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THE C^1 CLOSING LEMMA FOR NONSINGULAR ENDOMORPHISMS EQUIVARIANT UNDER FREE ACTIONS OF FINITE GROUPS

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Abstract. In this paper a closing lemma for C^1 nonsingular endomorphisms equivariant under free actions of finite-groups is proved. Hence a recurrent trajectory, as well as all of its symmetric conjugates, of a C^1 nonsingular endomorphism equivariant under a free action of a finite group can be closed up simultaneously by an arbitrarily small C^1 equivariant perturbation.

1. Introduction

A closing problem in dynamical system theory concerns whether or not a recurrent trajectory Γ of a dynamical system can be closed up by a nearby system. The affirmative answer of this problem, the so-called closing lemma, has its significance in the study of structural stability of dynamical systems. As defined by Smale, an axiom A system satisfies (a): nonwandering set is hyperbolic and (b): periodic points are dense in the nonwandering set. The famous structural stability conjecture states that C^1 structural stable dynamical systems are of axiom A. This conjecture has been proved both for diffeomorphisms and flows [5, 11]. In both of the proofs the closing lemma plays an important role.

The solution of the closing problem depends on the category of the concerned dynamical systems. Hence there are various versions of the closing lemma for different kinds of systems. The closing lemma was first announced by Pugh for Poisson stable points ([6]) and for nonwandering points ([7]) of C^1 flows with the additional assumption that the phase space is compact. The latter result was reproved by Liao [2] who made no assumption on the compactness of phase space. (See also [3] and [4] for simpler proofs.) Pugh and Robinson showed that the C^1 closing lemma is true in the categories of vector fields, flows, and diffeomorphisms [8]. Wen proved a C^1 closing lemma for endomorphisms which have at most finitely many singularities [9, 10].

Attention is also paid to the case that the phase space has some special structure, for example, a symplectic structure or a volume structure etc. It is interesting to investigate systems which preserve these structures. Pugh and Robinson showed that a C^1 closing lemma holds for symplectic diffeomorphisms, volume-preserving diffeomorphisms, and Hamiltonian vector fields [8]. In this paper we investigate systems

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with symmetry and show that the C^1 closing lemma is valid for nonsingular endomorphisms which commute free actions of finite groups on compact boundaryless manifolds. Hence a recurrent trajectory as well as all of its symmetric conjugates of an equivariant C^1 nonsingular endomorphism can be closed up simultaneously by an arbitrarily small C^1 equivariant perturbation under the assumption that the manifold is compact and boundaryless and the group action on the manifold is free.

Let M be a compact smooth Riemannian manifold without boundary. Let f be any map from M into itself. For any positive integer n we denote by f^n the nth iteration of f and for any subset $S \subset M$ we denote by $f^{-n}(S)$ the preimage of S under f^n. A point $p \in M$ is called a *periodic point* of f if $f^n(p) = p$ for some positive integer n. The set of all the periodic points of a map $f\colon M \to M$ is denoted as $\mathrm{Per}(f)$. A point $\sigma \in M$ is called a *nonwandering point* of f if for any neighborhood U of σ and any integer N there is an integer $n > N$ such that $f^n(U) \cap U \neq \varnothing$. The set of all nonwandering points of f is denoted as $\Omega(f)$.

Let G be a finite group acting on M. The G-action is *free* if for every $x \in M$ the isotropic subgroup

$$G_x := \{\gamma \in G | \gamma \cdot x = x\}$$

contains the identity only. Let G be a Lie group acting on manifolds M and N. Then a map $f\colon M \to N$ is said to be *G-equivariant* if

$$f(\gamma \cdot x) = \gamma \cdot f(x), \quad \forall x \in M \text{ and } \gamma \in G.$$

A C^1 map $f\colon M \to M$ is a *nonsingular endomorphism* if the tangent map $T_x f$ is injective at all $x \in M$. We denote by $\mathrm{NEnd}^1_G(M)$ the set of G-equivariant C^1 nonsingular endomorphisms endowed with the C^1 topology.

Our main result of this paper is

Theorem 1.1. *Let M be a compact smooth manifold without boundary and G be a finite group acting freely on M. Let $f\colon M \to M$ be a G-equivariant nonsingular C^1 map and $\sigma \in M$ be a nonwandering point of f. Then for any C^1 neighborhood $\mathcal{U}$ of f in $\mathrm{NEnd}^1_G(M)$ there exists a map $g \in \mathcal{U}$ such that $\sigma \in \mathrm{Per}(g)$.*

In Section 2 we summarize some facts on group actions. In Section 3 we recall some techniques used in the proof of closing lemma for endomorphisms ([9]) and establish their equivariant versions. In Section 4 we give a proof of Theorem 1.1.

2. Preliminaries on Lie group actions

In this section we recall some concepts and results about Lie group actions on manifolds.

Let M be a smooth manifold and G a Lie group acting on M. Denote by $\alpha\colon G \to \mathrm{Diff}^\infty(M)$ the G-action and $\gamma \cdot x := \alpha(\gamma)(x)$ for $\gamma \in G$ and $x \in M$. Then G acts on the tangent bundle TM as

$$\gamma \cdot v := D\alpha(\gamma)(x) \cdot v, \qquad \gamma \in G, v \in T_x M, x \in M. \tag{2.1}$$

The following theorem can be found in [1].

