels 2, 3 and 5 for $\delta = \delta_2$.

So we must consider the following:

1) Assume $\delta = a\delta_1 + a\delta_3 + a\delta_4$. By the last Lemma, $\dim(V/\left[V,Q_Y\right]) = 3$, so either $a_2 = 1$ or $a_3 = 1$, with all other $a_i = 0$ for $\alpha_i \in \Pi(L'_Y)$. By Lemma 2.4, $\dim(V^2(Q_Y)) \leq \dim(V^2(U_X)) = 3\dim(V^2(U_X)) \leq 3(4\dim(V^2(U_X))) = 12$. If $a_3 = 1$, then in $V^2(Q_Y)$ we have either $\lambda - \alpha_4$ (if $a_4 \neq 0$) or $\lambda - \alpha_3 - \alpha_4$ (if $a_4 = 0$) as a high weight in $V^2(Q_Y)$. The $L'_Y$-modules with these high weights have dimension greater than 12, however, so in fact $a_3 = 0$.

If $a_2 = 1$, then similar calculations give contradictions to $\dim(V^2(Q_Y)) \leq 12$ if $0 \neq a_1 \neq p - 2$ or $a_i \neq 0$ for some $i > 2$ or if $a > 1$ and $a_1 \neq 0$. So the only possibilities here are $a_2 = 1$ with $a_i = 0$ for $i \neq 2$ (i.e. $\lambda = \lambda_2$) and $a_1 = a_2 = 1$ with $a_i = 0$ for $i > 2$ (so $\lambda = (p - 2)\lambda_1 + \lambda_2$).

Assume $p = 2$. Then, since $\delta$ is restricted by Lemma 2.6, $\delta = \delta_1 + \delta_3 + \delta_4$. In characteristic 2, the $D_4$-module with this high weight has dimension 294, so $Y$ has type $C_{147}$ or $D_{147}$. By the previous paragraph, we have $a_1 = 0$. But then $\lambda = \lambda_2$ has $(2^{147} \cdot 147!)/(2 \cdot 2^{145} \cdot 145!) = 2 \cdot 147 \cdot 146 = 42,924$ conjugates. So $\dim(V) \geq 42,924$. The $T_X$-high weights of $V$ are restricted as well and not symmetric with respect to $s$, so the possibilities for these are very limited. In fact, any $D_4$-module with such a high weight has dimension at most 840, but then $42,924 \leq \dim(V) = 3\dim(V_1) \leq 3 \cdot 840$ gives a contradiction. So we may assume $p \neq 2$.

Assume $\lambda = \lambda_2$; this implies $V \cong \Lambda^2 W$ ([7, II.2.15]). If $w \in W_\delta$, $w_1 \in W_{\delta - \beta_1}$, $w_2 \in W_{\delta - \beta_3}$, and $w_3 \in W_{\delta - \beta_4}$, then $w \wedge w_1$, $w \wedge w_2$, and $w \wedge w_3$ are $T_X$-high weight vectors in $\Lambda^2 W$, of weights $(2a - 2)\delta_1 + \delta_2 + 2a\delta_3 + 2a\delta_4$, $2a\delta_1 + \delta_2 + (2a - 2)\delta_3 + 2a\delta_4$, and $2a\delta_1 + \delta_2 + 2a\delta_3 + (2a - 2)\delta_4$. Since we are assuming there are only three such $T_X$-high weights, $V$ must be the sum (as $X$-modules) of the three $D_4$-modules with these high weights.

