REMARKS ABOUT GLOBAL ANALYTIC HYPOELLIPTICITY

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Dedicated to Antonio Gilioli, in memoriam

Abstract. We present a characterization of the operators

\[ L = \partial_t + (a(t) + ib(t)) \partial_x \]

which are globally analytic hypoelliptic on the torus. We give information about the global analytic hypoellipticity of certain overdetermined systems and of sums of squares.

0. Introduction

The purpose of this work is to study the property of global analytic hypoellipticity for certain partial differential operators on the torus. An operator \( P \) is said to be globally analytic hypoelliptic on \( T^n \) (GAH) if the conditions \( u \in D'(T^n) \) and \( Pu \in C^\omega(T^n) \) imply \( u \in C^\omega(T^n) \).

The local version of this property was studied by Treves [T1] in the case of operators of principal type. For operators of the form \( L = \partial_t +(a(t) + ib(t)) \partial_x \), one has \( L \) analytic hypoelliptic if and only if \( b \) does not change sign (i.e., condition (P) holds) and \( b \) is not identically zero (i.e., condition (Q) holds); in particular, \( a(t) \) plays no role.

In the case of real constant coefficients Greenfield [G] showed that \( L = \partial_t + \alpha \partial_x \) is GAH on \( T^2 \) if and only if \( \alpha \) is an irrational number not too well approximable by rationals (in this paper we say that \( \alpha \) is not an exponential Liouville number (EL); see Definition 2.2). In Section 2 we show that

\[ L = \partial_t + (a(t) + ib(t)) \partial_x \]

(where \( a \) and \( b \) are real-valued, real-analytic functions on the unit circle \( S^1 \)) is GAH on \( T^2 \) if and only if \( a(t) \) is an irrational number not too well approximable by rationals (in this paper we say that \( \alpha \) is not an exponential Liouville number (EL); see Definition 2.2). In Section 2 we show that

\[ L = \partial_t + (a(t) + ib(t)) \partial_x \]

(when \( a \) and \( b \) are real-valued, real-analytic functions on the unit circle \( S^1 \)) is GAH on \( T^2 \) if and only if \( t \in S^1 \mapsto b(t) \in \mathbb{R} \) does not change sign (this is a global version of (P)) and, when \( b \equiv 0 \), the real number \( a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t)dt \) is neither rational nor exponential Liouville. The hardest part of the proof of this result is the construction of singular solutions when (P) is violated; the main tool here is the steepest descent method, as described in [deB, Ch. 5].

In Section 3 we study involutive systems of vector fields on \( T^{n+1} \), of the form \( L = (L_1, \ldots, L_n) \), where \( L_j = \partial_t + c_j(t_j) \partial_x \), \( j = 1, \ldots, n \), with each \( c_j = a_j + ib_j \) real-analytic. We show that \( L \) is GAH on \( T^{n+1} \) if and only if the set
\[ J = \{ j; b_j \text{ does not change sign} \} = \{ j_1, \ldots, j_k \} \text{ is nonempty and, when } b_j \equiv 0, \text{ for all } j \in J, \text{ the } \ell\text{-tuple } (a_{0j})_{j \in J} \text{ (where } a_{0j} = (2\pi)^{-1} \int_0^{2\pi} a_j(t_j) dt_j) \text{ is neither an exponential Liouville vector (see definition 3.2) nor an element of } \mathbb{Q}^\ell. \text{ What is involved here is the problem of simultaneous approximation, that is approximations of a point in } \mathbb{R}^\ell \text{ by elements } (p_1/q, \ldots, p_\ell/q) \text{ of } \mathbb{Q}^\ell, \text{ with all entries having the same denominators.}

Section 4 furnishes several examples illustrating the main theorems. We call attention to Example 4.9, where real numbers } \alpha, \beta, \text{ are constructed so that each of them is exponential Liouville, but } (\alpha, \beta) \text{ is not an exponential Liouville vector. One then has a system } L = (L_1, L_2), \text{ where } L_1 = \partial/\partial t_1 - \alpha \partial/\partial x, L_2 = \partial/\partial t_2 - \beta \partial/\partial x, \text{ such that } L \text{ is GAH on } T^3, \text{ even though neither } L_1 \text{ nor } L_2 \text{ is GAH (these are not GAH even if we consider them as operators acting only in two variables). The same phenomenon may occur locally, but never in real structures of codimension } \geq 1. \text{ In our construction, the theory of continued fractions is heavily relied upon.}

In Section 5 we comment on the connections of our work with [B], [BCM], [Ca-Ho], [Co-Hi], and [GPY]. A characterization of GAH for certain sums of squares arising from involutive systems of real vector fields is presented.

1. Asymptotic behavior of certain integrals

The following result is an important ingredient in the construction of singular solutions.

**Lemma 1.1.** Consider the integral

\[
J(n) = \int_{-a}^{a} \exp[-n(F(t) + KG(t))] dt, \quad n \in \mathbb{N},
\]

where } a > 0, K > 0 \text{ is a large parameter, and } F, G \in \mathcal{C}^\infty(\mathbb{R}) \text{ are such that } F(0) = \Re F'(0) = G(0) = G'(0) = 0; G''(0) = 1; G \text{ is strictly decreasing (resp. increasing) on } -a \leq t < 0 \text{ (resp. } 0 < t \leq a) \text{.}

Then there exist } R_0 > 0, K_0 > 0 \text{ such that for each } K \geq K_0, \text{ the holomorphic function } z \mapsto F'(z) + KG'(z) \text{ has exactly one zero, } z_0, \text{ in the (complex) disc } |z| \leq R_0. \text{ Furthermore,}

\[
J(n) = \gamma n^{-1/2} \exp[-n(F(z_0) + KG(z_0))] (1 + O(1/n)),
\]

as } n \to \infty, \text{ where } \gamma \text{ is a nonzero constant.}

We also have

\[
|J(n)| = \gamma_1 n^{-1/2} \exp(-\varepsilon n) (1 + O(1/n)),
\]

as } n \to \infty, \text{ where } \gamma_1 = |\gamma| > 0, \text{ and } \varepsilon = \Re(F(z_0) + KG(z_0)); \text{ finally, } \varepsilon \geq 0 \text{ and } \varepsilon = O(1/K), \text{ as } K \to \infty.

**Proof.** The proof relies on applying the so-called steepest descent method; the main point is to deform the integration contour into the complex plane so that, in the new contour, a single point of minimum of } \Re(F(z) + KG(z)) \text{ occurs.