Theorem 2.1. *Let M a smooth manifold and G be a compact Lie group acting on M. Then there is a Riemannian metric $\langle \cdot, \cdot \rangle$ on M such that G acts on M as isometries, i.e.,*

$$\langle \gamma \cdot v, \gamma \cdot w \rangle_{\gamma \cdot p} = \langle v, w \rangle_p, \quad \forall \gamma \in G, \forall p \in M, \forall v, w \in T_p M. \quad \square$$

Remark 2.2. Let $d\colon M \times M \to \mathbb{R}$ be the distance function on M which is induced by a G-invariant Riemannian metric. Then

$$d(\gamma \cdot x, \gamma \cdot y) = d(x, y), \quad \forall \gamma \in G, \forall x, y \in M.$$

The exponential map exp (w.r.t. a Riemannian metric) is defined on some neighborhood $W \subset TM$ of the 0-section such that for any tangent vector $v \in W$, $\exp v = c_v(1)$, where $c_v\colon I \to M$ is the unique geodesic with $c_v(0) = \pi(v)$ and $c_v'(0) = v$, $I = [0,1] \subset \mathbb{R}$, $\pi\colon TM \to M$ is the bundle projection. If G acts isometrically on M then the definition domain W of the exponential map is G-invariant and $\exp\colon W \to M$ is G-equivariant.

Lemma 2.3. *Let M be a compact smooth Riemannian manifold without boundary and G a finite group acting isometrically and freely on M. Then there is a number $\zeta > 0$ such that for every G-orbit A, the exponential map*

$$\exp\colon \{v \in T_A M\colon |v| \le \zeta\} \to M \tag{2.2}$$

is an equivariant embedding, where $T_A M$ is the restriction of the tangent bundle TM to A.

Proof. It is well known that there is a number $\zeta_1 > 0$ such that for all $p \in M$, $\exp_p\colon \{v \in T_p M\colon |v| \le \zeta_1\} \to M$ is an embedding because of the compactness of M. Define

$$D(x) := \min\{d(x, \gamma \cdot x) | \gamma \in G - \{1\}\}.$$

For any $x, y \in M$ we have

$$\begin{aligned}
-2d(x,y) &= -d(x,y) - d(\gamma \cdot x, \gamma \cdot y) \\
&\le d(y, \gamma \cdot y) - d(x, \gamma \cdot x) \\
&\le d(x,y) + d(\gamma \cdot x, \gamma \cdot y) = 2d(x,y).
\end{aligned}$$

Hence

$$\begin{aligned}
d(y, \gamma \cdot y) &\ge d(x, \gamma \cdot x) - 2d(x,y) \ge D(x) - 2d(x,y), \\
d(x, \gamma \cdot x) &\ge d(y, \gamma \cdot y) - 2d(x,y) \ge D(y) - 2d(x,y)
\end{aligned}$$

for all $\gamma \in G - \{1\}$. Therefore

$$-2d(x,y) \le D(y) - D(x) \le 2d(x,y),$$

i.e., D is uniformly continuous on M.

Note that $D(x) > 0$ for all $x \in M$ because G is finite and the G-action is free. By the compactness of M there is an $\zeta_2 > 0$ such that $D(x) > \zeta_2$ for all $x \in M$. Take $\zeta = \min\{\zeta_1, \zeta_2/2\}$. Then for any G-orbit A, exp maps $\{v \in T_A M\colon |v| \le \zeta\}$ diffeomorphically onto a union of disjoint balls in M whose centers lie on A. □

Throughout the rest of this paper we assume ζ is defined so as in Lemma 2.3.

3. Lifts, local linearizations and ε-kernel avoiding transitions

In this section we review three basic techniques used in [9, 10]: lift, local linearization, and ε-kernel avoiding transition. We establish the equivariant versions of the first two.

Denote by $\Gamma_G(A)$ the Banach space of all G-equivariant sections $v\colon A \to TM$ with the norm defined by

$$|v| := \sup_{x \in A} |v(x)|, \quad v \in \Gamma_G(A).$$

We note that $|v(x)|$ is in fact independent of $x \in A$ due to the symmetry.

Lemma 3.1 (equivariant ε-kernel lift). *For any $\eta > 0$ and any $f \in \mathrm{NEnd}^1_G(M)$, there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any G-orbit A, and any two G-equivariant sections $v_1, v_2 \in \Gamma_G(A)$ with $B(v_2, |v_1 - v_2|/\varepsilon) \subset \{v \in \Gamma_G(A) | \, |v| \le \zeta\}$, there is a G-equivariant diffeomorphism $h = h_{\varepsilon,A,v_1,v_2} \colon M \to M$, called a G-equivariant ε-kernel lift, such that:*

(a) *$h(\exp_p v_2(p)) = \exp_p v_1(p)$, for all $p \in A$;*

(b) *$\mathrm{supp}(h) \subset \bigcup_{p \in A} \exp_p B(v_2(p), |v_1 - v_2|/\varepsilon)$, here the support means the closure of the set where h differs from the identity;*

