

## MINIMAL LATTICE-SUBSPACES

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ABSTRACT. In this paper the existence of minimal lattice-subspaces of a vector lattice  $E$  containing a subset  $B$  of  $E_+$  is studied (a lattice-subspace of  $E$  is a subspace of  $E$  which is a vector lattice in the induced ordering). It is proved that if there exists a Lebesgue linear topology  $\tau$  on  $E$  and  $E_+$  is  $\tau$ -closed (especially if  $E$  is a Banach lattice with order continuous norm), then minimal lattice-subspaces with  $\tau$ -closed positive cone exist (Theorem 2.5).

In the sequel it is supposed that  $B = \{x_1, x_2, \dots, x_n\}$  is a finite subset of  $C_+(\Omega)$ , where  $\Omega$  is a compact, Hausdorff topological space, the functions  $x_i$  are linearly independent and the existence of finite-dimensional minimal lattice-subspaces is studied. To this end we define the function  $\beta(t) = \frac{r(t)}{\|r(t)\|_1}$  where  $r(t) = (x_1(t), x_2(t), \dots, x_n(t))$ . If  $R(\beta)$  is the range of  $\beta$  and  $K$  the convex hull of the closure of  $R(\beta)$ , it is proved:

- (i) There exists an  $m$ -dimensional minimal lattice-subspace containing  $B$  if and only if  $K$  is a polytope of  $\mathbb{R}^n$  with  $m$  vertices (Theorem 3.20).
- (ii) The sublattice generated by  $B$  is an  $m$ -dimensional subspace if and only if the set  $R(\beta)$  contains exactly  $m$  points (Theorem 3.7).

This study defines an algorithm which determines whether a finite-dimensional minimal lattice-subspace (sublattice) exists and also determines these subspaces.

### 1. INTRODUCTION

It is known that  $C[0, 1]$  is a universal Banach space in the sense that every separable Banach space is isometric to a closed subspace of  $C[0, 1]$ . In [11] it is shown that each separable Banach lattice is order-isomorphic to a closed lattice-subspace of  $C[0, 1]$ ; therefore  $C[0, 1]$  is also a universal Banach lattice. Since the sublattices of  $C[0, 1]$  are not enough for this representation, the lattice-subspaces seems to be the right class of subspaces for studying Banach lattices.

The structure of lattice-subspaces has not been systematically studied. In [7] it is shown that a subspace  $X$  of a vector lattice is a lattice-subspace if and only if there exists a positive projection from the vector sublattice generated by  $X$  onto  $X$ . In [10] and [11] the existence of positive bases in lattice-subspaces is studied. A survey of lattice-subspaces and positive projections, as well as some new results, is proved in [1]. In [12] the finite-dimensional lattice-subspaces of  $C(\Omega)$  are studied.

In the present paper the existence of minimal lattice-subspaces of a vector lattice  $E$  which contains a subset  $B$  of  $E_+$  is studied. In the theory of Banach lattices (and

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in applications) we are interested in a lattice-subspace of  $E$  containing  $B$  which is as “close” as possible to the linear subspace  $[B]$  generated by  $B$ .

Such a subspace is the sublattice  $S(B)$  generated by  $B$  (note that  $S(B)$  is the minimum sublattice containing  $B$  and also that  $S(B) = [B]^\vee - [B]^\vee$  where  $[B]^\vee$  is the set of finite supremum of the elements of  $[B]$ ) but  $S(B)$  is in general a “big” subspace which is “very far” from  $[B]$ . In Example 3.18  $[B]$  is 3-dimensional,  $S(B)$  is dense in  $C(\Omega)$  but a 4-dimensional lattice-subspace containing  $B$  exists. In Example 3.21 it is shown that a minimum lattice-subspace containing  $B$  does not always exist.

An important question is “how far” a minimal lattice-subspace is from  $[B]$ . Motivated by this question we study the existence of finite-dimensional minimal lattice-subspaces. Especially we suppose that  $B = \{x_1, x_2, \dots, x_n\}$  is a subset of  $C_+(\Omega)$ , the vectors  $x_i$  are linearly independent and we study the existence of finite-dimensional minimal lattice-subspaces of  $C(\Omega)$  containing  $B$ . In the framework of this problem we study also the question whether  $S(B)$  is a finite-dimensional subspace.

To study this problem we define the function  $\beta(t) = \frac{r(t)}{\|r(t)\|_1}$  where  $r(t) = (x_1(t), x_2(t), \dots, x_n(t))$ . This function defines a curve in the simplex  $\Delta_n$  of  $\mathbb{R}_+^n$  which we call basic curve of the functions  $x_i$  and is very important for our study.

In Theorem 3.7 it is proved that  $S(B)$  is finite-dimensional if and only if the range  $R(\beta)$  of  $\beta$  is finite and a positive basis of  $S(B)$  is also determined. Hence we can determine whether  $S(B)$  is finite-dimensional because it is very easy to check if  $R(\beta)$  is finite or not. By the property that  $S(B) = [B]^\vee - [B]^\vee$  we cannot conclude whether  $S(B)$  is finite-dimensional and also we cannot determine a positive basis of  $S(B)$ .

In Theorem 3.10 it is proved that if the convex hull  $K$  of the closure of  $R(\beta)$  is a polytope with  $m$  vertices, then an  $m$ -dimensional minimal lattice-subspace  $Y$  exists and a positive basis of  $Y$  is given. The determination of the basis of  $Y$  is based on the determination of the vertices of  $K$ .

In general it is difficult to study whether  $K$  is a polytope or not and determine its vertices. In Corollary 3.15 it is proved that if  $K$  is a polytope,  $\beta(t_0)$  a vertex of  $K$  and  $t_0$  an interior point of a curve  $c$  of  $\Omega$ , then the derivative at  $t_0$  (whenever it exists) of the restriction of  $\beta$  on  $c$  is equal to zero. If for example  $\Omega \subseteq \mathbb{R}^l$  and the function  $\beta$  is defined on the whole set  $\Omega$ , then the partial derivatives of  $\beta$  at  $t_0$  are equal to zero whenever  $t_0$  is an interior point of  $\Omega$  and the derivatives at  $t_0$  of the restriction of  $\beta$  on the parametric curves of  $\partial(\Omega)$  are equal to zero, if  $t_0 \in \partial(\Omega)$ . Hence  $t_0$  can be obtained as a solution of a system of equations.

This property helps us to determine a set of possible vertices of  $K$ , i.e., a subset  $G$  of  $\mathbb{R}^n$  which contains the vertices of  $K$ , whenever  $K$  is a polytope. After the determination of  $G$  it is easier to study if  $K$  is a polytope or not (see Algorithm 3.17 and Example 3.18).

An interesting remark on the structure of the lattice-subspaces is also that a minimal lattice-subspace containing  $B$  is not necessarily a subspace of  $S(B)$ , Example 3.21.

Recently lattice-subspaces have been employed in economics [2], [3].

Let  $E$  be a (partially) ordered vector space with positive cone  $E_+$  and  $X$  a subspace of  $E$ . The cone  $X \cap E_+$  will be called the *induced cone* of  $X$ , and the ordering defined in  $X$  by this cone the *induced ordering*. We will denote by  $X_+$  the

induced cone of  $X$ , i.e.,  $X_+ = X \cap E_+$ . An *ordered subspace* of  $E$  is a subspace of  $E$  ordered by the induced cone. A *lattice-subspace* of  $E$  is an ordered subspace of  $E$  which is also a vector lattice (Riesz space).

Let  $X$  be a lattice-subspace of  $E$ . Then, for each  $x, y \in X$  we will denote by  $x \nabla y$  (resp.  $x \triangle y$ ) the supremum (resp. infimum) of  $\{x, y\}$  in  $X$ . It is clear that

$$x \vee y \leq x \nabla y \quad \text{and} \quad x \triangle y \leq x \wedge y$$

whenever  $x \vee y, x \wedge y$  exist. If  $E$  is a vector lattice and  $x \nabla y = x \vee y$  for any  $x, y \in X$  then  $X$  is a sublattice (Riesz subspace) of  $E$ . Let  $E$  be an ordered Banach space with positive cone  $E_+$ . A sequence  $\{e_n\}$  is a *positive basis* of  $E$  if  $\{e_n\}$  is a (Schauder) basis of  $E$  and  $E_+ = \{x = \sum_{i=1}^{\infty} \lambda_i e_i \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$ . A positive basis  $\{e_n\}$  of  $E$  is unique (in the sense of a positive multiple). The following result (see [1] or [12]) is very important for the study of finite-dimensional lattice-subspaces. It can be proved either elementary or as a partial result of the Choquet-Kentall Theorem.

**Theorem 1.1.** *A finite-dimensional ordered vector space  $E$  is a vector lattice if and only if  $E$  has a positive basis.*

For notation and terminology not defined here we refer to [4, 6, 9].

## 2. MINIMAL LATTICE-SUBSPACES

Let  $E$  be a vector lattice and  $B \subseteq E_+, B \neq \emptyset$ . Let  $L$  be the set of lattice-subspaces of  $E$ , each of which contains  $B$ . If  $X \in L$  and for any  $Y \in L$  it holds:

$$Y \subseteq X \Rightarrow Y = X,$$

then we will say that  $X$  is a *minimal lattice-subspace* of  $E$  containing  $B$ .

If  $E$  is a vector lattice, then the sublattice generated by  $B$  is the minimum sublattice containing  $B$ .

As we will show later (Example 3.21) even if  $E = \mathbb{R}^m$  a minimum lattice-subspace of  $E$  containing  $B$  does not always exist. So we state the following question:

**Problem 2.1.** *Does a minimal lattice subspace of  $E$  containing  $B$  exist?*

Let  $P$  be a cone of a linear space  $F$  (i.e.,  $P$  is a convex subset of  $F, \lambda x \in P$  for each  $x \in P$  and  $\lambda \in \mathbb{R}_+$  and  $P \cap (-P) = \{0\}$ ). Suppose that  $x, y \in P$ . If there exists  $z \in P$  with the properties:  $z - x, z - y \in P$  and for each  $w \in P, w - x, w - y \in P$  imply that  $w - z \in P$ , then we will say that  $z$  is the supremum of  $\{x, y\}$  in  $P$  and we will denote

$$z = \sup_P \{x, y\}.$$

The infimum of  $\{x, y\}$  in  $P$  is defined analogously. If for each  $x, y \in P, z = \sup_P \{x, y\}$  exists, then  $\inf_P \{x, y\}$  also exists.

