

EVEN VALUATIONS ON CONVEX BODIES

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ABSTRACT. The notion of even valuation is introduced as a natural generalization of volume on compact convex subsets of Euclidean space. A recent characterization theorem for volume leads in turn to a connection between even valuations on compact convex sets and continuous functions on Grassmannians. This connection can be described in part using generating distributions for symmetric compact convex sets. We also explore some consequences of these characterization results in convex and integral geometry.

Recent interest in volume as a valuation on compact convex sets stems from Hilbert's Third Problem, which is actually an ancient problem recast in modern terms. Hilbert asked if two polytopes P and Q can be each cut into a finite number of pieces P_1, \dots, P_m and Q_1, \dots, Q_m with each P_i congruent to Q_i by a *rigid Euclidean motion*, provided that P and Q have the same volume [18]. This question was answered in the negative by Max Dehn [2, 4, 29], who found a functional on polytopes that is invariant under dissections over rigid motions, while varying in value among polytopes of equal volume. In other words, the Dehn invariant is a "simple rigid motion invariant valuation" on polytopes that is not equal to volume (under any normalization).

Dehn's solution left open the question of exactly what conditions on P and Q imply equidissectability over the group of rigid motions, although this problem was solved by Hadwiger in the case where only *translations* (and no rotations nor reflections) are permitted (see [2, 16, 17, 26, 29]).

In the course of studying this and related problems, Hadwiger discovered a characterization of Euclidean volume as a continuous rigid motion invariant *simple* valuation on compact convex sets, that is, a continuous rigid motion invariant valuation that vanishes on convex sets of less than full dimension. This result led in turn to a complete characterization of all continuous rigid motion invariant valuations on compact convex sets in \mathbb{R}^n as consisting of a real $(n + 1)$ -dimensional vector space spanned by the *intrinsic volumes* (or *Quermassintegrals*) [16] (also [20, 21, 31]). Since many standard functionals and integral operators can be interpreted as invariant valuations (such as intrinsic volumes, mean projections, Crofton and kinematic formulas), what came to be known as *Hadwiger's characterization theorem* proved to be a valuable tool for generating quick and effortless proofs of many formulas and equations in integral geometry.

Unfortunately Hadwiger's original proof was long and difficult [16]. While seeking a shorter proof of Hadwiger's volume characterization, the author discovered

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that in fact fewer conditions were sufficient to characterize volume than Hadwiger originally required. In particular, complete rigid motion invariance is not necessary; it is sufficient for a continuous simple valuation to be invariant under translations and reflection through the origin [20, 21]. The sufficiency of these invariance conditions to characterize volume motivates in turn the study of *even* valuations.

In the present work we pursue this line, begun in [20], by establishing a connection between continuous even valuations on compact convex sets and continuous functions on Grassmannians. We also examine some consequences to convex and integral geometry.

LIST OF SYMBOLS

| | |
|---------------------------------|---|
| \mathbb{R}^n | n -dimensional Euclidean space |
| \mathbb{S}^{n-1} | unit sphere in \mathbb{R}^n , centered at o |
| \mathcal{K}^n | set of compact convex subsets of \mathbb{R}^n |
| \mathcal{K}_c^n | set of symmetric sets in \mathcal{K}^n |
| h_K | support function of K |
| $K + L$ | Minkowski sum of K and L |
| $V_n(K)$ | n -volume of $K \in \mathcal{K}^n$ |
| V_m | m -th intrinsic volume |
| $\text{Polycon}(n)$ | set of finite unions of compact convex sets |
| $dS(K, \cdot)$ | Aleksandrov-Fenchel-Jessen measure of K |
| $V_n(K_1, \dots, K_n)$ | mixed n -volume of $K_1, \dots, K_n \in \mathcal{K}^n$. |
| \mathcal{A}^n | space of continuous, translation invariant, even valuations on \mathcal{K}^n |
| \mathcal{A}_m^n | space of valuations in \mathcal{A}^n that are homogeneous of degree m |
| $G(n, m)$ | set of m -dimensional subspaces of \mathbb{R}^n |
| $\text{Par}(n)$ | set of finite unions of parallelotopes (w.r.t. a given basis) |
| \bar{u} | line segment with endpoints u and $-u$ |
| \mathcal{U}^n | space of continuous translation invariant valuations on $\text{Par}(n)$ |
| \mathcal{U}_m^n | space of valuations in \mathcal{U}^n that are homogeneous of degree m |
| $\wedge^m(\mathbb{R}^n)$ | space of m -th exterior tensors over \mathbb{R}^n |
| Φ | injective mapping from valuations to generating functions (or vectors) |
| X^μ | generating vector for a valuation μ |
| $\wedge(\mathbb{R}^n)$ | exterior (Grassmann) algebra over \mathbb{R}^n |
| $C(G(n, m))$ | space of continuous real-valued functions on $G(n, m)$ |
| \mathcal{K}_E | set of compact convex subsets of E |
| f_μ | generating function of a valuation μ |
| $C(\mathbb{S}^{n-1})$ | space of continuous real-valued functions on \mathbb{S}^{n-1} |
| $\mathcal{D}(\mathbb{S}^{n-1})$ | space of smooth functions on \mathbb{S}^{n-1} |
| $\mathcal{E}(\mathbb{S}^{n-1})$ | space of functions on \mathbb{S}^{n-1} having generating measures |
| ρ_f | generating measure of a function f |
| ρ_K | generating measure of a convex set K |
| T_K | generating distribution of a convex set K (on \mathbb{S}^{n-1}) |
| T_K^m | analogous functional to T_K for functions on $G(n, m)$. |
| \mathcal{F}_m^n | domain of T_K^m |
| $\langle E, F \rangle$ | cosine of the angle between subspaces E and F |
| $K E$ | orthogonal projection of K onto E |
| \tilde{T}_K^m | an alternative version of T_K^m |
| \mathcal{E}_m^n | space of functions on $G(n, m)$ having generating measures; domain of \tilde{T}_K^m |
| μ_f | valuation on \mathcal{K}^n induced by f under \tilde{T}_K^m |
| $\text{Mod}(n)$ | set of all subspaces of \mathbb{R}^n |
| $C(\text{Mod}(n))$ | space of continuous real-valued functions on $\text{Mod}(n)$ |

1. EVEN VALUATIONS

Denote by \mathcal{K}^n the collection of all compact convex subsets of \mathbb{R}^n , that is, n -dimensional Euclidean space. The elements of \mathcal{K}^n are also known as *convex bodies*. A convex body K is *centered about the origin* if K is symmetric under reflection through the origin, that is, if $K = -K$. A convex body K is *symmetric* or *centered* if there exists a translate of K that is centered about the origin. Denote by \mathcal{K}_c^n the collection of all symmetric convex bodies in \mathcal{K}^n .

