A CLASSIFICATION OF BAIRE-1 FUNCTIONS

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ABSTRACT. In this paper we give some topological characterizations of bounded Baire-1 functions using some ranks. Kechris and Louveau classified the Baire-1 functions to the subclasses $\mathbb{B}_1^\xi(K)$ for every $\xi < \omega_1$ (where $K$ is a compact metric space). The first basic result of this paper is that for $\xi < \omega$, $f \in \mathbb{B}_1^{\xi+1}(K)$ iff there exists a sequence $(f_n)$ of differences of bounded semicontinuous functions on $K$ with $f_n \to f$ pointwise and $\gamma((f_n)) \leq \omega^\xi$ (where “$\gamma$” denotes the convergence rank). This extends the work of Kechris and Louveau who obtained this result for $\xi = 1$. We also show that the result fails for $\xi = \omega$. The second basic result of the paper involves the introduction of a new ordinal-rank on sequences $(f_n)$, called the $\delta$-rank, which is smaller than the convergence rank $\gamma$. This result yields the following characterization of $\mathbb{B}_1^\xi(K)$: $f \in \mathbb{B}_1^\xi(K)$ iff there exists a sequence $(f_n)$ of continuous functions with $f_n \to f$ pointwise and $\delta((f_n)) \leq \omega^{\xi-1}$ if $1 \leq \xi < \omega$, resp. $\delta((f_n)) \leq \omega^\xi$ if $\xi \geq \omega$.

INTRODUCTION

Let $K$ be a compact metric space and $C(K)$ the set of continuous real-valued functions on $K$. A function $f : K \to \mathbb{R}$ is Baire-1 if there exists a sequence $(f_n)$ in $C(K)$ that converges pointwise to $f$. Let $\mathbb{B}_1(K)$ be the set of bounded Baire-1 functions on $K$. Haydon, Odell and Rosenthal in [H-O-R] and Kechris and Louveau in [K-L] defined the oscillation rank $\beta(f)$ of a general function $f : K \to \mathbb{R}$ and proved that $f$ is Baire-1 iff $\beta(f) < \omega_1$. Also, for every ordinal $\xi < \omega_1$, the subclass $\mathbb{B}_1^\xi(K)$ was defined by Kechris and Louveau in [K-L] to be the set of all $f$ in $\mathbb{B}_1(K)$ such that $\beta(f) \leq \omega^\xi$, and it was proved that $f$ in $\mathbb{B}_1^\xi(K)$ iff $f$ is the uniform limit of differences of bounded semicontinuous functions on $K$ (Theorem 3). Theorem 3 was originally proved in [H-O-R] (where $\mathbb{B}_1^1(K)$ is called $\mathbb{B}_{1/2}(K)$). This is in fact stated in [K-L], just before the statement of their Theorem 1, Section 3.

In this paper we give a general result for $\mathbb{B}_1^\xi(K)$ which is analogous to the above result for $\mathbb{B}_1^1(K)$.

In Theorem 7, we obtain the result that for $\xi < \omega$, $f \in \mathbb{B}_1^{\xi+1}(K)$ iff there exists a sequence $(f_n)$ in DBSC$(K)$ with $f_n \to f$ pointwise and $\gamma((f_n)) \leq \omega^\xi$ (where “$\gamma$” denotes the convergence rank, whose definition is recalled below). This extends the work of [K-L], who obtained this result for $\xi = 1$. We also show in Corollary 9 that the result fails for $\xi \geq \omega$: indeed we obtain there that if $f_n \to f$ pointwise and $\gamma((f_n)) \leq \omega^\xi$ with $(f_n) \subset$ DBSC$(K)$, then also $\beta(f) \leq \omega^\xi$. Also Proposition 12 shows that Theorem 7 fails if we suppose in addition that $\sup_n |f_n|_D < \infty$. In Theorem 8 we obtain that if $f_n \to f$ pointwise, with $f_n$'s Baire-1 functions, $\lambda$ a

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limit ordinal, and \( m < \omega \), with \( \gamma((f_n)) \leq \omega^{\lambda+m} \) and \( \sup_n \beta(f_n) < \omega^{\lambda} \), then \( f \) is Baire-1 with \( \beta(f) \leq \omega^{\lambda+m} \). In Proposition 10 we show by example that this result fails, if we allow \( \sup_n \beta(f_n) = \omega^{\lambda} \) instead (for \( \lambda = \omega \)).

The final result of the paper, Theorem 17, involves the introduction of a new ordinal-rank on sequences \((f_n)\), called the \( \gamma \)-rank, which is smaller than the convergence rank \( \gamma \). This is motivated by a characterization of \( B_{1/4}(K) \) given in [H-O-R]. Theorem 17 yields the following characterization of \( B^\beta \) such that \( B \)

\[ \text{The sequence } (f_n) \text{ of continuous functions converging pointwise and } \delta((f_n)) \leq \omega^{\xi-1} \text{ if } 1 \leq \xi < \omega, \text{resp. } \delta((f_n)) \leq \omega^\xi \text{ if } \xi \geq \omega. \]

In fact, such a sequence \((f_n)\) may be chosen as convex blocks of any sequence \((g_n)\) of continuous functions converging pointwise to \( f \); the analogous result for the \( \gamma \)-rank is due to Kechris and Louveau, and used in a fundamental way in the proof.

1. Definition. Let \( K \) be a compact metric space, \( f : K \to \mathbb{R}, P \subset K \) and \( \varepsilon > 0 \). Let \( P^0_{\varepsilon,f} = P \) and for any ordinal number \( a \) let \( P^{a+1}_{\varepsilon,f} \) be the set of those \( x \in P^a_{\varepsilon,f} \) such that for every open set \( U \) around \( x \) there are two points \( x_1 \) and \( x_2 \) in \( P^a_{\varepsilon,f} \cap U \) such that \( |f(x_1) - f(x_2)| \geq \varepsilon \).

At a limit ordinal \( a \) we set

\[ P^a_{\varepsilon,f} = \bigcap_{\beta < a} P^\beta_{\varepsilon,f}. \]

Let

\[ \beta(f, \varepsilon) = \begin{cases} \text{the least } a \text{ with } K^a_{\varepsilon,f} = \emptyset \text{ if such an } a \text{ exists,} \\ \omega_1, \text{ otherwise.} \end{cases} \]

Define the oscillation rank \( \beta(f) \) of \( f \) by

\[ \beta(f) = \sup\{\beta(f, \varepsilon) : \varepsilon > 0\}. \]

The above rank is defined by Haydon, Odell and Rosenthal in [H-O-R] and Kechris and Louveau in [K-L].