Set } h(z) = F(z) + KG(z), \text{ and } H(z) = F(z)/K + G(z); \text{ then } h \text{ and } H \text{ are holomorphic in a neighborhood of } [-a, a]. \text{ The assumptions imply that, if } R_0 > 0 \text{ is small and } K_0 > 0, \text{ then for all } K \geq K_0, G' \text{ dominates } F'/K \text{ on } |z| = R_0; \text{ also } R_0 \text{ can be chosen so that } z = 0 \text{ is the only zero of } G'(z) \text{ on } |z| \leq R_0. \text{ Now, by Rouché’s theorem, } H'(z) \text{ and } h'(z) \text{ have exactly one zero, } z_0, \text{ on } |z| \leq R_0. \text{ Simple computations show that } |z_0| = O(1/K), \text{ and that } \varepsilon = \Re h(z_0) = O(1/K).
We now use Taylor’s formula to write $H(z) - H(z_0) = \tilde{H}_2(z)(z - z_0)^2$, where $\tilde{H}_2$ is holomorphic on $|z - z_0| < r = R_0 - |z_0|$. Simple computations show that $\tilde{H}_2$ has a square root, $H_2 = (\tilde{H}_2)^{1/2}$, on $|z - z_0| < r'$, for some $0 < r' < r$; we also have $H_2(z_0) = (\tilde{H}_2(z_0))^{1/2} = (1/\sqrt{2})(1 + O(1/K))$. We write $H_2(z)(z - z_0) = u + iv$, and so $H(z) - H(z_0) = (u + iv)^2$.

We are interested in the curves $\Re(H(z) - H(z_0)) = 0$, or $u = \pm v$.

Write $z = t + is$, $z_0 = t_0 + is_0$.

The above arguments show that $|H(z) - H(z_0) - \alpha(z - z_0)^2| \leq \delta|z - z_0|^2$, where $0 < \delta = O(1/K)$, and $\alpha = 1/\sqrt{2}$ for $|z - z_0| \leq r'$.

Thus, $u(t + is) = \pm v(t + is)$ implies $(1 - O(1/K))(t - t_0)^2 \leq (s - s_0)^2 \leq (1 + O(1/K))(t - t_0)^2$.

The point is that both curves $u = \pm v$ reach the real axis inside the disc $|z| \leq R_0$.

Furthermore, the same happens with the curves $\Re(H(z) - H(z_0)) = \delta'$, if $\delta' > 0$ is small enough.

Moreover, one of the curves $\Im(H(z) - H(z_0)) = 0$ is nearly horizontal. This allows us to deform the contour $[-a, a]$ into $P_1 P_2 P_3 z_0 P_4 P_5 P_6$, where: $P_1 = -a$, $P_6 = a$; $P_2$ and $P_3$ are on the real axis; $P_2 P_3$ and $P_4 P_5$ are pieces of $\Re(H(z) - H(z_0)) = \delta'$; $P_3 z_0 P_4$ is an arc of the horizontal piece of $\Im(H(z) - H(z_0)) = 0$.

We can now use the steepest descent method: indeed, $z_0$ is the only point of minimum of $\Re H(z)$ along the deformed curve. This completes the proof of Lemma 1.1.

\section{The case of a single vector field}

Our concern here is the study of vector fields of the form

$$(2.1) \quad L = \partial_t + c(t) \partial_x$$

where $c \in C^\infty(S^1)$, i.e., $c$ is a real-analytic, $2\pi$-periodic, complex-valued function of a single real variable. We may write $c(t) = a(t) + ib(t)$, with $a$ and $b$ real-valued, and we may also write

$$(2.2) \quad L = \partial_t + (a(t) + ib(t)) \partial_x.$$

We are interested in the action of $L$ on periodic (in both variables $t, x$) functions or distributions. More precisely, our aim is to give a characterization of those functions $c$ for which $L$ has the regularity property appearing in the following definition.
Definition 2.1. The vector field $L$ is said to be globally analytic hypoelliptic on the torus $\mathbb{T}^2$ (briefly: $L$ is GAH) if the conditions $u \in \mathcal{D}'(\mathbb{T}^2)$ and $Lu \in C^\omega(\mathbb{T}^2)$ imply $u \in C^\omega(\mathbb{T}^2)$.

We need one more definition, namely

Definition 2.2. An irrational number $\alpha$ is said to be an exponential Liouville number (briefly: $\alpha$ is EL) if there exists $\varepsilon > 0$ such that the inequality $|\alpha - p/q| \leq \exp(-\varepsilon q)$ has infinitely many rational solutions $p/q$, with $p \in \mathbb{Z}, q \in \mathbb{N}$.

We are now ready to state the main result of this section.

Theorem 2.3. The vector field $L = \partial_t + (a(t) + ib(t))\partial_x$ is globally analytic hypoelliptic on the torus if and only if the following conditions are satisfied:

(2.3) $b$ does not change sign;

(2.4) if $b \equiv 0$, then the real number $a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t)dt$ is neither rational nor exponential Liouville.

Before embarking on the proof of this theorem we will state, without proof, two results describing the possible solutions to the homogeneous equation $Lu = 0$, and, also, to the non-homogeneous equation

(2.5) $Lu = f$

where $u \in \mathcal{D}'(\mathbb{T}^2)$ and $f \in C^\omega(\mathbb{T}^2)$.

Set $c_0 = (2\pi)^{-1} \int_0^{2\pi} c(t)dt$; then $c_0 = a_0 + ib_0$, where $a_0, b_0 \in \mathbb{R}$ are given by $a_0 = (2\pi)^{-1} \int_0^{2\pi} a(t)dt$, and $b_0 = (2\pi)^{-1} \int_0^{2\pi} b(t)dt$.

Lemma 2.4. A distribution $u \in \mathcal{D}'(\mathbb{T}^2)$ is a solution to $Lu = 0$ if and only if one of the following situations occurs:

(i) $c_0 \notin \mathbb{Q}$ and $u$ is a constant;

(ii) $c_0 = p/q$ (lowest terms), with $p \in \mathbb{Z}, q \in \mathbb{N}$ and $u$ belongs to the closed span (in $\mathcal{D}'(\mathbb{T}^2)$) of the functions $\exp(ikq(x - \int_0^t c))$, $k \in \mathbb{Z}$.

Lemma 2.5. Assume that $c_0 \notin \mathbb{Q}$. If (2.5) has a solution, then necessarily

\[ \int_0^{2\pi} f_0(t)dt = 0. \]

If $u$ is a solution to (2.5), then its partial Fourier coefficients are given by

(2.6) $u_0(t) = \int_0^t f_0(s)ds + \text{ constant},$

and, for $n \in \mathbb{Z} \setminus \{0\}$, by

(2.7) $u_n(t) = (1 - \exp(-i2\pi nc_0))^{-1} \int_0^{2\pi} \exp(-in \int_{t-s}^t c) f_n(t-s)ds$;

here $f_n(t)$ denotes the $n^{\text{th}}$ partial (with respect to $x$) Fourier coefficient of $f$.

For $n \in \mathbb{Z} \setminus \{0\}$ we have the following alternative expression:

(2.8) $u_n(t) = (\exp(i2\pi nc_0) - 1)^{-1} \int_0^{2\pi} \exp(in \int_{t}^{t+s} c) f_n(t+s)ds.$

We now point out the following consequence of Lemma 2.4, which already proves part of Theorem 2.3.
Corollary 2.6. If $c_0 \in \mathbb{Q}$, then $L$ is not GAH.