Now switch parabolics, and embed the parabolic subgroup $P_X$ of $X$ corresponding to $\{\beta_1, \beta_3, \beta_4\}$ in a parabolic subgroup $P_Y$ of $Y$ via the $Q_X$-level construction. Then the factor $L_1$ of $L'_Y$ corresponding to level 0 is the only simple factor to act nontrivially on $V/\left[V,Q_Y\right]$ and is of type $A_1$ with $l = (a + 1)^3 - 1$. Now $V/\left[V,Q_Y\right] \cong L'_Y \Lambda^2 \left(W/\left[W,Q_Y\right]\right) = \Lambda^2$ (the natural module for $L_1$), and $(V/\left[V,Q_Y\right])_X$ has the three
high weights \((2a - 2, 2a, 2a)\), \((2a, 2a - 2, 2a)\), and \((2a, 2a, 2a - 2)\). But this gives
\[
\binom{(a + 1)^3}{2} = \dim(\wedge^2(W/[W, Q_Y])) = 3 \dim(V_1/[V_1, Q_X]) = 3((2a - 1)(2a + 1)^2),
\]
and this equation in \(a\) has no solutions in the positive integers. So \(\lambda \neq \lambda_2\).

Finally, assume \(a = 1\) and \(\lambda = (p - 2)\lambda_1 + \lambda_2\). Then \(\lambda_1 \supseteq \lambda_1 = \delta_1 + \delta_2 + \delta_3\) and as above we may assume that \(\lambda_2 \supseteq \lambda_2 = (2\lambda_1 - \alpha_1) \supseteq \lambda_3 = 2(\delta_1 + \delta_2 + \delta_3) - \beta_i\) for some \(i = 1, 3, 4\). So \(\lambda_2 \supseteq \lambda_2 = 2\delta_2 + 2\delta_4, 2\delta_1 + 2\delta_2 + 2\delta_3,\) or \(2\delta_1 + \delta_2 + 2\delta_3\). In any case, \(\lambda \supseteq \lambda = ((p - 2)\lambda_1 + \lambda_2) \supseteq \lambda_3 = X_T\)-high weight of \(V\) which is not restricted, contrary to our assumption. This eliminates the final possibility for case 1.

2) If \(\delta = \delta_2\), then \(P_Y\) corresponds to the indicated nodes of the Dynkin diagram for \(Y\), and the possibilities are that there is a label of 1 on one of the nodes in an \(A_2\) factor of \(L'\) or a label of 2 on one of the nodes corresponding to an \(A_1\) factor. We have \(\dim(V^2(Q_Y)) \leq 12\) by Lemma 2.4, and the same sorts of arguments as above give contradictions to this bound for any of these possibilities except \(a_4 = 1\) and \(a_3 = 1\); even further, we get contradictions to the bound unless \(a_1\) and \(a_3\) are not both 0. So assume \(a_1\) and \(a_3\) are both 0. Then \(\lambda = a_1\lambda_1 + a_2\lambda_2 = 1\) and \(a_3 = 1\). Further, we get contradictions to the bound unless \(a_4\) is also 0.

a) Suppose \(\delta = \delta_2\) and \(\lambda = a_1\lambda_1 + a_2\lambda_2 + a_3\). Then \(\lambda_1 \supseteq \lambda_1 = (2\lambda_1 - \alpha_1) \supseteq \lambda_3 = (3\lambda_1 - 2\alpha_1 - \alpha_3) \supseteq \lambda_2 = \{2\delta_1 + 2\delta_3, 2\delta_1 + 2\delta_4, 2\delta_1 + 2\delta_4\}. So \(\lambda\) restricts to \(T_X\) as \(a_2\delta_1 + a_1\delta_2 + (a_2 + 2)\delta_3 + (a_2 + 2)\delta_4\) or one of its \(s\)-conjugates.

If we now embed the parabolic subgroup \(P_X\) of \(X\) corresponding to \(\{\beta_1, \beta_2, \beta_3, \beta_4\} \subseteq \Pi(X)\) in a parabolic subgroup \(P_Y = QY - L_Y\) of \(Y\), then the first nontrivial factor \(L_1\) of \(L_Y\) corresponds to \(\{a_2, a_3, \ldots, a_8\} \subseteq \Pi(Y)\) (since \(Q_X\)-level 1 has dimension 8). Here \(L_X'\) is a product of three groups of type \(A_1\), and \(L/\{V^1(Q_Y)\}L_X'\) is the sum of three simple \(L_X\)-modules, with high weights \((a_2 + 2, a_2 + 2, a_2 + 2, a_2 + 2, a_2 + 2, a_2 + 2, a_2 + 2, a_2 + 2)\). So \(\dim(V^1(Q_Y)) = 3(a_2 + 1)(a_2 + 3)^2\). On the other hand, \(V^1(Q_Y)\) is isomorphic as an \(L_1\)-module to the \(A_7\)-module with high weight \(\gamma = a_2\gamma_1 + \gamma_2\), where the \(\gamma_i\) are the fundamental dominant weights for \(A_7\). Assume that \(a_2 > 15\). Then \(\gamma\) has \((a_2 - 6)\gamma_1\) as a subdominant weight, and the \(A_7\)-module with high weight \((a_2 - 6)\gamma_1\) has \((a_2 + 1)\) nonzero weights by Lemma 2.5. So \(\dim(V^1(Q_Y)) \geq (a_2 + 1)(a_2 + 3)^2\) for \(a_2 > 15\), which is a contradiction.

