QUASITRIANGULAR + SMALL COMPACT = STRONGLY IRREDUCIBLE

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Abstract. Let $T$ be a bounded linear operator acting on a separable infinite dimensional Hilbert space. Let $\epsilon$ be a positive number. In this article, we prove that the perturbation of $T$ by a compact operator $K$ with $\|K\| < \epsilon$ can be strongly irreducible if $T$ is a quasitriangular operator with the spectrum $\sigma(T)$ connected. The Main Theorem of this article nearly answers the question below posed by D. A. Herrero.

Suppose that $T$ is a bounded linear operator acting on a separable infinite dimensional Hilbert space with $\sigma(T)$ connected. Let $\epsilon > 0$ be given. Is there a compact operator $K$ with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible?

1. Introduction

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}$ be separable Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$. Denote by $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ the subset of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of all compact operators. We simply write $\mathcal{B}(H)$ and $\mathcal{K}(H)$ instead of $\mathcal{B}(H, H)$ and $\mathcal{K}(H, H)$ respectively. For $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, denote the kernel of $T$ and the range of $T$ by $\text{Ker}T$ and $\text{Ran}T$ respectively. If $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ (closed), we shall write $\mathcal{H}_0 \leq \mathcal{H}$. Let $T \in \mathcal{B}(H)$; we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_i(T)$, $\sigma_r(T)$, $\sigma_l(T)$, $\sigma_{lre}(T)$ and $\sigma_w(T)$ the spectrum, the point spectrum, the left spectrum, the right spectrum, the essential spectrum, the left essential spectrum, the Wolf spectrum and the Weyl spectrum of $T$ respectively. Denote by $\sigma_0(T)$ the set of all isolated points of $\sigma(T) \setminus \sigma_e(T)$. For $\lambda \in \rho_{S-F}(T)$ ($\text{def} \ C_0 \sigma_{lre}(T)$), $\text{ind}(T - \lambda) = \text{dim Ker}(T - \lambda) - \text{dim Ker}(T - \lambda)^*$ and $\text{min ind}(T - \lambda) = \text{min}\{\text{dim Ker}(T - \lambda), \text{dim Ker}(T - \lambda)^*\}$. For $-\infty \leq n \leq +\infty$, $\rho^{(n)}_{S-F}(T) = \{\lambda \in \rho_{S-F}(T) : \text{ind}(T - \lambda) = n\}$. $T$ is said to be quasitriangular if there is a sequence $\{P_n\}_{n \geq 1}$ of finite rank projections increasing to the unit operator $I$ with respect to the strong operator topology such that $\lim_{n \to \infty} \|(I-P_n)TP_n\| = 0$. It is well-known that $T$ is quasitriangular if and only if $\text{ind}(T - \lambda) \geq 0$ for all $\lambda \in \rho_{S-F}(T)$. $T$ is said to be strongly irreducible if there are no nontrivial idempotents commuting with $T$. A Cowen-Douglas operator is an operator $T$ satisfying the following conditions:

(i) There is a nonempty connected open subset $\Omega$ of $\rho^{(n)}_{S-F}(T)$ for a natural number $n$.
(ii) $T - \lambda$ is surjective for each $\lambda \in \Omega$.
(iii) $\bigvee \{\text{Ker}(T - \lambda) : \lambda \in \Omega\}$ is equal to the acting space of $T$.

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If the conditions above are satisfied, we shall write \( T \in B_n(\Omega) \). If \( T \in B_0(\Omega) \), then \( \sqrt{\{\text{Ker}(T - \lambda)^k : k \geq 1\}} \) is equal to the acting space of \( T \) for each \( \lambda \) in \( \Omega \).

Let \( \sigma \) be a compact subset of the complex field \( C \). A clopen \( \sigma_0 \) of \( \sigma \) is a subset of \( \sigma \) such that there are two disjoint open subsets \( \Omega_1, \Omega_2 \) of \( C \) such that \( \Omega_1 \supset \sigma_0 \) and \( \Omega_2 \supset (\sigma \setminus \sigma_0) \). If \( \sigma \) is a clopen of \( \sigma(T) \), then there is an analytic Cauchy domain \( \Omega \) such that \( \sigma(T) \cap \Omega = \sigma \) and such that \( \sigma(T) \cap \partial \Omega = \emptyset \), where \( \partial \Omega \) is the boundary of \( \Omega \). Thus \( E(\sigma, T) = \frac{1}{2\pi i} \int_{\Omega} (\lambda - T)^{-1}d\lambda \) is an idempotent commuting with \( T \). We call \( E(\sigma, T) \) the Riesz idempotent of \( T \) corresponding to \( \sigma \). Write \( \mathcal{H}(\sigma, T) = \text{Ran}E(\sigma, T) \). It follows from the classical Riesz decomposition theorem that \( T \) is not strongly irreducible if \( \sigma(T) \) is not connected. But the converse is not true. However, D. A. Herrero and C. L. Jiang obtained the approximate inverse of the Riesz decomposition theorem (see [3] or [6]):

**Theorem HJ.** The closure of the class of all strongly irreducible operators is the class of all those operators which have connected spectrum.

And then, D.A. Herrero posed the following question.

**Question H.** Let \( T \) be an operator with \( \sigma(T) \) connected. Given \( \epsilon > 0 \), can we find a compact operator \( K \) with \( \|K\| < \epsilon \) such that \( T + K \) is strongly irreducible?

C.L. Jiang, S.H. Sun and Z.Y. Wang (see [10]) proved that if \( T \) is essentially normal and if \( \sigma(T) \) is connected, then one can find a compact \( K \) such that \( T + K \) is strongly irreducible. (But \( \|K\| \) may be bigger than \( \epsilon \).) Y.Q. Ji, C.L. Jiang and Z.Y. Wang (see [8], [9]) proved that if \( T \) is an essentially normal quasitriangular operator with \( \sigma(T) \) and \( \sigma_w(T) \) connected, then there exists a compact operator \( K \) with \( \|K\| < \epsilon \) such that \( T + K \) is strongly irreducible. They (see [7]) also proved that if \( T \) is a Cowen-Douglas operator having unique (SI)-decomposition, then there exists a compact operator \( K \) with \( \|K\| < \epsilon \) such that \( T + K \) is strongly irreducible. C.L. Jiang, S. Power, and Z.Y. Wang (see [11]) proved that if \( T \) is a biquasitriangular operator with \( \sigma(T) \) connected, then there exists a compact operator \( K \) with \( \|K\| < \epsilon \) such that \( T + K \) is strongly irreducible.

The main result of this article is the theorem below.

**Main Theorem.** Let \( T \in B(\mathcal{H}) \) be a quasitriangular operator with \( \sigma(T) \) connected and let \( \epsilon > 0 \) be given. Then there exists a compact operator \( K \) with \( \|K\| < \epsilon \) such that \( T + K \) is strongly irreducible.

2. Preparation

In order to prove the Main Theorem, we need to prepare some lemmas.

**Lemma 2.1.** Let \( T \in B(\mathcal{H}) \) be an operator with \( \text{Ran}T \) nonclosed. Then there is an infinite dimensional subspace \( \mathcal{H}_0 \) (closed) of \( \mathcal{H} \) such that \( \mathcal{H}_0 \cap \text{Ran}T = \{0\} \).

**Proof.** We know that \( \text{Ran}T = \text{Ran}(TT^*)^{1/2} \). Without loss of generality, assume that \( T \) is positive and that \( \text{Ran}T \) is dense in \( \mathcal{H} \). Let \( E(\cdot) \) be the spectral measure of \( T \). It is easy to see that \( E((0, t]) \neq 0 \) for all \( t > 0 \) and that \( E((0, \|T\|]) = I \). Choose a sequence \( \{t_k\}_{k \geq 0} \) of positive numbers decreasing to zero such that \( t_0 = \|T\| \) and such that \( E((t_k, t_{k-1}]) \neq 0 \) for all \( k \geq 1 \). Write \( E_k = E((t_k, t_{k-1}]). \) Let \( \mathcal{H}_n = \sqrt{\text{Ran}E((2k-1)^n- : k \geq 1}) \). Then \( \{\mathcal{H}_n\}_{n \geq 1} \) is a pairwise orthogonal family of subspaces and \( \mathcal{H} = \bigoplus_{n \geq 1} \mathcal{H}_n \). Let \( P_n \) be the projection onto \( \mathcal{H}_n \), i.e.
that

\[ \bigvee_{n \geq 1} \alpha_n x_n, \]

where the entry omitted is 0. Suppose that

(i) \( \dim \mathrm{Ker} T_1 = 1 \)

(ii) \( \bigvee_{n \geq 1} \mathrm{Ker} T_1^n = \mathcal{H}_1 \)

(iii) \( \mathrm{Ker} T_1 \cap \mathrm{Ker} T_2 = \{0\} \)

(iv) \( \mathrm{Ran} T_1 \cap \mathrm{Ran} (T_1 |_{\mathrm{Ker} T_2}) = \{0\} \)

Then \( T \) is strongly irreducible.