Let \((f_n)\) be a sequence of real functions on \( K, P \subset K \) and \( \varepsilon > 0 \). Let \( P^0_{\varepsilon,(f_n)} = P \) and for any ordinal number \( a \) let \( P^{a+1}_{\varepsilon,(f_n)} \) be the set of those \( x \in P^a_{\varepsilon,(f_n)} \) such that for every open set \( U \) around \( x \) and any \( p \) in \( \mathbb{N} \), there are \( n \) and \( m \) in \( \mathbb{N} \) with \( n > m > p \) and there is \( x' \) in \( P \cap U \) with \( |f_n(x') - f_m(x')| \geq \varepsilon \).

At a limit ordinal \( a \) we set

\[ P^a_{\varepsilon,(f_n)} = \bigcap_{\beta < a} P^\beta_{\varepsilon,(f_n)}. \]

Let

\[ \gamma((f_n), \varepsilon) = \begin{cases} \text{the least } a \text{ with } K^a_{\varepsilon,(f_n)} = \emptyset \text{ if such an } a \text{ exists,} \\ \omega_1, \text{ otherwise.} \end{cases} \]

Define the convergence rank \( \gamma((f_n)) \) of \((f_n)\) by

\[ \gamma((f_n)) = \sup\{\gamma((f_n), \varepsilon) : \varepsilon > 0\}. \]

The derivative sets \( P^1_{\varepsilon,(f_n)} \) are defined by Zalcwasser in [Z], Gillespie and Hurwicz in [G-H]. The convergence rank is defined by Kechris and Louveau in [K-L].
Remark 1. (i) By compactness of $K$ it is easy to see that $\beta(f, \varepsilon)$ and $\gamma((f_n), \varepsilon)$ are isolated ordinals for all positive real numbers $\varepsilon$.

(ii) As in the proof of Corollary 4, section 2 of [K-L], it is easy to prove that $\beta(X_A) = \beta(X_A, 1/2)$ and hence $\beta(X_A)$ is an isolated ordinal.

2. Definition ([H-O-R], [K-L]). Let $K$ be a compact metric space.

(a) DBSC($K$) is the class of differences of two bounded semicontinuous real-valued functions on $K$. Without difficulty it can be shown that DBSC($K$) coincides with the class of those $F : K \to \mathbb{R}$ for which there exist $(f_n) \subset C(K)$ and $C \in \mathbb{R}$ such that $f_n = 0$, $f_n \to F$ pointwise, and $\sum_{n=0}^{\infty} |f_{n+1}(y) - f_n(y)| \leq C$ for all $y \in K$.

(b) We define $| \cdot |_D : DBSC(K) \to \mathbb{R}$ using (a) as follows: $|F|_D$ is the infimum of all positive numbers $C$ satisfying the condition in (a). Then $| \cdot |_D$ is a norm and DBSC($K$) with $| \cdot |_D$ is a Banach space.

3. Theorem ([K-L], Theorem 1, Section 3). $\mathbb{B}^1_1(K)$ is the sup-norm-closure of DBSC($K$).

4. Proposition ([K-L], Lemma 5, Section 2). Let $K$ be a compact metric space, $(f_n), (g_n)$ be the two sequences of functions on $K$, pointwise converging to $f$ and $g$ respectively. If $\xi < \omega_1$ is such that $\gamma((f_n)) \leq \omega^\xi$ and $\gamma((g_n)) \leq \omega^\xi$, then $\gamma((f_n + g_n)) \leq \omega^\xi$.

5. Theorem ([K-L], Theorem 3, Section 2). Let $(f_n)$ be a bounded sequence of continuous functions on $K$, pointwise converging to some (bounded) Baire-1 function $f$.

Then there exists a sequence $(g_n)$ of convex blocks of $(f_n)$ with $\gamma((g_n)) = \beta(f)$.

The following proposition is due to Kechris and Louveau, [K-L], Prop. 9, Section 2.

6. Proposition. Let $f \in \mathbb{B}_1(K)$, $f \geq 0$ and $\xi < \omega_1$ with $\beta(f) \leq \omega^\xi$ and $n \in \mathbb{N}$, $n > 2$. Then there are $n - 2$ sets $A_1, \ldots, A_{n-2}$ with $\beta(X_{A_k}) < \omega^\xi$, such that the function

$$g = \frac{\|f\|_\infty}{n} \sum_{k=1}^{n-2} X_{A_k}$$

satisfies $0 \leq g \leq f \leq g + 2\|f\|_\infty/n$.

7. Theorem ([K-N]). Let $K$ be a compact metric space, $\xi < \omega$ an ordinal and $f \in \mathbb{B}_1(K)$. Then $f \in \mathbb{B}^{\xi+1}_1(K)$ if and only if there is a sequence $(f_n) \subset$ DBSC($K$) converging pointwise to $f$ such that $\gamma((f_n)) \leq \omega^\xi$.

Proof. Necessity. Let $f \in \mathbb{B}^{\xi+1}_1(K)$. Then $\beta(f) \leq \omega^{\xi+1}$.

Case 1. We assume that $f = X_A$. Then by Remark 1(ii) $\beta(X_A)$ is isolated and hence $\beta(X_A) < \omega^{\xi+1}$. Then there is $k < \omega$ such that $\beta(X_A) < k\omega^\xi$. Then there is a decreasing sequence $(F_\eta)_{\eta < k\omega^\xi}$ of closed subsets of $K$ such that

$$A = \bigcup_{\eta < k\omega^\xi} \{ F_\eta \setminus F_{\eta+1} \}.$$ 

We set

$$A_i = \bigcup \{ (F_\eta \setminus F_{\eta+1}) : i\omega^\xi \leq \eta < (i + 1)\omega^\xi, \eta \text{ even} \} \quad \forall i = 0, 1, \ldots, k.$$
Then \(X_A = X_{A_1} + \cdots + X_{A_k}\). By Proposition 4 we shall show the conclusion for \(X_{A_i}, i = 0, 1, \ldots, k\).

Without loss of generality we can assume that \(k = 1\), that is,

\[
A = \bigcup_{\eta < \omega^\xi} (F_\eta \setminus F_{\eta+1}).
\]

Let \(\{\eta_1, \eta_2, \ldots, \eta_n, \ldots\}\) be an enumeration of the set \(\{\eta : \eta \text{ even with } 0 \leq \eta < \omega^\xi\}\).

For every \(n \in \mathbb{N}\) we set:

\[
A_n = \bigcup_{i=1}^{n} (F_{\eta_i} \setminus F_{\eta_i+1}).
\]

Then \(X_{A_n} \in \text{DBSC}(K)\) for every \(n \in \mathbb{N}\) and \(X_{A_n} \to X_A\) pointwise.