If \(a_2 \leq 15\), then we can check each case individually and we find more than \(3(a_2 + 1)(a_2 + 3)^2\) weights in \(M\), arriving at the same contradiction. So (a) is ruled out.

b) Suppose \(\delta = \delta_2\) and \(\lambda = \lambda_4\). We have \(\lambda_1 \supseteq \lambda_1 = (4\lambda_1 - 3\alpha_1 - 2\alpha_2 - \alpha_3) \supseteq \lambda_1 = \{4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2, 4\delta_2 - 3\delta_2 - \beta_1 - \beta_2\}\) as above. When we embed the parabolic subgroup \(P_X\) of \(X\) as above, we now have \(\dim(V^1(Q_Y) L_X') = 3(2 \cdot 2 \cdot 4) = 48\) on the one hand and \(\dim(V^1(Q_Y)) = \binom{8}{3} = 56\) on the other (since \(V^1(Q_Y)\) is isomorphic to the \(A_7\)-module with high weight \(\gamma_3\)). Again we have a contradiction, and (b) is eliminated.

So there are no examples with \(G = D_4(s)\).

6.2. \(G = X(s, t)\). Assume \(X = D_4\) with \(G = X(s, t)\). Lemma 6.2 tells us that we must have a \(U_X\)-level in \(W\) of dimension 6 or less. Lemma 6.3 shows that there are few \(T_X\)-high weights \(\delta\) of \(W\) which allow such a level (other than level 1). We must consider level 1 for all \(\delta\), level 2 for \(\delta = b_0\delta_2\) or \(\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4\), and the
three possible \( \delta \) not covered by the argument in the proof of that Lemma (\( \delta_2, 2\delta_2, \)
and \( \delta_1 + \delta_3 + \delta_4 \)).

As above, by Lemma 2.4 we have here \( \dim(V^2(Q_Y)) \leq 4 \dim(V^1(Q_Y)) = 24 \). This rules out most of the remaining possibilities.

Assume \( \delta = \delta_1 + \delta_3 + \delta_4 \). Then we check the dimensions of the levels directly. With \( W_i \) denoting \( U_X \)-level \( i \) of \( W \), we have \( \dim(W_0) = 1, \dim(W_1) = 3, \dim(W_2) = 6, \)
and \( \dim(W_i) \geq 7 \) for \( i \geq 3 \). So \( P_Y \) corresponds to the indicated nodes in the Dynkin diagram for \( Y \) in this picture:

\[
\begin{array}{cccccccc}
  a_2 & a_3 & a_5 & a_9 & \cdots \\
\end{array}
\]

Lemma 6.2 tells us that \( \dim(V^1(Q_Y)) = 6 \); the ways in which this can happen given the dimensions of \( U_X \)-levels above are a) \( a_2 = 2 \) and \( a_i = 0 \) for \( i \neq 2, \alpha_i \in \Pi(Y) - \Pi(L'_Y) \); b) \( a_3 = 2 \) and \( a_i = 0 \) for \( i \neq 3, \alpha_i \in \Pi(Y) - \Pi(L'_Y) \); c) \( a_5 = 1 \) and \( a_i = 0 \) for \( i \neq 5, \alpha_i \in \Pi(Y) - \Pi(L'_Y) \); and d) \( a_9 = 1 \) and \( a_i = 0 \) for \( i \neq 9, \alpha_i \in \Pi(Y) - \Pi(L'_Y) \). We consider each possibility in turn (the arguments below must be modified slightly in some small characteristics).