Proof. Pick $t_0 \in S^1$ such that $B(t_0) = \max\{B(t); t \in S^1\}$. Write $a_0 = p/q$ and set $u(t, x) = \sum_{k=1}^{\infty} \exp(ikq(x - \int_{t_0}^t c + iB(t_0)))$. It is easy to see that $u \in \mathcal{D}'(T^2) \setminus C^\omega(T^2)$ and $Lu = 0$.

Proof of Theorem 2.3. We begin by proving the necessity of conditions (2.3) and (2.4); the proof amounts to constructing singular solutions to $Lu = f$, when either (2.3) or (2.4) are not satisfied. Suppose first that (2.3) does not hold; thus $b(t) \neq 0$ and $b(t)$ changes sign. We will divide the proof into four cases, according to the nature of the complex number $c_0 = a_0 + ib_0$.

Case 1. $b_0 < 0$. Here we choose $f(t, x) = (2\pi)^{-1} \sum_{n=1}^{\infty} f_n(t) \exp(inx)$ with

$$f_n(t) = (1 - \exp(-i2\pi nc_0)) \exp(-n(A + KG(t) - \varepsilon))$$

where $G(t) = 1 - \cos(t - t_0 + s_0)$; in the sequel we will explain how to choose the real numbers $A, K, \varepsilon, t_0, s_0$.

A formal solution to $Lu = f$ is $u(t, x) = (2\pi)^{-1} \sum_{n=1}^{\infty} u_n(t) \exp(inx)$, where

$$u_n(t) = \int_0^{2\pi} \exp\left\{-n(A + KG(t - s) - \varepsilon + i \int_s^t c\right\} ds.$$ 

Note that each $u_n \in C^\omega(S^1)$.

We set $A = \max\left\{\int_{t-s}^t b; 0 \leq s, t \leq 2\pi\right\}$. This maximum is attained when, say, $t = t_0, s = s_0$; thus $A = \int_{t_0-s_0}^{t_0} b$. We may assume that $b(0) > 0$ and that $0 < t_0, s_0, t_0 - s_0 < 2\pi$. In what follows it will be important to consider the points belonging to the set $Y = \left\{t; 0 \leq t \leq 2\pi ; \int_{t_0-s_0}^t b = A\right\}$. Since $b$ is $C^\omega$ and $\neq 0$, $Y$ is a finite set; we write $Y = \{t_0, t_1, \ldots, t_r\}$.

Note that $\int_{t_k}^{t_0} b = 0$ for each $k$; note also that $b$ changes sign from $-\to +$ at $t = t_0 - s_0$, and from $+\to -$ at $t = t_k$, for $k = 0, \ldots, r$. This implies $\int_{t_k}^{t_0} b \leq 0$ if $|t - t_k| \leq \rho$, for $k = 0, \ldots, r$, if $\rho > 0$ is small. We also have $t_0 - s_0 \neq t_k$, for $k = 0, \ldots, r$.

We have, for all $n \in \mathbb{N}$, $0 < c_1 \doteq 1 - \exp(2\pi b_0) \leq |1 + \exp(-i2\pi nc_0)| \leq 1 + \exp(2\pi b_0) \doteq c_2 < \infty$.

The estimate $|f_n(t)| \leq c_2 \exp(-n(A - \varepsilon - K(\cosh(\delta) - 1)))$ holds for all $n \in \mathbb{N}, t \in S^1 + i(-\delta, \delta)$.

We will choose $K > 0$ very large; the value of $\varepsilon \geq 0$ will come out from applying Lemma 1.1, and we will have $\varepsilon = O(K^{-1})$. We will also require $\delta = O(K^{-1})$. Thus $K\delta^2 = O(K^{-1})$, and $A - \varepsilon - K(\cosh(\delta) - 1) = A - \varepsilon - 2K\delta^2 = A - O(K^{-1}) \geq A/2 > 0$, for large $K$.

Thus $|f_n(t)| \leq c_2 \exp(-An/2)$, for all $n \in \mathbb{N}, t \in S^1 + i(-\delta, \delta)$; since each $f_n \in C^\omega(S^1)$ we will have $f \in C^\omega(T^2)$.

We claim that there exist $C > 0, \gamma_1 > 0$ such that

$$|u_n(t)| \leq C n^{-1/2}, t \in S^1, n \in \mathbb{N},$$

and

$$|u_n(t_k)| = \gamma_1 n^{-1/2}(1 + O(n^{-1})), k = 0, \ldots, r, \text{ as } n \to \infty.$$


Assume, for a moment, that the claim has been proved. Then \( u \in \mathcal{D}'(\mathbf{T}^2) \) because of (2.11). On the other hand, \( u \not\in C^\omega(\mathbf{T}^2) \) because of (2.12); in fact, \((t_k, x)\) belongs to the analytic singular support of \( u \), for all \( k = 0, \ldots, r, \ x \in S^1 \).

We now proceed to prove our claim.

We may write
\[
u_n(t) = \tilde{u}_n(t) \exp \left\{ -n[A - \varepsilon + i \int_{\sigma_0}^{\sigma} c] \right\}, \text{ where } \sigma_0 = t_0 - s_0 \text{ and } \tilde{u}_n(t) = \int_{t-2\pi}^{t} \exp \left\{ -n(K[1 - \cos(\sigma - \sigma_0)] - i \int_{\sigma}^{\sigma_0} c) d\sigma \right\}.
\]

Pick \( \rho > 0 \) so that, when \( t \) belongs to \( D \equiv \{t; |t-t_k| \leq \rho, \text{ for some } k = 0, \ldots, r\} \), we have \( \int_{t_0}^t b \leq 0 \) and \( t \neq \sigma_0 \). For \( t \in D \) we also have \( t - 2\pi < \sigma_0 < t \).

Set \( h(\sigma) = K[1 - \cos(\sigma - \sigma_0)] - i \int_{\sigma}^{\sigma_0} c, t - 2\pi \leq \sigma \leq t \).

It is easy to see that, given \( C_1, C_2 \), there exists \( K_0 > 0 \) such that, for all \( K \geq K_0 \), we have: \( \Re h \) is strictly decreasing (resp. increasing) on \([\sigma_0 - \pi/2, \sigma_0] \cap [t - 2\pi, t] \) (resp. \([\sigma_0, \sigma_0 + \pi/2] \cap [t - 2\pi, t] \)). Furthermore, \( \Re h \geq C_1 + C_2 \max\{|b(t)|; 0 \leq t \leq 2\pi\} \) on \([\sigma - \sigma_0] \geq \pi/2 \) \( \cap [t - 2\pi, t] \).

We will first analyze the growth of \(|u_n(t)|\), as \( n \to \infty \), when \( t \in D \).

Pick \( \sigma_1, \sigma_2 \) so that \( \sigma_1 < \sigma_0 < \sigma_2, [\sigma_1, \sigma_2] \subset [t - 2\pi, t] \cap [\sigma_0 - \pi/2, \sigma_0 + \pi/2] \), and \([\sigma_1, \sigma_2] \cap D = \emptyset \).

Set \( J_n = \int_{\sigma_1}^{\sigma_2} \exp(-nh(\sigma)) d\sigma \).