We shall show that: \(\gamma((X_{A_n})) < \omega^\xi\).

Let \(0 < \varepsilon < 1\). We prove first that \(K^1_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta < \omega} F_\eta\).

Let \(x \in K^1_{\varepsilon,(X_{A_n})}\) such that \(x \notin \bigcap_{\eta < \omega} F_\eta\).

Then there exists an open neighborhood \(V\) of \(x\) such that \(V \cap \bigcap_{\eta < \omega} F_\eta = \emptyset\).

Since \(K\) is compact we have that \(V\) intersects at most finitely many \((F_\eta)_{\eta < \omega}\). Then since \((F_\eta)_{\eta < \omega}\) is decreasing we have that \(V\) intersects at most finite many \(F_{\eta_n} \setminus F_{\eta_n+1}, n = 1, 2, \ldots\). Hence there is \(n_0 \in \mathbb{N}\) such that \(X_{A_n} | V = X_{A_{n_0}} | V\) for every \(n \geq n_0\) which is a contradiction, because \(x \in K^1_{\varepsilon,(X_{A_n})}\).

By induction we have: \(K^1_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta < \omega^2} F_\eta\).

Again by induction, we have: \(K^\omega_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta < \omega^\xi} F_\eta\) for every \(n < \omega\).

Hence \(K^\omega_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta < \omega^\xi} F_\eta\) and since \(X_{A_n}(y) = 0\) for every \(y \in \bigcap_{\eta < \omega^\xi} F_\eta\) and \(n \in \mathbb{N}\) we have \(K^\omega_{\varepsilon,(X_{A_n})} = \emptyset\), that is, \(\gamma((X_{A_n})) = \omega^\xi + 1 < \omega^\xi\).

**Case 2.** Suppose that \(f \geq 0\). Then using Theorem 5 we find a sequence \((g_n)\) where \(0 \leq g_n = \sum_{i=1}^{k_n} \beta(X_{A_i}) \varepsilon(X_{A_i})\) with \(\beta(X_{A_i}) < \omega^\xi + 1\) for every \(i = 1, 2, \ldots, k_n, n \in \mathbb{N}\), such that:

\[
0 \leq g_1 + \cdots + g_n \leq f \leq g_1 + \cdots + g_n + \frac{\|f\|_\infty}{2^{n+2}} \quad \forall n \in \mathbb{N}.
\]

Then for every \(n > 1\) we have

\[
0 \leq g_n = g_1 + \cdots + g_n + \frac{\|f\|_\infty}{2^{n+1}} - g_1 - \cdots - g_{n-1} - \frac{\|f\|_\infty}{2^{n+1}} \leq f + \frac{\|f\|_\infty}{2^{n+1}} - f \leq \frac{\|f\|_\infty}{2^n}.
\]

Hence \(\|g_n\|_\infty \leq \|f\|_\infty 2^{-n}\) for any \(n > 1\). Without loss of generality we can assume that \(\|f\|_\infty \leq 1\). Then \(\|g_n\|_\infty \leq 2^{-n}\) for every \(n > 1\). Also \(f = \sum_{n=1}^{\infty} g_n\) uniformly.

Since \(\beta(X_{A_i}) < \omega^\xi + 1\) for every \(i = 1, 2, \ldots, k_n, n \in \mathbb{N}\), and by Case 1 and Proposition 4 we have that, for each \(n \in \mathbb{N}\), there is \((g^p_n) \subset \text{DBSC}(K)\) pointwise converging to \(g_n\) such that \(g^p_n \geq 0\) for every \(p \in \mathbb{N}\) and \(\gamma((g^p_n)) \leq \omega^\xi\). For \(\xi = 0\) this is proved by Kechris and Louveau (cf. [K-L]).

Since \(\|g^p_n\|_\infty \leq \|g_n\|_\infty \leq 2^{-n}\) for every \(n > 1\) and \(\|g^p_n\|_\infty \leq \|g_l\|_\infty\) for every \(p \in \mathbb{N}\), we have that for any \(p \in \mathbb{N}\) \(\sum_{n=1}^{\infty} g^p_n < \infty\) uniformly.

For any \(p \in \mathbb{N}\) we set \(g^p = \sum_{n=1}^{\infty} g^p_n\). Since \(g^p \in \text{DBSC}(K)\) for every \(n \in \mathbb{N}\) and the convergence of the series is uniform we have \(g^p \in \mathcal{B}^1_1(K)\) for every \(p \in \mathbb{N}\).
Then, by Theorem 3 we have that for every \( p \in \mathbb{N} \) there exists \( f_p \in \text{DBSC}(K) \) such that \( \|g^p - f_p\|_\infty < \frac{1}{p} \). Then since \( (g^p) \) is pointwise converging to \( f \) we have that and the sequence \( (f_p) \) is also pointwise converging to \( f \).

The proof of Case 2 can be finished by proving that \( \gamma((f_n)) \leq \gamma((g^p)) \leq \omega^\xi \).

We see this, as follows:

Let \( \varepsilon > 0 \), \( P \) be a closed subset of \( K \). We shall show that \( P_{\varepsilon, f_p}^1 \subset P_{\varepsilon/2, g^p}^1 \).

Let \( x \in P_{\varepsilon, f_p}^1 \setminus P_{\varepsilon/2, g^p}^1 \). Then there exists an open subset \( U \) of \( P \) with \( x \in U \) and \( p_0 \in \mathbb{N} \) such that

\[
|g^p(x') - g^p(x')| \leq \varepsilon/2 \quad \forall x' \in U, p, p' \geq p_0.
\]

Let \( p_1 \geq p_0 \) with \( \frac{1}{p_1} < \frac{\varepsilon}{2} \). Then for each \( p, p' \geq p_1 \) and \( x' \in U \) we have

\[
|f_p(x') - g^p(x')| \leq |f_p(x') - g^p(x')| + |g^p(x') - g^p(x')| + |g^p(x') - f_p(x')|
\]

\[
< \frac{1}{p} + \frac{\varepsilon}{2} + \frac{1}{p'} < \varepsilon,
\]

a contradiction since \( x \in P_{\varepsilon, f_p}^1 \). Hence \( \gamma((f_p)) \leq \gamma((g^p)) \).