*If  $P$  is a cone of a linear space  $F$  and for each  $x, y \in P$  the supremum of  $\{x, y\}$  exists in  $P$ , then we will say that  $P$  is a lattice cone of  $F$ .*

If  $x = x_1 - x_2$  where  $x_1, x_2 \in P$ , then it is easy to show that  $\sup\{x, 0\} = \sup_P \{x_1, x_2\} - x_2$  is the supremum of  $\{x_1, x_2\}$  in  $X = P - P$ . Therefore the following result holds.

*A cone  $P$  of a vector space  $F$  is a lattice-cone if and only if the subspace  $X = P - P$ , ordered by the cone  $P$ , is a vector lattice.*

In the next results of this paragraph we will suppose that  $E$  is a vector lattice equipped with a linear topology  $\tau$  with the properties:

- (i)  $E_+$  is  $\tau$ -closed;
- (ii) each increasing, order bounded net of  $E$  has a  $\tau$ -convergent subnet (i.e., the topology  $\tau$  is Lebesgue).

Property (i) implies also that  $\tau$  is Hausdorff because if we suppose that  $x \in E$ ,  $x \neq 0$  and  $0 \in x + V$  for each open symmetric neighborhood  $V$  of zero, then  $0 \in -x + V$ ; therefore  $x$  and  $-x$  belong to  $E_+$  and hence  $x = 0$ , contradiction.

If the topology  $\tau$  is order continuous (i.e., each decreasing net of  $E$  with infimum zero is  $\tau$ -convergent to zero) and  $E$  is Dedekind complete, then  $\tau$  satisfies (ii). If the order intervals of  $E$  are  $\tau$ -compact, the statement (ii) is also satisfied (for related results see [4, Theorem 11.13]). Hence, the weak star topology of a dual Banach lattice and the weak topology of a Banach lattice with order continuous norm [4, Theorem 12.9], have property (ii).

**Proposition 2.2.** *Let  $(P_i)_{i \in I}$  be a decreasing net of  $\tau$ -closed lattice cones of  $E_+$  (i.e.,  $P_i \subseteq E_+$  and  $i \preceq j \Rightarrow P_i \supseteq P_j$ ). Then  $P = \bigcap_{i \in I} P_i$  is a  $\tau$ -closed lattice cone of  $E$ .*

*Proof.*  $P$  is a  $\tau$ -closed cone of  $E_+$ . Let  $x, y \in P$ . Denote by  $z_i$  the supremum of  $\{x, y\}$  in  $P_i$ . For each  $i, j \in I$  with  $i \preceq j$  we have  $P_j \subseteq P_i \subseteq E_+$ ; therefore,

$$x, y \leq z_i \leq z_j \leq x + y.$$

Since  $\tau$  has property (ii), there exists a  $\tau$ -convergent subnet of  $(z_i)_{i \in I}$  which we will still denote by  $(z_i)_{i \in I}$ . This net is also increasing, and let  $z = \lim_{i \in I} z_i$ . Let  $i \in I$ . Then for each  $j \in I$  with  $i \preceq j$ , we have:

$$z_j, z_j - x, z_j - y \in P_j \subseteq P_i.$$

Since the cone  $P_i$  is  $\tau$ -closed, we have that

$$z, z - x, z - y \in P_i, \quad \text{for each } i \in I.$$

Therefore

$$z, z - x, z - y \in P.$$

Suppose that  $w \in P$  with  $w - x, w - y \in P$ . Since  $P \subseteq P_j$  we have that  $w - z_j \in P_j \subseteq P_i$  for each  $j \in I$  with  $i \preceq j$ . Hence  $w - z \in P_i$  for each  $i$ ; therefore  $w - z \in P$ . So we have proved that  $z = \sup_P \{x, y\}$ ; therefore  $P$  is a lattice cone.  $\square$

**Theorem 2.3.** *Let  $P \subseteq E_+$  be a cone and let  $\Phi(P)$  be the set of  $\tau$ -closed lattice cones of  $E_+$  each of which contains  $P$ . Then  $\Phi(P)$  has minimal elements.*

*Proof.*  $\Phi(P) \neq \emptyset$  because  $E_+ \in \Phi(P)$  and  $\Phi(P)$ , ordered by the relation " $\supseteq$ ", is a partially ordered set. Suppose that  $\mathcal{F}$  is a totally ordered subset of  $\Phi(P)$ . Then by the previous result  $Q = \bigcap_{A \in \mathcal{F}} A$  is a  $\tau$ -closed lattice cone of  $E$ . By Zorn's Lemma the theorem is true.  $\square$

**Proposition 2.4.** *Let  $(X_i)_{i \in I}$  be a decreasing net of lattice-subspaces of  $E$  with  $\tau$ -closed positive cones. Let  $X = \bigcap_{i \in I} X_i$ ,  $Y = X_+ - X_+$  and  $Y_+ = Y \cap E_+$ . Then*

- (i)  $X_+ = \bigcap_{i \in I} X_i^+$ .
- (ii)  $Y \subseteq X$ ,  $Y_+ = X_+$  and  $Y$  is a lattice-subspace of  $E$  with  $\tau$ -closed positive cone.

*Proof.* (i)  $X_+ = X \cap E_+ = (\bigcap_{i \in I} X_i) \cap E_+ = \bigcap_{i \in I} X_i^+$ .  
 (ii)  $Y = X_+ - X_+ \subseteq X$ .  $Y_+ \subseteq X \cap E_+ = X_+$ . Also  $X_+ = X_+ - \{0\} \subseteq Y$ ; therefore  $X_+ \subseteq Y_+$ . Hence  $X_+ = Y_+$ . The net  $(X_i^+)_{i \in I}$  is a decreasing net of  $\tau$ -closed lattice cones of  $E_+$ ; therefore  $Y_+$  is a  $\tau$ -closed lattice cone. Hence  $Y$ , is a lattice-subspace of  $E$ .  $\square$

**Theorem 2.5.** *Let  $B \subseteq E_+$  and*

$$l(B) = \{Y \subseteq E \mid Y \text{ is a lattice-subspace, } Y_+ \text{ is } \tau\text{-closed and } B \subseteq Y\}.$$

*Then  $l(B)$  has minimal elements.*

*Proof.* The set  $l(B)$  is nonempty because it contains  $E$ . The set  $l(B)$ , ordered by the relation “ $\supseteq$ ”, is a partially ordered set. Let  $\mathcal{F}$  be a totally ordered subset of  $l(B)$ . By the previous proposition there exists  $Y \in l(B)$  such that  $Y \subseteq A$  for each  $A \in \mathcal{F}$ . Therefore, by Zorn’s Lemma  $l(B)$  has minimal elements.  $\square$

**Corollary 2.6.** *Let  $E$  be a Banach lattice with order continuous norm and  $B \subseteq E_+$ . Then the set of lattice-subspaces of  $E$  with (norm) closed positive cone which contains  $B$  has minimal elements.*

### 3. THE FINITE-DIMENSIONAL CASE IN $C(\Omega)$

In this paper we shall denote by  $\Omega$  a compact, Hausdorff topological space and by  $C(\Omega)$  the Banach lattice of continuous real valued functions defined on  $\Omega$ .

We will also denote by  $x_1, \dots, x_n$ ,  $n$  fixed linearly independent positive elements of  $C(\Omega)$  and by  $X$  the subspace of  $C(\Omega)$  generated by  $x_1, \dots, x_n$ , i.e.,

$$X = [x_1, x_2, \dots, x_n].$$

In [12] necessary and sufficient conditions in order for  $X$  to be a lattice-subspace of  $C(\Omega)$  are given.

In this paper we study the problem:

**Problem 3.1.** *Does a finite-dimensional lattice-subspace (sublattice) of  $C(\Omega)$  containing  $x_1, x_2, \dots, x_n$  exist?*

For each  $x \in \mathbb{R}^m$  we will denote by  $x(i)$  the  $i$ -coordinate of  $x$ , by  $\|x\|_1$  the norm  $\|x\|_1 = \sum_{i=1}^m |x(i)|$ , by  $\{e_1, e_2, \dots, e_m\}$  the usual basis of  $\mathbb{R}^m$  and by  $\Delta_m$  the simplex (base) of  $\mathbb{R}_+^m$ , i.e.,

$$\Delta_m = \{x \in \mathbb{R}_+^m \mid \|x\|_1 = 1\}.$$

Also if  $x \in \mathbb{R}^m, y \in \mathbb{R}^l$  we shall denote by  $(x, y)$  the vector  $z$  of  $\mathbb{R}^{m+l}$  with  $z(i) = x(i)$  for  $i = 1, 2, \dots, m$  and  $z(m+i) = y(i)$  for  $i = 1, 2, \dots, l$ . If  $A$  is an  $m \times m$  matrix we shall denote by  $A^T$  the transpose and by  $A^{-1}$  the inverse matrix of  $A$ .

Let  $y_1, y_2, \dots, y_m \in C_+(\Omega)$ . Then we will call the function  $v(t) = (y_1(t), y_2(t), \dots, y_m(t)), t \in \Omega$ , the *curve* and the function  $\gamma(t) = \frac{v(t)}{\|v(t)\|_1}, t \in \Omega$ , with  $v(t) \neq 0$ , the *basic curve* of  $y_1, y_2, \dots, y_m$ . We will denote by  $D(\gamma)$  the domain and by  $R(\gamma)$  the range of  $\gamma$ . It is clear that  $D(\gamma)$  is an open subset of  $\Omega$  and  $R(\gamma) \subseteq \Delta_m$ .

In this paper we will denote by  $r$  the curve and by  $\beta$  the basic curve of  $x_1, x_2, \dots, x_n$ , i.e.,

$$r(t) = (x_1(t), x_2(t), \dots, x_n(t)), \quad t \in \Omega \quad \text{and} \quad \beta(t) = \frac{r(t)}{\|r(t)\|_1}.$$

As usual if  $K$  is a subset of a topological space  $F$ , we shall denote by  $\text{int}(K)$  the interior, by  $\bar{K}$  the closure and by  $\partial(K)$  the boundary of  $K$ . Also whenever  $F$  is a linear topological space we shall denote by  $\text{co } K$  the convex hull of  $K$ , by  $\bar{\text{co}}K$  the closure of  $\text{co } K$  and by  $\text{ep}(K)$  the set of extreme points of  $K$ .