Lemma 1.1 applies to give \( J_n = \gamma n^{-1/2}(1 + O(1/n)) \exp(-nh(\sigma_0)) \).

Set
\[
(2.13) \quad \varepsilon = \Re \left\{ K[1 - \cos(z_0 - \sigma_0)] - i \int_{\sigma_0}^{z_0} c \right\}.
\]

By taking \( K_0 \) sufficiently large we can be sure that (for all \( K \geq K_0 \)), when we deform the integration contour in order to bypass (via \( z_0 \)) the point \( \sigma_0 \), we get back to the real axis at points \( \sigma_1, \sigma_2 \) with \( \sigma_1 < \sigma_1 < \sigma_0 < \sigma_2 < \sigma_2 \); we then have \( \Re h(\sigma) \geq \varepsilon' > \varepsilon \), for all \( \sigma \in \{[\sigma - \sigma_0] \geq \rho \} \cap [t - 2\pi, t] \). Thus
\[
\left| \left\{ \int_{t-2\pi}^{\sigma_1} + \int_{\sigma_2}^{t} \right\} \right| \left( \exp(-nh(\sigma)) d\sigma \right| = O(\exp(-\varepsilon'n)),
\]
as \( n \to \infty \), uniformly in \( t \in D \).

Hence, with \( \gamma_1 = |\gamma| \), and for \( t \in D \),
\[
|u_n(t)| = O(\exp(-\varepsilon'n)) + \gamma_1 n^{-1/2}(1 + O(1/n)) \left| \exp \left\{ -n \left( A - \varepsilon + i \int_{\sigma_0}^{\sigma} c + h(z_0) \right) \right\} \right|.
\]

Now \( \Re \left\{ A - \varepsilon + i \int_{\sigma_0}^{\sigma} c + h(z_0) \right\} = A - \int_{\sigma_0}^{\sigma} b = A - \int_{\sigma_0}^{\sigma_0} b + \int_{\sigma_0}^{\sigma} b = -\int_{\sigma_0}^{\sigma} b \).

Recall that \( -\int_{t_0}^{t} b \geq 0 \) if \( t \in D \), and \( -\int_{t_0}^{t} b = 0 \) if \( t = t_k, k = 0, \ldots, r \).

Thus, for each \( t_k \), we have
\[
|u_n(t_k)| = O(\exp(-\varepsilon'n)) + \gamma_1 n^{-1/2}(1 + O(1/n)), \text{ as } n \to \infty,
\]
which implies the validity of (2.12).

On the other hand, we have
\[
(2.14) \quad |u_n(t)| = O(\exp(-\varepsilon'n)) + \gamma_1 n^{-1/2}(1 + O(1/n)) \exp \left( n \int_{t_0}^{t} b \right),
\]
for \( t \in D \), as \( n \to \infty \), which implies the validity of (2.11) when \( t \in D \).
It remains to estimate $|u_n(t)|$ when $t \in [0, 2\pi] \setminus D$, i.e., when $|t - t_k| \geq \rho$, for all $k = 0, \ldots, r$.

There exist $K_0 > 0, \eta > 0$ such that, for all $K > K_0$,

\begin{equation}
\Re \left( A - \varepsilon + i \int_{\sigma_0}^t c \right) \geq \eta/2, \quad t \in [0, 2\pi] \setminus D.
\end{equation}

Indeed take $\eta > 0$ such that $\int_{\sigma_0}^t b \leq A - \eta$, for $t \in [0, 2\pi] \setminus D$. Then

\begin{equation}
\Re \left( A - \varepsilon + i \int_{\sigma_0}^t c \right) = A - \varepsilon - \int_{\sigma_0}^t b \geq \eta - \varepsilon \geq \eta/2,
\end{equation}

since $\varepsilon = O(1/K)$.

We claim that, for $K_0$ sufficiently large, and for all $K \geq K_0$,

\begin{equation}
\Re h(\sigma) \geq 0, \quad \text{for all } \sigma \in [t - 2\pi, t], \quad t \in [0, 2\pi] \setminus D.
\end{equation}

We divide the proof of (2.16) into three cases, according to the location of $\sigma$; note that we always have $-2\pi \leq \sigma \leq 2\pi$ in (2.16).

Let $\rho'$ be such that $\int_{\sigma_0}^\sigma b \geq 0$ if $|\sigma - \sigma_0| \leq \rho'$.

If $|\sigma - \sigma_0| \leq \rho'$ we get (2.16) because $K[1 - \cos(\sigma - \sigma_0)] \geq 0$ for all $\sigma$.

Now if $|\sigma - (\sigma_0 - 2\pi)| \leq \rho'$ we have $\int_{\sigma_0}^\sigma b = \int_{\sigma_0}^{\sigma_0 - 2\pi} b + \int_{\sigma_0 - 2\pi}^\sigma b = -2\pi b_0 + \int_{\sigma_0 - 2\pi}^\sigma b > 0$, and (2.16) again follows.

Finally if $\sigma$ is such that $-2\pi \leq \sigma \leq 2\pi, |\sigma - \sigma_0| \geq \rho'$, and $|\sigma - (\sigma_0 - 2\pi)| \geq \rho'$ we have $1 - \cos(\sigma - \sigma_0) \geq \tilde{\eta}$, for some $\tilde{\eta} > 0$; thus $K\tilde{\eta} + \int_{\sigma_0}^\sigma b \geq K\tilde{\eta}/2$, provided $K$ is large. The proof of (2.16) is complete.

Now the conjunction of (2.15) and (2.16) shows that

\begin{equation}
|u_n(t)| = O(-\eta n/2), \quad t \in [0, 2\pi] \setminus D, \quad n \to \infty.
\end{equation}

Finally (2.14) and (2.17) together imply (2.11); this completes the proof in Case 1.

Case 2. $b_0 > 0$. Here the proof is entirely analogous to that of Case 1. We use (2.8) instead of (2.7), we let $n$ vary in $N$, take $A = -\min \{\int_{t-s}^t b; 0 \leq s, t \leq 2\pi\}$, and so on.

Case 3. $b_0 = 0$ and $a_0 \in \mathbb{R} \setminus \mathbb{Q}$. The proof is again similar to that of Case 1. Note that, for all $n \in \mathbb{Z}$, $|1 - \exp(-i2\pi n a_0)| = |1 - \exp(-i2\pi n a_0)| \leq 2$.

Case 4. $b_0 = 0$ and $a_0 \in \mathbb{Q}$. This was taken care of in Corollary 2.6.

This concludes the proof of the necessity of (2.3).

Suppose now that (2.3) holds but (2.4) does not. Thus $b(t) \equiv 0$, and $a_0$ is either rational or exponential Liouville.

We use the automorphism of $D'(T^2)$ (and, also, of $C^\infty(T^2)$) defined by $Su = v$, where the partial Fourier coefficients are related by

\begin{equation}
v_n(t) = u_n(t) \exp \left[ i n \left( \int_0^t a(s) ds - a_0 t \right) \right], \quad n \in \mathbb{Z}.
\end{equation}

The equation $Lu = f$ becomes $(\partial_t + a_0 \partial_x)u = q$, where $Su = v, Sf = q$. The conclusion is that, when $b \equiv 0$, $L$ is GAH if and only if $\partial_t + a_0 \partial_x$ is GAH if and only if $a_0$ is neither rational nor exponential Liouville ([G]). This concludes the proof of the necessity of (2.4), and also proves the sufficiency when $b \equiv 0$. 