Proof. Suppose \( T(x \oplus y) = 0 \), where \( x \in \mathcal{H}_1 \), \( y \in \mathcal{H}_2 \). Computation shows that

\[ T_2 y = 0 \] and \( T_1 x + T_1 T_2 y = 0 \). By (iv), \( T_1 x = 0 \), i.e. \( x \in \mathrm{Ker} T_1 \), and \( T_1 T_2 y = 0 \). It follows from (iii) that \( y = 0 \). So \( \mathrm{Ker} T = \mathrm{Ker} T_1 \). It is easy to show that \( \bigvee_{n \geq 1} \mathrm{Ker} T^n = \mathcal{H}_1 \). Suppose that \( P \) is an idempotent commuting with \( T \). Then

\[ P T^n = T^n P \]

for \( n \geq 1 \). So \( P (\mathrm{Ker} T^n) \subset \mathrm{Ker} T^n \) for all \( n \geq 1 \). Thus \( \mathcal{H}_1 \in \mathrm{Lat} P \).

Set

\[ P = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_2 \end{bmatrix} \]

Then \( P_2^n = P_2 \) and \( P_2 T_1 = T_1 P_2 \), \( i = 1, 2 \). By Lemma 2.3 and the condition (i), \( P_1 = I_{\mathcal{H}_1} \) or 0. Assume \( P_1 = 0 \) (otherwise, consider \( I - P \)). Computing the (1, 2)-entry shows that \( P_2 T_2 = T_1 P_{12} + T_2 P_2 \). Let \( y \in \mathrm{Ker} T_2 \). It follows that \( P_2 y \in \mathrm{Ker} T_2 \), and so \( P_2 T_2 = T_2 P_2 \). So \( T_1 P_{12} y = -T_1 P_2 y \in \mathrm{Ran} T_1 \cap \mathrm{Ran} (T_1 |_{\mathrm{Ker} T_2}) \). By the condition (iv), \( P_2 y = 0 \). Hence \( P_2 (\mathrm{Ker} T_2) = \{0\} \). If \( x \in \mathrm{Ker} T_2^n \), then \( T_2 x \in \mathrm{Ker} T_2 \).
This shows that $T_2P_2x = P_2T_2x = 0$, $P_2x \in \ker T_2$. Thus $P_2x = P_2(P_2x) = 0$. So $P_2(\ker T^2) = \{0\}$. Inductively, $P_2(\ker T^2) = \{0\}$ for all $n \geq 1$. By the condition (ii), $P_2 = 0$. So $P = P^2 = 0$, and $T$ is strongly irreducible. 

**Lemma 2.5.** Suppose that $T \in \mathcal{B}(\mathcal{H})$ and that $T$ satisfies the following conditions:

(i) $0 \in \partial \sigma(T)$ (the boundary of $\sigma(T)$),

(ii) $\ker T \subset \bigcap_{n \geq 1} \operatorname{Ran}T^n$, 

(iii) $\bigvee_{n \geq 1} \ker T^n = \mathcal{H}$.

Let $\epsilon > 0$ be given. Then there exists a compact operator $K$ with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

**Proof.** By Lemma 2.3, we only need to show this lemma in the case that $\dim \ker T > 1$. Choose $x_0 \in \ker T \setminus \{0\}$. Let $\mathcal{N}_\infty = \ker T \ominus Cx_0$. By the condition (ii), we can take $x_k \in \mathcal{N}_\infty$ such that $Tx_k = x_{k-1}$ for each $k \geq 1$. Let $\mathcal{H}_1 = \bigvee \{x_k : 1 \leq k < +\infty\}$. Then $\mathcal{H}_1 \in \text{Lat} T$ and $\mathcal{N}_\infty \subset \mathcal{H}_1^\perp$. Set

$$ T = \begin{bmatrix} T_1 & T_{12} \\ T_2 & \mathcal{H}_1 \end{bmatrix} \mathcal{H}_2 = \mathcal{H}_1^\perp $$

It is easy to see that the following hold:

(1) $\dim \ker T_1 = 1$, \quad $\ker T_1 \ominus \mathcal{H}_1 \cap \text{ran} T_1 = \{0\}$.

(2) $0 \in \partial \sigma(T_1)$ (this follows from $0 \in \partial \sigma(T)$).

(3) $\bigvee_{n \geq 1} \ker T_2^n = \mathcal{H}_2$ (It follows from that $\bigwedge_{n \geq 1} \text{ran} T_2^n \subset \bigwedge_{n \geq 1} \text{ran} T^n = \{0\}$).

(4) $\mathcal{N}_\infty \subset \ker T_2$ and $T_{12}(\mathcal{N}_\infty) = \{0\}$

(5) $\ker(T_{12}|_{\ker T_2 \ominus \mathcal{N}_\infty}) = \{0\}$ and $\text{ran} T_1 \cap T_{12}(\ker T_2 \ominus \mathcal{N}_\infty) = \{0\}$.

Let $A$ be an operator mapping $(\mathcal{H}_1 \ominus \ker T_1) \ominus (\mathcal{H}_2 \ominus \mathcal{N}_\infty)$ into $\mathcal{H}_1$ such that $A(x \oplus y) = T_1x + T_{12}y$. By (5) above, $\ker A = \{0\}$. Since $0 \in \partial \sigma(T_1)$ and $\text{ran} T_1 = \mathcal{H}_1$, $A|_{\mathcal{H}_1 \ominus \ker T_1}$ is unbounded from below. So $\text{ran} A$ is nonclosed. By Lemma 2.2, we can take a $B \in \mathcal{K}(\mathcal{N}_\infty, \mathcal{H}_1)$ with $\|B\| < \epsilon$ and $\ker B = \{0\}$ such that $\text{ran} B \cap \text{ran} A = \{0\}$. Define

$$ Kx = \begin{cases} Bx, & x \in \mathcal{N}_\infty, \\ 0, & x \in \mathcal{N}_\infty^\perp. \end{cases} $$

Then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \epsilon$. It is easy to see that $K$ satisfies $\ker C \cap \ker T_2 = \{0\}$ and $\text{ran} T_1 \cap C(\ker T_2) = \{0\}$. By Lemma 2.4, $T + K$ is strongly irreducible.

**Lemma 2.6.** Let $T$ be an operator acting on $\mathcal{H}$ satisfying the following conditions:

(i) $0 \in \partial \sigma(T)$ and $\bigvee_{n \geq 1} \ker T^n = \mathcal{H}$.

(ii) $\ker T \cap (\bigcap_{n \geq 1} \text{ran} T^n)$ is closed and $\dim \ker T \ominus (\ker T \cap (\bigcap_{n \geq 1} \text{ran} T^n)) < \infty$.

Let $\epsilon > 0$ be given. Then there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

**Proof.** Write $\mathcal{N}_\infty = \ker T \cap (\bigcap_{n \geq 1} \text{ran} T^n)$. Denote $\ker T \ominus \mathcal{N}_\infty = \mathcal{N}_0$. By the condition (ii), $\dim \mathcal{N}_0 < +\infty$. For $k \geq 1$, we can inductively define $\mathcal{N}_k = \{x:$
Take unit vectors \( f \in R_n \) all \( A_i \) that \( 0 \in \|A\| \). Then \( H_1 \in \text{Lat} T \) and \( \dim H_1 = m \). Set

\[
T = \begin{bmatrix} T_1 & T_{12} \\ \ast & \ast \ldots \ast & \ldots & \ast \\ \ast & \ast \ldots \ast & \ldots & \ast \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \ast & \ast \ldots \ast & \ldots & \ast \\ 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

It is not difficult to show that

1. \( \text{Ker} T_2 \subset \mathcal{N}_1 \subset \bigcap_{n \geq 1} \text{Ran} T_n \).
2. \( \bigvee_{n \geq 1} \text{Ker} T_n = \mathcal{H}_2 \).
3. \( \mathcal{T}_{an}^m = 0 \).

Choose \( C \in \mathcal{K}(\mathcal{H}) \) with \( \|C\| < \frac{\epsilon}{2} \) \( (T_1 + C)^{-1} \neq 0 \) and such that \( (T_1 + C)^m = 0 \).