**Proposition 3.2** ([12, Proposition 2.2]). *Let  $Y$  be a lattice-subspace of  $C(\Omega)$  with a positive basis  $\{b_1, b_2, \dots, b_n\}$ . Then  $Y$  is a sublattice of  $C(\Omega)$  if and only if the sets  $b_i^{-1}(0, +\infty) = \{t \in \Omega \mid b_i(t) > 0\}$ ,  $i = 1, 2, \dots, n$ , are pairwise disjoint.*

**Theorem 3.3** ([12, Theorem 3.6]). *The statements (i) and (ii) are equivalent:*

- (i)  $X$  is a lattice-subspace of  $C(\Omega)$ .
- (ii) *There exist  $n$  linearly independent vectors  $P_1, P_2, \dots, P_n$  of  $\mathbb{R}^n$ , belonging to the closure of the range of  $\beta$  such that for each  $t \in D(\beta)$  the vector  $\beta(t)$  is a convex combination of the vectors  $P_1, P_2, \dots, P_n$ .*

*If the statement (ii) is true,  $A$  is the  $n \times n$  matrix whose  $i$ th column is the vector  $P_i$  and  $b_1, b_2, \dots, b_n$  are the functions defined by the formula*

$$(1) \quad (b_1, b_2, \dots, b_n)^T = A^{-1}(x_1, x_2, \dots, x_n)^T,$$

*then  $\{b_1, b_2, \dots, b_n\}$  is a positive basis of  $X$ .*

**Lemma 3.4.** *The functions  $y_i \in C_+(\Omega)$ ,  $i = 1, 2, \dots, m$ , are linearly independent if and only if the space generated by the range of the basic curve  $\gamma$  of  $y_i$ ,  $i = 1, 2, \dots, m$ , is  $\mathbb{R}^m$ .*

*Proof.* Let  $W$  be the subspace of  $\mathbb{R}^m$  generated by  $R(\gamma)$ . Then  $W$  is also generated by the range of the curve  $v$  of  $y_i$ ,  $i = 1, 2, \dots, m$ . Let  $\{u_i = v(t_i) \mid i = 1, 2, \dots, l\}$  be a basis of  $W$ . Then  $l \leq m$ .

Suppose that the functions  $y_i$  are linearly independent. Then

$$v(t) = \sum_{i=1}^l \xi_i(t)u_i, \quad \text{for each } t \in \Omega;$$

therefore

$$(2) \quad y_j(t) = \sum_{i=1}^l \xi_i(t)u_i(j), \quad j = 1, 2, \dots, m,$$

where  $u_i(j)$  is the  $j$ -coordinate of  $u_i$ . For each  $t$ , the vector  $(\xi_1(t), \xi_2(t), \dots, \xi_l(t))$  is the unique solution of the system (2); therefore the functions  $\xi_i$  as linear combinations of the functions  $y_i$  belong to  $C(\Omega)$ . By (2) we have also that

$$y_i \in L = [\xi_1, \xi_2, \dots, \xi_l], \quad \text{for each } i;$$

therefore  $m \leq \dim L \leq l$ . Hence  $m = l$  and  $W = \mathbb{R}^m$ .

To prove the converse, suppose that  $l = m$  and

$$\sum_{i=1}^m \lambda_i y_i = 0.$$

Then

$$\sum_{i=1}^m \lambda_i y_i(t_j) = 0 \quad \text{for each } j = 1, 2, \dots, m.$$

Since the vectors  $v(t_i)$ ,  $i = 1, 2, \dots, m$ , are linearly independent, the system has the unique solution  $\lambda_i = 0$  for each  $i$ ; therefore the functions  $y_i$  are linearly independent.  $\square$

**Sublattices.**

**Theorem 3.5.** *Let  $R(\beta) = \{P_1, P_2, \dots, P_n\}$ . (By the previous lemma the vectors  $P_i$  are linearly independent and by Theorem 3.3  $X$  is a lattice-subspace.) Let  $\{b_1, b_2, \dots, b_n\}$  be the positive basis of  $X$  defined by (1) and let  $I_i = b_i^{-1}(0, +\infty)$ , for each  $i$ .*

*Then the following statements hold:*

- (i)  $X$  is a sublattice of  $C(\Omega)$ .
- (ii)  $I_i = \beta^{-1}(P_i)$  for each  $i$  and  $D(\beta) = \bigcup_{i=1}^n I_i$ .
- (iii) *If  $y_i$ ,  $i = 1, 2, \dots, m$ , are linearly independent elements of  $X_+$  and  $\gamma$  is the basic curve of  $y_i$ ,  $i = 1, 2, \dots, m$ , then there exists  $\Phi \subseteq \{1, 2, \dots, n\}$  such that*
  - (a)  $D(\gamma) = \bigcup_{i \in \Phi} I_i$ ,
  - (b) *the function  $\gamma$  is constant on  $I_i$  for each  $i \in \Phi$ ,*
  - (c)  $m \leq l \leq n$ , where  $l$  is the cardinal number of  $R(\gamma)$ .

*Proof.* Let  $z = \sum_{i=1}^n x_i$  and  $B_i = \beta^{-1}(P_i)$ ,  $i = 1, 2, \dots, n$ . Then the sets  $B_i$  are pairwise disjoint and  $D(\beta) = \bigcup_{i=1}^n B_i$ . By (1) we have that

$$\frac{1}{z(t)} (b_1(t), b_2(t), \dots, b_n(t))^T = A^{-1}(\beta(t))^T.$$

Since  $A^{-1} \cdot A = I$ , the dot-product of the  $j$ -row of  $A^{-1}$  and the vector  $P_i$  is equal to 1 if  $i = j$  and 0 whenever  $i \neq j$ ; therefore

$$A^{-1}(\beta(t))^T = (e_i)^T \quad \text{for each } t \in B_i,$$

where  $\{e_1, e_2, \dots, e_n\}$  is the usual basis of  $\mathbb{R}^n$ . Therefore

$$\frac{1}{z(t)} (b_1(t), b_2(t), \dots, b_n(t)) = e_i \quad \text{for each } t \in B_i.$$

Hence for each  $t \in B_i$  it holds:

$$z(t) = b_i(t) > 0 \quad \text{and} \quad b_j(t) = 0 \quad \text{for each } j \neq i.$$

So

$$B_i \subseteq I_i \quad \text{and} \quad B_i \cap I_j = \emptyset \quad \text{for each } j \neq i.$$

Suppose that  $t \in I_i \setminus B_i$ . Since  $D(\beta) = \bigcup_{k=1}^n B_k$ ,  $t \in B_j$  for exactly one  $j \neq i$ . Hence  $I_i \cap B_j \neq \emptyset$ , contradiction. Hence  $B_i = I_i$  for each  $i$ , and by Theorem 3.2,  $X$  is a sublattice. We have also shown the statement (ii).

The basic curve  $\gamma$  is

$$\gamma(t) = \frac{1}{y(t)} (y_1(t), y_2(t), \dots, y_m(t))$$

where  $y = \sum_{i=1}^m y_i$ . Let

$$y_j = \sum_{i=1}^n \mu_{ji} b_i, \quad j = 1, 2, \dots, m.$$

Then  $y = \sum_{i=1}^n \mu_i b_i$  where  $\mu_i = \sum_{j=1}^m \mu_{ji}$  for each  $i$ . Let  $\Phi = \{i \mid \mu_i > 0\}$ . Then it is clear that

$$D(\gamma) = \bigcup_{i \in \Phi} I_i.$$

If  $i \in \Phi$  and  $t \in I_i$ , then

$$\gamma(t) = \frac{1}{\mu_i} (\mu_{1i}, \mu_{2i}, \dots, \mu_{mi}) = Q_i;$$

hence  $\gamma$  is constant on  $I_i$ . Therefore

$$R(\gamma) = \{Q_i \mid i \in \Phi\}.$$

Since  $\Phi$  is a subset of  $\{1, 2, \dots, n\}$ , we have that  $l \leq n$  and by Lemma 3.4,  $m \leq l$ .  $\square$

**Theorem 3.6.** *The following statements are equivalent:*

- (i)  $X$  is a sublattice of  $C(\Omega)$ .
- (ii)  $R(\beta) = \{P_1, P_2, \dots, P_n\}$ .

*Proof.* Let  $X$  be a sublattice of  $C(\Omega)$  and let  $\{b_1, b_2, \dots, b_n\}$  be a positive basis of  $X$ . Let  $x_j = \sum_{i=1}^n \lambda_{ji} b_i$ . Then  $z = \sum_{j=1}^n x_j = \sum_{i=1}^n \lambda_i b_i$  where  $\lambda_i = \sum_{j=1}^n \lambda_{ji}$ . Then the sets

$$I_i = b_i^{-1}(0, +\infty), \quad i = 1, 2, \dots, n,$$

are pairwise disjoint by Proposition 3.2. Hence for each  $t \in I_k$  we have  $x_i(t) = \lambda_{ik} b_k(t)$  and  $x(t) = \lambda_k b_k(t)$ , and therefore

$$\beta(t) = \frac{1}{\lambda_k} (\lambda_{1k}, \lambda_{2k}, \dots, \lambda_{nk}) = P_k.$$

Also  $D(\beta) = \bigcup_{i=1}^n I_i$  because  $t \in D(\beta)$  iff  $z(t) > 0$  iff  $b_i(t) > 0$  for at least one  $i$ . Hence

$$R(\beta) = \{P_1, P_2, \dots, P_n\};$$

therefore the theorem is true.  $\square$

**Theorem 3.7.** *Let  $Z$  be the sublattice of  $C(\Omega)$  generated by  $x_1, x_2, \dots, x_n$  and let  $m \in \mathbb{N}$ . Then the statements (i) and (ii) are equivalent:*

- (i)  $\dim(Z) = m$ .
- (ii)  $R(\beta) = \{P_1, P_2, \dots, P_m\}$ .

If the statement (ii) is true, then  $Z$  is constructed as follows:

- (a) Enumerate  $R(\beta)$  so that its  $n$  first vectors are linearly independent. (Such an enumeration exists by Lemma 3.4.) Denote again by  $P_i$ ,  $i = 1, 2, \dots, m$ , the new enumeration and let  $I_i = \beta^{-1}(P_i)$ ,  $i = 1, 2, \dots, m$ .
- (b) Define the functions

$$x_{n+k}(t) = a_k(t) \|r(t)\|_1, \quad t \in \Omega, \quad k = 1, 2, \dots, m - n,$$

where  $a_k$  is the characteristic function of  $I_{n+k}$ .