To complete the proof of sufficiency it remains to consider the case where $b \neq 0$ and $b$ does not change sign. Here the results of Treves [T1] imply that $L$ is actually (locally) analytic hypoelliptic, hence also GAH.

The proof of Theorem 2.3 is complete.

3. A CLASS OF OVERDETERMINED SYSTEMS

Here we study systems of vector fields of the form $\mathbb{L} = (L_1, \ldots, L_n)$, where

$$L_j = \frac{\partial}{\partial t_j} + c_j(t_j) \frac{\partial}{\partial x}, \quad j = 1, \ldots, n,$$

with each $c_j \in C^\omega(S^1)$.

We use the notations: $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$, $x \in S^1$, $(t, x) \in \mathbb{T}^{n+1}$, $t_j \in S^1_j$.

We set $c_{j,0} = (2\pi)^{-1} \int_0^{2\pi} c_j(t_j) dt_j$; we also write $c_j = a_j + ib_j$ and $c_{j,0} = a_{j,0} + ib_{j,0}$.

We split the set $\{1, \ldots, n\}$ as $\{1, \ldots, n\} = J \cup K$, where $j \in J$ if and only if the function $b_j$ does not change sign; we write $J = \{j_1, \ldots, j_\ell\}$. We allow $\ell$ to be 0 (this means $J$ is empty) or $n$ (this means $K$ is empty).

We are now ready to state the main result of this section.

**Theorem 3.3.** Under the above assumptions and notations, $\mathbb{L}$ is GAH if and only if the following conditions are satisfied:

1. $J$ is nonempty;
2. if $b_j \equiv 0$ for all $j \in J$, then $(a_{j_1,0}, \ldots, a_{j_\ell,0})$ is neither EL nor an element of $Q^\ell$.

**Proof.** We begin by proving the necessity.

Suppose first that (3.2) is not satisfied, that is, $b_j$ changes sign, for all $j = 1, \ldots, n$.

The results of Section 2 apply to yield $u_j = u_j(t_j, x)$, with $u_j \in D'(S^1_j \times S^1_j)$ and $f_j \equiv L_j u_j \in C^\omega(S^1_j \times S^1_j)$, for $j = 1, \ldots, n$.

In Section 2 we constructed these objects and proved that the following estimates hold, for all $j = 1, \ldots, n$, and all $k \in \mathbb{N}$:

$$|\tilde{u}_j(t_j, k)| \leq C k^{-1/2}, \quad t_j \in S^1,$$

$$|\tilde{u}_j(t_{0,j}, k)| = C k^{-1/2} (1 + O(k^{-1})), \quad \text{for some } t_{0,j} \in S^1,$$
(3.6) \[ |\hat{f}_j(t_j, k)| \leq C \exp(-Bk), \quad t_j \in S^1 + i(-\delta, \delta), \text{ for some } B > 0. \]

We make the remark that a more careful analysis of the proof of (3.4) yields the following estimates:

(3.7) \[ |\tilde{u}_j(t_j, k)| \leq Ck^{-1/2} \exp(\varepsilon k), \quad t_j \in S^1 + i(-\delta, \delta) \]

where \( \varepsilon \) is small.

We set

(3.8) \[ u = u_1 \ast \cdots \ast u_n \]

where \( \ast \) denotes the convolution in the \( x \)-variable alone.

Now, by using basic properties of the convolution product together with the estimates (3.4)–(3.7), one sees that \( u \in D'(T^{n+1}) \) and that each \( L_j u \in C^\omega(T^{n+1}) \); this shows that (3.2) is indeed necessary.

Assume now that (3.2) holds but (3.3) does not. Then, after reordering the variables (if necessary) we may assume that there exists \( \ell \), with \( 1 \leq \ell \leq n \), such that

(i) \( b_j \equiv 0 \), for \( j = 1, \ldots, \ell \);
(ii) \( b_j \) changes sign, for \( j = \ell + 1, \ldots, n \);
(iii) \( a_0 \equiv (a_1, a_2, \ldots, a_{\ell+1}) \) is either EL or belongs to \( Q_\ell^0 \).

We will split the variables as \( j = 1, \ldots, n \), become \( L_j v = g_j \), \( j = 1, \ldots, n \), where \( v = Su \), \( g_j = Sf_j \), and \( L = \frac{\partial}{\partial t} + a_j \frac{\partial}{\partial x_j}, j = 1, \ldots, \ell \).

We set \( A = \{ (p, k) \in Z^\ell \times N : \max_{1 \leq j \leq \ell} |p_j + a_0_j k| < \exp(-\varepsilon(|p| + k)) \} \).

Note that \( A \) is an infinite set; indeed, when \( a_0 \) is EL this follows from the definition of EL; when \( a_0 \in Q_\ell^0 \), we may take \( k = qm, p_j = -qm a_0_j, m \in N \), where \( a_0 = \vec{r}/q \) (lowest terms).

We set \( w(t', x) = \sum_{(p, k) \in A} \exp(\varepsilon(p \cdot t' + kx)) \).

We have \( w \in D'(T'_\ell \times S^1_x) \setminus C^\omega(T'_\ell \times S^1_x) \), and \( L_j w = h_j \), where

(3.10) \[ h_j(p, k) = \begin{cases} i(p_j + a_0_j k), & \text{if } (p, k) \in A, \\ 0, & \text{otherwise.} \end{cases} \]

Thus \( h_j \in C^\omega(T'_\ell \times S^1_x) \), \( j = 1, \ldots, \ell \).

Now, as in the first part of this proof, for each \( j = \ell + 1, \ldots, n \), there exists \( u_j \in D'(S^1_{t_j} \times S^1_\ell) \setminus C^\omega(S^1_{t_j} \times S^1_\ell) \) such that \( L_j u_j \in C^\omega(S^1_{t_j} \times S^1_\ell) \).

We now define \( u = w \ast u_{\ell+1} \ast \cdots \ast u_n \), and the proof ends just like in the proof of the necessity of (3.2).

We now prove the sufficiency of (3.2)–(3.3).

Case 1. There exists \( j \in \{1, \ldots, n\} \) such that \( b_j \) does not change sign and \( b_j \neq 0 \). In this case \( L \) is even locally analytic hypoelliptic, hence GAH; indeed no local primitive of the 1-form \( b_1 dt_1 + \cdots + b_n dt_n \) has local extrema, and so the results of [BT] apply.

Case 2. Each \( b_j \) which does not change sign is \( \equiv 0 \). Then, after reordering the variables (if necessary) we may assume that there exists \( \ell \), with \( 1 \leq \ell \leq n \), such that
(i) \( b_j \equiv 0 \), for \( j = 1, \ldots, \ell \);
(ii) \( b_j \) changes sign, for \( j = \ell + 1, \ldots, n \);
(iii) \( \alpha_0 = (a_{1,0}, \ldots, a_{\ell,0}) \) is neither EL nor in \( Q^{\ell} \).