(c) *$d_1(hf_1, f_1) < \eta$ and $d_1(h^{-1} f_1 h, f_1) < \eta$ for any $f_1 \in \mathrm{NEnd}^1_G(M)$ with $d_1(f_1, f) \le 1$, where d_1 is the C^1 distance defined in $\mathrm{NEnd}^1_G(M)$.*

Proof. Let η be any positive number and f be any map in $\mathrm{NEnd}^1_G(M)$. Take an $\varepsilon_0 \in (0, 1/4)$ (determined later). Given any $\varepsilon \in (0, \varepsilon_0)$, any G-orbit A, and any sections $v_1, v_2 \in \Gamma_G(A)$ with $B(v_2, |v_1 - v_2|/\varepsilon) \subset \{v \in \Gamma_G(A) | |v| \le \zeta\}$. We assume that $v_1 \ne v_2$ since otherwise we can take h as the identity on M. Let $\alpha \colon \mathbb{R} \to [0,1]$ be a C^∞ bump-function such that

$$\alpha(s) = \begin{cases} 1, & \text{if } |s| \le 1/3; \\ 0, & \text{if } |s| \ge 1 \end{cases} \tag{3.1}$$

and $|\alpha'(s)| \le 2$ for all $s \in \mathbb{R}$. Define

$$H = H_{\varepsilon,A,v_1,v_2} \colon \Gamma_G(A) \to \Gamma_G(A), \quad v \mapsto v + \alpha(\varepsilon|v - v_2|/|v_1 - v_2|)(v_1 - v_2),$$

and

$$K \colon \Gamma_G(A) \times \Gamma_G(A) \to \Gamma_G(A), \ (w, v) \mapsto w - \alpha(\varepsilon|v - v_2|/|v_1 - v_2|)(v_1 - v_2).$$

Note that both H and K are of C^∞. For any two sections $u, v \in \Gamma_G(A)$ we have

$$|\alpha(\varepsilon|u - v_2|/|v_1 - v_2|)(v_1 - v_2) - \alpha(\varepsilon|v - v_2|/|v_1 - v_2|)(v_1 - v_2)| \le 2\varepsilon|u - v|. \tag{3.2}$$

This shows that $K(w, \cdot)$ is a contraction uniformly in w. Therefore H is C^∞ diffeomorphism and $H^{-1}(w)$ is the fixed point of $K(w, \cdot)$. For any $v, w \in \Gamma_G(A)$ we have

$$\begin{aligned}
|H(v) - v| &\le |v_1 - v_2| \le \zeta\varepsilon, \\
|DH(v)w - w| &= \begin{cases} 0, & \text{if } \frac{|v - v_2|}{|v_1 - v_2|} \le \frac{1}{4\varepsilon}, \\ \left| \frac{\varepsilon\alpha'(\varepsilon|v - v_2|/|v_1 - v_2|)}{|v - v_2|} \langle v - v_2, w \rangle \right|, & \text{otherwise}, \end{cases} \\
&\le 2\varepsilon|w|, \\
|H^{-1}(v) - v| &= |\alpha(\varepsilon|H^{-1}(v) - v_2|/|v_1 - v_2|)(v_1 - v_2)| \le |v_1 - v_2| \le \zeta\varepsilon, \\
|DH^{-1}(v)w - w| &= \begin{cases} 0, & \text{if } \frac{|u - v_2|}{|v_1 - v_2|} \le \frac{1}{4\varepsilon}, \\ \left| \frac{\varepsilon\alpha'(\varepsilon|u - v_2|/|v_1 - v_2|)}{|u - v_2|} \langle u - v_2, DH^{-1}(v)w \rangle \right|, & \text{otherwise}, \end{cases} \\
&\le 2\varepsilon|DH^{-1}(v)w| \le \frac{2\varepsilon}{1 - 2\varepsilon}|w| < 4\varepsilon|w|, \quad \text{where } u = H^{-1}(v).
\end{aligned}$$

For each $v_0 \in T_A M$ there is a unique section $v \in \Gamma_G(A)$, denoted as v_0^*, such that $v_0^*(\pi(v_0)) = v_0$. Here $\pi \colon TM \to M$ is the bundle projection. Define

$$E \colon T_A M \to A \times \Gamma_G(A), \quad E(v_0) = (\pi(v_0), v_0^*).$$

Then E is a vector bundle isomorphism. Let $U = \exp\{v \in T_A M \colon |v| \leq \zeta\}$. Then U is a G-invariant neighborhood of A. Define $h_{\varepsilon,A,v_1,v_2} \colon M \to M$ such that $h_{\varepsilon,A,v_1,v_2} =$ identity off U and the following diagram commutes (whenever the diagram makes sense):

(3.3)
$$\begin{array}{ccc} A \times \Gamma_G(A) & \xrightarrow{\mathrm{id}_A \times H_{\varepsilon,A,v_1,v_2}} & A \times \Gamma_G(A) \\ \downarrow{\scriptstyle E^{-1}} & & \downarrow{\scriptstyle E^{-1}} \\ T_A M & \xrightarrow{\widetilde{H}_{\varepsilon,A,v_1,v_2}} & T_A M \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ M & \xrightarrow{h_{\varepsilon,A,v_1,v_2}} & M \end{array}$$

It is easy to see that

(1) $h_{\varepsilon,A,v_1,v_2}(\exp v_2(x)) = \exp v_1(x)$, for all $x \in A$;

(2) $\operatorname{supp}(h_{\varepsilon,A,v_1,v_2}) \subset \bigcup_{x\in A} \exp_x B(v_2(x), |v_1 - v_2|/\varepsilon)$.