Note that for \( q, q', p > 1 \), we have

\[
\|g^q - g^{q'}\|_\infty \leq \left\| \sum_{n \leq p} g^q_n - \sum_{n \leq p} g^{q'}_n \right\|_\infty + 4.2^{-p}.
\]

Also, \( \gamma((g^q_n)) \leq \omega^\xi \) for all \( n \in \mathbb{N} \) and by Proposition 4 we have that \( \gamma((\sum_{n \leq p} g^q_n)) \leq \omega^\xi \) and hence by (\( \ast \)) this implies that \( \gamma((g^q)) \leq \omega^\xi \).

**Case 3.** (General case). If \( f \in B_{\xi+1}^+(K) \) then \( f = f^+ - f^- \) where \( f^+ = \max\{f, 0\} \) and \( f^- = -\min\{f, 0\} \). Then \( 0 \leq f^+, f^- \in B^\xi_1(K) \) and from Case 2 there are sequences \((f^n_1), (f^n_2)\) in \( \text{DBSC}(K) \) with \( (f^n_1) \) converging pointwise to \( f^+ \), \( (f^n_2) \) converging pointwise to \( f^- \), \( \gamma((f^n_1)) \leq \omega^\xi \) and \( \gamma((f^n_2)) \leq \omega^\xi \). Then \( f^n_1 - f^n_2 \in \text{DBSC}(K) \) for every \( n \in \mathbb{N} \), \( (f^n_1 - f^n_2) \) converges pointwise to \( f \) and by Proposition 4 we have that \( \gamma((f^n_1 - f^n_2)) \leq \omega^\xi \).

**Sufficiency.** Let \((f_n) \subset \text{DBSC}(K)\) be a sequence converging pointwise to \( f \) with \( \gamma((f_n)) \leq \omega^\xi \). We prove that \( \beta(f) \leq \omega^\xi \).

**Claim.** \( P_{\varepsilon, f}^1 \subset P_{\varepsilon/3, (f_n)}^1 \) for all closed subsets \( P \) of \( K \) and \( \varepsilon > 0 \).

**Proof of claim:** Let \( P \) be a closed subset of \( K \) and \( x \in P_{\varepsilon, f}^1 \setminus P_{\varepsilon/3, (f_n)}^1 \). Then choose an open subset \( V \) of \( P \) with \( x \in V \) and \( n_0 \in \mathbb{N} \) such that

\[
|f_n(y) - f_{n_0}(y)| \leq \varepsilon/3 \quad \forall y \in V, n \geq n_0.
\]

Then \( |f_{n_0}(y) - f_n(y)| \leq \varepsilon/3 \) for all \( y \in V \), all \( n \geq n_0 \) and since \( (f_n) \) converges pointwise to \( f \) we have that \( |f_{n_0}(y) - f(y)| \leq \varepsilon/3 \) for all \( y \in V \).

Then \( \overline{V}_{\varepsilon, f}^\eta \subset \overline{V}^\eta_{\varepsilon, f_{n_0}} \) for all \( \eta < \omega \). Since \( \beta(f_{n_0}) \leq \omega \) we have \( \overline{V}^\eta_{\varepsilon, f_{n_0}} = \emptyset \).

Then \( V \cap P_{\varepsilon^2, f}^1 \subset \overline{V}^\omega_{\varepsilon, f} \subset \overline{V}^\omega_{\varepsilon, f_{n_0}} = \emptyset \), a contradiction, since \( x \in V \cap P_{\varepsilon^2, f}^1 \). Hence the proof of the claim is finished.

By induction and applying the claim we get

\[
K_{\varepsilon, f}^{\omega^n} \subset K_{\varepsilon/3, (f_n)}^{\omega^{m-1}} \quad \forall m < \omega \Rightarrow K_{\varepsilon, f}^{\omega^2} \subset K_{\varepsilon/3, (f_n)}^{\omega^1}.
\]

Also, by induction we have \( K_{\varepsilon, f}^{\omega^{n+1}} \subset K_{\varepsilon/3, (f_n)}^{\omega^n} \) for all \( n < \omega \).
Hence \( K_{\varepsilon,f}^{\omega^{\xi+1}} \subseteq K_{\varepsilon,3,(f_n)}^{\omega^{\xi}} = \emptyset \) and hence \( \beta(f) \leq \omega^{\xi+1} \).

\[ \square \]

Remark 2. In Theorem 7, the sequence \((f_n)\) can in fact also be chosen uniformly bounded (as the proof shows).

For \( \xi = 1 \), Theorem 7 was proved by Kechris and Louveau in [K-L].

8. Theorem ([K-N]). Let \( K \) be a compact metric space, \( f, f_n \in \mathcal{B}_1(K), n \in \mathbb{N}, \) with \((f_n)\) converging pointwise to \( f, \lambda < \omega_1 \) a limit ordinal and \( m < \omega \) such that

\[ \operatorname{sup}\{\beta(f_n) : n \in \mathbb{N}\} < \omega^{\lambda} \quad \text{and} \quad \gamma((f_n)) \leq \omega^{\lambda+m}. \]

Then \( \beta(f) \leq \omega^{\lambda+m} \).

Proof. Since \( \lambda \) is a limit ordinal and \( \operatorname{sup}\{\beta(f_n) : n \in \mathbb{N}\} < \omega^{\lambda} \) we choose a strictly increasing sequence \((\lambda_n)\) such that \( \operatorname{sup}\lambda_n = \lambda \) and \( \operatorname{sup}\{\beta(f_n) : n \in \mathbb{N}\} < \omega^{\lambda_n} \).

Claim. \( P_{\varepsilon,\lambda} \subseteq P_{\varepsilon,3,(f_n)}^{1} \) for all closed subsets \( P \) of \( \varepsilon > 0 \).

[Proof of claim: Let \( P \subseteq K \) be closed, \( \varepsilon > 0 \) and \( x \in P_{\varepsilon,\lambda} \setminus P_{\varepsilon,3,(f_n)}^{1} \). Then there exists an open subset \( V \) of \( P \) with \( x \in V \) and \( n_0 \in \mathbb{N} \) such that

\[ |f_m(y) - f_n(y)| \leq \varepsilon/3 \ \forall y \in V, m \geq n_0. \]

Then \( |f_{m_0}(y) - f(y)| \leq \varepsilon/3 \forall y \in V, n \geq n_0 \) and hence \( V_{\varepsilon,f} \subseteq V_{\varepsilon,3,f_{m_0}}^0 \forall y < \omega^{\lambda_n} \).

Since \( \beta(f_{n_0}) \leq \omega^{\lambda_n} \) implies that \( V_{\varepsilon,3,f_{n_0}} = \emptyset \). Also \( V \cap P_{\varepsilon,\lambda} = \emptyset, \) a contradiction. Hence the proof of the claim is finished.]