Let \( u \in D(T^{n+1}) \) be given with \( f_j \doteq L_j u \in C^\infty(T^{n+1}) \), for \( j = 1, \ldots, n \). We proceed to show that \( u \in C^\infty(T^{n+1}) \).

As in the proof of the necessity of (3.3) we use the automorphism \( S \) and get the equations \( \tilde{L}_j = g_j, j = 1, \ldots, n \), where \( \tilde{L}_j = \frac{\partial}{\partial x_j} + a_{0,j} \frac{\partial}{\partial x_0} \), \( j = 1, \ldots, \ell \).

We use partial Fourier series in the variables \((t', x)\) in the first \( \ell \) equations and get
\[
(3.9) \quad i(p_j + a_{0,j}k)\tilde{v}(t', p, k) = \tilde{g}_j(t', p, k), \quad j = 1, \ldots, \ell.
\]

We get, for \((p, k) \in \mathbb{Z}^{\ell+1}\),
\[
(3.10) \quad \tilde{v}(t', p, k) = \tilde{g}_j(t', p, k)/i(p_j + a_{0,j}k),
\]
where \( j \) is any index with \( 1 \leq j \leq \ell \) and \( p_j + a_{0,j}k \neq 0 \) (such a \( j \) exists because \( \alpha_0 \notin Q^\ell \)).

When \((p, k) = (0, 0)\), we see that \( \tilde{v}(t', 0, 0) \) is not determined by (3.9), and we also see that \( \tilde{g}_j(t', 0, 0) \equiv 0 \), \( j = 1, \ldots, \ell \).

Set \( w(t, x) = v(t, x) - (2\pi)^{-\ell-1} \tilde{v}(t', 0, 0) \). From (3.10) we get, since \( g_1, \ldots, g_\ell \in C^\infty \) and \( a_0 \notin Q^\ell \cup EL \), that there exist \( \varepsilon > 0, C > 0 \) such that \( |\tilde{v}(t', p, k)| \leq C \exp(-\varepsilon(||p|| + |k|)) \), for \((p, k) \in \mathbb{Z}^{\ell+1} \setminus \{0\}, t' \in T^\infty, \ell, i\delta T^{n-\ell}, \) where \( \delta = (-1, 1) \).

It follows that \( w \in C^\infty(T^{n+1}) \).

Now, for \( j = \ell + 1, \ldots, m \), we have \( L_j w = L_j v - (2\pi)^{-\ell-1} \frac{\partial}{\partial t_j} \tilde{v}(t', 0, 0) \), and so \( \frac{\partial}{\partial t_j} \tilde{v}(t', 0, 0) \in C^\infty \), which implies \( \tilde{v}(t', 0, 0) \in C^\infty \). Thus \( v = w + (2\pi)^{-\ell-1}v_{0,0} \in C^\infty \), and also \( u \in C^\infty(T^{n+1}) \).

\[ \square \]

4. Examples

We begin with examples of vector fields on the torus \( T^2 \).

**Example 4.1.** \( L = \partial/\partial t + (\alpha \sin t + i(\beta - \cos t)) \partial/\partial x, \alpha, \beta \in \mathbb{R} \). Here \( L \) is GAH if and only if \( |\beta| \geq 1 \).

**Example 4.2.** \( L = \partial/\partial t + (\sqrt{2} + \alpha \sin t) \partial/\partial x, \alpha \in \mathbb{R} \). This \( L \) is GAH because \( \sqrt{2} \) is an algebraic number of degree 2 hence, by Liouville’s theorem, satisfies \(|\sqrt{2} - p/q| \geq Cq^{-2} \), for all \( p \in \mathbb{Z}, q \in \mathbb{N} \), and some \( C > 0 \).

**Example 4.3.** \( L = \partial/\partial t + (2 + \alpha \sin t) \partial/\partial x, \alpha \in \mathbb{R} \). This \( L \) is not GAH.

**Example 4.4.** \( L = \partial/\partial t + (\alpha + \beta \cos t) \partial/\partial x, \alpha, \beta \in \mathbb{R} \) and \( \alpha \) has a continued fraction \( \kappa_n^{-1}(1/a_n) \) with \( a_{n+1} > \exp q_n \), for all \( n \in \mathbb{N} \) (here \( q_n \) is the denominator of the \( n \)th convergent to \( \alpha \); see Example 4.9 for more details about continued fractions). This \( \alpha \) is EL and so \( L \) is not GAH (for another, more explicit, example one may take \( a_n = n!, n \in \mathbb{N} \); see [G]).

We now give examples of systems of two vector fields \( L = (L_1, L_2) \) on \( T^1 \); Theorem 3.3 implies that only the first two are GAH.

**Example 4.5.** \( L_1 = \partial/\partial t_1 + i(1 - \cos t_1) \partial/\partial x, \)
\( L_2 = \partial/\partial t_2 + i(1 - 2 \cos t_2) \partial/\partial x. \)
Example 4.6. \[ L_1 = \partial/\partial t_1 + \sqrt{2}\partial/\partial x, \]
\[ L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x. \]

Example 4.7. \[ L_1 = \partial/\partial t_1 + i(1 - 2\cos t_1)\partial/\partial x, \]
\[ L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x. \]

Example 4.8. \[ L_1 = \partial/\partial t_1 + \alpha\partial/\partial x, \]
\[ L_2 = \partial/\partial t_2 + i(1 - 2\cos t_2)\partial/\partial x, \]
where \( \alpha \) is as in Example 4.4.

In order to motivate the next example, we begin by remarking that, for each \( j \), the operator \( L_j = \partial/\partial t_j + c_j(t_j)\partial/\partial x \) may be considered as an operator acting on functions or distributions depending only on two variables, namely \((t_j, x) \in T^2\). Thus we may ask whether \( L_j \) is GAH on \( T^2 \) or not.

Note that in Examples 4.5 and 4.6, where \( L \) is GAH, one of the vector fields (namely \( L_1 \)) was also GAH on \( T^2 \). Hence it makes sense to ask if this is always the case; more precisely, does the fact that \( L \) is GAH on \( T^{n+1} \) imply that at least one of the vector fields \( L_j \) is GAH on \( T^2 \)? The answer is no, as the following example will show.

Example 4.9. \[ L_1 = \partial/\partial t_1 - \alpha\partial/\partial x, \]
\[ L_2 = \partial/\partial t_2 - \beta\partial/\partial x. \]

We are going to construct two exponential Liouville numbers \( \alpha, \beta \) such that \((\alpha, \beta)\) is not an exponential Liouville vector. The conclusion will be that \( L \) is GAH on \( T^4 \) even though neither \( L_1 \) nor \( L_2 \) is GAH on \( T^2 \).

The numbers \( \alpha, \beta \in (0, 1) \) will be constructed by means of their (simple) continued fractions, namely

\[ \alpha = \kappa_{n=1}^\infty(1/a_n), \quad \beta = \kappa_{n=1}^\infty(1/b_n), \]

where each \( a_n, b_n \in \mathbb{N} \).

