A SHARP VERSION OF ZHANG’S THEOREM ON TRUNCATING SEQUENCES OF GRADIENTS

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Abstract. Let $K \subset \mathbb{R}^{m \times n}$ be a compact and convex set of $m \times n$ matrices and let $\{u_j\}$ be a sequence in $W^{1,1}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ that converges to $K$ in the mean, i.e. \( \int_{\mathbb{R}^n} \text{dist}(Du_j, K) \to 0 \). I show that there exists a sequence $v_j$ of Lipschitz functions such that $\|\text{dist}(Dv_j, K)\|_{\infty} \to 0$ and $\mathcal{L}^n(\{u_j \neq v_j\}) \to 0$. This refines a result of Kewei Zhang (Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), 313-326), who showed that one may assume $\|Dv_j\|_{\infty} \leq C$. Applications to gradient Young measures and to a question of Kinderlehrer and Pedregal (Arch. Rational Mech. Anal. 115 (1991), 329-365) regarding the approximation of $\mathbb{R} \cup \{+\infty\}$ valued quasiconvex functions by finite ones are indicated. A challenging open problem is whether convexity of $K$ can be replaced by quasiconvexity.

1. Main results

Let $\{u_j\}$ be a sequence of weakly differentiable functions $u_j : \mathbb{R}^n \to \mathbb{R}^m$ whose gradients approach the ball $B(0, R)$ in the mean, i.e.

\[
\int_{\mathbb{R}^n} \text{dist}(Du_j, B(0, R))dx \to 0.
\]

Motivated by work of Acerbi and Fusco [1], [2], and Liu [13], Kewei Zhang showed that the sequence can be modified on a small set in such a way that the new sequence is uniformly Lipschitz. The following theorem is a slight variant of Lemma 3.1 in [21].

Theorem 1 (Zhang). There exists a constant $c(n, m)$ with the following property. If (1.1) holds, then there exists a sequence of functions $v_j : \mathbb{R}^n \to \mathbb{R}^m$ such that

\[
\|Dv_j\|_{\infty} \leq c(n, m)R, \quad \mathcal{L}^n(\{u_j \neq v_j\}) \to 0.
\]

In fact one has the seemingly stronger conclusions

\[
\mathcal{L}^n(\{u_j \neq v_j \text{ or } Du_j \neq Dv_j\}) \to 0, \quad \int_{\mathbb{R}^n} |Du_j - Dv_j|dx \to 0.
\]

For the first conclusion it suffices to note that for weakly differentiable functions $u$ and $v$ the implication

\[
u = v \text{ a.e. in } A \implies Du = Dv \text{ a.e. in } A
\]

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holds (see e.g. [8], Lemma 7.7). For the second conclusion observe that

\[|D u_j - D v_j| \leq \frac{1}{R} + \frac{1}{R} + \text{dist}(D u_j, B(0, R))\]

and integrate over the set \(\{D u_j \neq D v_j\}\).

Theorem 1 has found important applications to the calculus of variations, in particular the study of quasiconvexity, lower semicontinuity, relaxation and gradient Young measures ([9], [21]; see also Corollary 3). The purpose of the present work is to show that the constant \(c(n, m)\) can be chosen arbitrarily close to 1 and that the ball \(B(0, R)\) can be replaced by a compact, convex set.

**Theorem 2.** Let \(K\) be a compact, convex set in \(\mathbb{R}^{mn}\). Suppose \(u_j \in W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)\) and

\[
\int_{\mathbb{R}^n} \text{dist}(D u_j, K) dx \rightarrow 0.
\]

Then there exists a sequence \(v_j\) of Lipschitz functions such that

\[
||\text{dist}(D v_j, K)||_{\infty} \rightarrow 0, \quad \mathcal{L}^n\{u_j \neq v_j\} \rightarrow 0.
\]

**Remarks.**

1. A more natural and apparently much harder question is whether the same assertion holds if \(K\) is quasiconvex rather than convex.

2. Jan Kristensen pointed out to me that in the scalar case \(m = 1\) the assumption that \(K\) is convex can be dropped. Let \(C K\) denote the convex hull of \(K\) and \(C \text{dist}_K\) the convex envelope of the distance function. Kristensen’s proof uses (2.14), applied with \(C K\) and \((C K)_\gamma = C K_\gamma\), the identity \(C \text{dist}_K = \text{dist}(C K)_\gamma\), and the relaxation of nonconvex integral functionals (see e.g. [5]) to obtain

\[
\inf \{ \int_B \text{dist}(D v, K_\gamma) dy : v = u \text{ on } \partial B\}
\]

\[
= \inf \{ \int_B C \text{dist}(D w, K_\gamma) : w = u \text{ on } \partial B\}
\]

\[
\leq \int_B \text{dist}(D \tilde{u}, (C K)_\gamma)
\]

\[
\leq (1 - 3^{-n}) \int_B \text{dist}(D u, C K) dx.
\]

A similar argument can be applied for \(m > 1\) provided that a (somewhat artificial) condition holds which is slightly stronger than the requirement that \(C K\) agrees with the quasiconvex hull \(Q K\) of \(K\).