- (c)  $Z = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m]$ .

*Proof.* Suppose that (ii) is true and the assumptions (a), (b) are satisfied. We shall show that (c) is true. It is clear that  $m \geq n$ . The sets  $I_i$  are open subsets of  $D(\beta)$  because the sets  $\{P_i\}$  are open subsets of  $R(\beta)$ . Also  $D(\beta) = \bigcup_{i=1}^m I_i$ . Since  $D(\beta)$  is an open subset of  $\Omega$ , the sets  $I_i$  are open, nonempty subsets of  $\Omega$ . Also  $\partial(I_i) \cap I_j = \emptyset$ . Hence  $\partial(I_i) \subseteq \Omega \setminus D(\beta)$ ; therefore  $\|r(t)\|_1 = 0$  for each  $t \in \partial(I_i)$ . This implies that the functions  $x_{n+k}$  are continuous; therefore  $x_{n+k} \in C_+(\Omega)$  for each  $k$ .

Let  $v$  be the curve and  $\gamma$  the basic curve of  $x_i, i = 1, 2, \dots, m$ . Then by the definition of  $x_{n+k}$  we have that

$$v(t) = (r(t), 0) \quad \text{for each } t \in \bigcup_{i=1}^n I_i$$

and

$$v(t) = (r(t), \|r(t)\|_1 e_{i-n}) \quad \text{if } t \in I_i, i > n.$$

Let  $t \in I_i$ . Then

$$\gamma(t) = (\beta(t), 0) = (P_i, 0) = Q_i, \quad \text{if } i \leq n$$

and

$$\gamma(t) = \frac{1}{2} (\beta(t), e_{i-n}) = \frac{1}{2} (P_i, e_{i-n}) = Q_i, \quad \text{for each } i = n + 1, \dots, m.$$

Since  $D(\gamma) = D(\beta) = \bigcup_{i=1}^m I_i$ , we have that

$$R(\gamma) = \{Q_i \mid i = 1, 2, \dots, m\}.$$

The vectors  $Q_i, i = 1, 2, \dots, m$ , are linearly independent. Hence the functions  $x_i, i = 1, 2, \dots, m$ , are also linearly independent; therefore the subspace  $Y$  generated by  $x_i, i = 1, 2, \dots, m$ , is an  $m$ -dimensional sublattice of  $C(\Omega)$  by the previous theorem. Therefore  $Z \subseteq Y$ . Since  $x_i, i = 1, 2, \dots, n$ , are linearly independent elements of  $Z_+$  and the cardinal number of  $R(\beta)$  is  $m$ , by the statement (iii) of Theorem 3.5 we have that  $m \leq \dim Z$ . Therefore  $\dim Z = m$ ; hence  $Z = Y$ .

Suppose now that the statement (i) is true. Then  $x_i, i = 1, 2, \dots, n$ , are linearly independent elements of  $Z_+$ ; therefore by Theorem 3.5, there exist a nonempty subset  $\Phi$  of  $\{1, 2, \dots, m\}$  and nonempty, pairwise disjoint open subsets  $I_i, i \in \Phi$ , of  $\Omega$  such that  $D(\beta) = \bigcup_{i \in \Phi} I_i$  and  $\beta$  is constant on each  $I_i$ . Hence  $R(\beta) = \{P_1, P_2, \dots, P_l\}$  where  $l$  is the cardinal number of  $\Phi$ . By the same theorem we have also that  $n \leq l \leq m$ . As we have proved before, we can construct an  $l$ -dimensional sublattice  $Y$  of  $\Omega$  containing  $x_1, x_2, \dots, x_n$ ; therefore  $Z \subseteq Y$  and  $m \leq l$ . Hence  $l = m$  and therefore the statement (ii) is true. □

**Lattice-subspaces.** A subset  $K$  of  $\mathbb{R}^l$  is a *polytope* if  $K$  is the convex hull of a finite subset of  $\mathbb{R}^l$ . The extreme points of  $K$  are called vertices of  $K$ .

**Theorem 3.8.** *Let  $Y$  be an  $l$ -dimensional lattice-subspace of  $C(\Omega)$  containing  $x_1, x_2, \dots, x_n$ . Suppose that  $\{b_1, b_2, \dots, b_l\}$  is a positive basis of  $Y$ ,*

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, n,$$

$$\sigma_i = \sum_{j=1}^n \lambda_{ji}, \quad i = 1, 2, \dots, l,$$

$$\Phi = \{i \mid \sigma_i \neq 0\},$$

$$P_i = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi,$$

and  $K$  is the convex hull of  $\overline{R(\beta)}$ . Then

- (i)  $P_i \in \overline{R(\beta)}$  for each  $i \in \Phi$ .
- (ii)  $K$  is a polytope with vertices  $P_{i_1}, P_{i_2}, \dots, P_{i_m}$  where  $n \leq m \leq l$  and  $i_\nu \in \Phi$  for each  $\nu = 1, 2, \dots, m$ .

*Proof.* Let  $x_{n+1}, \dots, x_l \in Y_+$  such that

$$Y = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_l].$$

Let

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, l,$$

$$s_i = \sum_{j=1}^l \lambda_{ji}, \quad i = 1, 2, \dots, l,$$

and  $v(t) = (x_1(t), x_2(t), \dots, x_l(t))$ ,  $t \in \Omega$ . Then  $\|v(t)\|_1 = \sum_{i=1}^l s_i b_i$  and the function

$$\gamma(t) = \frac{v(t)}{\|v(t)\|_1}, \quad \|v(t)\|_1 \neq 0,$$

is the basic curve of  $x_1, x_2, \dots, x_l$ . By [12, Proposition 2.3], for each  $i = 1, 2, \dots, l$  there exists a sequence  $(\omega_{i\nu})$  of  $\Omega$  such that

$$\lim_{\nu \rightarrow \infty} \frac{b_j(\omega_{i\nu})}{b_i(\omega_{i\nu})} = 0, \quad \text{for each } j \neq i.$$

Then

$$\lim_{\nu \rightarrow \infty} \frac{x_j(\omega_{i\nu})}{\|v(\omega_{i\nu})\|_1} = \lim_{\nu \rightarrow \infty} \left( \frac{\sum_{k=1}^l \lambda_{jk} \frac{b_k}{b_i}}{\sum_{k=1}^l s_k \frac{b_k}{b_i}} \right) (\omega_{i\nu}) = \frac{\lambda_{ji}}{s_i},$$

therefore

$$(3) \quad \lim_{\nu \rightarrow \infty} \gamma(\omega_{i\nu}) = \frac{1}{s_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{li}) = M_i.$$

Let  $A$  be the  $l \times l$  matrix with columns the vectors  $M_i$ ,  $i = 1, 2, \dots, l$ . Then using the expansion of  $x_i$  relative to the positive basis of  $Y$  we get

$$(4) \quad (x_1, x_2, \dots, x_l)^T = A(s_1 b_1, s_2 b_2, \dots, s_l b_l)^T.$$

Since  $\{x_1, x_2, \dots, x_l\}$  is also a basis of  $Y$ , we have that  $\text{rank } A = l$ ; therefore the vectors  $M_i$ ,  $i = 1, 2, \dots, l$ , are linearly independent. Let

$$(5) \quad \gamma(t) = \sum_{i=1}^l \xi_i(t) M_i$$

be the expansion of  $\gamma(t)$  relative to the basis  $\{M_1, M_2, \dots, M_l\}$  of  $\mathbb{R}^l$ . Then

$$(\gamma(t))^T = A(\xi_1(t), \xi_2(t), \dots, \xi_l(t))^T$$

and by (4) we get

$$(\xi_1(t), \xi_2(t), \dots, \xi_l(t)) = \frac{1}{\|v(t)\|_1} (s_1 b_1(t), s_2 b_2(t), \dots, s_l b_l(t)).$$

Hence  $\xi_i(t) \in \mathbb{R}_+$  and  $\sum_{i=1}^l \xi_i(t) = 1$ . Therefore  $\gamma(t)$  is a convex combination of  $M_1, M_2, \dots, M_l$ . Therefore

$$R(\gamma) \subseteq \text{co}\{M_1, M_2, \dots, M_l\}.$$

Let  $P(x) = (x(1), x(2), \dots, x(n))$ ,  $x \in \mathbb{R}^l$ , be the natural projection of  $\mathbb{R}^l$  onto  $\mathbb{R}^n$ . Then

$$(6) \quad P\left(\frac{s_i}{\sigma_i} M_i\right) = P_i, \quad \text{for each } i \in \Phi.$$

If  $i \notin \Phi$ , then  $P(M_i) = 0$ , because  $\sigma_i = 0$  and therefore  $\lambda_{ki} = 0$  for each  $k = 1, 2, \dots, n$ . Also

$$\beta(t) = \frac{\|v(t)\|_1}{\|r(t)\|_1} P(\gamma(t)), \quad \text{for each } t \in D(\beta) \subseteq D(\gamma);$$

therefore by (5) we get

$$\beta(t) = \sum_{i \in \Phi} \frac{\|v(t)\|_1}{\|r(t)\|_1} \xi_i(t) \frac{\sigma_i}{s_i} P_i.$$