By induction and applying the claim we get

\[ K_{\varepsilon,f}^{\omega^{\lambda+1}} \subseteq K_{\varepsilon,3,(f_n)}^{\theta} \forall \theta < \omega^{\lambda} \]

and hence \( K_{\varepsilon,f}^{\omega^{\lambda+1}} = \bigcap_{n=1}^{\infty} K_{\varepsilon,f}^{\omega^{\lambda+n+1}} \subseteq \bigcap_{n=1}^{\infty} K_{\varepsilon,3,(f_n)}^{\omega^{\lambda+n}} = K_{\varepsilon,3,(f_n)}^{\omega^{\lambda}}. \)

By induction we have that

\[ K_{\varepsilon,f}^{\omega^{\lambda+n}} \subseteq K_{\varepsilon,3,(f_n)}^{\omega^{\lambda+n}} \forall n < \omega \quad \text{and hence} \quad K_{\varepsilon,f}^{\omega^{\lambda+1}} \subseteq K_{\varepsilon,3,(f_n)}^{\omega^{\lambda+1}}. \]

Also, by induction we get \( K_{\varepsilon,f}^{\omega^{\lambda+m}} \subseteq K_{\varepsilon,3,(f_n)}^{\omega^{\lambda+m}} = \emptyset \) and hence \( \beta(f) \leq \omega^{\lambda+m} \). \[ \square \]

Note. Theorems 7 and 8 are due jointly to Professor Negrepontis (cf. [K-N]). I am grateful to Professor Negrepontis for his kind permission to present some of our joint work here.

In the following corollary it is proved that the conclusion of Theorem 7 is not true for \( \xi \geq \omega \).

9. Corollary. Let \( K \) be a compact metric space, \( \omega \leq \xi < \omega_1, f \in \mathcal{B}_1(K) \) and \((f_n) \in \text{DBSC}(K)\) such that \((f_n)\) is pointwise converging to \( f \) and \( \gamma((f_n)) \leq \omega^{\xi}. \)

Then \( \beta(f) \leq \omega^{\xi}. \)

Proof. If \( \xi \geq \omega \) there is a limit ordinal \( \lambda \geq \omega \) and \( m < \omega \) such that \( \xi = \lambda + m \). Also \( \operatorname{sup}\{\beta(f_n) : n \in \mathbb{N}\} = \omega < \omega^{\mu} \leq \omega^{\lambda}. \) Hence by Theorem 8 we have \( \beta(f) \leq \omega^{\lambda}. \) \[ \square \]

10. Proposition. Let \( K \) be a scattered compact metric space with \( K^{(\omega+1)} \neq \emptyset \). Then there is a sequence \((f_n) \subseteq \mathcal{B}_1(K), f \in \mathcal{B}_1(K)\) such that \((f_n)\) is pointwise converging to \( f, \) \( \operatorname{sup}\{\beta(f_n) : n \in \mathbb{N}\} = \omega^{\omega}, \) \( \gamma((f_n)) \leq \omega^{\omega+1} \) and \( \beta(f) > \omega^{\omega+1}. \)

Proof. We set

\[ A = \bigcup\{(K^{(n)} \setminus K^{(n+1)}): \eta \text{ even and } \eta < \omega^{\omega+1}\}. \]
Then $\beta(X_A) = \omega^{\omega+1} + 1$. For every $n \in \mathbb{N}$ we set

$$A_n^k = \bigcup \{ (K^{(n)} \setminus K^{(n+1)} : \eta \text{ even and } (k-1)\omega \leq \eta \leq \omega^n + \omega^\eta \},$$

$$k = 1, 2, \ldots, n.$$

Then we have $\omega^n < \beta(X_{A_k}) \leq \omega^n + 1$ $\forall k = 1, 2, \ldots, n$, $n \in \mathbb{N}$.
We set $A_n = \bigcup_{k=1}^n A_n^k$ $\forall n \in \mathbb{N}$. Then $X_{A_n} = X_{A_1} + \cdots + X_{A_n}$ and hence $\omega^n < \beta(X_{A_n}) \leq \omega^{n+1}$ for all $n \in \mathbb{N}$.

Then $\sup \{ \beta(X_{A_n}) : n \in \mathbb{N} \} = \omega^\omega$. Also $(X_{A_n})$ is pointwise converging to $X_A$.

The proof will be finished by proving that $\gamma((X_{A_n})) \leq \omega + 1$.

To see this, if $\varepsilon > 0$ then $A_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega} K^{(\eta)}_{\omega+1}$ for all $m < \omega$ and hence $K^{(\omega)}_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega} K^{(\eta)}_{\omega+1}$. Since the functions $X_{A_n}$ are zero on $\bigcap_{\eta<\omega} K^{(\eta)}_{\omega+1}$ we have that $K^{(\omega)}_{\varepsilon,(X_{A_n})} = \emptyset$.

**Remark 3.** Proposition 10 is an example, showing that one of the conditions in Theorem 7 is best possible. Also, there is surely no need to assume $K$ scattered in the statement of the result. I thank the referee for this remark.

**11. Proposition ([H-O-R]).** Let $K$ be a compact metric space, $m \in \mathbb{N}$, $\delta > 0$ and a function $f : K \to \mathbb{R}$ is such that $K^{(m)}_{\varepsilon,f} \neq \emptyset$. Then $|f|_{D} \geq m\delta/4$.

**12. Proposition.** Let $K$ be a compact metric space, $f \in \mathbb{B}_1(K), \xi < \omega, (f_n) \subset \text{DBSC}(K)$ pointwise converging to $f$, $\gamma((f_n)) \leq \omega^{\xi}$ and $\sup_n |f_n|_{D} < \infty$.

Then $\beta(f) \leq \omega^{\xi}$.

**Proof.** Let $\varepsilon > 0$.

**Claim 1.** $\exists n_0 \in \mathbb{N} : \beta(f_{n_0}, \varepsilon/3) = \beta(f_{n_0}, \varepsilon/3) \forall n \geq n_0$.

[Proof of Claim 1. Let then $\beta(f_{n_0}, \varepsilon/3) = m_n + 1$, where $m_n, n \in \mathbb{N}$. Then $K^{(m_n)}_{\varepsilon,f_{n_0}} \neq \emptyset$ and hence by Proposition 10 we have that $|f_{n_0}|_{D} \geq m_n \varepsilon/12$. If the sequence $(m_n)$ is infinite then $\sup_n |f_{n_0}|_{D} = \infty$, a contradiction.

Thus there is $n_0 \in \mathbb{N}$ such that $m_n = m_{n_0}$ for all $n \geq n_0$.