A convex body $K \in \mathcal{K}^n$ is determined uniquely by its *support function*, $h_K : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$, defined by $h_K(u) = \max_{x \in K} \{x \cdot u\}$, where \cdot denotes the standard inner product on \mathbb{R}^n .

For all $K, L \in \mathcal{K}^n$ and all $\lambda \geq 0$, the *Minkowski sum* $K + \lambda L$ is defined by

$$K + \lambda L = \{x + \lambda y : x \in K \text{ and } y \in L\},$$

and has support function $h_{K+\lambda L} = h_K + \lambda h_L$.

A sequence of convex bodies K_i is said to *converge* to K in the *Hausdorff topology* if, for all $\epsilon > 0$, there exists $N > 0$ such that

$$K_i \subseteq K + \epsilon B \quad \text{and} \quad K \subseteq K_i + \epsilon B$$

for all $i > N$. In this case we write $K_i \rightarrow K$.

A function $\mu : \mathcal{K}^n \rightarrow \mathbb{R}$ is called a *valuation* on \mathcal{K}^n if $\mu(\emptyset) = 0$, where \emptyset is the empty set, and if

$$(1) \quad \mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L),$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$ as well.

The condition that $K \cup L$ be convex may seem excessively restrictive. However, any *continuous* valuation μ on \mathcal{K}^n can be extended in a unique way to the lattice $\text{Polycon}(n)$ of *polyconvex* subsets of \mathbb{R}^n , that is, the set of all *finite unions* of compact convex subsets of \mathbb{R}^n . (The lattice of polyconvex sets is also known as the *convex ring*.) The extension is constructed as follows: Given a valuation μ on \mathcal{K}^n , and a set $M \in \text{Polycon}(n)$, express M as a finite union of convex bodies,

$$(2) \quad M = K_1 \cup \cdots \cup K_m,$$

and then compute $\mu(K_1 \cup \cdots \cup K_m)$ by iterating (1). If μ is continuous, this extension of μ turns out to be well-defined [14]. In the arguments that follow, this unique extension of μ shall allow us to consider the value of μ on all finite unions of convex bodies, whether or not such unions are actually convex.

Note that a valuation μ on \mathcal{K}^n is said to be *continuous* if, for any convergent sequence $K_i \rightarrow K$ in \mathcal{K}^n ,

$$\lim_{i \rightarrow \infty} \mu(K_i) = \mu(K).$$

Although a given valuation may have a well-defined extension to all of $\text{Polycon}(n)$, we do not consider convergent sequences of non-convex sets when testing a valuation μ for continuity.

A valuation μ on \mathcal{K}^n is *translation invariant* if $\mu(K + x) = \mu(K)$ for all $x \in \mathbb{R}^n$. A valuation μ on \mathcal{K}^n is called *even* if $\mu(-K) = \mu(K)$ for all K . If a valuation μ vanishes on all convex bodies of dimension strictly less than n , then μ is called a *simple* valuation. Finally, a valuation μ on \mathcal{K}^n is said to be *homogeneous of degree* m , if

$$\mu(\alpha K) = \alpha^m \mu(K),$$

for all $\alpha \geq 0$ and all $K \in \mathcal{K}^n$.

Denote by \mathcal{A}^n the vector space of continuous, translation invariant, even valuations on \mathcal{K}^n , and let \mathcal{A}_m^n denote the subspace of valuations in \mathcal{A}^n that are homogeneous of degree m .

A well-known example of a continuous translation invariant even valuation on \mathcal{K}^n is the volume V_n . Another example is *surface area*.

It turns out that the intrinsic volumes V_0, V_1, \dots, V_n are continuous rigid motion invariant valuations on \mathcal{K}^n (see [23, 27], [31, p. 290]). The intrinsic volumes V_i give invariant extensions of i -dimensional volume (on i -planes) to polyconvex subsets of \mathbb{R}^n , where $n \geq i$. Denote by $G(n, i)$ the set of all i -dimensional subspaces of \mathbb{R}^n , equipped with the invariant (Haar) measure ξ_i normalized so that

$$\xi_i(G(n, i)) = \binom{n}{i} \frac{\omega_n}{\omega_i \omega_{n-i}},$$

where ω_i is the i -dimensional volume of the unit ball in \mathbb{R}^i . Denote by V_i the i -dimensional volume in \mathbb{R}^i . The i -volume V_i is extended to the i -th intrinsic volume (also denoted V_i) on all of \mathcal{K}^n by the mean projection formula:

$$V_i(K) = \int_{G(n, i)} V_i(K|E) d\xi_i(E),$$

where $K|E$ denotes the orthogonal projection of K onto the subspace E . Evidently the intrinsic volume V_i is homogeneous of degree i .

The valuation V_0 , which takes the value 1 on all non-empty compact convex sets, extends to the Euler characteristic on the lattice of polyconvex sets (see, for example, [21, 27, 31]).

Up to a constant factor, the volume V_n is the only continuous translation invariant even *simple* valuation on \mathcal{K}^n . Specifically, we have the following result [20]:

Theorem 1.1. *Suppose that μ is a simple valuation on \mathcal{K}^n that is continuous and translation invariant. Then there exists $c \in \mathbb{R}$ such that $\mu(K) + \mu(-K) = cV_n(K)$, for all $K \in \mathcal{K}^n$.*

The relevance of even valuations is demonstrated by the following equivalent statement of Theorem 1.1.

Theorem 1.2 (Volume Characterization Theorem). *Suppose that μ is a simple valuation on \mathcal{K}^n that is continuous, translation invariant and even. Then there exists $c \in \mathbb{R}$ such that $\mu(K) = cV_n(K)$, for all $K \in \mathcal{K}^n$.*

Theorem 1.2 fails for valuations that are not even. For example, let Δ denote the equilateral triangle in \mathbb{R}^2 with vertices at the points $(0, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The valuation τ on \mathcal{K}^2 defined by

$$(3) \quad \tau(K) = V_2(K + \Delta) - V_2(K + (-\Delta))$$

is continuous, translation invariant, and vanishes on convex bodies of dimension one. (Here V_2 denotes *area* in \mathbb{R}^2 .) Moreover τ vanishes on all centered bodies. But $\tau(K) \neq 0$ if K is not centered. This follows from the Brunn-Minkowski inequality [31, p. 309]. Alternatively, an easy direct calculation shows that $\tau(\Delta) \neq 0$. Consequently $\tau \neq cV_2$ for any $c \in \mathbb{R}$.