Since  $\beta(t)$  and  $P_i$  belong to the simplex  $\Delta_n$  of  $\mathbb{R}_+^n$ , we have that  $\beta(t)$  is a convex combination of the vectors  $P_i$ ,  $i \in \Phi$ ; hence

$$R(\beta) \subseteq \text{co}\{P_i \mid i \in \Phi\} = L.$$

Since  $\Phi$  is finite, the set  $L$  is closed; hence  $\overline{R(\beta)} \subseteq L$ . We shall show that  $P_i \in \overline{R(\beta)}$ , for each  $i \in \Phi$ . By (3) and (6) we have that  $P\left(\frac{s_i}{\sigma_i} \gamma(\omega_{i\nu})\right) \rightarrow P_i$ . Since  $P_i \neq 0$ , we have that  $P(\gamma(\omega_{i\nu})) \neq 0$ , for each  $\nu$ . Therefore  $r(\omega_{i\nu}) = \|v(\omega_{i\nu})\|_1 P(\gamma(\omega_{i\nu})) \neq 0$ ; hence  $\omega_{i\nu} \in D(\beta)$ , for each  $\nu$ . Similarly with the proof of (3) we can show that  $P_i = \lim \beta(\omega_{i\nu})$ . Hence  $P_i \in \overline{R(\beta)}$ ; therefore  $K = L$ . Also  $\text{ep}(K) \subseteq \{P_i \mid i \in \Phi\}$ . Hence

$$\text{ep}(K) = \{P_{i_1}, P_{i_2}, \dots, P_{i_m}\}$$

where  $i_\nu \in \Phi$  for  $\nu = 1, 2, \dots, m$ ; therefore

$$K = \text{co}\{P_{i_1}, P_{i_2}, \dots, P_{i_m}\}.$$

By Lemma 3.4, the subspace generated by  $R(\beta)$ , and therefore also by  $K$ , is the space  $\mathbb{R}^n$ . Hence  $\text{ep}(K)$  contains at least  $n$  vectors; therefore  $n \leq m \leq l$ .  $\square$

**Theorem 3.9** ([5, Theorem 2]). *Let  $d_1, d_2, \dots, d_m \in \mathbb{R}^l$  and let the polytope  $D = \text{co}\{d_1, d_2, \dots, d_m\}$ . Then there exist non-negative, real-valued continuous functions  $\xi_1, \xi_2, \dots, \xi_m$  defined on  $D$  such that  $x = \sum_{i=1}^m \xi_i(x) d_i$  and  $\sum_{i=1}^m \xi_i(x) = 1$ , for each  $x \in D$ .*

The previous result in a more general form is given also in [8].

**Theorem 3.10.** *Let the set  $K = \overline{\text{co}\{P_1, P_2, \dots, P_m\}}$  be a polytope with vertices  $P_1, P_2, \dots, P_m$ . Suppose that the  $n$  first vertices  $P_1, P_2, \dots, P_n$  of  $K$  are linearly independent<sup>1</sup>. Suppose also that  $\xi_i$ ,  $i = 1, 2, \dots, m$ , are positive continuous real-valued functions defined on  $D(\beta)$  such that  $\sum_{i=1}^m \xi_i(t) = 1$  and  $\beta(t) = \sum_{i=1}^m \xi_i(t) P_i$ , for each  $t \in D(\beta)$ .*

<sup>1</sup>A such enumeration of the vertices of  $K$  exists by Lemma 3.4.

Let  $x_{n+i}$ ,  $i = 1, 2, \dots, m - n$ , be the functions  $x_{n+i}(t) = \xi_{n+i}(t) \|r(t)\|_1$  for each  $t \in D(\beta)$  and  $x_{n+i}(t) = 0$  if  $t \notin D(\beta)$ . Then

$$Y = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m]$$

is a minimal lattice-subspace of  $C(\Omega)$  containing  $x_1, x_2, \dots, x_n$  and  $\dim Y = m$ .

A positive basis  $\{b_1, b_2, \dots, b_m\}$  of  $Y$  is given by the formula

$$(b_1, b_2, \dots, b_m)^T = A^{-1} (x_1, x_2, \dots, x_m)^T$$

where  $A$  is the  $m \times m$  matrix with columns the vectors  $R_i$ ,  $i = 1, 2, \dots, m$ , defined below, in (8).

*Proof.* We shall show that  $Y$  is a lattice-subspace of  $C(\Omega)$ . Let  $v(t) = (x_1(t), x_2(t), \dots, x_m(t))$ ,  $\gamma(t) = \frac{v(t)}{\|v(t)\|_1}$  and  $l = m - n$ . Then

$$\begin{aligned} v(t) &= (r(t), 0) + (0, \sum_{i=1}^l \xi_{n+i}(t) \|r(t)\|_1 e_i) \\ &= \|r(t)\|_1 \sum_{i=1}^m \xi_i(t) (P_i, 0) + \|r(t)\|_1 \sum_{i=1}^l \xi_{n+i}(t) (0, e_i) \\ (7) \quad &= \|r(t)\|_1 \sum_{i=1}^m \xi_i(t) M_i, \quad \text{for each } t \in D(\beta) \end{aligned}$$

where  $M_i$  are the following vectors of  $\mathbb{R}^m$ :

$$M_i = (P_i, 0) \quad \text{for } i = 1, 2, \dots, n$$

and

$$M_i = (P_{n+i}, e_i) \quad \text{for } i = 1, 2, \dots, l.$$

The vectors  $M_i$  are linearly independent with  $\|M_i\|_1 = 1$  for  $i = 1, 2, \dots, n$  and  $\|M_i\|_1 = 2$  for  $i = n + 1, \dots, m$ . Hence  $\|v(t)\|_1 = \|r(t)\|_1 g(t)$ , where  $g(t) = \sum_{i=1}^m \xi_i(t) \|M_i\|_1 = 1 + \sum_{i=n+1}^m \xi_i(t)$ . Therefore, by (7) we have

$$(8) \quad \gamma(t) = \frac{1}{g(t)} \sum_{i=1}^m \xi_i(t) \|M_i\|_1 R_i, \quad \text{where } R_i = \frac{M_i}{\|M_i\|_1}.$$

Hence  $\gamma(t)$  is a convex combination of  $R_i$ ,  $i = 1, 2, \dots, m$ . We shall show that  $R_i \in \overline{R(\gamma)}$  for each  $i$ . If  $P_i = \beta(t_i)$ , then  $P_i = \sum_{j=1}^m \xi_j(t_i) P_j$  and by our assumption that  $P_i$  is an extreme point of  $K$ , we have that  $\xi_i(t_i) = 1$  and  $\xi_j(t_i) = 0$  for each  $j \neq i$ . Hence by (8) we have

$$\gamma(t_i) = \frac{1}{g(t_i)} \|M_i\|_1 R_i = R_i.$$

If  $P_i \notin R(\beta)$ , then there exists a sequence  $(\omega_\nu)$  of  $D(\beta)$  such that

$$P_i = \lim_{\nu \rightarrow \infty} \beta(\omega_\nu).$$

Then

$$\beta(\omega_\nu) = \sum_{j=1}^m \xi_j(\omega_\nu) P_j.$$

Since  $0 \leq \xi_j(\omega_\nu) \leq 1$ , there exists a subsequence of  $(\omega_\nu)$ , which we will denote again by  $(\omega_\nu)$  such that

$$\lambda_j = \lim_{\nu \rightarrow \infty} \xi_j(\omega_\nu), \quad \text{for each } j = 1, 2, \dots, m.$$

Hence

$$P_i = \sum_{j=1}^m \lambda_j P_j,$$

which implies that  $\lambda_i = 1$  and  $\lambda_j = 0$  for each  $j \neq i$ , because  $P_i$  is an extreme point of  $K$ . By (8) and the definition of  $g$  we have that

$$\lim_{\nu \rightarrow \infty} \gamma(\omega_\nu) = R_i.$$

So by Theorem 3.3,  $Y$  is a lattice-subspace and a positive basis of  $Y$  is as in the formulation of the theorem.

Suppose that  $Z \subseteq Y$  is a lattice-subspace containing  $x_1, x_2, \dots, x_n$  and let  $\dim Z = l$ . Then  $l \leq m$ . By Theorem 3.8 the number  $m$  of vertices of  $K$  is less than or equal to  $l$ ; therefore  $m = l$ . Hence  $Z = Y$ ; therefore  $Y$  is minimal.  $\square$

**Definition 3.11.** Let  $C$  be a convex subset of a normed space  $E$ . We shall say that  $x_0$  is a conic point of  $C$  if  $x_0$  is an extreme point of  $C$ ,  $C \setminus \{x_0\} \neq \emptyset$ , and there exists a real number  $\rho > 0$  such that

$$x_0 + \rho \frac{x - x_0}{\|x - x_0\|} \in C, \quad \text{for each } x \in C, x \neq x_0.$$

**Proposition 3.12.** Let  $D$  be a convex subset of a normed space  $E$  and  $x_0 \in E$ . If  $d = d(x_0, D) > 0$  and  $C = \text{co}(\{x_0\} \cup D)$ , then  $x_0$  is a conic point of  $C$ . (If  $D$  is bounded and closed, then  $C$  is also bounded and closed.)

*Proof.* Let  $x \in C, x \neq x_0$ . Then  $x = \lambda x_0 + (1 - \lambda)y$ , where  $y \in D$  and  $\lambda \in [0, 1]$ . Hence  $x - x_0 = (1 - \lambda)(y - x_0)$ ; therefore

$$\|x - x_0\| = (1 - \lambda) \|y - x_0\| \geq (1 - \lambda) d.$$

Also  $x_0 + l(y - x_0) \in C$  for each  $l \in [0, 1]$ . Therefore

$$x_0 + d \frac{x - x_0}{\|x - x_0\|} = x_0 + \frac{d(1 - \lambda)}{\|x - x_0\|} (y - x_0) \in C.$$

To show that  $x_0$  is an extreme point of  $C$  suppose that  $x_0 = \frac{x_1 + x_2}{2}$  where  $x_1, x_2 \in C$  and  $x_1, x_2 \neq x_0$ . Then  $x_i = \lambda_i x_0 + (1 - \lambda_i)y_i$  with  $\lambda_i \in (0, 1)$  and  $y_i \in D$ . Then  $x_0 = \frac{1}{2 - \lambda_1 - \lambda_2} ((1 - \lambda_1)y_1 + (1 - \lambda_2)y_2) \in D$ , contradiction. Hence  $x_0$  is a conic point of  $C$ .  $\square$

**Example 3.13.** (i) For each cone  $P \neq \{0\}$  of a normed space,  $0$  is a conic point of  $P$ .

(ii) Let  $C$  be a closed, convex, bounded subset of a Banach space  $E$  and let  $x_0$  be an extreme point of  $C$ . If  $C = \overline{\text{co}} \text{ep}(C)$  (i.e.,  $C$  is the closure of the convex hull of the extreme points of  $C$ ) and  $x_0 \notin D = \overline{\text{co}}(\text{ep}(C) \setminus \{x_0\})$ , then  $C = \text{co}(\{x_0\} \cup D)$ ; therefore  $x_0$  is a conic point of  $C$ .

(iii) Each vertex of a polytope  $C$  of  $\mathbb{R}^m$  is a conic point of  $C$ .