In the language of Young measures (see [9], [10] for the relevant definitions) one can deduce the following.

**Corollary 3.** Let \(K\) be a compact, convex set in \(\mathbb{R}^{mn}\) and let \(\Omega \subset \mathbb{R}^n\) be open, let \(p \geq 1\), and suppose that \(\{u_j\}\) generates a \(W^{1,p}\) gradient Young measure \(\nu = \{\nu_x\}_{x \in \Omega}\) and that

\[
\text{supp} \nu_x \subset K \text{ for a.e. } x \text{ in } \Omega.
\]
Then there exists a sequence \( \{v_j\} \) that generates the same gradient Young measure and satisfies
\[
\| \text{dist}(Dv_j, K) \|_\infty \to 0.
\]

**Warning:** There are slightly different definitions of \( W^{1,1} \) gradient Young measures in use. Above we have adopted the convention that those measures are generated by sequences for which \( \{Du_j\} \) is equi-integrable (and not merely bounded in \( L^1 \)). No ambiguities arise for \( p > 1 \).

Using Corollary 3, one can simplify the theory of \( W^{1,\infty} \) gradient Young measures and answer some of the questions raised in [9] (see Corollary 9 below).

A version of Corollary 3 for Young measures with finite \( p \)th moment was discovered by Kristensen [11] and later independently in [7]. It can be used to obtain a simpler approach to \( W^{1,p} \) gradient Young measures ([16], [17]).

For \( \Omega \neq \mathbb{R}^n \), Corollary 3 requires a local version of Theorem 2.

**Theorem 4.** Let \( K \) be a compact, convex set in \( \mathbb{R}^{mn} \), let \( \Omega \subset \mathbb{R}^n \) be open and let \( \{u_j\} \) be a sequence in \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^m) \) that satisfies
\[
(1.4) \quad u_j \to u_0 \text{ in } L^1_{\text{loc}}(\Omega; \mathbb{R}^m),
\]
\[
(1.5) \quad \text{dist}(Du_j, K) \to 0 \text{ in } L^1_{\text{loc}}(\Omega).
\]
Then there exists an increasing sequence of open sets \( U_j \), compactly contained in \( \Omega \), and functions \( v_j \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^m) \) such that
\[
(1.6) \quad v_j = u_0 \text{ on } \Omega \setminus U_j,
\]
\[
(1.7) \quad \mathcal{L}^n(\{u_j \neq v_j\} \cap U_j) \to 0,
\]
\[
(1.8) \quad \| \text{dist}(Dv_j, K) \|_{\infty, \Omega} \to 0.
\]

**Remarks.** 1. If \( \Omega \) has finite volume, we have \( \mathcal{L}^n(\Omega \setminus U_j) \to 0 \), and thus \( \mathcal{L}^n(\{u_j \neq v_j\}) \to 0 \).

2. If \( u_j \to u_0 \) in \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^m) \), then (1.4) holds by the compact Sobolev embedding. In fact, (1.4) and (1.5) imply weak convergence in \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^m) \) (see the proof).

3. Condition (1.6) is a statement of the fact that \( u_0 \) and \( v_j \) satisfy the same ‘boundary condition’ (traces may not exist, since we assumed no regularity of \( \Omega \)).

**2. Proofs in \( \mathbb{R}^n \)**

In order to remove the small ‘bad’ region where \( \text{dist}(Du_j, K) > \epsilon \) we locally mollify \( u_j \). A key point is to use different mollification radii in different regions of \( \mathbb{R}^n \) (I learned about the use of \( x \)-dependent mollifiers through the papers [18] and [19] of Schoen and Uhlenbeck). Each mollification step reduces the \( L^1 \) norm of the distance function by a fixed factor, but slightly increases the \( L^\infty \) norm on the good set. Careful iteration shows, however, that the latter effect can be controlled.

A more precise outline of the proof is as follows. In Lemma 5 we obtain quantitative estimates for mollification on a ball. In Lemma 6 we combine these estimates with a covering argument to achieve the desired reduction of the \( L^1 \) norm. Theorem 7 contains the result of the iteration procedure. Finally, Theorem 2 is an immediate consequence of Theorem 7.
In the following $K$ always denotes a compact, convex set in $M := M^{m \times n} = \mathbb{R}^{mn}$. We use the operator norm $|F| := \sup\{|Fx| : |x| = 1\}$ on $M$. The distance function
$$\text{dist}(A, K) = \min\{|A - F| : F \in K\}$$
is 1-Lipschitz and convex, since $K$ is convex. Its sublevel sets
$$K_{\gamma} := \{A \subset M : \text{dist}(A, K) \leq \gamma\}$$
are compact and convex, and for $\gamma > 0$ and $\delta > 0$ one has
$$\text{dist}(A, K_{\gamma}) \leq (\text{dist}(A, K) - \gamma)^+, \quad \text{where } a^+ = \max(a, 0).$$
If we let
$$|K|_{\infty} := \max\{|A| : A \in K\},$$
we have
$$|K_{\gamma}|_{\infty} = |K|_{\infty} + \gamma, \quad |A| \leq |K|_{\infty} + \text{dist}(A, K).$$
By $\int_E f dx$ we denote the mean value $(\mathcal{L}^n(E))^{-1} \int_E f dx$.