Note that $\widetilde{H}(v) = H(v^*)(\pi(v))$, $v \in T_A M$. For any $\gamma \in G$ and any $v, w \in T_A M$

$$\begin{aligned} \widetilde{H}_{\varepsilon,A,v_1,v_2}(\gamma \cdot v) &= H_{\varepsilon,A,v_1,v_2}((\gamma \cdot v)^*)(\pi(\gamma \cdot v)) \\ &= H_{\varepsilon,A,v_1,v_2}(v^*)(\gamma \cdot \pi(v)) \\ &= \gamma \cdot H_{\varepsilon,A,v_1,v_2}(v^*)(\pi(v)) \\ &= \gamma \cdot \widetilde{H}_{\varepsilon,A,v_1,v_2}(v), \\ |\widetilde{H}_{\varepsilon,A,v_1,v_2}(v) - v| &= |H_{\varepsilon,A,v_1,v_2}(v^*)(\pi(v)) - v^*(\pi(v))| \\ &\leq |\alpha(\varepsilon|v^* - v_2|/|v_1 - v_2|)(v_1 - v_2)| \\ &\leq |v_1 - v_2| \leq \varepsilon\zeta, \\ |D\widetilde{H}_{\varepsilon,A,v_1,v_2}(v)(w) - w| &\leq \varepsilon|\alpha'(\varepsilon|v^* - v_2|/|v_1 - v_2|)|\,|w| \leq 2\varepsilon|w|, \end{aligned}$$

i.e., $\widetilde{H}_{\varepsilon,A,v_1,v_2}$ is G-equivariant and $|\widetilde{H}_{\varepsilon,A,v_1,v_2} - \mathrm{id}\,|_{C^1} \leq \max\{2,\zeta\}\varepsilon$. Similarly $\widetilde{H}^{-1}_{\varepsilon,A,v_1,v_2}$ is G-equivariant and $|\widetilde{H}^{-1}_{\varepsilon,A,v_1,v_2} - \mathrm{id}\,|_{C^1} \leq \max\{4,\zeta\}\varepsilon$. Since exp is G-equivariant and $D\exp 0 = \mathrm{id}_{TM}$ the diffeomorphism $h_{\varepsilon,A,v_1,v_2}$ is G-equivariant and $d_1(h_{\varepsilon,A,v_1,v_2}, \mathrm{id}) \leq M\varepsilon$ and $d_1(h^{-1}_{\varepsilon,A,v_1,v_2}, \mathrm{id}) \leq M\varepsilon$ for some constant $M > 0$ which is independent of ε, A, v_1, v_2. Hence there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any G-orbit A, any two G-equivariant sections $v_1, v_2 \in \Gamma_G(A)$ with $B(v_2, |v_1 - v_1|/|\varepsilon) \subset \{v \in \Gamma_G(A) \,|\, |v| \leq \zeta\}$, and any $f_1 \in \mathrm{NEnd}^1_G(M)$ with $d_1(f_1, f) \leq 1$, we have

$$d_1(h_{\varepsilon,A,v_1,v_2}, f_1, f_1) < \eta$$

and

$$d_1(h^{-1}_{\varepsilon,A,v_1,v_2} f_1 h_{\varepsilon,A,v_1,v_2}, f_1) < \eta. \quad \Box$$

Let A be a G-orbit and $\mu \geq 1$ be an integer. A neighborhood W of A is called a *G-invariant μ-dynamical neighborhood* of A if W is G-invariant and each component of $\bigcup_{n=0}^{\mu} f^{-n}(W)$ is a neighborhood of a unique point $q \in \bigcup_{n=0}^{\mu} f^{-n}(A)$, denoted as $W_f(q)$ and called the W-component at q, and if f^n maps $W_f(q)$ onto $W_f(p)$ whenever $f^n(q) = p \in A$, $n = 1, 2, \ldots, \mu$. When f is clear we often omit it and write $W(q) := W_f(q)$. For the trivial group $G = 1$, G-invariant dynamical neighborhoods are exactly dynamical neighborhoods defined in [9].

Lemma 3.2 (equivariant local linearization)**.** *Let $f \in \mathrm{NEnd}^1_G(M)$, A be a G-orbit and $\mu \geq 1$ be an integer such that $f^{-n}(A) \cap f^{-m}(A) = \varnothing$ whenever $0 \leq n \neq m \leq \mu + 1$. Then for any $\eta > 0$ and any $\lambda_0 > 0$ there is a λ with $0 < \lambda < \lambda_0$ and an $f_1 \in \mathrm{NEnd}^1_G(M)$, called a G-equivariant and local linearization of f, with the following properties* (a)-(e)*.*

Write $W = \exp\{v \in T_A M||v| \leq \lambda\}$ and $V = \exp\{v \in T_A M||v| \leq \lambda/4\}$.