---

<sup>2</sup>With  $d(x_0, D)$  we denote the distance from  $x_0$  to  $D$ .

We prove below that the tangent vector of a curve of  $C$  at a conic point of  $C$  is equal to zero.

**Proposition 3.14.** *Let  $C$  be a closed, convex subset of a normed space  $E$  and  $x_0$  be a conic point of  $C$ . Let  $\phi : (-\epsilon, \epsilon) \rightarrow C$  be a function with  $\phi(0) = x_0$  where  $\epsilon$  is a positive real number. Then*

$$\phi'(0) = 0,$$

whenever the derivative  $\phi'(0)$  exists.

*Proof.* Let  $\phi'(0) = \lim_{t \rightarrow 0} \frac{\phi(t) - \phi(0)}{t} \neq 0$ . Then there exists  $\delta > 0$  such that  $\phi(t) \neq \phi(0)$  for each  $|t| < \delta$ . Hence

$$\lim_{t \rightarrow 0^+} \frac{\phi(t) - \phi(0)}{\|\phi(t) - \phi(0)\|} = \lim_{t \rightarrow 0^+} \frac{\phi(t) - \phi(0)}{t} \cdot \lim_{t \rightarrow 0^+} \frac{1}{\left\| \frac{\phi(t) - \phi(0)}{t} \right\|} = \frac{\phi'(0)}{\|\phi'(0)\|},$$

and similarly

$$\lim_{t \rightarrow 0^-} \frac{\phi(t) - \phi(0)}{\|\phi(t) - \phi(0)\|} = -\frac{\phi'(0)}{\|\phi'(0)\|}.$$

Since  $x_0$  is a conic point of  $C$ , there exists  $\rho > 0$  such that

$$x_0 + \rho \frac{x - x_0}{\|x - x_0\|} \in C, \quad \text{for each } x \in C, x \neq x_0.$$

Therefore

$$\lim_{\nu \rightarrow \infty} \left( \phi(0) + \rho \frac{\phi(1/\nu) - \phi(0)}{\|\phi(1/\nu) - \phi(0)\|} \right) = x_0 + \rho \frac{\phi'(0)}{\|\phi'(0)\|} = z_1 \in C$$

and

$$\lim_{\nu \rightarrow \infty} \left( \phi(0) + \rho \frac{\phi(-1/\nu) - \phi(0)}{\|\phi(-1/\nu) - \phi(0)\|} \right) = x_0 - \rho \frac{\phi'(0)}{\|\phi'(0)\|} = z_2 \in C.$$

Hence  $x_0 = \frac{1}{2}(z_1 + z_2)$ , contradiction. Therefore  $\phi'(0) = 0$ .  $\square$

**Corollary 3.15.** *Let the set  $K = \overline{\text{co } R(\beta)}$  be a polytope of  $\mathbb{R}^n$  and let  $\beta(t_0)$  be a vertex of  $K$ . If  $\epsilon$  is a positive real number and  $g : (-\epsilon, \epsilon) \rightarrow \Omega$  is a function with  $g(0) = t_0$  and  $\phi(\lambda) = \beta(g(\lambda))$ , then*

$$\phi'(0) = 0,$$

whenever the derivative exists.

*Remark 3.16.* Suppose that there exists a finite-dimensional lattice-subspace of  $C(\Omega)$  containing  $X$ . Then  $K$  is a polytope of  $\mathbb{R}^n$ . Suppose that  $\beta(t_0)$  is a vertex of  $K$ . If  $c$  is a curve of  $\Omega$  and  $t_0$  an interior point of  $c$ , then the derivative at  $t_0$  of the restriction of  $\beta$  on the curve  $c$  is equal to zero.

If for example  $\Omega \subseteq \mathbb{R}^l$ , then the partial derivatives of  $\beta$  at  $t_0$  are equal to zero whenever  $t_0 \in \text{int}(\Omega)$ . If  $t_0 \in \partial(\Omega)$ , the derivatives at  $t_0$  of the restriction of  $\beta$  on the parametric curves of  $\partial(\Omega)$  are equal to zero.

**Algorithm 3.17.** *Theorem 3.10 and Corollary 3.15 define a process which in many cases, especially when  $\Omega \subseteq \mathbb{R}^l$ , determines whether a finite dimensional minimal lattice-subspace exists and determines also a positive basis of these subspaces. To study this problem we study if  $K$  is a polytope or not.*

If the set  $R(\beta)$  is closed, then each extreme point (vertex)  $P_0$  of  $K = \text{co} R(\beta)$  belongs to  $R(\beta)$ ; therefore  $P_0 = \beta(t_0)$ . Also the geometry of the boundary of  $D(\beta)$  and the differentiability of the functions  $x_i$  are very important for this study.

Let  $\Omega = [a, b]$ , the functions  $x_i$  are differentiable and  $D(\beta) = \Omega$ . Suppose that the set  $K$  is a polytope with vertices  $\beta(t_i)$ ,  $i = 1, 2, \dots, m$ . Then at least  $m - 2$  of  $t_i$  belong to  $(a, b)$ ; therefore the equation

$$(9) \quad \beta'(t) = 0,$$

where  $\beta'$  is the derivative of  $\beta$ , has at least  $m - 2$  roots in  $(a, b)$ . Hence the vertices of  $K$  belong to the set

$$G = \{\beta(t) | t = a, t = b, \text{ or } t \text{ is a root of (9)}\}$$

which we call the set of possible vertices of  $K$ . Let  $D = \text{co} G$ . It is easy to show that  $K$  is a polytope if and only if  $D$  is a polytope and  $R(\beta) \subseteq D$ .

Hence in this case the algorithm is the following:

- (i) Determine equation (9). If this equation does not have at least  $n - 2$  roots in  $(a, b)$ , then  $K$  is not a polytope.
- (ii) Determine the roots  $t_i$  of (9) in  $(a, b)$ .
- (iii) We study whether  $R(\beta) \subseteq D$ . So we study whether  $\beta(t)$  is a convex combination of  $\beta(a), \beta(b), \beta(t_i)$ , for each  $i$ . If  $R(\beta) \not\subseteq D$ , then  $K$  is not a polytope.
- (iv) Determine the vertices of  $K$  and a positive basis of the minimal lattice-subspace, in accordance with Theorem 3.10.

We give three examples below. In (i) it is shown that a finite-dimensional minimal lattice-subspace does not always exist. In (ii) we consider three elements  $x_1, x_2, x_3$  of  $C(\Omega)$ , where  $\Omega$  is a square of  $\mathbb{R}^2$ . We show that a 4-dimensional minimal lattice-subspace  $Y$  exists and a positive basis of  $Y$  is determined. We also remark that the sublattice generated by the elements  $x_i$  is dense in  $C(\Omega)$ . In (iii) the functions  $x_i$  are as in (ii), but  $\Omega$  is a circle of  $\mathbb{R}^2$ . It is shown that a finite-dimensional minimal lattice-subspace does not exist. This difference between (ii) and (iii) depends on the geometry of the boundary of  $\Omega$ .

**Example 3.18.** (i) Let  $\Omega = [0, 1]$ ,  $x_1(t) = 1, x_2(t) = t, x_3(t) = t^2$ . Then

$$\beta(t) = \left( \frac{1}{1+t+t^2}, \frac{t}{1+t+t^2}, \frac{t^2}{1+t+t^2} \right), \quad t \in [0, 1],$$

is the basic curve of  $x_1, x_2, x_3$  and  $\beta'(t) \neq 0$  for each  $t \in (0, 1)$ . Suppose that  $Y$  is a finite-dimensional lattice-subspace of  $C(\Omega)$  containing the functions  $x_i$ . Then  $\dim Y \geq 3$ , and therefore by Theorem 3.8  $K$  is a polytope of  $\mathbb{R}^3$  with at least three vertices,  $\beta(t_1), \beta(t_2), \beta(t_3)$ . Hence  $\beta'(t) = 0$  for at least one point of  $(0, 1)$ , contradiction.

(ii) Let  $\Omega = [0, 1] \times [0, 1]$ ,  $x_1(u, v) = 1, x_2(u, v) = u, x_3(u, v) = v$  and  $X = [x_1, x_2, x_3]$ . Then

$$\beta(u, v) = \left( \frac{1}{1+u+v}, \frac{u}{1+u+v}, \frac{v}{1+u+v} \right), \quad (u, v) \in \Omega,$$

is the basic curve of  $x_1, x_2, x_3$  and let  $K = \text{co} R(\beta)$ . Since the range of  $\beta$  is not finite, the sublattice  $Z$  generated by  $x_1, x_2, x_3$  is an infinite-dimensional subspace of  $C(\Omega)$ , Theorem 3.7. In this example we can also show that  $Z$  is dense in  $C(\Omega)$  because  $Z$  is a sublattice of  $C(\Omega)$  and  $Z$  contains the constant functions.