\textbf{Lemma 5.} If $u \in W^{1,1}(B(a, r); \mathbb{R}^m)$ and if
$$\Theta \geq \frac{1}{|K|_{\infty}} \int_{B(a, r)} \text{dist}(Du, K) dx, \quad \Theta < 8^{-(n+1)}, \quad \gamma := 9\Theta \pi^{n+1} |K|_{\infty},$$
then there exists $\tilde{u} \in W^{1,1}(B(a, r))$ such that
$$\tilde{u} = u \text{ on } \partial B(a, r),$$
$$\int_{B(a, r)} \text{dist}(D\tilde{u}, K_{\gamma}) dx \leq (1 + \Theta \pi^{n+1}) \int_{B(a, r) \setminus B(a, r/2)} \text{dist}(Du, K) dx.$$ 

\textbf{Proof.} The statement is invariant under the rescaling
$$u \rightarrow \frac{r}{|K|_{\infty}} u\left(\frac{x - a}{r}\right), \quad K \rightarrow \frac{K}{|K|_{\infty}}, \quad \gamma \rightarrow \frac{\gamma}{|K|_{\infty}}.$$ 
We may thus assume $|K|_{\infty} = 1, a = 0, r = 1$, and we write $B := B(0, 1), B_\rho := B(0, \rho)$. Let $\epsilon \in (0, 1/8)$ (a specific choice will be made below), and for $x \in B_{7/8}$ let
$$v(x) = \int_{B(x, \epsilon)} u dy = \int_{B_\epsilon} u(x + z)dz.$$ 

Then
$$Dv(x) = \int_{B(x, \epsilon)} Du dy$$
and, by convexity of the distance function,
$$\text{dist}(Dv(x), K) \leq \int_{B(x, \epsilon)} \text{dist}(Du, K) dy \leq \epsilon^{-n} \Theta.$$ 
Let $\varphi : B \rightarrow [0, 1]$ be a cut-off function that satisfies
$$\varphi \in W^{1,\infty}_0(B_{7/8}), \quad \varphi \equiv 1 \text{ on } B_{5/8}, \quad |D\varphi| \leq 8,$$
and define
\[ \tilde{u} = (1 - \varphi)u + \varphi v. \]
Then \( \tilde{u} = u \) on \( B \setminus B_{7/8} \) and
\[ D\tilde{u} = (1 - \varphi)Du + \varphi Lv + (v - u) \otimes Du \varphi \text{ in } B. \]
Thus
\[ D\tilde{u} = Du \text{ in } B \setminus B_{7/8}, \]
(2.4) \[ \text{dist}(D\tilde{u}, K) = \text{dist}(Du, K) \leq \epsilon^{-n}\Theta \text{ in } B_{5/8}. \]

We next estimate \( v - u \). In view of (2.3) and the assumption \( \|K\| = 1 \) we have for a.e. \( x \in B_{7/8} \)
\[ |v - u|(x) \leq \int_{B(x,\epsilon)} |u(y) - u(x)|dy \]
\[ \leq \frac{1}{E^n(B_\epsilon)} \int_0^\epsilon \int_{S^{n-1}} |u(x + \rho e) - u(x)|dH^{n-1}(e)\rho^{n-1}d\rho \]
\[ \leq \frac{1}{E^n(B_\epsilon)} \int_0^\epsilon \int_0^\rho 1dH^{n-1}(e)\rho^{n-1}d\rho \]
\[ + \frac{1}{E^n(B_\epsilon)} \int_0^\epsilon \int_0^\rho \text{dist}(Du, K)(x + te)dtdH^{n-1}(e)\rho^{n-1}d\rho \]
\[ =: T_1(x, \epsilon) + T_2(x, \epsilon). \]
We have \( T_1(x, \epsilon) = \epsilon^{n+1} \leq \epsilon \), and thus Fubini’s theorem yields
\[ \int_{B_{7/8} \setminus B_{5/8}} (|v - u| - \epsilon)dx \leq \int_{B_{7/8} \setminus B_{5/8}} T_2(x, \epsilon)dx \]
(2.5) \[ \leq \frac{1}{E^n(B_\epsilon)} \int_0^\epsilon \int_0^\rho \left( \int_{B(B_{1/2})} \text{dist}(Du, K)dx \right)dtdH^{n-1}(e)\rho^{n-1}d\rho \]
\[ \leq \epsilon \int_{B(B_{1/2})} \text{dist}(Du, K)dx. \]
Since the distance function is convex and 1-Lipschitz, we have
\[ \text{dist}(D\tilde{u}, K) \leq \varphi \text{ dist}(Du, K) + (1 - \varphi) \text{ dist}(Lv, K) + |v - u|D\varphi \]
\[ \leq \text{dist}(Du, K) + \epsilon^{-n}\Theta + 8\epsilon + 8(|v - u| - \epsilon). \]
Let \( \epsilon = \Theta^{-1}. \) Then \( \epsilon^{-n}\Theta + 8\epsilon = \gamma \) and
\[ \text{dist}(D\tilde{u}, K) \leq \text{dist}(Du, K) + \gamma + 8(|v - u| - \epsilon) \text{ in } B_{7/8} \setminus B_{5/8}. \]
The estimate (2.4) gives
\[ \text{dist}(D\tilde{u}, K) < \gamma \text{ in } B_{5/8}. \]
Since $D\tilde{u} = Du$ in $B \setminus B_{7/8}$, the assertion follows from (2.5), (2.1), and the definition of $\epsilon$. \hfill \Box