(a) *$d_1(f_1, f) < \eta$ and for $1 \leq n \leq \mu+1$, $f^{-n}(A) = f_1^{-n}(A)$;*

(b) *W and V are G-invariant $(\mu+1)$-dynamical neighborhoods both for f and f_1, and $W_f(q) = W_{f_1}(q)$ for each $q \in \bigcup_{n=1}^{\mu+1} f^{-n}(A)$;*

(c) *$f_1 = \exp_{f(q)} \circ (T_q f) \circ \exp_q^{-1}$ on $V_{f_1}(q)$ for each $q \in \bigcup_{n=1}^{\mu} f^{-n}(A)$;*

(d) *$f_1^{\mu+1} = f^{\mu+1}$ on $W_f(q)$ for each $q \in f^{-\mu-1}(A)$. In particular, $f_1 = \exp_{f(q)} \circ (T_{f(q)} f^{\mu})^{-1} \circ \exp_p^{-1} \circ f^{\mu+1}$ on $V(q)$ ($V_{f_1}(q) = V_f(q)$ here) for each $q \in f^{-\mu-1}(A)$;*

(e) *$f_1 = f$ on $M - \bigcup\{W(q)|q \in \bigcup_{n=1}^{\mu+1} f^{-n}(A)\}$.*

Proof. Let p be any point in A. Since $f^{-n}(A) \cap f^{-m}(A) = \varnothing$ for $0 \leq n \neq m \leq \mu+1$, there exists a small $\zeta_1 > 0$ such that $U = \exp\{v \in T_A M||v| < \zeta_1\}$ is a G-invariant $(\mu+1)$-dynamical neighborhood for f. Given any $\lambda_0 > 0$. By the construction presented in [9], there is a $0 < \lambda < \min\{\zeta_1, \lambda_0\}$ together with a nonsingular C^1 map $\tilde{f}_1 \colon M \to M$ such that for

$$W = \exp\{v \in T_A M||v| \leq \lambda\} \quad \text{and} \quad V = \exp\{v \in T_A M||v| \leq \lambda/4\}:$$

(a′) $d_1(\tilde{f}_1, f) < \eta$ and for $1 \leq n \leq \mu+1$, $f^{-n}\{p\} = \tilde{f}_1^{-n}\{p\}$;

(b′) $W(p)$ and $V(p)$ are $(\mu+1)$-dynamical neighborhoods both for f and $\tilde{f}_1$, and $W_f(q) = W_{\tilde{f}_1}(q)$ for each $q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{p\}$;

(c′) $\tilde{f}_1 = \exp_{f(q)} \circ (T_q f) \circ \exp_q^{-1}$ on $V_{\tilde{f}_1}(q)$ for each $q \in \bigcup_{n=1}^{\mu} f^{-n}\{p\}$;

(d′) $\tilde{f}_1^{\mu+1} = f^{\mu+1}$ on $W_f(q)$ for each $q \in f^{-\mu-1}\{p\}$. In particular, $\tilde{f}_1 = \exp_{f(q)} \circ (T_{f(q)} f^{\mu})^{-1} \circ \exp_p^{-1} \circ f^{\mu+1}$ on $V(q)$ ($V_{\tilde{f}_1}(q) = V_f(q)$ here) for each $q \in f^{-\mu-1}\{p\}$;

(e′) $\tilde{f}_1 = f$ on $M - \bigcup\{W(q)|q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{p\}\}$.

Define

$$f_1(x) = \begin{cases} \gamma \cdot \tilde{f}_1(\gamma^{-1} \cdot x), & \text{on } \bigcup\{W_f(q)|q \in f^{-n}\{\gamma \cdot p\}, 1 \leq n \leq \mu+1\}, \gamma \in G; \\ f(x), & \text{otherwise.} \end{cases}$$

Then f_1 gives the desired equivariant local linearization. □

As defined in [9], a *complete tree* is a pair $\mathcal{T} = (Q, f)$ such that:

1. Q is a union of an infinite sequence of disjoint nonempty finite sets $Q_0, Q_1, \ldots, Q_n, \ldots$, where Q_0 consists of a single point q_0;
2. $f \colon Q - \{q_0\} \to Q$ maps Q_n onto Q_{n-1}.

A *branch* of a complete tree $\mathcal{T} = (Q, f)$ is an infinite sequence $\{q_n\}_{n=0}^{\infty}$ in Q such that $q_n \in Q_n$ and $f(q_{n+1}) = q_n$ for all $n \geq 0$. If we associate with each $q \in Q$ an m-dimensional inner product space V_q and a linear isomorphism $T_q \colon V_q \to V_{q_0}$, where T_{q_0} is the identity on V_{q_0}, then we call $(\mathcal{T}, \{T_q|q \in Q\})$ a *complete tree of isomorphisms*.