The best rational approximations to an irrational number, say \( \alpha \), are in a certain sense, the convergents \( p_n/q_n \); these are given recursively by \( p_1 = 1, q_1 = a_1, p_2 = a_2, q_2 = a_2a_1 + 1 \), and, for \( n \geq 3 \), \( p_n = a_np_{n-1} + p_{n-2}, q_n = a_nq_{n-1} + q_{n-2} \).

Legendre’s theorem says that if \( |\alpha - p/q| < 1/2q^2 \), then \( p/q = p_n/q_n \), for some \( n \).

On the other hand, the convergents satisfy the following:

\[ \frac{1}{(2 + a_{n+1})q_n^2} \leq \frac{1}{q_n(q_n + q_{n+1})} \leq |\alpha - p_n/q_n| \leq \frac{1}{q_nq_{n+1}} \leq \frac{1}{a_{n+1}q_n^2}. \]

Let us agree to say that \( p/q \) is a good approximation to \( \alpha \) when \( |\alpha - p/q| < 1/3q^2 \).

Note that, when \( a_{n+1} = 1 \), \( p_n/q_n \) is not a good approximation to \( \alpha \). On the other hand, when \( a_{n+1} = \lfloor \exp(q_n) \rfloor + 1 \) (\( \lfloor x \rfloor \) denoting the largest integer not exceeding the real number \( x \)), as will be the case below for certain values of \( n, p_n/q_n \), will be a (very) good approximation to \( \alpha \); we will need to know that, for large \( t \in \mathbb{N} \), \( tp_n/tq_n \) is not a good approximation to \( \alpha \). We have, if \( t \geq t_n = 1 + \max\{2, \lfloor \exp(q_n/2) \rfloor \} \),

\[ |\alpha - tp_n/tq_n| = |\alpha - p_n/q_n| \geq \frac{1}{(2 + a_{n+1})q_n^2} \]
\[ \geq \frac{1}{(3 + \exp q_n)q_n^2} \geq \frac{1}{(3 + t^2)q_n^2} \]
\[ \geq \frac{1}{2t^2q_n^2} > \frac{1}{3(tq_n)^2}; \]

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hence, for such \( t \), \( tp_n/tq_n \) is not a good approximation to \( \alpha \).

The convergents to \( \beta \) (resp.\( \alpha \)) will be denoted \( r_n/s_n \) (resp. \( p_n/q_n \)).

To construct \( \alpha \) (resp. \( \beta \)) we will choose most of the \( a_n \) (resp. \( b_n \)) equal to 1; the remaining \( a_n \) (resp. \( b_n \)) will be very large. The continued fractions (with a new notation) are

\[
\alpha = [1, 1, a_{k+1}, 1, \ldots, 1, a_{k_3+1}, 1, \ldots],
\]

\[
\beta = [1, 1, 1, b_{k_2+1}, 1, \ldots, 1],
\]

where \( k_1 < k_2 < \cdots \), and

\[
a_{k_1+1} = \lfloor \exp(q_{k_1}) + 1, \ell = 1, 3, 5, \ldots, \]

\[
b_{k_2+1} = \lfloor \exp(s_{k_2}) + 1, \ell = 2, 4, 6, \ldots, \]

These conditions imply

\[
|\alpha - p_{k_1}/q_{k_1}| \leq \frac{1}{q_{k_1}q_{k_1+1}} \leq \frac{1}{a_{k_1+1}q_{k_1}^2} \leq \exp(-q_{k_1})
\]

and, similarly, \( |\beta - r_{k_2}/s_{k_2}| \leq \exp(-s_{k_2}) \), hence both \( \alpha \) and \( \beta \) are EL.

In order to achieve the goal of having \( (\alpha, \beta) \) not EL we must make a very careful choice of the sequence \( k_1, k_2, \ldots \); basically, it will have to grow very fast.

We define \( k_1 < k_2 < k_3 < \cdots \) recursively by setting \( k_1 = 2 \), and for \( \ell \geq 2 \), we require \( k_\ell \) to be such that

\[
s_{k_\ell} \geq q_{\ell-1} = \lfloor \exp(q_{k_{\ell-1}/2}) + 1, \ell \text{ is even},
\]

and such that

\[
q_{k_\ell} \geq q'_{\ell-1} = \lfloor \exp(s_{k_{\ell-1}/2}) + 1, \ell \text{ is odd}.
\]

We get \( q_{k_1} < q'_{1} < s_{k_2} < s'_{2} < q_{k_3} < q'_{3} < \cdots \). Set

\[
I_\ell = \{ q \in \mathbb{N} : q_k \leq q \leq q'_{\ell} \}, \quad \ell = 1, 3, 5, \ldots,
\]

and

\[
J_\ell = \{ q \in \mathbb{N} : s_k \leq q \leq s'_{\ell} \}, \quad \ell = 2, 4, 6, \ldots.
\]

Note that \( I_\ell \cap J_m = \emptyset \), for all \( \ell = 1, 3, 5, \ldots, m = 2, 4, 6, \ldots \).

We claim that the good approximations to \( \alpha \) (resp. \( \beta \)) have denominators belonging to some \( I_\ell \) (resp. \( J_m \)); this will imply that there are no good approximations to \((\alpha, \beta)\) with the same denominator, i.e., one has \( |(\alpha, \beta) - (p/q, r/q)| \geq 1/(3q^2) \), for all \( p, r \in \mathbb{Z}, q \in \mathbb{N} \); this will imply that \( (\alpha, \beta) \) is not even a Liouville vector; indeed, for any \( \varepsilon > 0 \), the inequality \( \exp(-\varepsilon q) > 1/3q^2 \) has (at most) a finite number of solutions \( q \in \mathbb{N} \); in other words, one has \( |(\alpha, \beta) - (p/q, r/q)| \geq \exp(-\varepsilon q) \), except for a finite number of \((p/q, r/q)\).

It remains to prove our claim; we will prove it only for \( \alpha \). We have, by Legendre’s theorem, \( |\alpha - p/q| \geq 1/2q^2 \), provided \( p/q \neq p_n/q_n, n = 1, 2, \ldots \).

Now, when \( n_{\ell+1} = 1 \), we have \( |\alpha - p_n/q_n| \geq 1/3q_n^2 \); thus we also have, for all \( t \in \mathbb{N}, |\alpha - p_n/q_n| \geq 1/3tq_n^2 \).

Finally, when \( n = \ell, \ell = 1, 3, \ldots \), we have seen that \( t \geq t_{\ell} \) implies \( |\alpha - p_{k_{\ell}}/q_{k_{\ell}}| \geq 1/3tq_{k_{\ell}}^2 \).

The conclusion is that if \( p/q \) is a good approximation to \( \alpha \), then \( p = tp_{k_\ell}, q = tq_{k_\ell}, \) for some \( \ell = 1, 3, \ldots \) and some \( t \in \mathbb{N} \) with \( 1 \leq t \leq t_{\ell} \); in other words, if \( p/q \) is a good approximation to \( \alpha \), then necessarily \( q_{k_\ell} \leq q \leq q'_{\ell}, \) for some \( \ell = 1, 3, \ldots \).
This concludes the analysis of Example 4.9.