**Claim 2.** If $m = \beta(f_{n_0}, \varepsilon/3)$ then $P^{(m)}_{\varepsilon,f} \subset P^{(m)}_{\varepsilon/3,(f_{n_0})}$ for each closed subset $P$ of $K$.

[Proof of Claim 2. Let $x \in P^{(m)}_{\varepsilon,f} \setminus P^{(m)}_{\varepsilon/3,(f_{n_0})}$. Then there are an open neighborhood $V$ of $x$ in $P$ and $n_0 \in \mathbb{N}$ such that $|f_{m}(y) - f_{n_0}(y)| \leq \varepsilon/3 \ \forall n, m \geq n_0, y \in V$.

Then, $|f_{n_0}(y) - f(x)| \leq \varepsilon/3$ for all $y \in V$ and hence $P^{(m)}_{\varepsilon,f} \subset P^{(m)}_{\varepsilon/3,f_{n_0}}$.

Finally, by induction we get $P^{(m)}_{\varepsilon,f} \subset P^{(m)}_{\varepsilon/3,f_{n_0}} = \emptyset$. Since $V \cap P^{(m)}_{\varepsilon,f} \subset P^{(m)}_{\varepsilon,f}$ we have $V \cap P^{(m)}_{\varepsilon,f} = \emptyset$, a contradiction since $x \in V \cap P^{(m)}_{\varepsilon,f}$.

Since $\gamma((f_n)) \leq \omega^{\xi}$ we have that $\gamma((f_{n_0}), \varepsilon/3) < \omega^{\xi}$ and hence there is $k < \omega$ such that $\gamma((f_{n_0}), \varepsilon/3) < k\omega^{\xi-1}$. Applying Claim 1 we have

$$K^{(k)}_{\varepsilon,f} \subset K^{(k)}_{\varepsilon/3,(f_{n_0})} \subset K^{(k)}_{\varepsilon/3,(f_{n_0})} \subset K^{(k\omega^{\xi-1})}_{\varepsilon/3,(f_{n_0})} = \emptyset.$$ 

Then $\beta(f, \varepsilon) \leq k\omega^{\xi-1} < \omega^{\xi}$. Hence it is proved that $\beta(f) \leq \omega^{\xi}$.

**13. Definition ([H-O-R]).** Define $\mathbb{B}_1(K)$ to be the set of those $f$ in $\mathbb{B}_1(K)$ for which there is a sequence $(f_n)$ in $\text{DBSC}(K)$ that converges uniformly to $f$ and is such that $\sup_n |f_n|_{D} < \infty$. 

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14. **Theorem** ([H-O-R], Th. 6.1). Let $K$ be a compact metric space and let $f \in B_1(K)$. Then $f \in B_{1/4}(K)$ iff there exists a $C < \infty$ such that for all $\varepsilon > 0$ there exists a sequence $(s_n)_{n=0}^{\infty} \subset C(K)$, $s_0 = 0$, with $(s_n)$ converging pointwise to $f$ and such that for all subsequences $(n_i)$ of $\{0\} \cup \mathbb{N}$ and $x \in K$,

$$\sum_{j \in B((n_i),x)} |s_{n_{j+1}}(x) - s_{n_j}(x)| \leq C,$$

where $B((n_i),x) = \{ j : |s_{n_{j+1}}(x) - s_{n_j}(x)| \geq \varepsilon \}$.

The above result gave the idea for the definition of the rank $\delta$ (cf. [K-N]). I am grateful to Professor Negrepontis who gave me this idea.

15. **Definition.** Let $K$ be a compact metric space, $f,s_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, real-valued functions with $s_0 = 0$ such that $(s_n)$ is pointwise converging to $f$. For each closed subset $P$ of $K$ and $\varepsilon > 0$ we set:

$$P^0((s_n),\varepsilon) = P,$$

$$P'((s_n),\varepsilon) = \left\{ x \in P : \forall 0 < C < \infty, \forall m \in \mathbb{N}, \forall U \subset K \text{ open neighborhood of } x \exists j_0 > \cdots > j_m \geq m \text{ and } x' \in U \cap P \text{ such that } |s_{j_{i+1}}(x') - s_{j_i}(x')| > \varepsilon \text{ for } i = 1,2,\ldots,p \text{ and } \sum_{i=1}^{p} |s_{j_{i+1}}(x') - s_{j_i}(x')| > C \right\}.$$

For each ordinal $a < \omega_1$ we set

$$P^{a+1}((s_n),\varepsilon) = (P^a((s_n),\varepsilon))'(s_n),\varepsilon).$$

If $\beta$ is a limit ordinal, we set

$$P^\beta((s_n),\varepsilon) = \bigcap_{a < \beta} P^a((s_n),\varepsilon).$$

We set

$$\delta((s_n),\varepsilon) = \left\{ \begin{array}{ll}
\text{the least ordinal } a < \omega_1 \text{ such that } K^a((s_n),\varepsilon) = \emptyset & \text{if such an } a \text{ exists}, \\
\omega_1, & \text{otherwise}.
\end{array} \right.$$ 

and

$$\delta((s_n)) = \sup\{\delta((s_n),\varepsilon) : \varepsilon > 0\}.$$ 

**Remark 4.** $\delta((s_n)) \leq \gamma((s_n)).$

We see this as follows: Let $P$ be a closed subset of $K$, $\varepsilon > 0$ and $x \in P \setminus P^1_{\varepsilon,(s_n)}$. Then there are an open neighborhood of $x$ in $P$ and $p \in \mathbb{N}$ such that for every $y \in U$ and $m,n \in \mathbb{N}$ with $m,n \geq p$ we have $|f_m(y) - f_n(y)| \leq \varepsilon$. By definition of $P'((s_n),\varepsilon)$ we have that $x \notin P'((s_n),\varepsilon)$. Hence $P'((s_n),\varepsilon) \subset P^1_{\varepsilon,(s_n)}$.

16. **Proposition** ([H-O-R]). Let $X$ be a Banach space and $C,D$ be convex subsets of $X$. Then

$$\inf\{\|c - d\| : c \in C, d \in D\} = \inf\{\|c - d\| : c \in \bar{C}, d \in \bar{D}\},$$

where $\bar{C}$ and $\bar{D}$ are the $w^*$-closure of $C$ and $D$ in $X^{**}$.  

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17. Theorem. Let $K$ be a compact metric space, $f \in \mathcal{B}_1(K)$, a sequence $(f_n) \subset C(K)$ pointwise converging to $f$ and $\xi < \omega_1$.