Lemma 3.3 (ε-kernel avoiding transition, [9])**.** *Given any complete tree of isomorphisms $(\mathcal{T}, \{T_q|q \in Q\})$ and any $\varepsilon > 0$. There is a number $\rho > 2$ and an integer $\mu \geq 1$ such that: for any finite ordered set $P = \{p_0, p_1, \ldots, p_t\}$ in V_{q_0}, there is a* *point* $y \in P \cap B(p_t, \rho|p_0 - p_t|)$ *such that for any branch* $\Sigma = \{q_0, q_1, \dots, q_n, \dots\}$ *of* $\mathcal{T}$*, there is a point* $w \in P \cap B(p_t, \rho|p_0 - p_t|)$*, where* w *is before* y *in the order of* P*, together with* $\mu + 1$ *points* $c_0, c_1, \dots, c_\mu$ *in* $B(p_t, \rho|p_0 - p_t|)$*, not necessarily distinct, satisfying the following two conditions* (a) *and* (b)*.*

(a) $c_0 = w, c_\mu = y$*; and*

(b) $|T_{q_n}^{-1}(c_n) - T_{q_n}^{-1}(c_{n+1})| \le \varepsilon d(T_{q_n}^{-1}(c_{n+1}), T_{q_n}^{-1}(\mathcal{A}))$ *for* $n = 0, 1, \dots, \mu - 1$*, where* $\mathcal{A} = P(w, y) \cup \partial B(p_t, \rho|p_0 - p_t|)$*,* $P(w, y) = \{p \in P | p$ *is after* w *and before* $y\}$*, and* d *is the distance on* V_{q_n}*.* □

4. Proof of the main theorem

In this section we first prove the following "weak form" of closing lemma and then prove the implication: "weak form" (Theorem 4.1)⇒"strong form" (Theorem 1.1).

Theorem 4.1. *Let* M *be a compact smooth manifold without boundary and* G *a finite group acting freely on* M*. Let* $f \colon M \to M$ *be a* G*-equivariant nonsingular* C^1 *map and* $\sigma \in M$ *a nonwandering point of* f*. Then for any* C^1 *neighborhood* $\mathcal{U}$ *of* f *in* $\mathrm{NEnd}_G^1(M)$ *and any neighborhood* U *of* σ *in* M*, there exists a map* $g \in \mathcal{U}$ *and a point* $p \in U$ *such that* $p \in \mathrm{Per}(g)$*.*

Proof. Let $\mathcal{U}$ be any C^1 neighborhood of f in $\mathrm{NEnd}_G^1(M)$ and U be any neighborhood of σ in M. Take an $\eta \in (0,1)$ such that $\mathcal{U}$ contains the η-ball in $\mathrm{NEnd}_G^1(M)$ centered at f. By Lemma 3.1, there is an $\varepsilon > 0$ such that

$$d_1(hf_1, f_1) < \eta/2$$

for any $f_1 \in \mathrm{NEnd}_G^1(M)$ with $d_1(f_1, f) < 1$ and any G-equivariant ε-kernel lift h.

Denote by $A = G(\sigma)$ the group orbit passing σ. If $f^m(x) \in A$ and $f^n(x) \in A$ for some $x \in M$ and some integers m and n with $m > n \ge 0$, then there exist $\gamma_1, \gamma_2 \in G$ such that $f^n(x) = \gamma_1 \cdot \sigma$ and $f^m(x) = \gamma_2 \cdot f^n(x)$. Let k be the order of γ_2. Then $k < \infty$ since G is finite. Hence $f^{k(m-n)}(\sigma) = \gamma_1^{-1} \cdot f^{k(m-n)+n}(x) = \gamma_1^{-1} \cdot f^n(x) = \sigma$. This shows that σ is a periodic point of f and the theorem is valid by taking $g = f$ and $p = \sigma$. Therefore we suppose in the following context that for all integers $m, n \ge 0$:

$$f^{-m}(A) \cap f^{-n}(A) \neq \varnothing \quad \text{if and only if } m = n.$$

Take $Q_n = f^{-n}\{\sigma\}$, $V_q = T_q M$, and $T_q = T_q f^n$ for all $q \in Q_n$ and $n \ge 0$. Then $\mathcal{T} = (Q, f)$ is a complete tree, where $Q = \mathrm{Orb}_f^-(\sigma)$, and $(\mathcal{T}, \{T_q | q \in Q\})$ is a complete tree of isomorphisms. For $(\mathcal{T}, \{T_q | q \in Q\})$ and $\varepsilon > 0$ there exists a number $\rho > 2$ and an integer $\mu \ge 1$ with the property described in Lemma 3.3.

Take a λ_0 such that $0 < \lambda_0 < \zeta$ and $\exp\{v \in T_A M | |v| \le \lambda_0\} \subset \bigcup_{\gamma \in G} \gamma U$. For f, A, μ, η, and λ_0, by Lemma 3.2, there is a λ with $0 < \lambda < \lambda_0$ together with a G-equivariant local linearization $f_1 \in \mathrm{NEnd}_G^1(M)$ such that $d_1(f_1, f) \le \eta/2$. Let $W = \exp\{v \in T_A M | |v| \le \lambda\}$ and $V = \exp\{v \in T_A M | |v| \le \lambda/4\}$. Hence $W(\sigma) \subset U$. For each $x \in A$ we introduce a new distance function d'_x on $W(x)$:

$$d'_x(p, q) = |\exp_x^{-1} p - \exp_x^{-1} q|.$$

Since σ is a nonwandering point for f there is a point p and an integer $\psi \ge 1$ such that

$$B(f^\psi(p), \rho d'_\sigma(p, f^\psi(p)); d'_\sigma) \subset V(\sigma).$$

We may suppose that $p \notin \mathrm{Per}(f)$, otherwise we may take $g = f$. Denote