**Lemma 6.** There exist positive constants $\alpha(n) < 1, c_2(n) < 1/8$, with the following property. If $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$, $\gamma \in (0, 9c_2|K|_{\infty})$ and

$$
\lambda := \frac{1}{|K|_{\infty}} \int_{\mathbb{R}^n} \text{dist}(Du; K) < \infty,
$$

then there exists a function $\tilde{u} \in W^{1,1}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$
\frac{1}{|K|_{\infty}} \int_{\mathbb{R}^n} \text{dist}(Du, K_\gamma) \leq \alpha(n) \lambda,
$$

(2.6)

$$
\mathcal{L}^n(\{u \neq \tilde{u}\}) \leq 2^n \left( \frac{9|K|_{\infty}}{\gamma} \right)^{n+1} \lambda.
$$

(2.7)

**Remark.** If $Du \in K$ on $R^n \setminus V$, then

$\{u \neq \tilde{u}\} \subset V_\rho = \{x : \text{dist}(x, V) \leq \rho\};$

$$
\rho = c_7(|K|_{\infty}^{n+1} \gamma^{-\gamma(n+1)} \lambda)^{1/n}.
$$

**Proof.**

1. We may suppose $|K|_{\infty} = 1$. Let

$$
\Theta := \left( \frac{\gamma}{9} \right)^{n+1} < c_2^{n+1} < 8^{-\gamma(n+1)},
$$

$$
E_\Theta := \{x \in \mathbb{R}^n : \sup_r \int_{B(x, r)} \text{dist}(Du, K) dy > \Theta\}.
$$

Note that by the Lebesgue point theorem

$$
\text{dist}(Du, K) \leq \Theta \quad \text{a.e. in } \mathbb{R}^n \setminus E.
$$

Since $c_2 \leq 1$ we have

$$
\Theta \leq \left( \frac{\gamma}{9} \right)^{n} \gamma \leq \gamma.
$$

(2.9)

2. **Claim:** For each $x \in E_\Theta$ there exists a radius $R(x) > 0$ such that

$$
\int_{B(x, R(x))} \text{dist}(Du, K) dy \leq \int_{B(x, R(x)/2)} \text{dist}(Du, K) dy = \Theta.
$$

(2.10)

To prove the claim, consider the function

$$
h(r) := \int_{B(x, r)} \text{dist}(Du, K) dy
$$

and let

$$
R(x) = 2 \sup\{r \in (0, \infty) : h(r) \geq \Theta\}.
$$

Then $R(x) < \infty$ since $\lambda < \infty$, and $h(R(x)/2) = \Theta$ by continuity of $h$. The claim is proved.

3. For $R(x)$ as above, consider the family of closed balls

$$
\mathcal{F} = \{\overline{B(x, R(x))} : x \in E_\Theta\}.
$$
By the Besicovitch covering theorem there exist at most \(k(n)\) (countable) subfamilies \(\mathcal{F}^{(j)}\) of disjoint balls such that the union of the sets
\[
A^{(j)} = \bigcup_{B \in \mathcal{F}^{(j)}} B
\]
covers \(E_\Theta\). Thus there exists a subfamily \(\mathcal{F}'\) of disjoint balls such that the set
\[
A = \bigcup_{B \in \mathcal{F}'} B
\]
satisfies
\[
\int_A \operatorname{dist}(Du, K) \, dy \geq \frac{1}{k(n)} \int_{E_\Theta} \operatorname{dist}(Du, K) \, dy.
\]