Take unit vectors \( f \in \mathcal{H}_1 \cap \text{Ran}(T_1 + C) \) and \( e \in \mathcal{N}_\infty \). Set

\[
K_1 = \begin{bmatrix} C & \frac{\epsilon}{2} f \otimes e \\ 0 & 0 \\
\end{bmatrix}
\]

where \( (f \otimes e) x = (x, e) f \). Then \( K_1 \in \mathcal{K}(\mathcal{H}) \) and \( \|K_1\| < \frac{\epsilon}{4} \). It is not difficult to show that \( \text{Ker}(T + K_1) = \text{Ker}(T_1 + C) \otimes (\mathcal{N}_\infty \otimes \mathcal{C}) \subset \bigcap_{n \geq 1} \text{Ran}(T + K_1)^n \). It is clear that \( 0 \in \partial \sigma(T + K_1) \). By Lemma 2.5, one can find a \( K_2 \in \mathcal{K}(\mathcal{H}) \) with \( \|K_2\| < \frac{\epsilon}{4} \) such that \( T + K_1 + K_2 \) is strongly irreducible. Let \( K = K_1 + K_2 \in \mathcal{K}(\mathcal{H}) \). Then \( \|K\| < \epsilon \).

Let \( T \in \mathcal{K}(\mathcal{H}) \) have the following form:

\begin{align*}
T = \begin{bmatrix} 0 & A_1 & \ast & \cdots & \ast \\ 0 & A_2 & * & \cdots & \ast \end{bmatrix}
& \quad \text{Ker} T \\
& \quad \text{Ker} T^2 \otimes \text{Ker} T \\
& \quad \text{Ker} T^3 \otimes \text{Ker} T \\
& \quad \cdots \\
& \quad \text{Ker} T^n \otimes \text{Ker} T^n \\
& \quad \cdots \\
& \quad \text{Ker} T \end{align*}

It is easy to see that \( \text{Ker} T_i = \{0\} \) and \( \text{Ker} T \cap \text{Ran} \text{T}^i = \text{Ran}(A_1 A_2 \cdots A_i) \) for all \( i \geq 1 \). Thus \( \text{Ker} T \cap \bigcap_{n \geq 1} \text{Ran} T^n = \bigcap_{n \geq 1} \text{Ran}(A_1 A_2 \cdots A_i) \). It follows that \( \text{Ker} T \cap \bigcap_{n \geq 1} \text{Ran} T^n \) is closed when \( \text{Ran} A_1 \) is closed for each \( n \geq 1 \).

**Lemma 2.7.** Let \( T \) be as above. Suppose that \( n_0 \) is a natural number and suppose that \( \mathcal{M} \) is an infinite dimensional subspace of \( \text{Ker} T^{n_0} \otimes \text{Ker} T^{n_0} \) such that \( \mathcal{M} \cap \text{Ran} A_{n_0} = \{0\} \). Let \( \epsilon > 0 \) be given. Then there exists a compact operator \( K \) with \( \|K\| < \epsilon \) such that \( T + K \) is strongly irreducible.

**Proof.** Without loss of generality, assume that \( n_0 > 1 \) and that \( A_i \) has closed range with finite codimension for each \( i < n_0 \). Let \( T_1 = T|_{\text{Ker} T^{n_0 - 1}} \), i.e.

\[
T_1 = \begin{bmatrix} 0 & A_1 & \cdots & \ast & \ast \\ 0 & \ast & \cdots & \ast & \ast \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \ast & \ast \ldots \ast & \ldots & \ast \\ 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

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Since \( \text{Ran } A_i \) is closed for each \( i < n_0 \), \( T_1 \) is similar to \( \bigoplus_{j=1}^{\infty} J_j \), where \( J_j \) is a Jordan block for each \( j \). So we can find \( C_1 \in K(\text{Ker}T_{n_0-1}) \) with \( \|C_1\| < \frac{\varepsilon}{2} \) such that \( \dim\text{Ker}(T_1 + C_1) = 1 \) and \( \text{Ker}(T_1 + C_1)^n = \text{Ker}T_{n_0-1} \). It is clear that

\[
\text{Ran}(T_1 + C_1) \neq \overline{\text{Ran}(T_1 + C_1)} = \text{Ker}T_{n_0-1}.
\]

Let \( P \) be the projection onto \( \text{Ker}T_{n_0-1} \). Write \( T_2 = PT|_{\text{Ran}P} \). Then

\[
T_2 = \begin{bmatrix}
0 & A_{n_0} & * & \cdots \\
0 & A_{n_0+1} & * & \ddots \\
& & & & \ddots \\
& & & & & & \\
\end{bmatrix}
\]

Decompose \( \mathcal{N}_1 \) as \( \mathcal{M} \oplus (\mathcal{N}_1 \oplus \mathcal{M}) \). Then

\[
T_2 = \begin{bmatrix}
0 & 0 & B_2 & * & \cdots \\
0 & B_1 & * & \cdots \\
& & 0 & A_{n_0+1} & \cdots \\
& & & & \ddots \\
& & & & & & \\
\end{bmatrix}
\]

where \( \begin{bmatrix} B_2 \\ B_1 \end{bmatrix} = A_{n_0} \). For each \( 0 \neq x \in \mathcal{N}_2 \), \( A_{n_0}x = B_2x + B_1x \notin \mathcal{M} \). So \( B_1x \neq 0 \).

Set

\[
T_3 = \begin{bmatrix}
0 & B_1 & * & \cdots \\
0 & A_{n_0+1} & * & \ddots \\
& & 0 & A_{n_0+1} & \cdots \\
& & & & \ddots \\
& & & & & & \\
\end{bmatrix}
\]

Then \( \text{Ker}T_3 = \mathcal{N}_1 \oplus \mathcal{M} \), and \( T \) can be written as

\[
T = \begin{bmatrix}
T_1 & T_{12} & * \\
0 & T_{23} & \\
T_3 & & \\
\end{bmatrix}
\]

\( \mathcal{H}_1 = \text{Ker}T_{n_0-1} \)

\( \mathcal{H}_2 = \mathcal{H} \oplus (\mathcal{H}_1 \oplus \mathcal{M}) \)

where \( T_{23}|_{\text{Ker}T_3} = 0 \). Take \( C_2 \in K(\mathcal{M}) \) with \( \|C_2\| < \frac{\varepsilon}{2} \) such that \( \dim\text{Ker}C_2 = 1 \) and such that \( \sqrt[n\geq1]{\text{Ker}C_2^n} = \mathcal{M} \). By Lemma 2.2, choose \( C_3 \in K(\mathcal{H}_2, \mathcal{M}) \) with \( \|C_3\| < \frac{\varepsilon}{2} \) such that \( \text{Ran}C_3 \cap \text{Ran}C_2 = \{0\} \) and such that \( \sqrt[n\geq1]{\text{Ker}C_3} = \mathcal{H}_2 \oplus \text{Ker}T_3 \). Take a rank one operator \( C_1 \in K(\mathcal{M}, \mathcal{H}_1) \) with \( \|C_4\| < \frac{\varepsilon}{16} \) such that \( \text{Ker}C_4 = \mathcal{M} \oplus \text{Ker}C_2 \), \( \text{Ker}(T_{12} + C_4) \cap \text{Ker}C_2 = \{0\} \) and \( (T_{12} + C_4)(\text{Ker}C_2) \cap \text{Ran}(T_1 + C_1) = \{0\} \).

Set

\[
K = \begin{bmatrix}
C_1 & C_2 & 0 \\
C_2 & C_3 & \mathcal{M} \\
0 & \mathcal{H}_2 & \\
\end{bmatrix}
\]

Then \( K \in K(\mathcal{H}) \) and \( \|K\| < \varepsilon \), and so

\[
T + K = \begin{bmatrix}
T_1+C_1 & T_{12} + C_4 & * \\
C_2 & T_{23} + C_3 & \mathcal{M} \\
T_3 & \mathcal{H}_2 & \\
\end{bmatrix}
\]

Similarly to the proof of Lemma 2.4, one can with no difficulty verify that \( T + K \) is strongly irreducible. \( \square \)
Lemma 2.8. Suppose that $T$ has the form (3) and that the following conditions are satisfied:

(i) $\text{Ran} A_i = \text{Ran} A_i$ and $\dim \ker A_i < +\infty$ for each $i$.
(ii) $\dim \ker T \cap \bigcap_{n \geq 1} (\text{Ran} T^n) = +\infty$.
(iii) $\dim \ker T \cap (\text{Ker} T \cap \bigcap_{n \geq 1} (\text{Ran} T^n)) = +\infty$.