(4.1)
$$\mathcal{P} := \{x \in V | x = f^k(\gamma \cdot p), \text{ for some } \gamma \in G \text{ and some integer } k \text{ with } 1 \le k \le \psi\},$$

$$P := \mathcal{P} \cap V(\sigma). \tag{4.2}$$

If there is some integer $k > 0$ and some $\gamma_1, \gamma_2 \in G$ such that $f^k(\gamma_1 \cdot p) \in P$ and $f^k(\gamma_2 \cdot p) \in P$, then we have $\gamma_1^{-1}V(\sigma) \cap \gamma_2^{-1}V(\sigma) \neq \varnothing$ and hence $\gamma_1 = \gamma_2$. Therefore we can introduce an order $\prec$ on P:

$$f^{k_1}(\gamma_1 \cdot p) \prec f^{k_2}(\gamma_2 \cdot p) \text{ if and only if } k_1 < k_2, \tag{4.3}$$

where $f^{k_1}(\gamma_1 \cdot p)$, $f^{k_2}(\gamma_2 \cdot p) \in P$. Rewrite P as $\{p_0, \dots, p_t\}$, where $p_0 \prec p_1 \prec \cdots \prec p_t$. Obviously, $p_0 = p$ and $p_t = f^\psi(p)$. Denote $P' = \exp_\sigma^{-1} P$ and $p'_i = \exp_\sigma^{-1} p_i$. Then $B(p'_t, \rho|p'_0 - p'_t|) \subset \exp_\sigma^{-1} V(\sigma)$.

By Lemma 3.3 there exists a point $y' \in P' \cap B(p'_t, \rho|p'_0 - p'_t|)$ such that: for any branch $\Sigma = \{q_0 = \sigma, q_1, \dots, q_n, \dots\}$ of $\mathrm{Orb}_f^-(\sigma)$ there exists a $w'(\Sigma) \in P' \cap B(p'_t, \rho|p'_0 - p'_t|)$, which is before y' in the order of P', and $\mu + 1$ points $c'_0(\Sigma), c'_1(\Sigma), \dots, c'_\mu(\Sigma) \in B(p'_t, \rho|p'_0 - p'_t|)$ such that

(a) $c'_0(\Sigma) = w'(\Sigma)$, $c'_\mu(\Sigma) = y'$; and

(b)
$$\begin{aligned}|(T_{q_n} f^n)^{-1}(c'_n(\Sigma)) - (T_{q_n} f^n)^{-1}(c'_{n+1}(\Sigma))| \\ \le \varepsilon d((T_{q_n} f^n)^{-1}(c'_{n+1}(\Sigma)), (T_{q_n} f^n)^{-1}(\mathcal{A}))\end{aligned}$$

for $n = 0, 1, \dots, \mu - 1$, where $\mathcal{A} = P'(w'(\Sigma), y') \cup \partial B(p'_t, \rho|p'_0 - p'_t|)$.

Let $w(\Sigma) = \exp_\sigma w'(\Sigma)$, $y = \exp_\sigma y'$. Then $w(\Sigma)$, $y \in P$ and there exists an integer $\phi(\Sigma) \ge 1$ and a $\gamma(\Sigma) \in G$ such that

$$f^{\phi(\Sigma)}(\gamma(\Sigma) \cdot w(\Sigma)) = y. \tag{4.4}$$

By the above construction we have $\phi(\Sigma) \ge \mu + 1$. Let $z(\Sigma) = f^{\phi(\Sigma) - \mu - 1}(\gamma(\Sigma) \cdot w(\Sigma))$. Then $f^{\mu+1}(z(\Sigma)) = y$.

Suppose $y = f^{t_1}(\gamma_1 \cdot p)$ and $w(\Sigma) = f^{t_2(\Sigma)}(\gamma_2(\Sigma) \cdot p)$. Then $t_1 > t_2(\Sigma)$ and by (4.4) we have

$$\gamma(\Sigma)\gamma_2(\Sigma) \cdot f^{\phi(\Sigma) + t_2(\Sigma)}(p) = \gamma_1 \cdot f^{t_1}(p). \tag{4.5}$$

Then from (4.5) we have $\gamma_1 = \gamma(\Sigma)\gamma_2(\Sigma)$ and $t_1 = \phi(\Sigma) + t_2(\Sigma)$ and hence

$$z(\Sigma) = f^{\phi(\Sigma) - \mu - 1}(\gamma(\Sigma) \cdot f^{t_2(\Sigma)}\gamma_2(\Sigma) \cdot p) = f^{t_1 - \mu - 1}(\gamma_1 \cdot p).$$

Note that both t_1 and γ_1 are determined by y and hence independent of Σ. Therefore $z(\Sigma)$ is actually independent of the choice of the branch Σ. We denote it simply by z. Then there is a unique $\sigma_{\mu+1} \in f^{-\mu-1}\{\sigma\}$ such that $z \in V(\sigma_{\mu+1})$.

Let Δ be any branch of $\mathrm{Orb}_f^-(\sigma)$ which contains $\sigma_{\mu+1}$, $\Delta = \{\sigma_0, \sigma_1, \dots, \sigma_n, \dots\}$. Take $w' = w'(\Delta)$, $c'_i = c'_i(\Delta)$ for $i = 0, 1, \dots, \mu$, and $\phi = \phi(\Delta)$, $\gamma = \gamma(\Delta)$. Then $\phi \ge \mu + 1$, $f^\phi(\gamma \cdot w) = y$, where $w = \exp_\sigma w' \in U$.