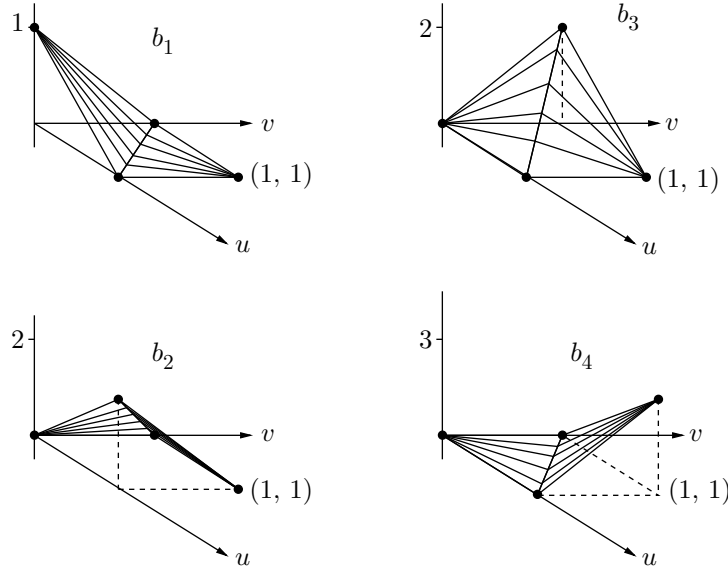


FIGURE 1

In order to study the existence of minimal lattice-subspaces we study whether the set  $K$  is a polytope of  $\mathbb{R}^3$ . To this end suppose that  $K$  is a polytope. Then by Theorem 3.8,  $K$  has at least three vertices and let  $\beta(t_0)$  be a vertex of  $K$ . Then  $t_0$  is also a vertex of  $\Omega$  because in the contrary case  $t_0$  will be an interior point of a line segment parallel to an axis of  $\mathbb{R}^2$ ; therefore, and by the previous corollary, at least one of the partial derivatives of  $\beta$  at  $t_0$  will be equal to zero, contradiction. Hence the points  $P_1 = \beta(0, 0) = (1, 0, 0)$ ,  $P_2 = \beta(1, 0) = (1/2, 1/2, 0)$ ,  $P_3 = \beta(0, 1) = (1/2, 0, 1/2)$  and  $P_4 = \beta(1, 1) = (1/3, 1/3, 1/3)$  define the set of possible vertices of  $K$ . Let  $D = \text{co}\{P_1, P_2, P_3, P_4\}$ . From the above remarks we have that  $K$  is a polytope if and only if  $K = D$  or equivalently if  $R(\beta) \subseteq D$ . It is easy to show that

$$\beta(u, v) = \sum_{i=1}^4 \xi_i(u, v) P_i,$$

where  $\xi_1 \in C(\Omega)$ ,  $\xi_2(u, v) = 2\left(\frac{1-v}{1+u+v} - \xi_1(u, v)\right)$ ,  $\xi_3(u, v) = 2\left(\frac{1-u}{1+u+v} - \xi_1(u, v)\right)$  and  $\xi_4(u, v) = 3\left(\frac{u+v-1}{1+u+v} + \xi_1(u, v)\right)$ .

Since  $\beta(u, v)$  and the points  $P_i$  belong to the plane  $x(1) + x(2) + x(3) = 1$  of  $\mathbb{R}^3$  we have that  $\sum_{i=1}^4 \xi_i(u, v) = 1$ . If  $\xi(u, v) = \frac{1-u-v}{1+u+v}$  and if we put  $\xi_1 = \xi^+$ , then the functions  $\xi_i$ ,  $i = 1, 2, 3, 4$ , are positive and continuous; therefore  $R(\beta) \subseteq D$ . Hence  $K$  is a polytope with vertices  $P_i$ ,  $i = 1, 2, 3, 4$ , and the three first of them are linearly independent. By Theorem 3.10,

$$Y = [x_1, x_2, x_3, x_4],$$

where  $x_4(u, v) = \xi_4(u, v) \|r(u, v)\|_1 = 3(1 - u - v)^+$ , is a minimal lattice-subspace containing  $x_1, x_2, x_3$ .

A positive basis  $\{b_1, b_2, b_3, b_4\}$  of  $Y$  is given by the formula

$$(b_1, b_2, b_3, b_4)^T = A^{-1} (x_1, x_2, x_3, x_4)^T,$$

where  $A$  is the  $4 \times 4$  matrix with columns the vectors  $R_i = \frac{M_i}{\|M_i\|_1}$ ,  $i = 1, 2, 3, 4$ , and  $M_1 = (P_1, 0) = (1, 0, 0, 0)$ ,  $M_2 = (P_2, 0) = (1/2, 1/2, 0, 0)$ ,  $M_3 = (P_3, 0) = (1/2, 0, 1/2, 0)$ ,  $M_4 = (P_4, e_1) = (1/3, 1/3, 1/3, 1)$ .

After the computations we get

$$\begin{aligned}
 b_1(u, v) &= x_1 - x_2 - x_3 + \frac{1}{3}x_4 = \begin{cases} 1 - u - v & | u + v \leq 1, \\ 0 & | u + v > 1, \end{cases} \\
 b_2(u, v) &= 2x_2 - \frac{2}{3}x_4 = \begin{cases} 2u & | u + v \leq 1, \\ 2(1 - v) & | u + v > 1, \end{cases} \\
 b_3(u, v) &= 2x_3 - \frac{2}{3}x_4 = \begin{cases} 2v & | u + v \leq 1, \\ 2(1 - u) & | u + v > 1, \end{cases} \\
 b_4(u, v) &= 2x_4 = \begin{cases} 0 & | u + v \leq 1, \\ 3(u + v - 1) & | u + v > 1 \end{cases} \quad (\text{Figure 1}).
 \end{aligned}$$

(iii) Let  $\Omega = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 \leq 1\}$  and let  $x_i$ ,  $i = 1, 2, 3$ , be the functions of the previous example. Suppose that  $K$  is a polytope and  $\beta(t_0)$  a vertex of  $K$ . As before we have that  $t_0 \in \partial(\Omega)$  and let  $t_0 = (\cos \theta_0, \sin \theta_0)$ . Then by the corollary we have  $\phi'(\theta_0) = 0$  where  $\phi(\theta) = \beta(\cos \theta, \sin \theta)$ . This is a contradiction because  $\phi'(\theta) \neq 0$  for each  $\theta$ . Therefore a finite-dimensional lattice-subspace containing the functions  $x_i$  does not exist.

To study subspaces of  $\mathbb{R}^l$ ,  $l > 1$ , suppose that  $\Omega = \{1, 2, \dots, l\}$ . Then  $C(\Omega) = \mathbb{R}^l$ ,

$$x_i = (x_i(1), x_i(2), \dots, x_i(l)), \quad i = 1, 2, \dots, n,$$

are linearly independent, positive elements of  $\mathbb{R}^l$  and

$$X = [x_1, x_2, \dots, x_n].$$

The curve  $r$  and the basic curve  $\beta$  of the vectors  $x_i$ ,  $i = 1, 2, \dots, n$ , are the functions:

$$r(i) = (x_1(i), x_2(i), \dots, x_n(i)), \quad i = 1, 2, \dots, l,$$

and

$$\beta(i) = \frac{r(i)}{\|r(i)\|_1}, \quad \text{for each } i \text{ with } \|r(i)\|_1 \neq 0.$$

Let  $m$  be the cardinal number of  $R(\beta)$ . Then  $m \leq l$  and by Lemma 3.4,  $n \leq m$ ; therefore  $n \leq m \leq l$ . Let  $K$  be the convex hull of  $R(\beta)$ . Then  $K$ , as the convex hull of a finite subset of  $\mathbb{R}^n$ , is a polytope with  $d$  vertices. It is clear that

$$n \leq d \leq m \leq l$$

and that each vertex of  $K$  belongs to  $R(\beta)$ . Let

$$R(\beta) = \{P_1, P_2, \dots, P_m\}$$

be an enumeration of  $R(\beta)$  such that:

- (i) the vectors  $P_i$ ,  $i = 1, 2, \dots, n$ , are linearly independent and
- (ii) the points  $P_i$ ,  $i = 1, 2, \dots, d$ , are the vertices of  $K$ .

As an application of Theorems 3.6, 3.3, 3.7 and 3.10 we obtain the following:

**Theorem 3.19** (The case of  $\mathbb{R}^l$ ). *Suppose that  $\Omega = \{1, 2, \dots, l\}$  and that the above assumptions are satisfied. Then*

- (i)  $X$  is a sublattice of  $\mathbb{R}^l$  if and only if  $R(\beta)$  contains exactly  $n$  points (i.e.,  $m = n$ ).
- (ii)  $X$  is a lattice-subspace of  $\mathbb{R}^l$  if and only if the polytope  $K$  has  $n$  vertices (i.e.,  $d = n$ ).
- (iii) Let  $m > n$ . If  $I_k = \beta^{-1}(P_k)$ , and

$$x_k = \sum_{i \in I_k} \|r(i)\|_1 e_i, \quad k = n+1, n+2, \dots, m,$$

then

$$Z = [x_1, \dots, x_n, x_{n+1}, \dots, x_m]$$

is the sublattice generated by  $x_1, x_2, \dots, x_n$  and  $\dim Z = m$ .

- (iv) Let  $d > n$ . If  $\xi_i : D(\beta) \rightarrow \mathbb{R}_+$ ,  $i = 1, 2, \dots, d$ , such that  $\sum_{i=1}^d \xi_i(j) = 1$  and  $\beta(j) = \sum_{i=1}^d \xi_i(j) P_i$  for each  $j \in D(\beta)$ , and  $x_{n+i}$ ,  $i = 1, 2, \dots, d-n$ , are the following vectors of  $\mathbb{R}^l$ :

$$x_{n+i} = \sum_{j \in D(\beta)} \xi_{n+i}(j) \|r(j)\|_1 e_j,$$

then

$$Y = [x_1, \dots, x_n, x_{n+1}, \dots, x_d]$$

is a minimal lattice-subspace of  $\mathbb{R}^l$  containing  $x_1, x_2, \dots, x_n$  and  $\dim Y = d$ .

In the following result  $\Omega$  is again a compact, Hausdorff, topological space.

**Theorem 3.20.** Let  $K = \text{co } \overline{R(\beta)}$  and let  $L$  be the set of finite-dimensional minimal lattice-subspaces of  $C(\Omega)$  containing  $x_1, x_2, \dots, x_n$ . Then the following statements are equivalent:

- (i)  $K$  is a polytope with  $m$  vertices.
- (ii)  $L \neq \emptyset$  and  $\dim Y = m$ , for each  $Y \in L$ .
- (iii)  $L \neq \emptyset$ .