4. In view of (2.10) we may apply Lemma 5 successively to each of the disjoint balls \(B(x_i, R_i) \in \mathcal{F}'\) to obtain a function \(\tilde{u} \in W^{1,1}_\text{loc}(\mathbb{R}^n; \mathbb{R}^m)\) that satisfies
\[
\tilde{u} = u \quad \text{in} \quad \mathbb{R}^n \setminus A,
\]
(2.12)
\[
\int_{B(x_i, R_i)} \operatorname{dist}(\tilde{D}\tilde{u}, K_\gamma) \, dy \leq (1 + \Theta \frac{1}{n+1}) \int_{B(x_i, R_i) \setminus B(x_i, R_i/2)} \operatorname{dist}(Du, K) \, dy.
\]
(2.13)

The definition of \(R_i = R(x_i)\) (see (2.10)) implies that
\[
\int_{B(x_i, R_i/2)} \operatorname{dist}(Du, K) \, dy \geq 2^{-n} \int_{B(x_i, R_i)} \operatorname{dist}(Du, K) \, dy.
\]
Hence (2.13) yields
\[
\int_{B(x_i, R_i)} \operatorname{dist}(\tilde{D}\tilde{u}, K_\gamma) \, dy \leq (1 - 2^{-n})(1 + \Theta \frac{1}{n+1}) \int_{B(x_i, R_i)} \operatorname{dist}(Du, K) \, dy.
\]
(2.14)

Let \(c_2 = \min(\bar{c}_2, 1/9)\), where \(\bar{c}_2\) is defined by the equation
\[
(1 - 2^{-n})(1 + c_2) = (1 - 3^{-n}).
\]
Then the definition of \(\Theta\) implies that
\[
(1 - 2^{-n})(1 + \Theta \frac{1}{n+1}) \leq (1 - 3^{-n}).
\]
Since the balls in \(\mathcal{F}'\) are disjoint and their union is \(A\), we deduce that
\[
\int_A \operatorname{dist}(\tilde{D}\tilde{u}, K_\gamma) \, dy \leq (1 - 3^{-n}) \int_A \operatorname{dist}(Du, K) \, dy.
\]
(2.15)

On the other hand, (2.12) yields, in combination with (2.8) and (2.9),
\[
\operatorname{dist}(\tilde{D}\tilde{u}, K_\gamma) = \operatorname{dist}(Du, K_\gamma) \quad \text{in} \quad \mathbb{R}^n \setminus A,
\]
\[
\operatorname{dist}(Du, K_\gamma) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus E_\Theta.
\]
Thus
\[
\int_{\mathbb{R}^n \setminus A} \operatorname{dist}(\tilde{D}\tilde{u}, K_\gamma) \, dy \leq \int_{E_\Theta \setminus A} \operatorname{dist}(Du, K) \, dy.
\]
If we add this to (2.15), use (2.11) and define
\[(2.16) \quad \alpha(n) = 1 - \frac{3^n}{k(n)},\]
we finally obtain
\[\int_{\mathbb{R}^n} \text{dist}(D\tilde{u}, K_\gamma)dy \leq \alpha(n) \int_{E_\delta} \text{dist}(Du, K)dy.\]
This proves the first assertion of the lemma.

5. To finish the proof it only remains to estimate \(L^n(A)\). One has
\[L^n(A) = \sum_{B(x_i, R_i) \in \mathcal{F}} L^n(B(x_i, R_i/2))\]
\[= 2^n \sum_{B(x_i, R_i/2)} \int \text{dist}(Du, K)dy\]
\[\leq 2^n \lambda = 2^n \left( \frac{9}{\gamma} \right)^{n+1} \lambda.\]
Hence (2.7) holds, and the lemma is proved.

Proof of the Remark. The function \(u\) is only modified on the balls \(B(x_i, R_i)\), and one has (see point 5, above)
\[L^n(B(x_i, R_i)) \leq 2^n 9^{n+1} \gamma^{-(n+1)} \lambda,\]
\[2R_i \leq c_7(\gamma^{-(n+1)})^{1/n} = \rho.\]
Now we must have \(x_i \in V_{R_i} \subset V_{\rho/2}\) since otherwise \(\int_{B(x_i, R_i)} \text{dist}(Du, K)dx = 0\).
Thus \(B(x_i, R_i) \subset V_\rho\). 

Let \(c_2 = c_2(n)\) be as in Lemma 6.

**Theorem 7.** There exists a constant \(\bar{c}(n)\) with the following property. Suppose that \(K\) is a compact, convex set in \(\mathbb{R}^{mn}\), \(u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)\), \(\gamma \in (0, 9c_2|K|_\infty)\) and \(\lambda := \frac{1}{|K|_\infty} \int_{\mathbb{R}^n} \text{dist}(Du, K)dx < \infty\).