Then the following equivalences are satisfied:

(i) If $1 \leq \xi < \omega$, then $\beta(f) \leq \omega^\xi$ if and only if there exists a sequence $(s_n)$ of convex blocks of $(f_n)$ with $\delta((s_n)) \leq \omega^{\xi-1}$.

(ii) If $\xi \geq \omega$, then $\beta(f) \leq \omega^\xi$ if and only if there exists a sequence $(s_n)$ of convex blocks of $(f_n)$ with $\delta((s_n)) \leq \omega^\xi$.

Proof. (i). Necessity. Let $1 \leq \xi < \omega$ and $\beta(f) \leq \omega^\xi$. Then by Theorem 7 we have there is a sequence $(F_n) \subset \text{DBSC}(K)$ pointwise converging to $f$ and $\gamma((F_n)) \leq \omega^\xi$.

Let $\varepsilon > 0$. Then $\gamma((F_n), \frac{\varepsilon}{2}) = \theta + 1 < \omega^\xi$.

For every $\eta \leq \theta$ we set $K_\eta = K_{\varepsilon/4}(F_n)$. Then for every $\eta \leq \theta$ and $x \in K_\eta \setminus K_{\eta+1}$ there are an open neighborhood $U_{x, \eta}$ of $x$ in $K_\eta$ and $n \in \mathbb{N}$ such that $|F_n(y) - f(y)| \leq \varepsilon / 4$ for every $y \in U_{x, \eta}$. Since $K$ is a compact metric space we have that for every $\eta \leq \theta$ there exists a countable subset $\{U_{n, k} : k \in \mathbb{N}\}$ of $\{U_{x, \eta} : x \in K_\eta \setminus K_{\eta+1}\}$ such that

$\bigcup \{U_{n, k} : k \in \mathbb{N}\} = \bigcup \{U_{x, \eta} : x \in K_\eta \setminus K_{\eta+1}\}$.

Let $\{U_{n, k, i} : i \in \mathbb{N}\}$ be an enumeration of $\{U_{n, k} : \eta \leq \theta, k \in \mathbb{N}\}$. Then for every $i \in \mathbb{N}$ $\exists n_1 \in \mathbb{N}$ such that

$(\ast) \quad \|f - F_{n_1}\|_{U_{n, k, i}} < \varepsilon / 4$.

(If $M$ is a subspace of $K$ we set $\| \|_M$ the supremum norm on $C(M)$.)

For every $i \in \mathbb{N}$ let $(f^i_{m, n})_{m=0}^\infty \subset C(K)$ with $f^i_0 = 0$, $(f^i_{m, n})_{m=0}^\infty$ is pointwise converging to $F_{n_1}$ and

$\sum_{m=0}^\infty |f^i_{m, n}(y) - f^j_{m, n}(y)| \leq |F_{n_1}|_D \quad \forall y \in U_{n, k, i}$.

By $(\ast)$ and Proposition 16 we have that there exist a sequence $(g^i_{m, n})$ of convex blocks of $(f^i_{m, n})$ and a sequence $(h^1_{m, n})$ of convex blocks of $(f^i_{m, n})$ such that

$(\ast \ast) \quad \|g^i_{m, n} - h^1_{m, n}\|_{U_{n, k, i}} < \varepsilon / 4 \quad \forall m \in \mathbb{N}$.

Then for every $m_1, \ldots, m_p \in \mathbb{N}$ with $m_1 < \cdots < m_p$ and $y \in U_{n, k, i}$ with $|g^i_{m_{i+1}, n}(y) - g^i_{m_i, n}(y)| \geq \varepsilon$ for all $i = 1, \ldots, p$ we have

$(\ast \ast \ast) \quad \sum_{i=1}^p |g^i_{m_{i+1}, n}(y) - g^i_{m_i, n}(y)| \leq \sum_{j=0}^\infty |f^j_{i+1}(y) - f^j_{i}(y)| + \frac{\varepsilon}{2} \|F_{n_1}|_D \leq 2|F_{n_1}|_D$.

[We see this as follows: Let $p, q \in \mathbb{N}$ and $y \in U_{n, k, i}$ with $|g^i_{m_{i+1}, n}(y) - g^i_{m_i, n}(y)| \geq \varepsilon$.

Then by $(\ast \ast)$ we have

$(1) \quad \varepsilon \leq |h^i_{p, n}(y) - h^i_{q, n}(y)| \leq \frac{\varepsilon}{2} + |h^i_{p, n}(y) - h^i_{q, n}(y)| \Rightarrow |h^i_{p, n}(y) - h^i_{q, n}(y)| \geq \frac{\varepsilon}{2}$.

Also, $\sum_{i=1}^p |g^i_{m_{i+1}, n}(y) - g^i_{m_i, n}(y)| \leq \sum_{j=0}^\infty |f^j_{i+1}(y) - f^j_{i}(y)| + \frac{\varepsilon}{2} \rho \leq |F_{n_1}|_D + \frac{\varepsilon}{2} \rho$. By

$(1) \quad \rho \frac{\varepsilon}{2} \leq \sum_{i=1}^p |h^i_{m_{i+1}, n}(y) - h^i_{m_i, n}(y)| \leq |F_{n_1}|_D \Rightarrow \rho \leq \frac{2}{\varepsilon} |F_{n_1}|_D$.

Hence the proof of $(\ast \ast \ast)$ is finished.]
By induction, for every $i \in \mathbb{N}$ we get a sequence $(g^{i+1}_m)$ of convex blocks of $(g^i_m)$ such that $\forall p \in \mathbb{N}$, $m_1, \ldots, m_\rho \in \mathbb{N}$ with $m_1 < \cdots < m_\rho$ and $y \in U_{m_\rho, k_\rho}$ with $|g^{i+1}_{m_{j+1}}(y) - g^{i+1}_{m_j}(y)| \geq \varepsilon$ for all $j = 1, \ldots, \rho$ we have

$$\sum_{j=1}^\rho |g^{i+1}_{m_{j+1}}(y) - g^{i+1}_{m_j}(y)| \leq 2|F_{n+1}|D.$$ 

We set $s_0 = 0$ and $s_n = g^n_n$ for all $n \in \mathbb{N}$. Then $(s_n)$ is pointwise converging to $f$ and $K^\eta((s_n), \varepsilon) \subset K_\eta$ for all $\eta \leq \theta + 1$. Hence $K^{\theta+1}((s_n), \varepsilon) = \emptyset$ and hence $\delta((s_n)) \leq \omega^{\xi-1}$.