*Proof.* Suppose that (i) is true. Then by Theorem 3.10, there exists  $Y \in L$  with  $\dim Y = m$ . Suppose that  $Z \in L$  and  $\{b_1, b_2, \dots, b_l\}$  is a positive basis of  $Z$ . Let

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, n,$$

$$\sigma_j = \sum_{i=1}^n \lambda_{ij}, \quad j = 1, 2, \dots, l,$$

$$\Phi = \{j \mid \sigma_j \neq 0\} \quad \text{and}$$

$$P_i = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi.$$

Then by Theorem 3.8  $P_i \in K$  for each  $i \in \Phi$  and the vertices of  $K$  are among the points  $P_i$ ,  $i \in \Phi$ ; therefore there exist  $i_1, i_2, \dots, i_m \in \Phi$  such that  $P_{i_1}, P_{i_2}, \dots, P_{i_m}$

are the vertices of  $K$ . Also  $n \leq m \leq l$ . Let  $T : Z \rightarrow \mathbb{R}^l$  such that  $T(\sum_{i=1}^l \xi_i b_i) = \sum_{i=1}^l \xi_i e_i$  and let  $y_i = T(x_i)$ ,  $i = 1, 2, \dots, n$ . The basic curve  $b$  of  $y_1, y_2, \dots, y_n$  is:

$$b(i) = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi,$$

with range

$$R(b) = \{P_i \mid i \in \Phi\}.$$

So  $R(b)$  is a subset of  $K$  containing the vertices of  $K$ ; therefore

$$K = \text{co } R(b).$$

Hence  $\text{co } R(b)$  is a polytope with vertices  $P_{i1}, P_{i2}, \dots, P_{im}$ . By the previous theorem, there exists an  $m$ -dimensional lattice-subspace  $F$  of  $\mathbb{R}^l$  containing  $y_1, y_2, \dots, y_n$ . If  $G = T^{-1}(F)$ , then  $G$  is a lattice-subspace of  $Z$  and therefore also of  $C(\Omega)$  containing  $x_1, x_2, \dots, x_n$ . Since  $Z$  is minimal, we have that  $G = Z$ , and therefore  $\dim Z = \dim F = m$ . Hence we have shown that (i)  $\Rightarrow$  (ii).

Suppose now that the statement (ii) is true. Let  $Y \in L$  and  $K = \text{co } \overline{R(\beta)}$ . Then by Theorem 3.8,  $K$  is a polytope with  $k$  vertices and

$$n \leq k \leq m.$$

By Theorem 3.10 there exists  $Z \in L$  with  $\dim Z = k$ . By our assumption we have that  $k = m$ ; therefore  $K$  has  $m$  vertices. Hence (ii)  $\Rightarrow$  (i).

Also (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (i) by Theorem 3.8. □

In the following example we construct the sublattice  $Z$  generated by a four-dimensional subspace  $X$  of  $\mathbb{R}^7$  as well as two minimal lattice-subspaces  $Y$  and  $Y'$  which contain  $X$ . It is remarkable that  $Y \cap Y'$  is not a lattice-subspace as well as that both  $Y$  and  $Y'$  are not subspaces of  $Z$ .

**Example 3.21.** Let

$$\begin{aligned} x_1 &= (1, 2, 1, 0, 1, 1, 4), \\ x_2 &= (0, 1, 1, 1, 1, 0, 2), \\ x_3 &= (2, 1, 0, 1, 1, 1, 2), \\ x_4 &= (1, 0, 1, 1, 1, 0, 0), \end{aligned}$$

and let  $X = [x_1, x_2, x_3, x_4]$ . Let  $r$  be the curve and  $\beta$  the basic curve of  $x_i$ ,  $i = 1, 2, 3, 4$ . Then  $r(1) = (1, 0, 2, 1)$ ,  $r(2) = (2, 1, 1, 0)$ ,  $r(3) = (1, 1, 0, 1)$ ,  $r(4) = (0, 1, 1, 1)$ ,  $r(5) = (1, 1, 1, 1)$ ,  $r(6) = (1, 0, 1, 0)$ ,  $r(7) = (4, 2, 2, 0)$  and  $\beta(1) = \frac{1}{4}(1, 0, 2, 1)$ ,  $\beta(2) = \beta(7) = \frac{1}{4}(2, 1, 1, 0)$ ,  $\beta(3) = \frac{1}{3}(1, 1, 0, 1)$ ,  $\beta(4) = \frac{1}{3}(0, 1, 1, 1)$ ,  $\beta(5) = \frac{1}{4}(1, 1, 1, 1)$ ,  $\beta(6) = \frac{1}{2}(1, 0, 1, 0)$ . In order to enumerate  $R(\beta)$  as in Theorem 3.19 we remark the following:

- (i) The vectors  $P_1 = \beta(4)$ ,  $P_2 = \beta(1)$ ,  $P_3 = \beta(6)$  and  $P_4 = \beta(3)$  are linearly independent.
- (ii) Let  $\beta(2) = P_5$ . Then it is easy to show that for any proper subset  $\Phi$  of  $\{P_1, P_2, P_3, P_4, P_5\}$ ,  $\text{co } \Phi \neq \text{co}\{P_1, P_2, P_3, P_4, P_5\} = K$ ; therefore  $P_i$ ,  $i = 1, 2, 3, 4, 5$ , are vertices of the polytope  $K$ .
- (iii) It is easy also to show that

$$(10) \quad \beta(5) = \frac{3(1-\theta)}{8}P_1 + \theta P_2 + \frac{1-5\theta}{4}P_3 + \frac{3(1-\theta)}{8}P_4 + \theta P_5.$$

Hence for any  $\theta \in [0, \frac{1}{5}]$  the vector  $P_6 = \beta(5)$  is a convex combination of  $P_i$ ,  $i = 1, 2, 3, 4, 5$ ; therefore  $P_6 \in K$ .

Hence

$$R(\beta) = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

and in accordance with the notations of Theorem 3.19,  $n = 4$ ,  $d = 5$  and  $m = 6$ . Since  $n < d$ ,  $X$  is not a lattice-subspace and therefore also  $X$  is not a sublattice of  $\mathbb{R}^7$ . Let  $Z$  be the sublattice of  $\mathbb{R}^7$  generated by  $x_1, x_2, x_3, x_4$ . In order to determine  $Z$  we define the sets

$$I_5 = \beta^{-1}(P_5) = \{2, 7\}, \quad I_6 = \beta^{-1}(P_6) = \{5\}$$

and the vectors

$$x_5 = \|r(2)\|_1 e_2 + \|r(7)\|_1 e_7 = 4e_2 + 8e_7$$

and

$$x_6 = \|r(5)\|_1 e_5 = 4e_5.$$

Then by the theorem

$$Z = [x_1, x_2, x_3, x_4, x_5, x_6].$$

By Theorem 3.3 a positive basis  $\{b_1, b_2, b_3, b_4, b_5, b_6\}$  of  $Z$  is given by the formula

$$(b_1, b_2, b_3, b_4, b_5, b_6)^T = A^{-1} (x_1, x_2, x_3, x_4, x_5, x_6)^T,$$

where  $A$  is the  $6 \times 6$  matrix with columns the vectors  $\gamma(i)$ ,  $i = 1, 2, \dots, 6$ , and  $\gamma$  is the basic curve of the vectors  $x_i$ ,  $i = 1, 2, \dots, 6$ . So after the computations we find that  $b_1 = 4e_1$ ,  $b_2 = 8e_2 + 16e_7$ ,  $b_3 = 3e_3$ ,  $b_4 = 3e_4$ ,  $b_5 = 8e_5$  and  $b_6 = 2e_6$ .

To determine a minimal lattice-subspace define the vectors  $\xi_i$ ,  $i = 1, 2, 3, 4, 5$ , of  $\mathbb{R}^7$  such that

$$\sum_{i=1}^5 \xi_i(j) = 1 \quad \text{and} \quad \beta(j) = \sum_{i=1}^5 \xi_i(j) P_i, \quad \text{for each } j = 1, 2, \dots, 7.$$

$$\begin{aligned} \beta(1) = P_2 = \sum_{i=1}^5 \xi_i(1) P_i &\Rightarrow \xi_2(1) = 1 \quad \text{and} \quad \xi_k(1) = 0 \quad \text{for } k \neq 2. \\ \beta(2) = P_5 = \sum_{i=1}^5 \xi_i(2) P_i &\Rightarrow \xi_5(2) = 1 \quad \text{and} \quad \xi_k(2) = 0 \quad \text{for } k \neq 5. \\ \beta(3) = P_4 = \sum_{i=1}^5 \xi_i(3) P_i &\Rightarrow \xi_4(3) = 1 \quad \text{and} \quad \xi_k(3) = 0 \quad \text{for } k \neq 4. \\ \beta(4) = P_1 = \sum_{i=1}^5 \xi_i(4) P_i &\Rightarrow \xi_1(4) = 1 \quad \text{and} \quad \xi_k(4) = 0 \quad \text{for } k \neq 1. \\ \beta(5) = P_6 = \sum_{i=1}^5 \xi_i(5) P_i &\Rightarrow \xi_1(5) = \xi_4(5) = \frac{3(1-\theta)}{8}, \quad \xi_2(5) = \xi_5(5) = \theta, \\ &\quad \xi_3(5) = \frac{1-5\theta}{4}, \quad \text{by (10)}. \\ \beta(6) = P_3 = \sum_{i=1}^5 \xi_i(6) P_i &\Rightarrow \xi_3(6) = 1 \quad \text{and} \quad \xi_k(6) = 0 \quad \text{for } k \neq 3. \\ \beta(7) = P_2 = \sum_{i=1}^5 \xi_i(7) P_i &\Rightarrow \xi_2(7) = 1 \quad \text{and} \quad \xi_k(7) = 0 \quad \text{for } k \neq 2. \end{aligned}$$

Define also the vector

$$\begin{aligned} y_5 &= \sum_{j=1}^7 \xi_5(j) \|r(j)\|_1 e_j = \|r(2)\|_1 e_2 + \theta \|r(5)\|_1 e_5 \\ &= 4e_2 + 4\theta e_5, \quad \theta \in [0, 1/5]. \end{aligned}$$

Suppose that  $\theta > 0$  in  $y_5$  and that  $y'_5$  is the vector corresponding to  $\theta = 0$ , i.e.,  $y'_5 = 4e_2$ . Then the subspaces

$$Y = [x_1, x_2, x_3, x_4, y_5] \quad \text{and} \quad Y' = [x_1, x_2, x_3, x_4, y'_5]$$

are minimal lattice-subspaces containing the vectors  $x_i$ . Since the vectors  $x_1, x_2, x_3, x_4, y_5, y'_5$  are linearly independent, we have  $Y \neq Y'$ . Also  $X = Y \cap Y'$  is not a lattice-subspace. An important remark is that the vectors  $y_5, y'_5$  do not belong to  $Z$ . To show this suppose that  $y_5 \in Z$ . Then  $y_5 \in Z_+$ , and therefore

$$y_5 = \sum_{i=1}^6 \lambda_i b_i, \quad \text{with } \lambda_i \in \mathbb{R}_+ \text{ for each } i.$$

This implies that  $\lambda_2 = 1/2$  and  $\lambda_2 = 0$ , contradiction. Hence  $y_5 \notin Z$ . Also  $y'_5 \notin Z$ . Therefore  $Y, Y'$  are not subspaces of  $Z$ .

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