Then there exist \(v \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^m)\) such that
\[Dv \in K_\gamma \quad \text{a.e.,} \quad L^n\{u \neq v\} \leq \bar{c}(n)|K|_{\infty}^{n+1} \gamma^{-(n+1)} \lambda.\]

**Corollary 8.** If \(Du \in K\) on \(\mathbb{R}^n \setminus V\), then
\[\{u \neq v\} \subset V_\rho, \quad \rho = c_9(|K|_\infty^{n+1} \gamma^{-(n+1)} \lambda)^{1/n}.\]
In fact, values of \(u\) outside \(V_\rho\) play no rôle in the construction of \(v\).

**Proof of Theorem 7.** By scaling we may suppose \(|K|_\infty = 1\). The proof is based on a simple iteration of Lemma 6. Let \(\alpha = \alpha(n)\) denote the constant in (2.16) and inductively define
\[K_0 = K, \quad K_{i+1} = (K_i)_\gamma, \quad M_i = |K_i|_\infty,\]
\begin{equation}
\gamma_i = \delta \alpha \frac{\gamma_i}{M_i}.
\end{equation}

The value of \( \delta > 0 \) will be chosen below. We have
\[ \ln \frac{M_{i+1}}{M_i} = \ln \frac{M_i + \gamma_i}{M_i} \leq \delta \alpha \frac{\gamma_i}{M_i}, \quad M_0 = 1, \]
and hence
\[ 1 \leq M_i \leq e^{c_3 \delta} =: \bar{M}, \quad \sum_{i=0}^{\infty} \gamma_i \leq c_4 \delta e^{c_3 \delta} =: \bar{\gamma}. \]

Construct a sequence \( u_i \) by successive application of Lemma 6, starting with \( u_0 = u \). Let
\[ \lambda_i = \frac{1}{M_i} \int_{\mathbb{R}^n} \text{dist}(Du_i, K_i) dy, \quad \mu_i = \mathcal{L}^n\{u_{i+1} \neq u_i\}. \]

By Lemma 6,
\[ \lambda_{i+1} \leq \alpha \lambda_i, \quad \mu_i \leq 2^n 9^n + 1 \bar{M}^{n+1} \gamma_i^{-(n+1)} \lambda_i. \]

Thus
\[ \lambda_i \leq \alpha^i \lambda, \quad \mu_i \leq c_5 \bar{M}^{n+1} \delta^{-(n+1)} \alpha^{i/2} \lambda, \]
where \( c_3, c_4 \) and \( c_5 \) depend only on the space dimension \( n \). Since \( \sum \mu_i < \infty \), it follows from the definition of \( \mu_i \) and (1.2) that
\[ u_i \to v, \quad Du_i \to h \quad \text{in measure.} \]

Moreover,
\[ \int_{\mathbb{R}^n} \text{dist}(Du_i, K_{\bar{\gamma}}) \leq \bar{M} \lambda_i \to 0. \]  

Application of the dominated convergence theorem with majorant
\[ |K_{\bar{\gamma}}| + \sum_i \text{dist}(Du_i, K_{\bar{\gamma}}) \]
shows that \( Du_i \to h \) in \( L^1_{\text{loc}}(\mathbb{R}^n; M) \), and by testing with smooth, compactly supported test functions we deduce that
\[ u_i \to v \quad \text{in} \quad W^{1,1}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m). \]

Moreover, by (2.17),
\[ Du \in K_{\bar{\gamma}} \quad \text{a.e.} \]

and
\[ \mathcal{L}^n(\{u \neq v\}) \leq \sum_{i=0}^{\infty} \mu_i \leq c_6 \bar{M}^{n+1} \delta^{-(n+1)} \lambda. \]

Now choose \( \delta \) such that
\[ \gamma = \bar{\gamma} = c_4 \delta e^{c_3 \delta}. \]
Since \( \gamma \leq 9c_2 \), we have \( \delta \leq 9c_2 c_4^{-1} \) and \( \delta \geq \gamma c_4^{-1} \exp(9c_2 c_3 c_4^{-1}) \), and now the choice
\[ \tilde{c}(n) \leq c_6 c_4^{n+1} \exp(18(n+1) \frac{c_2 c_3}{c_4}) \]
gives the desired estimate for \( \mathcal{L}^n(\{u \neq v\}) \).
Proof of Corollary 8. Let \( u_i \) be as in the proof of Theorem 7, and let
\[
V_i = V \cup \{ u_i \neq u \}, \quad \rho_i = c_7 (\bar{M}^{n+1} \gamma_i^{-(n+1)} \lambda_i)\frac{1}{n}.
\]
We have \( Du_i \in K \) in \( \mathbb{R}^n \setminus V_i \), and the remark after Lemma 6 yields
\[
V_{i+1} \subset (V_i)_{\rho_i}.
\]
Since \( \lambda_i \leq \alpha \lambda \), the definition of \( \lambda_i \) implies that
\[
\sum \rho_i \leq c_8 \bar{M}^{n+1} \gamma^{-\frac{n+1}{n}} \lambda^\frac{1}{n}.
\]
The assertion now follows from (2.18).

3. LOCAL ESTIMATES

Proof of Theorem 4. We may suppose \( |K|_\infty = 1 \).