In a similar vein, a valuation μ is defined to be *odd* if $\mu(-K) = -\mu(K)$ for all $K \in \mathcal{K}^n$. In [32] Schneider proved the following theorem:

Theorem 1.3. *Suppose that μ is a simple valuation on \mathcal{K}^n that is continuous, translation invariant and odd. Then there exists a continuous odd function $g : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ such that*

$$\mu(K) = \int_{\mathbb{S}^{n-1}} g(u) dS(K, u)$$

for all $K \in \mathcal{K}^n$.

Here \mathbb{S}^{n-1} denotes the unit sphere in \mathbb{R}^n , and $S(K, \cdot)$ denotes the Aleksandrov-Fenchel-Jessen measure associated with K .

Since every valuation on \mathcal{K}^n can be expressed as the sum of an even valuation and an odd valuation, Theorems 1.2 and 1.3 together characterize all continuous translation invariant simple valuations on \mathcal{K}^n .

The following theorem of McMullen [24, 26, 27] will be of use to us in the sequel.

Theorem 1.4. *Let μ be a continuous translation invariant valuation on \mathcal{K}^n , homogeneous of degree m . If $K_1, \dots, K_p \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_p > 0$, then*

$$(4) \quad \mu(\lambda_1 K_1 + \dots + \lambda_p K_p) = \sum_{i_1, \dots, i_m=1}^p \mu(K_{i_1}, \dots, K_{i_m}) \lambda_{i_1} \dots \lambda_{i_m},$$

where each symmetric coefficient $\mu(K_{i_1}, \dots, K_{i_m})$ depends only on the bodies K_{i_1}, \dots, K_{i_m} . Moreover, each coefficient $\mu(K_{i_1}, \dots, K_{i_m})$ is a continuous translation invariant function of the bodies K_{i_1}, \dots, K_{i_m} .

An important special case of Theorem 1.4 is Minkowski's expansion of the volume of a Minkowski linear combination: If $K_1, \dots, K_p \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_p > 0$, then the volume V_n is a homogeneous polynomial in the positive variables $\lambda_1, \dots, \lambda_p$; that is,

$$(5) \quad V_n(\lambda_1 K_1 + \dots + \lambda_p K_p) = \sum_{i_1, \dots, i_n=1}^p V_n(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \dots \lambda_{i_n}.$$

The following theorem of McMullen implies that in order to understand and characterize translation invariant valuations, it is sufficient to consider the translation invariant *homogeneous* valuations [24].

Theorem 1.5. *Let μ be a continuous translation invariant valuation on \mathcal{K}^n . Then there exist unique continuous translation invariant valuations $\mu_0, \mu_1, \dots, \mu_n$, such that each μ_i is homogeneous of degree i , and such that*

$$\mu(K) = \sum_{i=0}^n \mu_i(K)$$

for all $K \in \mathcal{K}^n$.

Note that if μ is even, then

$$\sum_{i=0}^n \mu_i(-K) \epsilon^i = \mu(-\epsilon K) = \mu(\epsilon K) = \sum_{i=0}^n \mu_i(K) \epsilon^i,$$

for all $\epsilon > 0$. It follows that if μ is even, so are the homogeneous component valuations μ_i .

It also follows easily from Theorem 1.5 that if μ is translation invariant and homogeneous of degree m , then m must be an integer. In particular, if $\mu(K) \neq 0$ for some set $K \in \mathcal{K}^n$, then necessarily $m \in \{0, 1, \dots, n\}$. Said differently, if μ is

translation invariant and homogeneous of degree $m > n$, then $\mu(K) = 0$ for all $K \in \mathcal{K}^n$.

While Theorems 1.4 and 1.5 are well-known for more general cases (i.e. translation invariant homogeneous valuations on compact polytopes) [26, 27], the following is true in general only for even valuations:

Proposition 1.6. *Suppose that $\mu \in \mathcal{A}_n^n$. Then there exists $c \in \mathbb{R}$ such that $\mu(K) = cV_n(K)$ for all $K \in \mathcal{K}^n$.*

Proof. Since μ is homogeneous of degree n , the valuation μ restricts to the zero valuation on $(n-1)$ -dimensional subspaces by Theorem 1.5. In other words, μ is simple. Proposition 1.6 then follows immediately from Theorem 1.2. \square

For $K, L \in \mathcal{K}^n$ it is well-known that the mixed volume $V_n(K, \dots, K, L)$ satisfies the equation

$$(6) \quad nV_n(\underbrace{K, \dots, K}_{n-1}, L) = \int_{\mathbb{S}^{n-1}} h_L(u) dS(K, u).$$

See, for example, [31, p. 275].

Theorem 1.7. *Suppose that $\mu \in \mathcal{A}^n$, and suppose that μ vanishes in dimensions less than $n-1$. Then there exists $c \in \mathbb{R}^n$ and sequences $\{L_j\}, \{M_j\}$ of convex bodies centered at the origin such that*

$$\mu(K) = cV_n(K) + \lim_{j \rightarrow \infty} [V_n(\underbrace{K, \dots, K}_{n-1}, L_j) - V_n(\underbrace{K, \dots, K}_{n-1}, M_j)]$$

for all $K \in \mathcal{K}^n$.

Proof. For each hyperplane $E \in G(n, n-1)$, let \mathcal{K}_E denote the set of all compact convex subsets of E . Since the restriction of μ to \mathcal{K}_E satisfies the conditions of Theorem 1.2 (in dimension $n-1$), there exists $c_E \in \mathbb{R}$ such that

$$\mu(K) = c_E V_{n-1}(K),$$

for all $K \in \mathcal{K}_E$.

Define $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ by the equation

$$f(u) = \frac{nc_{u^\perp}}{2}.$$

Because the function f is continuous and even, there exists a sequence of pairs of convex bodies $\{L_j, M_j\}$ centered about the origin such that the sequence of differences of support functions $\{h_{L_j} - h_{M_j}\}$ converges uniformly to f (see [30] or [31, p. 45]).

For $j > 0$ let $\mu_j(K) = V_n(K, \dots, K, L_j) - V_n(K, \dots, K, M_j)$. Then for $K \in \mathcal{K}_{u^\perp}$, the surface measure $dS(K, \cdot)$ is concentrated at the points $u, -u \in \mathbb{S}^n$ with mass $V_{n-1}(K)$ at each point. Since L_j, M_j are centered at the origin, we have

$h_{L_j}(u) = h_{L_j}(-u)$, and similarly for h_{M_j} . We therefore apply (6) to compute:

$$\begin{aligned} \mu_j(K) &= V_n(\underbrace{K, \dots, K}_{n-1}, L_j) - V_n(\underbrace{K, \dots, K}_{n-1}, M_j) \\ &= \frac{1}{n} V_{n-1}(K)(2h_{L_j}(u) - 2h_{M_j}(u)) \\ &\rightarrow V_{n-1}(K) \left(\frac{2f(u)}{n} \right) \text{ as } j \rightarrow \infty \\ &= c_{u^\perp} V_{n-1}(K) \\ &= \mu(K). \end{aligned}$$