For $n = 0, 1, \dots, \mu - 1$, define sections $v_n, u_n \in \Gamma_G(G(\sigma_n))$:

$$v_n(\gamma \cdot \sigma_n) := \gamma \cdot (T_{\sigma_n} f^n)^{-1}(c'_n), \qquad \gamma \in G; \tag{4.6}$$

$$u_n(\gamma \cdot \sigma_n) := \gamma \cdot (T_{\sigma_n} f^n)^{-1}(c'_{n+1}), \qquad \gamma \in G. \tag{4.7}$$

Then

$$\begin{aligned}|v_n - u_n| &= |(T_{q_n} f^n)^{-1}(c'_n) - (T_{q_n} f^n)^{-1}(c'_{n+1})| \\ &\le \varepsilon d((T_{q_n} f^n)^{-1}(c'_{n+1}), (T_{q_n} f^n)^{-1}(\mathcal{A})),\end{aligned}$$

where $0 \le n < \mu$ and $\mathcal{A} = P'(w', y') \cup \partial B(p'_t, \rho|p'_0 - p'_t|)$. Let $h_n\colon M \to M$ be the G-equivariant ε-kernel lift corresponding to $G(\sigma_n), \varepsilon, v_n$, and u_n such that $h_n(\exp u_n) = \exp v_n$ and define $g\colon M \to M$ as

$$g = \begin{cases} h_n \circ f_1, & \text{on } \bigcup_{\gamma\in G} W(\gamma \cdot \sigma_{n+1}),\ 0 \le n < \mu; \\ f_1, & \text{on the rest of } M. \end{cases} \tag{4.8}$$

Then $g \in \mathrm{NEnd}^1_G(M)$ and $g \in \mathcal{U}$ since $d_1(g, f) \le d_1(g, f_1) + d_1(f_1, f) < \eta$.

It is easy to see that the lifts give $g^{\mu+1}(z) = w$. By the construction of linearization we also have $f_1^{\phi-\mu-1}(w) = f^{\phi-\mu-1}(w) = \gamma^{-1} \cdot z$. The f_1-orbit from w to $\gamma^{-1} \cdot z$ never touches the supports of these lifts. Hence $g^{\phi-\mu-1}(w) = f_1^{\phi-\mu-1}(w) = \gamma^{-1} \cdot z$, and then

$$w = \gamma \cdot g^{\phi}(w) = g^{k\phi}(w)$$

where k is the order of γ. Therefore w is a periodic point of g in U. □

Proof of the Implication "Theorem 4.1⇒Theorem 1.1". Let $\mathcal{U}$ be any C^1 neighborhood of f in $\mathrm{NEnd}^1_G(M)$. Take an $\eta \in (0, 1)$ such that $\mathcal{U}$ contains the η-ball in $\mathrm{NEnd}^1_G(M)$ centered at f. Let $A = G(\sigma)$ and $v_2 = 0 \in \Gamma_G(A)$. By Lemma 3.1 there exists an $\varepsilon > 0$ such that for any $v_1 \in \Gamma_G(A)$ with $|v_1| < \varepsilon\zeta$ there is a G-equivariant diffeomorphism $h_{\varepsilon,A,v_1,v_2}\colon M \to M$ such that:

(a) $h_{\varepsilon,A,v_1,v_2}(q) = \exp_p v_1(q)$, for all $q \in A$;

(b) $\mathrm{supp}(h_{\varepsilon,A,v_1,v_2}) \subset \bigcup_{q\in A} \exp_q B(0, |v_1|/\varepsilon)$, here the support means the closure of the set where $h_{\varepsilon,A,v_1,v_2}$ differs from identity;

(c) $d_1(h_{\varepsilon,A,v_1,v_2} f_1, f_1) < \eta/2$ and $d_1(h^{-1}_{\varepsilon,A,v_1,v_2} f_1 h_{\varepsilon,A,v_1,v_2}, f_1) < \eta/2$ for any $f_1 \in \mathrm{NEnd}^1_G(M)$ with $d_1(f_1, f) \le 1$.

Denote $V = \exp\{v \in T_A M \,||v| \le \varepsilon\zeta\}$. By Theorem 4.1 there is an $f_1 \in \mathrm{NEnd}^1_G(M)$ and a point $p \in V \cap \exp\{v \in T_\sigma M \,||v| \le \varepsilon\zeta\}$ such that $d_1(f_1, f) < \eta/2$ and $p \in \mathrm{Per}(f_1)$. Define $v_1 \in \Gamma_G(A)\colon v_1(\gamma\sigma) = \gamma \cdot \exp_\sigma^{-1} p, \gamma \in G$. And define

$$g\colon M \to M, \quad g(x) = h^{-1}_{\varepsilon,A,v_1,v_2} \circ f_1 \circ h_{\varepsilon,A,v_1,v_2}(x). \tag{4.9}$$

Then $g \in \mathrm{NEnd}^1_G(M)$ and satisfies $d_1(g, f) < \eta$ and $\sigma \in \mathrm{Per}(g)$. □

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