Let $\epsilon > 0$ be given. Then there exists a compact operator $K$ with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

Proof. Write $N_\infty = \ker T \cap \bigcap_{n \geq 1} (\text{Ran} T^n)$ and $N_0 = \ker T \cap N_\infty$. For $1 \leq k < +\infty$, inductively define $N_k = \{x : T x \in N_{k-1}, x \perp N_\infty\}$, and set $\bigvee\{N_k : 0 \leq k < +\infty\} = \mathcal{H}_1$. By (i) and (iii), there is an infinite dimensional linear submanifold $X$ of $\mathcal{H}_1$ such that $X \cap \text{Ran} T = \{0\}$. It is clear that $\mathcal{H}_1 \in \text{Lat} T$. Write

$$T = \begin{bmatrix} T_1 & T_{12} & \mathcal{H}_1 \\ T_2 & \mathcal{H}_1 \end{bmatrix} \mathcal{H}_1$$

It is easy to see that

1. $\ker T_1 = N_0$ and $\bigvee_{n \geq 1} \ker T^n = \mathcal{H}_1$,
2. $N_\infty \subset \ker T_2$ and $\bigvee_{n \geq 1} \ker T^n = \mathcal{H}_1$.

Write $\mathcal{M} = A_1^{-1}(N_\infty)$. Since Ran $A_1$ is closed and $\ker A_1 = \{0\}$, we can find a positive number $r$ such that $r \|x\| \leq \|A_1 x\|$ for $x \in \mathcal{M}$. Write $\mathcal{L} = P_{\mathcal{H}_1} \mathcal{M}$, where $P_{\mathcal{H}_1}$ is the projection onto $\mathcal{H}_1$. Suppose $x = x_1 \oplus x_2 \in \mathcal{M}$, $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{L}$.

$$\|P_{\mathcal{H}_1} x\| = \|x_2\| \geq \frac{\|T_2 x_2\|}{\|T_2\|} = \frac{\|A_1 x_2\|}{\|T_2\|} \geq \frac{r}{\|T_2\|} \|x_2\|.$$ 

So $\mathcal{L}$ is closed. Moreover, $T_{12} y \in \text{Ran} T_1$ for all $y \in \mathcal{L}$. Write $\mathcal{H}_2 = \mathcal{H}_1 \oplus N_\infty$. Set

$$T_2 = \begin{bmatrix} 0 & T_{23} & N_\infty \\ T_3 & \mathcal{H}_2 \end{bmatrix} \mathcal{H}_2$$

It is easy to see that $\ker T_3 = \mathcal{L} \oplus (\ker T_2 \oplus N_\infty)$ and that $\bigvee_{n \geq 1} \ker T^n = \mathcal{H}_2$. If $0 \neq y \in \ker T_3 \oplus \mathcal{L}$, then $T_{12} y \notin \text{Ran} T_1$. Notice that $X \cap (\text{Ran} T_1 + T_{12} (\ker T_3 \oplus \mathcal{L})) \subset X \cap \text{Ran} T = \{0\}$ and that

$$T = \begin{bmatrix} T_1 & 0 & T_{12} & \mathcal{H}_1 \\ 0 & T_{23} & \mathcal{H}_2 \end{bmatrix} \mathcal{H}_1 = \begin{bmatrix} 0 & T_{23} & \mathcal{H}_1 \\ T_3 & \mathcal{H}_2 \end{bmatrix} \mathcal{H}_1$$

Similarly to the proof of Lemma 2.5, by Lemma 2.2 there is a $C \in \mathcal{K}(\mathcal{H}_2, \mathcal{H}_1)$ with $\|C\| < \frac{r}{2}$ such that $\ker C = \mathcal{H}_2 \oplus \mathcal{L}$ and $\ker C \cap (\text{Ran} T_1 + T_{12} (\ker T_3 \oplus \mathcal{L})) = \{0\}$. Write $B = C + T_{12} \mathcal{H}_2$. Then $\text{Ran} T_1 \cap B(\ker T_3) = \{0\}$. Take $T_0 \in \mathcal{K}(N_\infty)$ with $\|T_0\| < \frac{r}{2}$ such that $\dim \ker T_0 = 1$ and such that $\bigvee_{n \geq 1} \ker T^n = N_\infty$. Since $\dim N_\infty = +\infty$, $\text{Ran} T_0 \neq \overline{\text{Ran} T_0} = N_\infty$. By Lemma 2.2, we can find a $D \in \mathcal{K}(\mathcal{H}_1, N_\infty)$ with $\|D\| < \frac{r}{8}$ such that $\text{Ran} D \cap \text{Ran} T_0 = \{0\}$ and $\ker D = \mathcal{H}_1 \ominus \ker T_1$. Set

$$K = \begin{bmatrix} T_0 & D & 0 \\ 0 & C & \mathcal{H}_1 \\ 0 & \mathcal{H}_2 \end{bmatrix}$$
Then \( K \in \mathcal{K}(\mathcal{H}) \), \( \| K \| < \epsilon \) and

\[
T + K = \begin{bmatrix}
T_0 & D & 

T_1 & B & 

T_3 & 

\end{bmatrix}
\]

Similarly to the proof of Lemma 2.4, one can show that \( T + K \) is strongly irreducible.

By the equivalence of \((\text{str-v})_m\) and \((\text{str-vi})_m\) of Theorem 1.2 in [4], we have the following lemma.

**Lemma 2.9** ([4]). Let \( m \) be a natural number and let \( T \in \mathcal{B}(\mathcal{H}) \) be a quasitriangular operator with \( \sigma(T) \) and \( \sigma_w(T) \) connected. Let \( \epsilon > 0 \) and \( \lambda \in \sigma_r(T) \cup (\bigcup_{k \geq m} \rho_{S^{-F}}(T)) \) be given. Then there exist a \( K \in \mathcal{K}(\mathcal{H}) \) with \( \| K \| < \epsilon \) and a sequence \( \{ P_n \}_{n \geq 0} \) of finite rank projections increasing to \( I \) with respect to the strong operator topology with \( \text{rank} P_n = mn \) such that \( (I - P_n)(T - \lambda + K)P_n = 0 \) for all \( n \geq 0 \), i.e.

\[
T + K - \lambda = \begin{bmatrix}
0 & * & * & \cdots & \text{Ran} P_1 \\
0 & * & * & \cdots & \text{Ran}(P_2 - P_1) \\
0 & * & \cdots & \text{Ran}(P_3 - P_2) \\
0 & \cdots & \text{Ran}(P_4 - P_3) \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

**Theorem 2.1.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be a quasitriangular operator with \( \sigma(T) \) and \( \sigma_w(T) \) connected. Let \( \epsilon > 0 \) be given. Then there exists a compact operator \( K \) with \( \| K \| < \epsilon \) such that \( T + K \) is strongly irreducible.

**Proof.** Without loss of generality, assume that \( 0 \in \partial \sigma(T) \). By Lemma 2.9, find \( K_1 \in \mathcal{K}(\mathcal{H}) \) with \( \| K_1 \| < \frac{\epsilon}{4} \) and a sequence \( \{ P_n^{(1)} \}_{n \geq 0} \) of finite rank projections increasing to \( I \) with respect to the strong operator topology so that

\[
T + K_1 = \begin{bmatrix}
0 & * & \cdots & \text{Ran} P_1^{(1)} \\
0 & * & \cdots & \text{Ran}(P_2^{(1)} - P_1^{(1)}) \\
0 & \cdots & \text{Ran}(P_3^{(1)} - P_2^{(1)}) \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

It is obvious that \( \sigma_r((T + K_1)^*) \subset \{ 0 \} \). While \( 0 \in \sigma_{irr}(T) \),

\[
\sigma(T + K_1) = \sigma_w(T + K_1) = \sigma_w(T).
\]

Write \( P_1 = P_1^{(1)} \) and \( N_1 = \text{Ran} P_1^{(1)} \). Let \( T_1 = (I - P_1)(T + K_1)|_{\text{Ran}(I - P_1)} \). It is not difficult to show that \( \sigma(T_1) = \sigma_w(T_1) = \sigma_w(T) \), \( \sigma_{irr}(T_1) = \sigma_{irr}(T) \) and \( \text{ind}(T_1 - \lambda) = \text{ind}(T - \lambda) \) for all \( \lambda \in \rho_{S^{-F}}(T_1) \). Applying Lemma 2.9 to \( T_1 \), one can find a compact operator \( K^{(2)} \in \mathcal{K}(\mathcal{H}) \) with \( \| K^{(2)} \| < \frac{\epsilon}{8} \) and a sequence \( \{ P_n^{(2)} \}_{n \geq 0} \) of finite rank projections increasing to \( I|_{\text{Ran}(I - P_1)} \) with respect to the...
strong operator topology such that \( \text{rank} P_1^{(2)} = 2 \text{rank} P_1 \) and

\[
T_1 + K^{(2)} = \begin{bmatrix}
0 & * & * & \cdots \\
0 & \ast & \ast & \cdots \\
0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \begin{bmatrix}
\text{Ran} P_1^{(2)} \\
\text{Ran} (P_2^{(2)} - P_1^{(2)}) \\
\vdots \\
\vdots \\
\end{bmatrix}
\]