**Sufficiency.** Let $\delta((s_n)) \leq \omega^{\xi-1}$. We shall show that $\gamma((s_n)) \leq \omega^\xi$ and since $\beta(f) \leq \gamma((s_n))$, we have that $\beta(f) \leq \omega^\xi$. Hence we shall show that $P^\omega_{\varepsilon, (s_n)} \subset P'((s_n), \varepsilon/2)$ for all closed subsets $P$ of $K$.

Let $P$ be a closed subset of $K$ and let $x \in P^\omega_{\varepsilon, (s_n)} \setminus P'((s_n), \varepsilon/2)$. Then there are a positive real number $C$, an open neighborhood $U$ of $x$ in $P$ and $m \in \mathbb{N}$ such that $\forall y \in \mathbb{N}$, $n_1, \ldots, n_p \in \mathbb{N}$ with $n_p > \cdots > n_1 \geq m$ and $x \in U$ with $|s_{n_{i+1}}(y) - s_{n_i}(y)| \geq \varepsilon/2$ for all $i = 1, \ldots, p$ we have $\sum_{i=1}^p |s_{n_{i+1}}(y) - s_{n_i}(y)| \leq C$.

Then $p < \frac{\omega}{\varepsilon}$. Let $n \in \mathbb{N}$ with $n > \frac{\omega}{\varepsilon}$. Then $x \in P^\omega_{\varepsilon, (s_n)}$. We shall show that there are $y \in U$ and $m_1, \ldots, m_{n+1} \in \mathbb{N}$ with $m_{n+1} \geq m_1 \geq m$ such that $|s_{m_{j+1}}(y) - s_{m_j}(y)| > \varepsilon/2$ for all $j = 1, \ldots, n$, and we shall terminate in a contradiction.

We see this as follows:

$$x \in P^\omega_{\varepsilon, (s_n)} \cap U \Rightarrow \exists x_1 \in P^\omega_{\varepsilon, (s_n)} \cap U \text{ and } m_1, m_2 \in \mathbb{N} \text{ with } m_2 > m_1 \geq m \text{ and } |s_{m_2}(x_1) - s_{m_1}(x_1)| > \varepsilon > \varepsilon/2.$$ 

We set $V_1 = \{y \in U : |s_{m_2}(y) - s_{m_1}(y)| > \varepsilon/2\}$. $V_1$ is open and $x_1 \in V_1 \cap P^\omega_{\varepsilon, (s_n)}$; hence $\exists x_2 \in P^\omega_{\varepsilon, (s_n)} \cap V_1$ and $m_3 \in \mathbb{N}$ such that $m_3 > m_2$ and $|s_{m_3}(x_2) - s_{m_2}(x_2)| > \varepsilon/2$ (since if $|s_{m}(y) - s_{m_2}(y)| \leq \varepsilon/2$ for every $m \geq m_2$ and $y \in P^\omega_{\varepsilon, (s_n)} \cap V_1$, then $|s_{m}(y) - s_k(y)| \leq \varepsilon$ for all $m, k \geq m_2$ and $y \in P^\omega_{\varepsilon, (s_n)} \cap V_1$, that is, $x_1 \notin P^\omega_{\varepsilon, (s_n)}$ which is a contradiction).

We set $V_2 = \{y \in V_1 : |s_{m_3}(y) - s_{m_2}(y)| > \varepsilon/2\}$. $V_2$ is open in $P$ and $x_2 \in V_2 \subset V_1$.

By induction we get $m_1, \ldots, m_n \in \mathbb{N}$ with $m_n > \cdots > m_1 \geq m$ and $V_1, \ldots, V_{n-1}$ open subsets of $P$ with $V_{n-1} \subset \cdots \subset V_1 \subset U$ and $x_1 \in P^\omega_{\varepsilon, (s_n)} \cap V_{n-1}$ for all $i = 1, \ldots, n$ (where $V_0 = U$) such that $|s_{m_{j+1}}(y) - s_{m_j}(y)| > \varepsilon/2$ for all $y \in V_i$, $i = 1, \ldots, n - 1$. We set $V_n = \{y \in V_{n-1} : |s_{m_n}(y) - s_{m_{n-1}}(y)| > \varepsilon/2\}$. $V_n$ is open in $P$ and $x_{n-1} \in P^\omega_{\varepsilon, (s_n)} \cap V_n$; hence there is $y \in V_n$ and $m_{n+1} > m_n$ such that $|s_{m_{j+1}}(y) - s_{m_j}(y)| > \varepsilon/2$.

Then $|s_{m_{j+1}}(y) - s_{m_j}(y)| > \varepsilon/2$ for all $j = 1, \ldots, n$. Hence the proof of (i) is finished.

(ii) **Necessity.** By Theorem 5 we have that if $f \in \mathcal{B}_1(K)$ with $\beta(f) \leq \omega^\xi$ then there is a sequence $(s_n)$ of convex blocks of $(f_n)$ with $\gamma((s_n)) \leq \omega^\xi$.

Then by Remark 4 we get a conclusion.

**Sufficiency.** As in (i) we prove that $P^\omega_{\varepsilon, (s_n)} \subset P'((s_n), \varepsilon/2)$ for all closed subsets $P$ of $K$ and $\varepsilon > 0$. Then by induction we get $K^\omega_{\varepsilon, (s_n)} \subset K^\omega((s_n), \varepsilon/2)$ for all
n ∈ ℕ and hence $K_\xi^ω((s_n),\varepsilon/2)$. Finally, by induction we get $K_\xi^ω((s_n),\varepsilon/2)$ for all $\varepsilon > 0$.

Remark 5. If $(s_n)$ is a sequence of continuous real-valued functions on $K$ with $\delta((s_n)) < \omega_1$, then $(s_n)$ converges pointwise.

[We see this is follows: As is proved in the demonstration of the sufficiency of Theorem 17 (i) we have that $P_{\varepsilon,\varepsilon,\xi}((s_n),\varepsilon/2)$ for all closed subsets $P$ of $K$ and hence $K_{\varepsilon,\xi}^ω((s_n),\varepsilon/2)$ for all $\xi < \omega_1$.

Assume that $\delta((s_n)) < \omega_1$. Then there is a $\xi < \omega_1$ such that $\delta((s_n)) < \omega^\xi$; hence $K_\xi^ω((s_n),\varepsilon) = \emptyset$ for all $\varepsilon > 0$ and thus $K_{\varepsilon,\xi}^ω((s_n),\varepsilon) = \emptyset$ for all $\varepsilon > 0$. Then $\gamma((s_n),\varepsilon) < \omega^{\xi+1}$ for all $\varepsilon > 0$; hence $\gamma((s_n)) \leq \omega^{\xi+1} < \omega_1$ and thus the sequence $(s_n)$ converges pointwise (cf. [K-L]).]

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