1. Claim: \( u_j \rightharpoonup u_0 \) in \( W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^m) \), \( Du_0 \in K \) a.e.

Proof. Let \( U \) be open, \( U \subset \subset \Omega \) (as usual, this notion indicates that \( \bar{U} \) is compact and contained in \( \Omega \)). For \( A \in \mathbb{R}^{mn} \) let \( PA \) denote the best approximation of \( A \) in the convex, compact set \( K \). The sequence \( PDu_j \) is bounded in \( L^\infty(U) \), and hence there exists a subsequence that has a weak \( * \) limit \( h \) in \( L^\infty(U) \). Since \( U \) is bounded, in particular
\[
PDu_{j_k} \rightharpoonup h \quad \text{in} \quad L^1(U).
\]
Now
\[
|Du_{j_k} - PDu_{j_k}| = \text{dist}(Du_{j_k}, K) \to 0 \quad \text{in} \quad L^1(U)
\]
and hence \( Du_{j_k} \to h \) in \( L^1(U) \). The usual argument yields \( h = Du_0 \), and uniqueness of the limit implies that the whole sequence converges. Convexity of the distance function and Mazur’s and Fatou’s lemmas (or standard lower semicontinuity results) show that \( \text{dist}(Du_0, K) = 0 \) a.e. in \( U \), and hence a.e. in \( \Omega \) by arbitrariness of \( U \).

2. Let \( V \subset \subset U \subset \subset \Omega \). We construct \( v_j \) that almost satisfy (1.7) and (1.8). The proof will then be finished by a diagonalization argument. Let \( \varphi \in C_0^\infty(V) \), \( 0 \leq \varphi \leq 1 \), and define
\[
w_j = \varphi u_j + (1 - \varphi) u_0.
\]
Then
\[
Dw_j = \varphi Du_j + (1 - \varphi) Du_0 + (u_j - u_0) \otimes D\varphi.
\]
In particular,
\[
Dw_j \in K \quad \text{in} \quad \Omega \setminus V,
\]
\[
\lambda_j := \int_\Omega \text{dist}(Dw_j, K) dx \leq \int_V \text{dist}(Du_j, K) dx + \int_V |u_j - u_0| |D\varphi| dx.
\]
By the assumptions, \( \lambda_j \to 0 \). Let \( \delta > 0 \). In view of Theorem 6 and Corollary 8 there exists \( j_0 = j_0(U, V, \varphi, \delta) \) such that for all \( j \geq j_0 \) there exist \( v_j \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m) \) that satisfy
\[
\{ v_j \neq w_j \} \subset U, \quad \mathcal{L}^n(v_j \neq w_j) < \delta, \quad Du_j \in K_\delta \text{ a.e.}
\]
It follows that
\[
v_j = u_0 \quad \text{in} \quad \Omega \setminus U,
\]
\[
\mathcal{L}^n(\{ v_j \neq u_j \} \cap U) < \delta + \mathcal{L}^n(\{ \varphi \neq 1 \} \cap V) + \mathcal{L}^n(U \setminus V),
\]

\[\text{dist}(Dv_j, K) \leq \delta.\]

3. Let \( \{ \tilde{U}_k \} \) be an increasing sequence of open sets \( \tilde{U}_k \subset \Omega \) whose union exhausts \( \Omega \). Let \( V_k \subset \tilde{U}_k \) and \( \varphi_k \in C_0^\infty(V_k) \) be such that

\[
\mathcal{L}^n(\tilde{U}_k \setminus V_k) < \frac{1}{k}, \quad \mathcal{L}^n(\{ \varphi_k \neq 1 \} \cap V) < \frac{1}{k},
\]

\[0 \leq \varphi_k \leq 1, \quad \text{and let } \delta_k < \frac{1}{k}.\]

By point 2, there exists \( j_k \) such that for \( j \geq j_k \) there exist functions \( v_j \) that satisfy

\[v_j = u_0 \quad \text{in } \Omega \setminus \tilde{U}_k,\]

\[
\mathcal{L}^n(\{ v_j \neq u_j \}) \cap \tilde{U}_k < \frac{3}{k}, \quad \text{dist}(Dv_j, K) < \frac{1}{k}.
\]

We may suppose without loss of generality that \( j_k \) is (strictly) increasing. To finish the proof, define

\[U_j = \tilde{U}_k \quad \text{if } j_k \leq j < j_k + 1.\]

4. APPLICATION TO QUASICONVEX FUNCTIONS

A function \( f \) from the \( m \times n \) matrices \( \mathbb{R}^{mn} \) to \( \mathbb{R} \cup \{-\infty, \infty\} \) is called quasiconvex if for all bounded domains \( U \subset \mathbb{R}^n \) with \( \mathcal{L}^n(\partial U) = 0 \) and all \( F \in \mathbb{R}^{mn} \)

\[
\int_U f(F + D\eta)dx \geq \int_U f(F)dx = \mathcal{L}^n(U)f(F) \quad \forall \eta \in W_0^{1,\infty}(U; \mathbb{R}^m),
\]

whenever the integral on the left exists.