Since \( \text{rank} P_2^{(1)} < +\infty \), there is a natural number \( n_1 \) such that \( \| (I - P_2) P_2^{(1)} \| < \frac{1}{2} \), where \( P_2 \) is the projection onto \( \text{Ran} P_1 \oplus \text{Ran} P_2^{(2)} \). Write \( N_2 = \text{Ran} P_1^{(2)} \) and \( N_j = \text{Ran} (P_j^{(2)} - P_{j-1}^{(2)}) \) for \( 2 < j \leq n_1 + 1 \). Set

\[
K_2 = \begin{bmatrix}
0 & \vdots \\
\vdots & K^{(2)} \\
\vdots & \vdots \\
\end{bmatrix} \begin{bmatrix}
\text{Ran} P_1 \\
\text{Ran} (I - P_1) \\
\vdots \\
\vdots \\
\end{bmatrix}
\]

Then \( K_2 \in \mathcal{K}(\mathcal{H}) \) and \( \| K_2 \| < \frac{1}{2} \). Let \( T_2 = (I - P_2)(T + K_1 + K_2) |_{\text{Ran}(I - P_2)} \). One can show that \( \sigma(T_2) = \sigma_w(T_2) = \sigma_w(T), \sigma_{\text{tr}e}(T_2) = \sigma_{\text{tr}e}(T) \) and \( \text{ind}(T_2 - \lambda) = \text{ind}(T - \lambda) \) for all \( \lambda \in \rho_{\mathcal{F}}(T_2) \).

Repeatedly using the process above, we can inductively choose a sequence \( \{ n_i \}_{i \geq 1} \) of natural numbers, a sequence \( \{ N_k \}_{k \geq 1} \) of pairwise orthogonal finite dimensional subspaces, an increasing sequence \( \{ P_k \} \) of finite rank projections and a sequence \( \{ K_n \}_{n \geq 1} \) of compact operators such that

(i) \( \dim N_k \leq \dim N_{k+1} < +\infty \) (\( k \geq 1 \)),

(ii) \( \text{Ran} P_k = \bigoplus \{ N_j : j \leq 1 + \sum_{i=1}^{k-1} n_i \} \) (\( k \geq 1 \)),

(iii) \( \dim \bigoplus_{i=1}^{k-1} N_{n_i} = 2^k \text{rank} P_k \) (\( k \geq 1 \)),

(iv) \( \| (I - P_n) P_n^{(1)} \| < \frac{1}{n+1} \) for \( n \geq 1 \), and hence \( \bigoplus_{1 \leq k < +\infty} N_k = \mathcal{H} \).

(v) \( \| K_n \| < \frac{\varepsilon}{2^{n+1}} \), hence \( K_1 = \sum_{1 \leq k < +\infty} K_k \in \mathcal{K}(\mathcal{H}) \) and \( \| K_1 \| < \frac{\varepsilon}{2} \).

(vi)

\[
T + K_1 = \begin{bmatrix}
0 & B_1 & * & * & \cdots \\
0 & B_2 & * & \ast & \cdots \\
0 & B_3 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \begin{bmatrix}
N_1 \\
N_2 \\
N_3 \\
N_4 \\
\vdots \\
\end{bmatrix}
\]

Since \( \dim N_k \leq \dim N_{k+1} < +\infty \), we can choose \( C_k \in \mathcal{K}(N_{k+1}, N_k) \) with \( \| C_k \| < \frac{\varepsilon}{k+3} \) so that \( B_k + C_k \) is surjective.

Set

\[
C = \begin{bmatrix}
0 & C_1 \\
0 & C_2 \\
0 & \cdots \cdot \\
\end{bmatrix} \begin{bmatrix}
N_1 \\
N_2 \\
N_3 \\
\vdots \\
\end{bmatrix}
\]

Then \( C \in \mathcal{K}(\mathcal{H}) \) and \( \| C \| < \frac{\varepsilon}{3} \). Write \( K_2 = K_1 + C \). It is easy to see that

\[
\dim (\ker (T + K_2) \cap ( \bigcap_{n \geq 1} \text{Ran} (T + K_2)^n )) = +\infty
\]
and that \( \bigvee_{n \geq 1} \ker(T + \overline{K}_2)^n = \mathcal{H} \). Set \( T + \overline{K}_2 \) in the form (3):

\[
T + \overline{K}_2 = \begin{bmatrix}
0 & A_1 & * & \cdots \\
0 & 0 & A_2 & \\
& & & \ddots \\
& & & & \ddots \\
\end{bmatrix}
\]

Consider each \( A_i \). By Lemmas 2.6–2.8, we can find \( \overline{K}_3 \in \mathcal{K}(\mathcal{H}) \) with \( \|K_3\| < \frac{1}{3} \) such that \( T + \overline{K}_2 + \overline{K}_3 \) is strongly irreducible. Let \( K = \overline{K}_2 + \overline{K}_3 \in \mathcal{K}(\mathcal{H}) \). Then \( \|K\| < \epsilon \). This completes the proof of Theorem 2.1.

**Remark.** In fact, Theorem 2.1 can be strengthened to the theorem below, and this will be useful in answering Question H.

**Theorem 2.1'.** Let \( T \) be a quasitriangular operator with \( \sigma(T) \) and \( \sigma_w(T) \) connected. Given \( \epsilon > 0 \), then there exists a compact operator \( K \) with \( \|K\| < \epsilon \) such that

(i) \( T + K \) is strongly irreducible,

(ii) \( \sigma_p((T + K)^*) = \emptyset \),

(iii) \( \ker_B (T + K) = \{(0) \text{ if } \sigma_p(B) = \emptyset, \text{ where } \tau_{B,T+K} \text{ is the Rosenblum operator.} \)

**Proof.** Look back to the proof of Theorem 2.1. It is easy to see that \( T + \overline{K}_2 \) has dense range. We know that if \( A = \{a_{ij}\}_{i,j} \) is a triangular operator with respect to a suitable orthonormal basis, then \( \sigma_p(A^*) \subset \{\overline{a}_{ii} : i \} \). Now we recall the proof of Lemmas 2.5–2.8.

(a) Look back to the formula (1) in the proof of Lemma 2.5. If \( \text{Ran}T \) is dense in \( \mathcal{H} \), then \( \sigma_p(T_1^*) = \sigma_p(T_2^*) = \emptyset \). To see (2), note that \( \sigma_p((T + K)^*) = \emptyset \). Look back to the proof of Lemma 2.6. If \( \text{Ran}T \) is dense in \( \mathcal{H} \), then \( \text{Ran}(T + K_1) \) is dense. Thus \( \sigma_p((T + K)^*) = \emptyset \).

(b) Now look at (4), in the proof of Lemma 2.7. If \( \text{Ran}T = \mathcal{H} \), then \( \text{Ran}T_3 = \mathcal{H}_2 \) and \( \sigma_p(T_3^*) = \emptyset \). In (5), it is clear that \( \sigma_p((T_1 + C_1)^*) = \sigma_p(C_2^*) = \emptyset \). So \( \sigma_p((T + K)^*) = \emptyset \).

(c) Look back to (6) in the proof of Lemma 2.8. Write \( W = T_{12}P_L \) and \( V = T_{12}P_{\mathcal{H}_2 \oplus \mathcal{L}} \). Then \( T_{12}\mathcal{H}_2 = W + V \). Look at (7). Set

\[
S = \begin{bmatrix}
T_1 & B \\
T_3 & \mathcal{H}_2
\end{bmatrix}
\]

If \( \text{Ran}T = \mathcal{H} \), then \( \text{Ran}T_3 = \mathcal{H}_2 \). So \( \sigma_p(T_3^*) = \emptyset \). If \( S^*(x \oplus y) = 0 \), where \( x \in \mathcal{H}_1 \) and \( y \in \mathcal{H}_2 \), then \( T_3^*x = 0 \) and \( B^*x + T_3^*y = 0 \). Notice that \( B = C + W + V \). Since \( \text{Ran}W \subset \text{Ran}T_1 \), \( W^*x = 0 \). Since \( \text{Ran}C^* \subset \mathcal{L} \) and \( \text{Ran}V^* \subset \mathcal{H}_2 \subset \mathcal{L} \), it follows that \( C^*x = 0 \). Hence \( V^*x + T_3^*y = 0 \), i.e. \( T^*(x \oplus y) = 0 \). While \( \text{Ran}T = \mathcal{H} \), \( x \oplus y = 0 \). Thus \( \sigma_p(S^*) = \emptyset \). It is clear that \( \sigma_p(T_0^*) = \emptyset \). So \( \sigma_p((T + K)^*) = \emptyset \).