For $K \in \mathcal{K}^n$ define $\eta(K) = \lim_{j \rightarrow \infty} \mu_j(K)$. The valuation η satisfies the equations

$$\eta(K) = \lim_{j \rightarrow \infty} [V_n(\underbrace{K, \dots, K}_{n-1}, L_j) - V_n(\underbrace{K, \dots, K}_{n-1}, M_j)] = \int_{\mathbb{S}^{n-1}} f(u) dS(K, u).$$

Therefore, η is continuous, translation invariant, even, and

$$\eta(K) = \mu(K)$$

for all K of dimension less than n . It follows that the valuation $\mu - \eta$ satisfies the conditions of Theorem 1.2, so that there exists $c \in \mathbb{R}$ such that $\mu(K) - \eta(K) = cV_n(K)$ for all $K \in \mathcal{K}^n$. Thus

$$\mu(K) = cV_n(K) + \lim_{j \rightarrow \infty} [V_n(\underbrace{K, \dots, K}_{n-1}, L_j) - V_n(\underbrace{K, \dots, K}_{n-1}, M_j)].$$

□

The following corollary is actually a special case of a theorem of McMullen [25].

Corollary 1.8. *Suppose that $\mu \in \mathcal{A}_{n-1}^n$. Then there exist sequences $\{L_j\}, \{M_j\}$ of centered convex bodies such that*

$$\mu(K) = \lim_{j \rightarrow \infty} [V_n(\underbrace{K, \dots, K}_{n-1}, L_j) - V_n(\underbrace{K, \dots, K}_{n-1}, M_j)]$$

for all $K \in \mathcal{K}^n$.

Proof. Theorem 1.7 implies that there exists $c \in \mathbb{R}^n$ and sequences $\{L_j\}, \{M_j\}$ of centered convex bodies such that

$$(7) \quad \mu(K) - \lim_{j \rightarrow \infty} [V_n(\underbrace{K, \dots, K}_{n-1}, L_j) - V_n(\underbrace{K, \dots, K}_{n-1}, M_j)] = cV_n(K)$$

for all $K \in \mathcal{K}^n$. Since $\mu \in \mathcal{A}_{n-1}^n$, the left hand expression in (7) gives a valuation homogeneous of degree $n - 1$. It follows that $c = 0$. □

Corollary 1.8 can be used to show that every centered generalized convex body has a continuous even support function (see [22]).

In [9], Goodey and Weil give a similar classification for continuous valuations that are homogeneous of degree 1.

Theorem 1.9. *Suppose that μ is a continuous translation invariant valuation on \mathcal{K}^n . Then μ is homogeneous of degree 1 if and only if there exist sequences $\{L_j\}_{j=0}^\infty$ and $\{M_j\}_{j=0}^\infty$ in \mathcal{K}^n such that, for all $\delta > 0$,*

$$(8) \quad \mu(K) = \lim_{j \rightarrow \infty} [V_n(\underbrace{L_j, \dots, L_j}_{n-1}, K) - V_n(\underbrace{M_j, \dots, M_j}_{n-1}, K)]$$

uniformly for all convex bodies $K \subseteq \delta B$.

If we add the condition that μ is an *even* valuation, it is not difficult to show that (8) holds with sequences $\{L_j\}$ and $\{M_j\}$ of *symmetric* bodies. For example, if L_j and M_j are not symmetric, then replace them with the Blaschke sums $(1/2)(L_j \# (-L_j))$ and $(1/2)(M_j \# (-M_j))$. If μ is even, this substitution will not change the limit in (8). (For a discussion of Blaschke sums, see [31, p. 394] or [5, p. 122].)

Theorems 1.5, 1.7, and 1.9 together give a characterization of all continuous translation invariant even valuations on \mathcal{K}^n , for $n = 1, 2, 3$. However, the situation becomes more complicated for dimension $n \geq 4$, in part because of complications in the structure of the Grassmannian $G(n, k)$ for $n \geq 4$ and $2 \leq k \leq n - 2$ (as compared to $G(n, 1) \cong G(n, n - 1)$ and \mathbb{S}^{n-1}). In order to shed some light on these intermediate cases, we shall examine a connection between even valuations and functions on Grassmannians. In preparation for this, it will be useful first to understand the behavior and characterization of even valuations on a special class of compact convex sets, namely, the lattice of finite unions of parallelotopes in a fixed frame.

2. VALUATIONS ON PARALLELOTOPES

Choose an ordered basis of unit vectors u_1, \dots, u_n for \mathbb{R}^n , which shall remain fixed throughout this section, and let $\text{Par}(n)$ denote the family of sets that are obtained by taking finite unions and intersections of parallelotopes with sides parallel to the vectors u_1, \dots, u_n . Note that $\text{Par}(n)$ is closed under finite unions and intersections. This follows from the fact that the intersection of two parallelotopes in $\text{Par}(n)$ is a parallelotope. In other words, $\text{Par}(n)$ is a distributive lattice under the operations \cup and \cap .

A valuation μ is said to be *continuous* on $\text{Par}(n)$, provided that

$$\mu(P_n) \longrightarrow \mu(P)$$

whenever P_n, P are *parallelotopes* (and *not* just finite unions) and $P_n \longrightarrow P$ in the Hausdorff topology inherited from \mathcal{K}^n .

The object of this section is to classify all continuous translation invariant homogeneous valuations defined on $\text{Par}(n)$ (see also [21]).

When studying valuations on $\text{Par}(n)$ we may restrict our attention to the generating set of parallelotopes in \mathbb{R}^n with edges parallel to the coordinate axes. Specifically, Groemer [14] (see also [21]) has shown that a valuation μ defined on parallelotopes with edges parallel to the coordinate axes admits a unique extension to a valuation on the lattice $\text{Par}(n)$.

The restriction of the volume V_n to the lattice $\text{Par}(n)$ is characterized by the following theorem.

Theorem 2.1 (Volume Characterization for $\text{Par}(n)$). *Let μ be a continuous translation invariant simple valuation defined on $\text{Par}(n)$. Then there exists $c \in \mathbb{R}$ such*

that $\mu(P) = cV_n(P)$ for all $P \in \text{Par}(n)$; that is, μ is equal to the volume, up to a constant factor.

Proof. For $w \in \mathbb{R}^n$, denote by \overline{w} the closed line segment with endpoints at w and $-w$. Let P_0 denote the unit parallelotope in $\text{Par}(n)$, that is, the Minkowski sum $\overline{u}_1 + \cdots + \overline{u}_n$. Define a continuous invariant valuation ν on $\text{Par}(n)$ by the equation

$$\nu(P) = \mu(P) - \frac{\mu(P_0)}{V_n(P_0)}V_n(P).$$

Note that $\nu(P)$ vanishes in dimensions less than n and that $\nu(P_0) = 0$. We need to show that $\nu(P) = 0$ for all $P \in \text{Par}(n)$.