Quasiconvexity is the fundamental notion in the vector-valued calculus of variations (see [14], [15], [3], [4], [6], [20]). It states that affine functions minimize the functional \( u \mapsto \int_U f(Du) \) subject to their own boundary conditions. Quasiconvexity is difficult to handle, however, since no local characterization is known for \( n, m > 1 \) (and cannot exist for \( m \geq 3, n \geq 2 \); see [12]). Even the approximation of general quasiconvex functions by finite ones is a largely open question. As a corollary of Theorem 2 we obtain at least the following result, which answers the question in [9], p. 350, equation (5.19) (see pp. 342 and 345 for the relevant definitions). We remark that every \( \mathbb{R} \)-valued quasiconvex function is continuous and even locally Lipschitz, since it is rank-1 convex (see e.g. [4]).

**Corollary 9.** Let \( K \subset \mathbb{R}^{mn} \) be a convex, compact set with non-empty interior. Let \( f : \mathbb{R}^{mn} \to \mathbb{R} \cup \{-\infty, \infty\} \) be a quasiconvex function that satisfies

\[f \in C(K; \mathbb{R}), \quad f = +\infty \quad \text{on } \mathbb{R}^{mn} \setminus K.\]

Then, for all \( F \in K \),

\[f(F) = \sup\{g(F)|g : \mathbb{R}^{mn} \to \mathbb{R}, \ g \leq f \text{ on } K, \ g \text{ quasiconvex}\}.\]

**Proof.** 1. We may assume \( 0 \in \text{int } K \), since quasiconvexity is invariant under translation in \( \mathbb{R}^{mn} \). We have

\[K \subset \lambda \text{int } K, \quad \forall \lambda > 1.\]
Indeed, if $A \in \partial K$, then $tA + (1 - t)B \in K$ for all $t \in (0, 1)$ and all $B$ in a small neighbourhood of $0$. Hence $tA \in \text{int } K$, for all $t \in (0, 1)$. Thus (4.2) holds.

2. Let $G_{\infty}$ denote the right hand side of (4.1) and let $P$ denote the nearest neighbour projection onto $K$. For $k \in \mathbb{N} \cup \{0\}$ define

$$h_k(F) = f(PF) + k \text{ dist}(F, K) \leq f(F).$$

Let $g_k = h_k^{qc}$ denote the quasiconvex hull of $h_k$, i.e. the largest quasiconvex function below $h_k$. Thus $g_k(F) \leq G_{\infty}$. On the other hand, by standard relaxation results (see e.g. [4], Chapter 5, Theorem 1.1)

$$g_k(F) = \inf \left\{ \int_Q h_k(Du) dx : u - Fx \in W^{1, \infty}_0(Q, \mathbb{R}^m) \right\},$$

where $Q = (0, 1)^n$. Hence there exist Lipschitz functions $u_k$ such that

$$\text{(4.3)} \quad \limsup_{k \to \infty} \int_Q h_k(Du_k) dx \leq G_{\infty}, \quad u_k = Fx \text{ on } \partial Q.$$

In particular,

$$\int_Q \text{dist}(Du_k, K) \to 0.$$ 

Hence $Du_k$ is bounded in $L^1$, and after possible passage to a subsequence we may assume that $u_k \to u_0$ in $L^1$.

3. By Theorem 4 there exist $v_k \in W^{1, \infty}(Q, \mathbb{R}^m)$ which satisfy

$$\text{(4.4)} \quad \mathcal{L}^n(\{u_k \neq v_k\}) \to 0, \quad v_k = Fx \text{ on } \partial Q,$$

$$\text{(4.5)} \quad \|\text{dist}(Dv_k, K)\|_{\infty} \to 0.$$

Taking into account (1.2), the uniform continuity of $h_0$ and the inequality $h_0 \leq h_k$, we see that

$$\limsup_{k \to \infty} \int_Q h_0(Dv_k) dx = \limsup_{k \to \infty} \int_Q h_0(Du_k) dx \leq G_{\infty}.$$

In view of (4.2) and (4.5) there exist $\lambda_k \searrow 1$ such that $\lambda_k^{-1}Dv_k \in K$, $\lambda_k^{-1}F \in K$. Using the uniform continuity of $h_0$ as well as quasiconvexity and continuity of $f$, we obtain

$$f(F) = \lim_{k \to \infty} f(\lambda_k^{-1}F) \leq \limsup_{k \to \infty} \int_Q f(\lambda_k^{-1}Dv_k) dx$$

$$= \limsup_{k \to \infty} \int_Q h_0(\lambda_k^{-1}Dv_k) dx \leq G_{\infty}.$$

The proof is finished. \[ \Box \]

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