Summarily, in the proof of Theorem 2.1, because \( \text{Ran}(T + \overline{K}_2) \) is dense in \( \mathcal{H} \), not only is \( T + K \) strongly irreducible, but also \( \sigma_p((T + K)^*) = \emptyset \).

Now we are going to prove (iii). Without loss of generality, assume that \( T + K \) (def \( A \)) can be written as

\[
A = \begin{bmatrix}
T_1 & * & * \\
T_2 & * & \mathcal{H}_2 \\
T_3 & \mathcal{H}_3
\end{bmatrix}
\]
where $\bigvee_{i=1}^{n} \ker T_{i}^{n} = H_{i}$, $i = 1, 2, 3$. Suppose $BX - XA = 0$. Write $X = (X_{1}, X_{2}, X_{3})$ where $X_{i} = X|_{H_{i}}$. Thus $BX_{1} - X_{1}T_{1} = 0$ and hence $B^{n}X_{1} = X_{1}T_{1}^{n}$ for all $n \geq 1$. If $y \in \ker T_{1}^{n}$, then $B^{n}X_{1}y = 0$. By $\sigma_{p}(B) = \emptyset$, $X_{1}y = 0$. So $X_{1} = 0$. Similarly, $X_{2} = 0$ and $X_{3} = 0$, i.e. $X = 0$.

By the upper semi-continuity of the spectrum, the continuity of index and Theorem 2.2 of [1] or Theorem 3.49 of [3], we have the following lemma.

**Lemma 2.10** ([1], [3]). Suppose $\emptyset \neq \Gamma \subset \sigma_{te}(T)$ and $\epsilon > 0$. Then there exists a compact operator $K$ with $\|K\| < \epsilon$ such that

$$T + K = \begin{bmatrix} N & \ast \\ \hat{T} \end{bmatrix} \mathcal{H}_{2} \mathcal{H}_{1}$$

where $N$ is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{te}(N) = \Gamma$, $\sigma(\hat{T}) = \sigma(T)$, $\sigma_{te}(\hat{T}) = \sigma_{te}(T)$, and $\text{ind}(\hat{T} - \lambda) = \text{ind}(T - \lambda)$ for all $\lambda \in \rho_{S^{(n)}}(T)$.

**Lemma 2.11** ([2], [3]). Let $A, B$ be two operators and let $\tau_{A,B}$ be the Rosenblum operator. Then the followings are equivalent:

(i) $\sigma_{r}(A) \cap \sigma_{l}(B) = \emptyset$.

(ii) $\tau_{A,B}$ is surjective.

(iii) $\text{Ran}\tau_{A,B}$ contains all compact operators.

By Corollary 2.4 of [4], it is not difficult to prove the following lemma.

**Lemma 2.12** ([4], [5]). Suppose that $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular and that $\sigma(T) = \sigma_{w}(T)$. Let $\Gamma = \{\lambda_{n}\}_{n \geq 1} \subset \sigma(T)$ satisfying the following conditions:

(i) $\text{Card}\{n : \lambda_{n} = \lambda_{j}\} = +\infty$ for all $j \geq 1$.

(ii) Each clopen of $\sigma(T)$ intersects with $\Gamma$.

Let $\epsilon > 0$ be open. Then there exists a compact operator $K$ with $\|K\| < \epsilon$ such that $\bigvee \{\ker(T + K - \lambda_{n})^{k} : n \geq 1, k \geq 1\} = \mathcal{H}$, $\Gamma \subset \sigma_{p}(T + K)$ and $\sigma_{p}(\{T + K\}^{*}) = \emptyset$.

Moreover, if $\sigma(T)$ and $\sigma_{w}(T)$ are connected, and if $\rho_{S^{(n)}}(T)$ contains a nonempty connected open subset $\Omega$, then $K$ can be chosen so that $T + K \in B_{n}(\Omega)$.

**Lemma 2.13** ([11]). Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $\sigma_{0}(T) = \emptyset$ and $\epsilon > 0$. Then there exists a compact operator $K$ with $\|K\| < \epsilon$ such that

(i) $\sigma(T + K) = \sigma(T)$,

(ii) $\min \text{ind}(T + K - \lambda) = \begin{cases} 0, & \lambda \in \rho_{S_{-F}}^{(0)}(T), \\ 1, & \lambda \in \rho_{S_{-F}}^{(0)}(T) \cap \sigma(T). \end{cases}$

**Lemma 2.14.** Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\sigma(T) \cap \rho_{S_{-F}}(T) = \rho_{S_{-F}}^{(0)}(T)$. Let $\{\Omega_{j}\}_{j}$ be the connected components of $\rho_{S_{-F}}^{(1)}(T)$. Suppose that $\bigcup_{j} \Omega_{j}$ intersects with arbitrary clopen of $\sigma(T)$. Let $\epsilon > 0$ be given. Then there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that

(i) $\bigvee \{\ker(T + K - \lambda) : \lambda \in \bigcup_{j} \Omega_{j}\} = \mathcal{H}$ and $\sigma_{p}((T + K)^{*}) = \emptyset$. 


(ii) $T + K$ has the form

$$T + K = \begin{bmatrix}
B_1 & ** & B_2 \\
** & * & B_3 \\
\vdots & \vdots & \ddots \\
** & & B_\infty
\end{bmatrix}
\begin{bmatrix}
\mathcal{M}_1 \\
\mathcal{M}_2 \\
\mathcal{M}_3 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
K_1 \\
K_2 \\
K_3 \\
\vdots
\end{bmatrix}
$$

where each $B_j$ ($j < \infty$) is a Cowen-Douglas operator with index 1, $\sigma(B_j) \cap \sigma(B_j) = \emptyset$ ($i \neq j, i, j < +\infty$) and $\sqrt{} \{\mathcal{M}_j : k \leq j \leq +\infty\}$ is invariant under the commutant of $T + K$ ($1 \leq k \leq +\infty$).

**Proof.** Let $\sigma_j$ be the maximal connected closed subset of $\sigma(T)$ containing $\Omega_j$ for each $j$. Without loss of generality, assume that $\sigma_i \cap \sigma_j = \emptyset$ when $i \neq j$. Let $\Phi_j$ be the interior of the closure of $\Omega_j$. Take $\alpha_j \in \Omega_j$. Let $\mu_j$ be the probability measure supported by $\partial \Phi_j$ such that $\int \varphi(z) d\mu_j(z) = \varphi(\alpha_j)$ for those functions analytic in a neighbourhood of $\Phi_j$. Let $M_j$ be the operator of multiplication by $z$ on $L^2(\mu_j)$. Let $H^2(\mu_j)$ be the span of all rational functions with poles outside $\Phi_j$. Set

$$M_j = \begin{bmatrix}
M_j^+ & * \\
M_j^- & H^2(\mu_j)
\end{bmatrix}
\begin{bmatrix}
H^2(\mu_j) \\
H^2(\mu_j)
\end{bmatrix}
$$

It is easy to show that

1. $M_j$ is normal and $\sigma(M_j) = \sigma_{tr}(M_j) = \partial \Phi_j$,
2. $\sigma(M_j^-) = \Phi_j$ and $M_j^- \in b_1(\Phi_j)$.

Applying Lemma 2.10 to $T^*$, one can take $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \frac{\xi}{\delta}$ such that

3. $T + K_1 = \begin{bmatrix}
T_1 & * \\
\oplus & N_j
\end{bmatrix}$,

4. $\sigma(T_1) = \sigma(T, \sigma_{tr}(T_1) = \sigma_{tr}(T)$ and $\text{ind}(T - \lambda) = 1$ for $\lambda \in \bigcup \Omega_j$.

5. $N_j$ is diagonal normal and $\sigma(N_j) = \sigma_{tr}(N_j) = \partial \Phi_j$ for each $j$.