a) If \( a_2 = 2 \), then either \( \lambda - \alpha_1 \) (if \( a_1 \neq 0 \)) or \( \lambda - \alpha_2 - \alpha_3 \) (if \( a_1 = 0 \)) is a high weight in \( V^2(Q_Y) \), giving a composition factor there of dimension at least 7, and either \( \lambda - \alpha_4 \) (if \( a_4 \neq 0 \)) or \( \lambda - \alpha_2 - \alpha_3 - \alpha_4 \) (if \( a_4 = 0 \)) is another high weight, giving a factor of dimension at least 18. This contradicts \( \dim(V^2(Q_Y)) \leq 24 \). So this case does not arise.

b) If \( a_3 = 2 \), then \( \lambda - \alpha_4 \) (if \( a_4 \neq 0 \)) or \( \lambda - \alpha_3 - \alpha_4 \) gives a composition factor in \( V^2(Q_Y) \) of dimension at least 42; this again is a contradiction.

c) If \( a_5 = 1 \), then either \( \lambda - \alpha_4 \) (if \( a_4 \neq 0 \)) or \( \lambda - \alpha_4 - \alpha_5 \) (if \( a_4 = 0 \)) is a high weight in \( V^2(Q_Y) \), giving \( \dim(V^2(Q_Y)) \geq 45 \). This again contradicts \( \dim(V^2(Q_Y)) \leq 24 \).

d) Similarly, here we obtain a contradiction since either \( \lambda - \alpha_1 \) or \( \lambda - \alpha_9 - \alpha_{10} \)
is a high weight in \( V^2(Q_Y) \), giving \( \dim(V^2(Q_Y)) \geq 105 \).

So the embedding \( \delta = \delta_1 + \delta_3 + \delta_4 \) gives no examples.

Almost identical arguments rule out all possibilities for \( \delta = 2\delta_2 \), so this embedding gives no examples.

For \( \delta = \delta_2 \), there are many more possibilities for the labelling of \( \lambda \) on \( \Pi(L'_Y) \). Here the picture of \( P_Y \) is

\[
\begin{array}{cccccccc}
  a_2 & a_3 & a_5 & a_9 & \cdots \\
\end{array}
\]

for \( p \neq 2, \) and

\[
\begin{array}{cccccccc}
  a_2 & a_3 & a_5 & a_9 & \cdots \\
\end{array}
\]

for \( p = 2 \) (in [9] it is calculated that this embedding is actually into \( D_{13}, \) not \( C_{13} \)). Once again, however, arguments like those above give contradictions to \( \dim(V^2(Q_Y)) \leq 24 \) for all possible \( \lambda \) except perhaps \( \lambda = \alpha_1 \lambda_1 + \alpha_2 \lambda_2 + 2\lambda_3, \) with
\(a_1 a_2 = 0\). So we may assume \(\lambda\) has this form and \(p \neq 2\) since \(\lambda\) has \(\lambda_3\)-coefficient 2.

If \(a_1 = 0 \neq a_2\) and \(\lambda = a_2 \lambda_2 + 2 \lambda_3\), then we use the \(s\)-stable minimal parabolic subgroup \(P_X\) of \(X\) corresponding to \(\{ \beta_2 \} \subseteq \Pi(X)\). We embed \(P_X\) in a parabolic subgroup \(P_Y\) of \(Y\) corresponding to the indicated nodes in the picture

Now by Lemma 2.4, we have \(\dim(V^2(Q_Y)) \leq 6 \dim(V^1(Q_Y)) = 36\). But \(\lambda - \alpha_2\) and \(\lambda - \alpha_3 - \alpha_4 - \alpha_5\) are high weights in \(V^2(Q_Y)\), giving composition factors of dimension 20 (if \(p \neq 3\)) and 18, respectively; this is a contradiction (if \(p = 3\), there is another composition factor of high weight \(\lambda - \alpha_2 - \alpha_3\) which we must include to obtain the contradiction).