If $n = 1$, then the result follows readily, since a parallelotope in \mathbb{R} is merely a closed line segment. Since ν vanishes on the closed line segment $P_0 = \overline{u}_1$ and on sets consisting of a single point, it follows from the inclusion-exclusion principle (1) that ν must vanish on all closed line segments of rational length. It then follows from continuity that ν vanishes on all closed line segments.

Suppose $n > 1$. Recall that ν is translation invariant and vanishes on lower dimensions. Since $\nu(P_0) = 0$, a simple cut-and-paste argument shows that $\nu(\frac{1}{k}P_0) = 0$ for all integers $k > 0$. Therefore, $\nu(P) = 0$ for every parallelotope P in $\text{Par}(n)$ having rational edge lengths. This follows from the fact that such a box can be built up by stacking parallelotopes of the form $\frac{1}{k}P_0$ for some $k > 0$. The continuity of ν now implies that $\nu(P) = 0$ for every parallelotope P in $\text{Par}(n)$. It then follows from the inclusion-exclusion principle that $\nu(P) = 0$ for all $P \in \text{Par}(n)$. \square

For an alternative proof of Theorem 2.1, see also [15, p. 47].

A valuation μ on $\text{Par}(n)$ is said to be *homogeneous* of degree $k > 0$ if

$$\mu(\alpha P) = \alpha^k \mu(P)$$

for all $P \in \text{Par}(n)$ and all $\alpha \geq 0$. Using Theorem 2.1 we are now able to index all continuous valuations on $\text{Par}(n)$ that are invariant under translation and are homogeneous of degree m .

Denote by \mathcal{U}^n the vector space of continuous translation invariant valuations on $\text{Par}(n)$, and let \mathcal{U}_m^n denote the subspace of valuations in \mathcal{U}^n that are homogeneous of degree m . Recall that the exterior tensor space $\wedge^m(\mathbb{R}^n)$ is an $\binom{n}{m}$ -dimensional space, spanned by the basis:

$$(9) \quad \{u_{i_1} \wedge \cdots \wedge u_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}.$$

Theorem 2.2. *There exists a bijective linear map $\Phi : \mathcal{U}_m^n \rightarrow \wedge^m(\mathbb{R}^n)$, denoted $\Phi(\mu) = X^\mu$, such that for all $1 \leq i_1 < \cdots < i_m \leq n$ and all $P \in \text{Span}\{u_{i_1}, \dots, u_{i_m}\}$,*

$$\mu(P) = X_{\{i_1, \dots, i_m\}}^\mu V_m(P).$$

Here $X_{\{i_1, \dots, i_m\}}^\mu$ denotes the $u_{i_1} \wedge \cdots \wedge u_{i_m}$ -coordinate of the vector X^μ with respect to the basis (9). We call X^μ the *generating vector* for the valuation μ .

Proof. For each linear coordinate subspace $E_{i_1, \dots, i_m} = \text{Span}\{u_{i_1}, \dots, u_{i_m}\}$ let $\text{Par}(E_{i_1, \dots, i_m})$ denote the set of all $P \in \text{Par}(n)$ with $P \subset E_{i_1, \dots, i_m}$. Since the restriction of μ to $\text{Par}(E_{i_1, \dots, i_m})$ satisfies the conditions of Theorem 2.1 (in dimension m), there exists $c_{i_1, \dots, i_m} \in \mathbb{R}$ such that

$$\mu(P) = c_{i_1, \dots, i_m} V_m(P),$$

for all $P \in \text{Par}(E_{i_1, \dots, i_m})$. Define X^μ by the equation

$$X_{\{i_1, \dots, i_m\}}^\mu = c_{i_1, \dots, i_m}.$$

Then define $\Phi(\mu) = X^\mu$.

Evidently Φ is a linear map. To see that Φ is injective, suppose $\Phi(\mu) = 0$. Then $\mu = 0$ when restricted to subspaces of dimension m , so that μ satisfies the hypotheses of Theorem 2.1 in dimension $m + 1$. It follows that μ is equal to a constant multiple of the $(m + 1)$ -dimensional volume V_{m+1} when restricted to a subspace of dimension $m + 1$. But μ is homogeneous of degree m , while V_{m+1} is homogeneous of degree $m + 1$. Consequently, $\mu = 0$ when restricted to subspaces of dimension $m + 1$, so that μ satisfies the hypotheses of Theorem 2.1 in dimension $m + 2$. After $n - m$ iterations of this argument it follows that $\mu = 0$ on $\text{Par}(n)$.

To see that Φ is surjective, suppose $X = (X_{\{i_1, \dots, i_m\}}) \in \wedge^m(\mathbb{R}^n)$. Define a valuation ν on $\text{Par}(n)$ by

$$\nu(P) = \sum_{1 \leq i_1 < \dots < i_m \leq n} X_{\{i_1, \dots, i_m\}} V_m(\alpha_{i_1} \bar{u}_{i_1} + \dots + \alpha_{i_m} \bar{u}_{i_m}),$$

for all $P = \alpha_1 \bar{u}_1 + \dots + \alpha_n \bar{u}_n$ (up to translation). Evidently $\nu \in \mathcal{U}_m^n$, and $\Phi(\nu) = X$. \square

The following corollary follows immediately from the injectivity of the map Φ .

Corollary 2.3. *Let $\mu, \nu \in \mathcal{U}_m^n$ and suppose that $\mu(P) = \nu(P)$ for all parallelotopes P of dimension m . Then $\mu(P) = \nu(P)$ for all $P \in \text{Par}(n)$.*

Remark. From the proof of Theorem 2.2 it is clear that if μ and ν are known to agree on only one m -dimensional parallelotope P in each coordinate m -subspace E , then $\mu = \nu$ on all of $\text{Par}(n)$.

While the proof of McMullen's spectral theorem 1.5 is difficult for the general case of valuations on polytopes (see [24]), an analogous result for continuous valuations on the lattice $\text{Par}(n)$ is easy to prove:

Theorem 2.4. *Let μ be a continuous translation invariant valuation on $\text{Par}(n)$. Then there exist unique continuous translation invariant valuations $\mu_0, \mu_1, \dots, \mu_n$, such that each μ_m is homogeneous of degree m , and such that*

$$\mu(P) = \sum_{m=0}^n \mu_m(P)$$

for all $P \in \text{Par}(n)$.