Since $N_j, M_j$ are normal and $\sigma(N_j) = \sigma_{tr}(N_j) = \sigma_{tr}(M_j) = \sigma(M_j)$, there exists a compact operator $K_j$ with $\|K_j\| < \frac{\xi}{\delta}$ such that $N_j + K_j \cong M_j$, where $\cong$ is the unitary equivalence relation. Thus there is a $K_2 \in \mathcal{K}(\mathcal{H})$ with $\|K_2\| < \frac{\xi}{\delta}$ such that

$$T + K_1 + K_2 \cong \begin{bmatrix}
T_1 & * \\
\oplus & M_j
\end{bmatrix}
= \begin{bmatrix}
T_1 & * \\
\oplus & M_j^+
\end{bmatrix}
\begin{bmatrix}
* & * \\
\oplus & M_j^-
\end{bmatrix}
\overset{\text{def}}{=} \begin{bmatrix}
T_2 & T_{12} \\
\oplus & M_j^-
\end{bmatrix}
$$

By Theorem 3.48 of [3], choose $K_3 \in \mathcal{K}(\mathcal{H})$ with $\|K_3\| < \frac{\xi}{\delta}$ such that

$$T + \sum_{j=1}^3 K_j \cong \begin{bmatrix}
T_2 + C_1 & * \\
\oplus & M_j^-
\end{bmatrix}$$
and \( \sigma(T_2 + C_1) = \sigma_w(T_2 + C_1) = \sigma(T) \setminus \bigcup_j \Omega_j \). Notice that each clopen \( \sigma \) of \( \sigma(T_2 + C_1) \) intersects with the closure of \( \bigcup_j \Omega_j \). There is a subset \( \{ \lambda_k \}_{k \geq 1} \subset \bigcup_j \Omega_j \) such that

(6) \( \sup_k \text{dist}(\lambda_k, \sigma(T_2 + C_1)) < \frac{\epsilon}{16} \) and \( \lim_k \text{dist}(\lambda_k, \sigma(T_2 + C_1)) = 0 \).

(7) Each clopen of \( \sigma(T_2 + C_1) \) contains limit points of \( \{ \lambda_k \}_{k \geq 1} \).

Let \( \Gamma = \{ \mu_k \}_{k \geq 1} \) be a dense subset of all limit points of \( \{ \lambda_k \}_{k \geq 1} \). By Lemma 2.12, find \( K_4 \in \mathcal{K}(\mathcal{H}) \) with \( \| K_4 \| < \frac{\epsilon}{16} \) such that

\[
T + \sum_{j=1}^{4} K_j \cong \begin{bmatrix} T_2 + C_1 + C_2 & \ast \\ \ast & M_j^- \end{bmatrix}
\]

where

\[
T_2 + C_1 + C_2 = \begin{bmatrix} v_1 & \ast & \ast & \cdots \\ v_2 & \ast & \ast & \cdots \\ v_3 & & & \cdots \\ & & & \cdots \end{bmatrix}
\]

with respect to a suitable orthonormal basis, where \( v_i \in \Gamma \), and Card\( \{ n : v_n = \mu_j \} = +\infty \) for all \( j \geq 1 \). Choose \( \lambda_k \) such that \( |\lambda_k - v_j| < \frac{\epsilon}{16 \sqrt{j}} \) and \( \lambda_k \notin \{ \lambda_i \}_{i < j} \). Perturb \( v_j \) by \( \lambda_k - v_j \). Then one can find \( K_5 \in \mathcal{K}(\mathcal{H}) \) with \( \| K_5 \| < \frac{\epsilon}{16} \) such that

\[
T + \sum_{j=1}^{5} K_j \cong \begin{bmatrix} T_2 + \sum_{i=1}^{3} C_i & \ast \\ \ast & M_j^- \end{bmatrix}
\]

where \( \sigma_0(T_2 + \sum_{i=1}^{3} C_i) = \{ \lambda_{k_i} \}_{j \geq 1} \), \( \forall \text{ Ker}(T_2 + \sum_{i=1}^{3} C_i - \lambda_{k_j}) \) is equal to the acting space of \( T_2 + \sum_{i=1}^{3} C_i \), and rank\( E(\lambda_{k_j}, T_2 + \sum_{i=1}^{3} C_i) = 1 \). Write \( T = T_2 + \sum_{i=1}^{3} C_i \).

Without loss of generality, assume that \( \lambda_{k_j} = \lambda_j \) and that

\[
T + \sum_{j=1}^{5} K_j = \begin{bmatrix} T & \ast \\ \ast & M_j^- \end{bmatrix} \mathcal{M}^\perp \mathcal{M}
\]

Notice that \( \sigma_p((T + \sum_{i=1}^{5} K_i)^*) \subset \{ \lambda_j : j \geq 1 \} \). If \( \sigma_p((T + \sum_{i=1}^{5} K_i)^*) \) is nonempty, denote it by \( \{ \pi_j : j \} \). Write \( \mathcal{H}_1 = \bigcup \{ \text{ Ker}(T - a_j) : j \} \), \( \mathcal{H}_2 = \mathcal{M}^\perp \ominus \mathcal{H}_1 \). Then \( T + \sum_{j=1}^{5} K_j \) can be written as

\[
T + \sum_{j=1}^{5} K_j = \begin{bmatrix} A_1 & A_{12} & A_{13} \\ A_{12} & A_2 & A_{23} \\ & & \ast \end{bmatrix} \mathcal{H}_1 \mathcal{H}_2 \mathcal{M}
\]

Notice that \( \bigcup_k \text{ Ran}(A_1 - a_k) \neq \mathcal{H}_1 \) and that \( \bigcup_k \text{ Ran}(\bigoplus_j M_j^- - a_k)^* \neq \mathcal{M} \). Choose unit vectors \( e \in \mathcal{H}_1 \setminus \bigcup_k \text{ Ran}(A_1 - a_k) \) and \( f \in \mathcal{M} \setminus \bigcup_k \text{ Ran}(\bigoplus_j M_j^- - a_k)^* \). Define
$K_6x = \frac{1}{64}(x, f)e$, where $(x, f)$ is the scalar product of $x$ and $f$. Let $K = \sum_{j=1}^{6} K_j \in K(H)$. Then $\|K\| < \epsilon$. It is an exercise to show that $\sigma_p((T + K)^*) = \emptyset$ and that $\bigvee \{\text{Ker}(T + K - \lambda) : \lambda \in \bigcup_{j} \Omega_j\} = H$. Let $N_j = \bigvee \{\text{Ker}(T + K - \lambda) : \lambda \in \bigcup_{i \geq j} \Omega_j\}$ and $M_\infty = \bigcap_{j < \infty} N_j$. Write $M_j = N_j \oplus N_{j+1}$ for $1 \leq j < \infty$.

Then $T + K = \begin{bmatrix} B_1 & \cdots & B_3 \\ \vdots & \ddots & \vdots \\ B_\infty & \cdots & B_\infty \end{bmatrix} \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \end{bmatrix}$.

It is clear that $B_j \in B_1(\Omega_j)$ for $j < \infty$ and that $\sigma(B_j) \subset \sigma_j$. Furthermore, if $X$ commutes with $T + K$, then $X$ has the form

$$X = \begin{bmatrix} X_1 & \cdots & X_3 \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & X_\infty \end{bmatrix} \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \end{bmatrix}.$$ 

3. Proof of Main Theorem

By Theorem 2.1 and Lemma 2.13, assume that $\sigma_w(T)$ is nonconnected and

$$\min \text{ ind}(T - \lambda) = \begin{cases} 0, & \lambda \in \rho_{S_F}(T), \\ 1, & \lambda \in \sigma(T) \cap \rho_{S_F}(T). \end{cases}$$

Suppose $\{\Omega_j\}$ are the connected components of $\sigma(T) \cap \rho_{S_F}(T)$. Let $H_l = \bigvee \{\text{Ker}(T - \lambda)^* : \lambda \in \bigcup_{j} \Omega_j\}$. Then $T$ can be written as

$$T = \begin{bmatrix} T_1 & * \\ T_2 & \end{bmatrix} \begin{bmatrix} \mathcal{H}_l^\perp \\ \mathcal{H}_l \end{bmatrix}.$$ 

By Lemma 2.10, choose a $\mathbf{K}_1 \in \mathcal{K}(\mathcal{H}_l)$ with $\|\mathbf{K}_1\| < \frac{\epsilon}{2}$ such that

$$T_2 + \mathbf{K}_1 = \begin{bmatrix} N & * \\ \mathbf{T}_3 & \end{bmatrix} \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_l \end{bmatrix}.$$

where $N$ is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{\text{irr}}(N) = \sigma_{\text{irr}}(T_2)$, $\sigma(T_3) = \sigma(T_2)$, $\sigma_{\text{irr}}(\mathbf{T}_3) = \sigma_{\text{irr}}(T_2)$ and $\text{ind}(T_3 - \lambda) = \text{ind}(T_2 - \lambda) = -1$ for $\lambda \in \bigcup_{j} \Omega_j$. Write $\mathcal{H}_1 = \mathcal{H}_l^\perp \oplus \mathcal{H}_0$. Set

$$\mathbf{K}_1 = \begin{bmatrix} 0 \\ \overline{K}_1 \end{bmatrix} \begin{bmatrix} \mathcal{H}_l^\perp \\ \mathcal{H}_l \end{bmatrix}.$$
Then $K_1$ is compact and $\|K_1\| < \frac{\varepsilon}{4}$. Look at $T + K_1$:

$$T + K_1 = \begin{bmatrix} T_1 & * & * & \cdots & * \\
N & \mathcal{H}_i^+ & \mathcal{H}_0 & \mathcal{H}_i \oplus \mathcal{H}_0 & \mathcal{H}_i \\
T_3 & \mathcal{H}_i \oplus \mathcal{H}_0 & \mathcal{H}_0 & \mathcal{H}_i & \mathcal{H}_i \oplus \mathcal{H}_0
\end{bmatrix}$$