### Table 1. Examples arising from the connected case.

| No. | \(X\) | \(Y\) | \(W|_X\) | \(V|_X\) | \(V|_Y\) | char\((K)\) |
|-----|-------|-------|--------|--------|--------|----------|
| I_4 | \(D_n\) | \(A_{2n-1}\) | \(\delta_1\) | \(n \geq 4\) | \(n - 1 > k \geq 2\) | \(p \neq 2\) |
| I_5 | \(D_n\) | \(A_{2n-1}\) | \(\delta_1\) | \(n \geq 4\) | \(1\) | \(p \neq 2\) |
| I_6 | \(A_3\) | \(A_5\) | \(\delta_2\) | \(1\) | \(1\) | \(p \neq 2\) |
| II_1 | \(A_5\) | \(C_{10}\) | \(\delta_3\) | \(1\) | \(1\) | \(p \neq 2\) |
| S_1 | \(A_2\) | \(B_3\) | \(\delta_1 + \delta_2\) | \(2\) | \(p = 3\) |
| S_7 | \(A_3\) | \(D_7\) | \(\delta_1 + \delta_3\) | \(1\) | \(1\) | \(p = 2\) |
| S_8 | \(D_4\) | \(D_{13}\) | \(\delta_2\) | \(s, t \in \bar{Y}\) | \(1\) | \(p = 2\) |
| MR_4 | \(D_n\) | \(C_n\) | \(\delta_1\) | \(c_1\) | \(c_2\) | \(p = 2\) |
Table 2. New Examples.

| No. | X     | Y     | $W|_X$ | $V_1|_X$ | $V|_Y$ | char($K$) |
|-----|-------|-------|-------|---------|--------|-----------|
| U₁  | $A_{n-1}$ | $A_n$  | usual | ![Diagram](chart1) | ![Diagram](chart2) | any |
|     | (n odd) |        |       | $a_{1} \alpha_{n-2}$ | $a_{1} a_{n-1}$ | below |
| U₂  | $D_n$  | $B_n$  | usual | ![Diagram](chart3) | ![Diagram](chart4) | any |
|     |        |        |       | $a_{i} a_{j}$ | $a_{i}$ | any |
|     |        |        |       | ![Diagram](chart5) | ![Diagram](chart6) | ![Diagram](chart7) | $p \neq 2, 3, 5, 7$ |
|     |        |        |       | ![Diagram](chart8) | ![Diagram](chart9) | ![Diagram](chart10) | $p = 2$ |
| U₇  | $A_2$  | $A_5$  | $2\delta_1$ | ![Diagram](chart11) | ![Diagram](chart12) | ![Diagram](chart13) | $p \neq 2, 3$ |
| U₈  | $A_3$  | $A_9$  | $2\delta_1$ | ![Diagram](chart14) | ![Diagram](chart15) | ![Diagram](chart16) | $p \neq 2, 5$ |
| U₉  | $A_4$  | $A_9$  | $\delta_2$ | ![Diagram](chart17) | ![Diagram](chart18) | ![Diagram](chart19) | $p \neq 2, 5$ |

If $a_2 = 0$ and $a_1 \neq 0$ (so $\lambda = a_1 \lambda_1 + 2\lambda_3$), then we can examine the embedding of the $s$-stable parabolic subgroup of $X$ corresponding to $\{\beta_1, \beta_3, \beta_4\} \subseteq \Pi(L'_Y)$ in a parabolic subgroup $P_Y = Q_YL_Y$ of $Y$. Now the first nontrivial factor $L_1$ of $L'_Y$ corresponds to $\{\alpha_2, \ldots, \alpha_8\}$ (as $Q_X$-level 1 has dimension 8), and the second factor $L_2$ corresponds to $\{\alpha_{10}, \ldots, \alpha_{14}\}$. The first, $L_1$, is the only one to act nontrivially on $V^1(Q_Y)$, which is isomorphic to the $A_7$-module with high weight $2\gamma_2$ (where $\gamma_i$ are the fundamental dominant weights of $A_7$). We find (using the Weyl character formula and the Andersen-Jantzen sum formula) that $\dim(V^1(Q_Y)) = 336$ if $p \neq 3$. 