Proof. Let μ be a continuous translation invariant valuation on $\text{Par}(n)$. Let E_j denote the $(n - 1)$ -hyperplane in \mathbb{R}^n spanned by the coordinate vectors $u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n$. The restriction of μ to E_j is a translation invariant valuation on parallelotopes in E_j . Proceeding by induction, we may assume that, for all $P \in \text{Par}(E_j)$,

$$\mu(P) = \sum_{m=0}^{n-1} \mu_m(P),$$

where μ_m is a homogeneous valuation on $\text{Par}(E_j)$ of degree m . Thus μ_m is inductively defined in each coordinate subspace E_j .

To extend μ_m to all of $\text{Par}(n)$, define

$$\mu_m(P) = \sum_{1 \leq i_1 < \dots < i_m \leq n} \mu_m(\alpha_{i_1} \bar{u}_{i_1} + \dots + \alpha_{i_m} \bar{u}_{i_m}),$$

for all parallelotopes $P = \alpha_1 \bar{u}_1 + \dots + \alpha_n \bar{u}_n$, and then extend μ_m to $\text{Par}(n)$ by inclusion-exclusion. The valuation

$$\mu - \sum_{m=0}^{n-1} \mu_m$$

vanishes on all lower dimensional parallelotopes in $\text{Par}(n)$, since any such parallelotope is contained in a hyperplane parallel to one of the hyperplanes E_j . By Theorem 2.1,

$$\mu - \sum_{m=0}^{n-1} \mu_m = c_n V_n,$$

for some constant $c \in \mathbb{R}$. Setting $\mu_n = c_n V_n$ gives

$$\mu = \sum_{m=0}^n \mu_m.$$

□

Denote by $\wedge(\mathbb{R}^n)$ the exterior algebra (or Grassmann algebra) on \mathbb{R}^n , that is, the direct sum $\bigoplus_{m=0}^n \wedge^m(\mathbb{R}^n)$. Combining Theorem 2.4 with Theorem 2.2, we obtain the following:

Theorem 2.5. *There exists a bijective linear map $\Phi : \mathcal{U}^n \longrightarrow \wedge(\mathbb{R}^n)$, denoted $\Phi(\mu) = X^\mu$, such that*

$$\mu(P) = \sum_{m=0}^n \sum_{1 \leq i_1 < \dots < i_m \leq n} X_{\{i_1, \dots, i_m\}}^\mu V_m(\alpha_{i_1} \bar{u}_{i_1} + \dots + \alpha_{i_m} \bar{u}_{i_m}),$$

for all $P = \alpha_1 \bar{u}_1 + \dots + \alpha_n \bar{u}_n$ (up to translation).

Here $X_{\{i_1, \dots, i_m\}}^\mu$ denotes the $u_{i_1} \wedge \dots \wedge u_{i_m}$ -coordinate of the $\wedge^m(\mathbb{R}^n)$ -component of the vector X^μ with respect to the basis (9) of $\wedge^m(\mathbb{R}^n)$. The proof of Theorem 2.5 is omitted.

3. THE GENERATING FUNCTION OF AN EVEN VALUATION

While some of the results of the previous sections are already known for more general valuations, the results of this section are specific to even valuations. Theorem 1.2 will lead to a fundamental connection between even valuations and continuous functions on Grassmannians.

Let $C(G(n, m))$ denote the set of all real-valued continuous functions on the Grassmannian $G(n, m)$; i.e. the set of all m -dimensional vector subspaces of \mathbb{R}^n . For each subspace $E \in G(n, m)$, let \mathcal{K}_E denote the set of all compact convex subsets of E .

Theorem 3.1. *There exists an injective linear map $\Phi : \mathcal{A}_m^n \longrightarrow C(G(n, m))$, denoted $\Phi(\mu) = f_\mu$, such that for all $E \in G(n, m)$ and all $K \in \mathcal{K}_E$,*

$$\mu(K) = f_\mu(E) V_m(K).$$

Note that, unlike the case of valuations on $\text{Par}(n)$, we do not claim surjectivity for the map Φ . The image of Φ in $C(G(n, m))$ is the topic of Section 4.

Proof. Suppose $E \in G(n, m)$. Since the restriction of μ to \mathcal{K}_E satisfies the conditions of Theorem 1.2 (or Prop. 1.6) in dimension m , there exists $c_E \in \mathbb{R}$ such that

$$\mu(K) = c_E V_m(K),$$

for all $K \in \mathcal{K}_E$. Define $f_\mu : G(n, m) \rightarrow \mathbb{R}$ by the equation

$$f_\mu(E) = c_E,$$

and define $\Phi(\mu) = f_\mu$.

Evidently Φ is a linear map. To see that Φ is injective, suppose $\Phi(\mu) = 0$. Then $\mu = 0$ when restricted to subspaces of dimension m , so that μ satisfies the hypotheses of Theorem 1.2 in dimension $m + 1$. It follows that μ is equal to a constant multiple of the $(m + 1)$ -dimensional volume V_{m+1} when restricted to a subspace of dimension $m + 1$. But μ is homogeneous of degree m , while V_{m+1} is homogeneous of degree $m + 1$. Consequently, $\mu = 0$ when restricted to subspaces of dimension $m + 1$, so that μ satisfies the hypotheses of Theorem 1.2 in dimension $m + 2$. After $n - m$ iterations of this argument it follows that $\mu = 0$ on \mathcal{K}^n . \square

For $\mu \in \mathcal{A}_m^n$ the function $\Phi(\mu)$ is called the *generating function* of μ . Theorem 3.1 implies that a valuation $\mu \in \mathcal{A}_m^n$ is uniquely determined by its generating function.

The following is an immediate corollary of Theorem 3.1.

Corollary 3.2. *Suppose $\mu, \nu \in \mathcal{A}_m^n$, and suppose that $\mu(K) = \nu(K)$ whenever $\dim(K) \leq m$. Then $\mu(K) = \nu(K)$ for all $K \in \mathcal{K}^n$.*

Proof. Since μ and ν agree when restricted to any subspace of dimension m , it follows that $\Phi(\mu) = \Phi(\nu)$. Since Φ is injective (by Theorem 3.1), $\mu = \nu$. \square

In fact, we can do even better:

Corollary 3.3. *Suppose $\mu, \nu \in \mathcal{A}_m^n$, and suppose that $\mu(P) = \nu(P)$ whenever P is a parallelotope of dimension m . Then $\mu(K) = \nu(K)$ for all $K \in \mathcal{K}^n$.*

Proof. If P has dimension m , then some translate of P lies in a subspace $E \in G(n, m)$. By Theorem 3.1,

$$\Phi(\mu)(E)V_m(P) = \mu(P) = \nu(P) = \Phi(\nu)(E)V_m(P),$$

so that $\Phi(\mu)(E) = \Phi(\nu)(E)$ for all $E \in G(n, m)$. In other words, $\Phi(\mu) = \Phi(\nu)$. Since Φ is injective (by Theorem 3.1), it follows that $\mu = \nu$. \square

Remark. From the proof of Corollary 3.3 it is clear that if μ and ν are known to agree on only one m -dimensional parallelotope P in each $E \in G(n, m)$, then $\mu = \nu$ on all of \mathcal{K}^n .