It is clear that $\sigma(T_1) = \sigma_w(T_1) = \sigma(T_1)$, $\text{ind}(T_1 - \lambda) > 0$ for $\lambda \in \rho_{S-F}(T_1) \cap \sigma(T_1)$ and $\text{ind}(T_1 - \lambda) = 1$ for $\lambda \in \bigcup_j \Omega_j$. By Lemma 2.12, take $K_2 \in \mathcal{K}(\mathcal{H}_1)$ with $\|K_2\| < \frac{\varepsilon}{8}$ such that $A = T_1 + K_2 \in \mathcal{B}(\Omega_1)$. Hence $A$ is strongly irreducible. Notice that $\mathcal{H}_i = \bigvee \{\text{Ker}(T - \lambda)^*: \lambda \in \bigcup_j \Omega_j\}$. Each clopen of $\sigma(T_2)$ intersects with some $\Omega_j$. So each clopen of $\sigma(T_3)$ contains some $\Omega_j$. Notice that $\sigma(T_3^*) \cap \rho_{S-F}(T_3) = T_3^*$ and $\text{ind}(T_3 - \lambda)^* = 1$ for $\lambda \in \bigcup_j \Omega_j$. Applying Lemma 2.14 to $T_3$, find a compact operator $K_3$ with $\|K_3\| < \frac{\varepsilon}{16}$ such that $\sigma_p(T_3 + K_3) = \emptyset$ and $T_3 + K_3$ can be written as

$$T_3 + K_3 = \begin{bmatrix} B_1 & * & * & \cdots & * \\
B_2 & \cdots & B_3 & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& & & & B_{\infty}
\end{bmatrix} \begin{bmatrix} \mathcal{H}_1 \\
\mathcal{H}_1 \oplus \mathcal{H}_0
\end{bmatrix}$$

where each $B_j^*$ ($j < +\infty$) is a Cowen-Douglas operator with index 1, $\sigma(B_j) \cap \sigma(B_i) = \emptyset$ when $i \neq j$, $i, j < +\infty$, and $\bigoplus_{i=1}^k \mathcal{M}_j$ is invariant under the commutant of $T_3 + K_3$ for each $1 \leq k \leq +\infty$. Set

$$K_2 = \begin{bmatrix} K_2 \\
K_3
\end{bmatrix} \begin{bmatrix} \mathcal{H}_1 \\
\mathcal{H}_1 \oplus \mathcal{H}_0 \end{bmatrix} \in \mathcal{K}(\mathcal{H})$$

Then $\|K_2\| < \frac{\varepsilon}{8}$ and

$$T + K_1 + K_2 = \begin{bmatrix} A & * & * & \cdots & * \\
A_{11} & A_{12} & A_{13} & \cdots & \cdots \\
B_1 & B_{12} & B_{13} & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& & & & B_{\infty}
\end{bmatrix} \begin{bmatrix} \mathcal{H}_1 \\
\mathcal{H}_1 \oplus \mathcal{H}_0
\end{bmatrix}$$

Because $\sigma(B_i) \cap \sigma(B_j) = \emptyset$ for $i \neq j$ and $\sigma_{irr}(A) \cap \sigma_{irr}(B_j) = \sigma_{irr}(B_j) \neq \emptyset$ for all $j < +\infty$, by Lemma 2.11, we can inductively find $C_j \in \mathcal{K}(\mathcal{M}_j, \mathcal{H}_1)$ with $\|C_j\| < \varepsilon/2^{j+4}$ such that $A_{i1} + C_1 \notin \text{Ran}_{A,B_1}$ and

$$\begin{bmatrix} A_{i,j+1} + C_{j+1} \\
B_{i,j+1} \\
& \cdots \\
& & B_{j,j+1}
\end{bmatrix} \notin \text{Ran}_{A,B_{j+1}}$$
where

\[
A_j = \begin{bmatrix}
A & A_{11} + C_1 & \cdots & A_{1,j} + C_j \\
B_1 & \cdots & B_{1,j} \\
\ddots & \vdots & \ddots \\
B_j & & & \ddots \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{M}_1 \\
\vdots \\
\mathcal{M}_j \\
\end{bmatrix}
\]

Write \( D_j = A_{1,j} + C_j \). Define \( K_3 x = \sum_{j<+\infty} C_j P_{M_j} x \), where \( P_{M_j} \) is the projection onto \( M_j \) for each \( j < +\infty \). Then \( K = K_1 + K_2 + K_3 \) is compact and \( \| K \| < \epsilon \).

Moreover,

\[
T + K = \begin{bmatrix}
A & D_1 & D_2 & \cdots \\
B_1 & B_{1,2} & \cdots & B_{1,j} \\
\ddots & \ddots & \ddots & \ddots \\
B_j & & & B_{j,\infty} \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{M}_1 \\
\vdots \\
\mathcal{M}_\infty \\
\end{bmatrix}
\]

Now we are going to prove that \( T + K \) is strongly irreducible. Suppose that \( P \) is a non-zero idempotent operator commuting with \( T + K \). Set

\[
P = \begin{bmatrix}
P_0 & Q_{10} & P_{10} \\
Q_{10} & \mathcal{H}_1 & \mathcal{H}_1^+ \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{M}_1 \\
\mathcal{M}_2 \\
\vdots \\
\mathcal{M}_\infty \\
\end{bmatrix}
\]

It is easy to see that \( (\mathcal{T}_3 + \mathcal{K}_3)Q_{10} = Q_{10}A \). So \( (\mathcal{T}_3 + \mathcal{K}_3 - \lambda)Q_{10} = Q_{10}(A - \lambda) \) for \( \lambda \in \Omega_1 \). Since \( A \in \mathcal{B}(\Omega_1) \) and \( \sigma_p(\mathcal{T}_3 + \mathcal{K}_3) = \emptyset \), \( Q_{10} = 0 \). Furthermore, \( \mathcal{P} \) is an idempotent operator commuting with \( \mathcal{T}_3 + \mathcal{K}_3 \). So \( \mathcal{P} \) has the form

\[
\mathcal{P} = \begin{bmatrix}
P_1 & P_{1,2} & P_{1,3} & \cdots \\
P_2 & P_{2,3} & \cdots & \cdots \\
P_3 & & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
P_\infty \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{M}_1 \\
\mathcal{M}_2 \\
\vdots \\
\mathcal{M}_\infty \\
\end{bmatrix}
\]

It is clear that \( P_i \) is an idempotent operator commuting with \( B_i \) for each \( 1 \leq i \leq +\infty \), and that \( P_0 \) is idempotent and commutes with \( A \). Since all \( B_i \) (\( j < +\infty \)) and \( A \) are strongly irreducible, \( P_i \) is equal to either zero or the unit operator on its acting space for each \( 0 \leq j < +\infty \). Set

\[
P = \begin{bmatrix}
P_0 & P_{01} & P_{02} & \cdots \\
P_1 & P_{1,2} & \cdots & \cdots \\
P_2 & & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
P_\infty \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{M}_1 \\
\vdots \\
\mathcal{M}_\infty \\
\end{bmatrix}
\]

If \( P_0 = 0 \), then \( AP_{01} + D_1 P_1 = P_{01} B_1 \). Since \( D_1 \notin \text{Ran}_{A,B_1} \), \( P_1 = 0 \). It follows from \( P^2 = P \) that \( P_{01} = 0 \). Similarly, one can inductively prove that \( P_j = 0 \) for \( j < +\infty \) and that \( P_{i,j} = 0 \) for \( i, j < \infty \). Thus

\[
\overline{P} = \begin{bmatrix}
0 & * \\
P_\infty \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_1^+ \ominus \mathcal{M}_\infty \\
\mathcal{M}_\infty \\
\end{bmatrix}
\]
It is clear that $\text{Ran}\overline{P}^* \cap \text{Ker}(\overline{T}_3 + \overline{K}_3 - \lambda)^* = \{0\}$ for all $\lambda \in \bigcup_j \Omega_j$. So $\overline{P} = 0$.

Hence $P = 0$. If $P_0$ is the unit operator acting on $\mathcal{H}_1$, then one can show that $P = I$. So $T + K$ is strongly irreducible.

References


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