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and \( \dim(V^1(Q_Y)) = 266 \) if \( p = 3 \). By Lemma 2.4, we have \( \dim(V^2(Q_Y)) \leq 8 \dim(V^1(Q_Y)) \). With \( \lambda \) as above, the two high weights \( \lambda - \alpha_1 \) and \( \lambda - \alpha_3 - \cdots - \alpha_9 \) in \( V^2(Q_Y) \) give, for \( p \neq 3 \), two composition factors, each of dimension 1680; for \( p = 3 \), these two composition factors have dimension 1624 and 1120, respectively. This contradicts \( \dim(V^2(Q_Y)) \leq 8 \dim(V^1(Q_Y)) \).

The remaining possibility is \( \lambda = 2\lambda_3 \). We know that \( \lambda_1|_{T_X} = \delta_2, \alpha_1|_{T_X} = \beta_2 \) by [9, 3.4(i)], and \( \alpha_2|_{T_X} = \beta_1 \) for \( i = 1, 3 \) or 4, by the same arguments used in the proof of [9, 3.4]. Since \( \lambda_3 = 3\lambda_1 - 2\alpha_1 - \alpha_2 \), this gives \( \lambda_3|_{T_X} \) of the form \( 2\delta_i + 2\delta_j \) for \( (i, j) = (1, 3), (1, 4), \) or \( (3, 4) \). But we have noted that the \( \delta_1, \delta_3 \), and \( \delta_4 \)-coefficients of \( \lambda|_{T_X} \) must be distinct in this case \( G = D_4(s,t) \), since \( V \) is irreducible as a \( G \)-module. So the case \( \delta = \delta_2 \) does not occur.

Remaining to be considered are level 1 for all \( \delta \), and level 2 for \( \delta = \delta_1 + \delta_2 + \delta_3 + \delta_4 \) or \( \delta = b\delta_2 \) (with \( b > 1 \), as the case \( \delta = \delta_2 \) was examined thoroughly above).

First we consider level 2. If \( \delta = \delta_1 + \delta_2 + \delta_3 + \delta_4 \), then \( W_2 \) in fact has dimension 9 unless \( p = 3 \), in which case it has dimension 6 and we have the possibilities \( a_6 \neq 0 \) and \( a_{10} \neq 0 \). But these cases are easily ruled out by arguments based on \( \dim(V^2(Q_Y)) \leq 24 \), as in several cases above. If \( \delta = b\delta_2 \), then \( \dim(W_2) = 4 \) and we have the possibility \( a_4 \neq 0 \); this is again ruled out by the same sorts of arguments.

Finally, assume \( \lambda \) has a nonzero coefficient for a fundamental weight corresponding to some \( \alpha_i \) in \( \Pi(L_1) \), where \( L_1 \) is the simple factor of \( L_Y \) corresponding to the \( U_X \)-level 1 of \( W \) (in other words, we are in the remaining level 1 case). Then if \( a \neq 0 \neq b \), \( \dim(W_1) = 4 \) and we have the possibility \( a_3 = 1 \). If \( a \neq 0 = b \), then \( \dim(W_1) = 3 \) and we have the possibilities \( a_2 = 2 \) and \( a_3 = 2 \). These all fail quickly to arguments based on the fact that \( \dim(V^2(Q_Y)) \leq 24 \) as above. If \( b \neq 0 = a \), then \( \dim(W_1) = 1 \) and \( L_1 \) is trivial, so we do not have a possibility. This rules out level 1.

We have eliminated all possible high weights \( \delta \) for \( W \). This completes the proof of Theorem 6.1, which in turn completes the proof of Theorem 1.

References


Department of Mathematics, University of Washington, Box 354350, Seattle, Washington 98195-4350

Current address: Department of Mathematics, Sonoma State University, Rohnert Park, California 94928

E-mail address: ben.ford@sonoma.edu