Theorem 3.1 and Corollary 3.2 fail for valuations that are not even. For example, the valuation τ of (3) is continuous, translation invariant, homogeneous of degree 1, and restricts to zero on all 1-dimensional subspaces. But $\tau(K) \neq 0$ if K is not centered.

We continue this section with an application of Corollary 3.3 to mixed volumes. For $i \leq n$ and $K_1, \dots, K_i \in \mathcal{K}^n$, denote by

$$V_i(K_1|E, \dots, K_i|E)$$

the (i -dimensional) mixed volume of the projections of K_1, \dots, K_i onto an i -dimensional subspace E of \mathbb{R}^n .

Theorem 3.4. *Let $K_1, \dots, K_i, K'_1, \dots, K'_i \in \mathcal{K}_c^n$. The following statements are equivalent:*

- (i) $V_i(K_1|E, \dots, K_i|E) = V_i(K'_1|E, \dots, K'_i|E)$, for all $E \in G(n, i)$.
- (ii) $V_n(K_1, \dots, K_i, L_1, \dots, L_{n-i}) = V_n(K'_1, \dots, K'_i, L_1, \dots, L_{n-i})$, for all $L_1, \dots, L_{n-i} \in \mathcal{K}^n$.

Compare to Lemma 2.1 of Chakerian and Lutwak in [3].

Proof. For $E \in G(n, i)$ denote by C_{E^\perp} the $(n - i)$ -dimensional unit cube in the subspace E^\perp . Recall (or see in [31]) that

$$(10) \quad V_n(K_1, \dots, K_i, \underbrace{C_{E^\perp}, \dots, C_{E^\perp}}_{n-i}) = \binom{n}{i}^{-1} V_i(K_1|E, \dots, K_i|E).$$

Therefore, (ii) implies (i).

To show that (i) implies (ii), define valuations ν_1 and ν_2 on \mathcal{K}^n by

$$\nu_1(L) = V_n(K_1, \dots, K_i, L, \dots, L)$$

and

$$\nu_2(L) = V_n(K'_1, \dots, K'_i, L, \dots, L)$$

for $L \in \mathcal{K}^n$. Since the volume V_n is a valuation, it follows from (5) that ν_1 and ν_2 are valuations. It also follows from elementary properties of V_n and the mixed volumes that ν_1 and ν_2 are translation invariant, continuous, and homogeneous of degree $n - i$. Since the bodies K_1, \dots, K_i and K'_1, \dots, K'_i are symmetric, the valuations ν_1, ν_2 are also even.

Meanwhile, the identities (10) imply that

$$\nu_1(C_{E^\perp}) = \binom{n}{i}^{-1} V_i(K_1|E, \dots, K_i|E),$$

for all $E \in G(n, i)$, and similarly for ν_2 . It then follows from (i) that $\nu_1(C_{E^\perp}) = \nu_2(C_{E^\perp})$ for all $E \in G(n, i)$. In other words, we have $\nu_1(C_F) = \nu_2(C_F)$ for all $F \in G(n, n - i)$. Corollary 3.3 now implies that $\nu_1 = \nu_2$. Since the mixed volume V_n is linear with respect to Minkowski linear combinations in each parameter, (ii) then follows for all $L_1, \dots, L_{n-i} \in \mathcal{K}^n$. \square

It is well-known that if $K, L \in \mathcal{K}_c^n$, then $K = L$ up to translation if and only if

$$V_m(K|E) = V_m(L|E),$$

for all $E \in G(n, m)$ (see for example [31]). Theorem 3.1 implies a similar assertion for general even valuations.

Theorem 3.5. *Suppose $\mu \in \mathcal{A}_m^n$, and let $K, L \in \mathcal{K}^n$. If $V_m(K|E) = V_m(L|E)$ for all $E \in G(n, m)$, then $\mu(K|E) = \mu(L|E)$ for all $E \in G(n, m)$.*

If $\Phi(\mu)$ is a non-zero function (almost everywhere) on $G(n, m)$, then the converse also holds.

Proof. This is an immediate consequence of the assertion of Theorem 3.1, namely, that

$$\mu(K|E) = \Phi(\mu)(E)V_m(K|E),$$

for all $K \in \mathcal{K}^n$ and all $E \in G(n, m)$. \square

Corollary 3.6. *Suppose $\mu \in \mathcal{A}_m^n$, and suppose that $\Phi(\mu)$ is a non-zero function (almost everywhere) on $G(n, m)$. Let $K, L \in \mathcal{K}_c^n$. Then $K = L$ up to translation if and only if $\mu(K|E) = \mu(L|E)$ for all $E \in G(n, m)$.*

Proof. Corollary 3.6 follows directly from Theorem 3.5 and the fact that when K and L are symmetric we have $K = L$ up to translation if and only if $V_m(K|E) = V_m(L|E)$ for all $E \in G(n, m)$. \square

Remark. Corollary 3.6 does not hold for all (possibly non-symmetric) $K, L \in \mathcal{K}^n$.

4. THE GENERATING DISTRIBUTION OF A CONVEX SET

The conclusion of Theorem 3.1 motivates the following questions: Which functions in $C(G(n, i))$ are generating functions of valuations in \mathcal{A}_m^n ? And how does one reconstruct a valuation, given its generating function? In order to investigate the range of the injective linear map Φ of Theorem 3.1, we recall a few facts about the *generating distribution* of a convex body K , a construction due to Weil [33, 34, 35].

Denote by $\mathcal{E}(\mathbb{S}^{n-1})$ the space of all functions $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ such that there exists a signed measure ρ_f on \mathbb{S}^{n-1} with

$$f(u) = \int_{\mathbb{S}^{n-1}} |u \cdot v| d\rho_f(v).$$

In this case ρ_f is called a *generating measure* of the function f .

Denote by $C(\mathbb{S}^{n-1})$ the space of all continuous functions $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$, and denote by $\mathcal{D}(\mathbb{S}^{n-1})$ the space of all smooth even functions $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$. Blaschke [1] (see also [34, 35]) has shown that

$$\mathcal{D}(\mathbb{S}^{n-1}) \subset \mathcal{E}(\mathbb{S}^{n-1}) \subset C(\mathbb{S}^{n-1}).$$

Recall that a compact convex set K is called a *zonotope* if K can be expressed as a finite Minkowski sum of line segments. A set K is a *zonoid* if K can be approximated by zonotopes in the Hausdorff topology on \mathcal{K}^n . Zonoids having non-empty interior are also sometimes called *projection bodies*. Indeed, a set $K \in \mathcal{K}^n$ with non-empty interior is a zonoid if and only if there exists $L \in \mathcal{K}^n$ and a translate K' of K such that $h_{K'}(u) = V_{n-1}(L|u^\perp)$ for all $u \in \mathbb{S}^{n-1}$. In this case we write $K' = \Pi L$ and say that K' is the *projection body* of L .