**Example 4.10** (systems with constant coefficients). \( L_j = \partial/\partial t_j + (a_j + ib_j)\partial/\partial x, j = 1, \ldots, n. \) Here \( L \) is GAH if and only if either some \( b_j \neq 0 \) or else \((a_1, \ldots, a_n)\) is neither EL nor an element of \( Q^n. \)

**Example 4.11** (systems of real vector fields). \( L_j = \partial/\partial t_j + a_j(t)\partial/\partial x, j = 1, \ldots, n, \) where each \( a_j \) is real-analytic, real-valued; we also assume that the system is involutive, i.e., \( \partial a_j/\partial t_k = \partial a_k/\partial t_j, \) for all \( j, k. \) Here, the conjugation with \( \exp(\text{in}A(t)), \) where \( A(t) = \sum_{j=1}^n \int_0^t a_j(s)\,ds_j - a_0 \cdot t, \) where \( a_0 = (a_{01}, \ldots, a_{0n}) \) and \( a_{0j} = (2\pi)^{-1} \int_0^\pi a_j(t)\,dt, \) reduces \( L \) to the constant coefficient system \( L = (L_1, \ldots, L_n), \) where \( L_j = \partial/\partial t_j + a_{0j}\partial/\partial x. \) Thus \( L \) is GAH if and only if \( a_0 \) is neither EL nor an element of \( Q^n. \)

**Example 4.12**. \( L_j = \partial/\partial t_j + ib_j(t_j)\partial/\partial x, j = 1, \ldots, n \) with each \( b_j \neq 0. \) Here \( L \) is GAH if and only if some \( b_j \) does not change sign. Also, \( L \) is GAH if and only if \( L \) is (locally) analytic hypoelliptic.

5. Concluding remarks

The paper [GPY] studies global regularity for several classes of operators, especially second-order ones. The authors announce a result about global Gevrey hypoellipticity, for all Gevrey indices \( \sigma \) with \( 1 \leq \sigma < \infty, \) for the first-order operators (2.1). They prove the sufficiency in all cases and the necessity when \( 1 < \sigma < \infty, \) where cut-off functions can be used. Our proof of Theorem 2.3 may be viewed as a completion of the proof of Theorem 3.4 in [GPY].

The article [Ca-Ho] contains a theorem with the same statement as that of our Theorem 2.3. However, what is proved there is that (2.3)–(2.4) are equivalent to the following notion of GAH: the conditions \( u \in C^\infty_c(S^1_1; D'(S^1_1)) \) and \( Lu \in C^\infty_c(S^1_1; C^\omega(S^1_1)) \) imply \( u \in C^\infty_c(S^1_1; C^\omega(S^1_1)) \). In this context, one is free to use cut-off functions in the \( t \)-variable; this renders the construction of singular solutions a simple matter.

A crucial point in the construction of singular solutions, namely in the case when \( b(t) \) changes sign, was the choice of a right-hand side \( f \in C^\omega(T^2) \) which extended holomorphically, in the \( x \)-variable, to a strip of finite width, namely \( |3x| < A - \varepsilon. \) We remark that, were \( b(t) \) and \( f = Lu \) entire, then the distribution \( u \) would automatically be entire as well (\( L^{-1} \) destroys only a finite width). Thus we can say that, when \( b(t) \neq 0, \) \( L \) is always globally entirely hypoelliptic.

The paper [Co-Hi] studies GAH for sums of squares of real vector fields, under the assumption that each point is of finite type. We consider the sum of squares \( P = L_1^2 + \cdots + L_n^2, \) where \( L = (L_1, \ldots, L_n) \) is an involutive system of real vector fields, and remark that it is always of infinite type.

We claim that \( P \) is GAH on \( T^{n+1} \) if and only if \( a_0 \) is neither EL nor an element of \( Q^n. \) In view of Example 4.11, this is the same as saying that \( P \) is GAH if and only if \( L \) is GAH. It is well-known that \( P \) GAH implies \( L \) GAH; we must prove the converse.

Let \( u \in D'(T^{n+1}) \) be such that \( Pu = f \in C^\omega(T^{n+1}). \) Set \( v_j = L_ju, j = 1, \ldots, n. \) We see that \( v_1, \ldots, v_n, \) must verify \( L_1v_1 + \cdots + L_nv_n = f \) and \( L_jv_k - L_kv_j = 0, \) \( j, k = 1, \ldots, n. \)

It suffices to show that these equations have, up to a constant, a unique distribution solution, \((v_1, \ldots, v_n)\) which, furthermore, is real-analytic.
For simplicity, we will assume \( n = 2 \), the case of general \( n \) being similar. We must solve \( L_1 v_1 + L_2 v_2 = f \), \(-L_2 v_1 + L_1 v_2 = 0\).

Set \( S v = w \), \( S f = g \), where \( S \) is the automorphism in Example 4.11. We now have to solve \( L_1 w_1 + L_2 w_2 = g \), \(-L_2 w_1 + L_1 w_2 = 0\).

By taking Fourier series (in all variables) we see that we must solve

\[
\begin{align*}
\{ k_1 + a_{01j}]\overline{w}_1(j, k) + [k_2 + a_{02j}]\overline{w}_2(j, k) &= -i\hat{g}(j, k), \\
-[k_2 + a_{02j}]\overline{\hat{w}}_1(j, k) + [k_1 + a_{01j}]\overline{\hat{w}}_2(j, k) &= 0.
\end{align*}
\]

We have, for any \( \epsilon > 0 \), \( D_{jk} \doteq (k_1 + a_{01j})^2 + (k_2 + a_{02j})^2 \geq \exp(-\epsilon(|j| + k)) \), for \( |j| + |k| \) large.

It is now easy to finish the proof.

A study of global hypoellipticity (GH), i.e., \( C^\infty \) rather than \( C^\omega \) regularity, was carried out in [BCM] in a more general context. The techniques of the present paper, coupled with the use of cut-off functions, furnish results of GH for a class not covered in [BCM]. In fact, if we allow \( b_j(t_j) \) to be \( C^\infty \) we get the perfect analogue of Theorem 3.3: it suffices to replace GH for GAH and Liouville vector for EL.

Perturbations of a non-GAH vector field by a term of order zero may turn out to be GAH, in contrast with what happens in the usual (local) analytic hypoellipticity for operators of principal type (see [T1]). Actually, for any \( \alpha \in \mathbb{R} \), most (in the sense of Lebesgue measure) perturbations \( \partial_t - \alpha \partial_x - \lambda \), \( \lambda \in \mathbb{R} \), are indeed GAH. Furthermore, by mimicking the constructions (via continued fractions) in [B], one can produce two exponential Liouville numbers \( \alpha, \beta \) such that \( \partial_t - \alpha \partial_x - 1/2 \) is GAH but \( \partial_t - \beta \partial_x - 1/2 \) is not GAH.

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