If $K \in \mathcal{K}_c^m$ and if there exist zonoids L, M such that $K + L = M$, then K is called a *generalized zonoid*. If K is a generalized zonoid centered about the origin, then its support function h_K has a signed generating measure ρ_{h_K} on \mathbb{S}^{n-1} . In this case, the generalized zonoid K is a zonoid if and only if its generating measure ρ_{h_K} is a positive measure. Generalized zonoids are of special interest because of the following fact: If K is a symmetric compact convex set in \mathbb{R}^n , then there exists a sequence $\{K_i\}$ of generalized zonoids such that $K_i \rightarrow K$.

If K is a generalized zonoid centered about the origin, then $h_K \in \mathcal{E}(\mathbb{S}^{n-1})$, and we denote by ρ_K the generating measure of h_K (instead of ρ_{h_K}). Note that if K is a projection body, with $K = \Pi L$ for some convex body L , then $\rho_K = S(L, \cdot)$, the Aleksandrov-Fenchel-Jessen measure associated with L . For a thorough treatment of zonoids, generalized zonoids, and projection bodies, see [5, 10, 31, 36].

In order to generalize the notion of generating measure to all symmetric compact convex sets, we move to the level of distributions. If K is centered about the origin,

denote by T_K the distribution defined by

$$(11) \quad T_K(f) = \rho_f(h_K) = \int_{\mathbb{S}^{n-1}} h_K \, d\rho_f,$$

for all $f \in \mathcal{E}(\mathbb{S}^{n-1})$. For $K \in \mathcal{K}_c^n$, we define T_K similarly, after first translating the center of K to the origin. Weil [33, 34] has shown that T_K is well-defined and is continuous with respect to K . We call T_K the *generating distribution* of the compact convex set K . Evidently $T_K = \rho_K$ if K is a generalized zonoid, since in this case

$$\begin{aligned} T_K(f) &= \rho_f(h_K) \\ &= \int_{\mathbb{S}^{n-1}} h_K(u) \, d\rho_f(u) \\ &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |u \cdot v| \, d\rho_K(v) \, d\rho_f(u) \\ &= \int_{\mathbb{S}^{n-1}} f(v) \, d\rho_K(v) \\ &= \rho_K(f). \end{aligned}$$

The generating distribution T_K is a fundamental tool for the construction of a partial inverse to the mapping $\Phi : \mathcal{A}_m^n \rightarrow C(G(n, m))$ of Theorem 3.1. For a straightforward and elegant treatment of the theory of distributions, see [28]. See also [19, pp. 33-53].

For $u_1, \dots, u_n \in \mathbb{S}^{n-1}$ denote by $D(u_1, \dots, u_n)$ the absolute value of determinant of the collection (u_1, \dots, u_n) , that is, the volume of the parallelotope spanned by u_1, \dots, u_n . Denote by $u_1 \wedge \dots \wedge u_m$ the wedge product of the vectors u_i in the exterior tensor space $\wedge^m(\mathbb{R}^n)$. If $u_1 \wedge \dots \wedge u_m \neq 0$, then we abuse notation to denote by $u_1 \wedge \dots \wedge u_m$ the m -subspace spanned by the vectors u_i . Finally let $U \subseteq \mathbb{S}^{n-1} \times \dots \times \mathbb{S}^{n-1}$ denote the (open dense) set of all linearly independent m -tuples of unit vectors.

For $g \in \mathcal{D}(G(n, m))$, define a function, $\tilde{g} : \mathbb{S}^{n-1} \times \dots \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ by

$$(12) \quad \tilde{g}(u_1, \dots, u_m) = \frac{1}{m!} g(u_1 \wedge \dots \wedge u_m),$$

for $(u_1, \dots, u_m) \in U$ and by setting $\tilde{g} = 0$ otherwise. For $w_1, \dots, w_m \in \mathbb{R}^n - \{o\}$, we denote the value $\tilde{g}(w_1/|w_1|, \dots, w_m/|w_m|)$ by $\tilde{g}(w_1, \dots, w_m)$ for notational convenience.

Recall that if $K \in \mathcal{K}_c^n$, then there exists a sequence of generalized zonoids K_i , having generating measures ρ_{K_i} , such that $K_i \rightarrow K$. Define

$$(13) \quad T_K^m(g) = \lim_{i \rightarrow \infty} \underbrace{\int_{\mathbb{S}^{n-1}} \dots \int_{\mathbb{S}^{n-1}}}_{m} \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) \, d\rho_{K_i} \dots d\rho_{K_i},$$

provided this limit exists and is well-defined (independently of the choice of sequence $\{K_i\}$ converging to K). Denote by \mathcal{F}_m^n the space of all functions $g : G(n, m) \rightarrow \mathbb{R}$ such that the limit (13) is well-defined for all $K \in \mathcal{K}_c^n$.

Remark. Note that, unlike the functional T_K (on \mathbb{S}^{n-1}), the linear functional $T_K^m(g)$ is *not* necessarily a distribution in the traditional sense of the term, since there is no reason to believe a priori that $T_K^m(g)$ is defined for all C^∞ functions g . In some

cases a problem may arise, since the product $\tilde{g}D$ will not necessarily be smooth. Determination of the domain and the precise meaning of the functional $T_K^m(g)$ is the object of the remainder of this section.

Similarly, suppose $K_1, \dots, K_m \in \mathcal{K}_c^n$ with generalized zonoids $\{K_{ij}\}$ such that

$$\lim_{i \rightarrow \infty} K_{ij} = K_j$$

for each $j = 1, \dots, m$. For $g \in \mathcal{F}_m^n$ define

$$(14) \quad T_{K_1, \dots, K_m}(g) = \lim_{i \rightarrow \infty} \underbrace{\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}}}_m \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_{K_{i1}} \cdots d\rho_{K_{im}}.$$

Recall that for all generalized zonoids K, L we have $\rho_{K+L} = \rho_K + \rho_L$. An elementary combinatorial argument then implies that the limit (14) exists and that $T_{K_1, \dots, K_m}(g)$ is well-defined if and only if $g \in \mathcal{F}_m^n$. Note that

$$T_{K_1, \dots, K_m}(g) = T_K^m(g)$$

for all $g \in \mathcal{F}_m^n$.

If K is a generalized zonoid with generating measure ρ_K , then

$$(15) \quad T_K^m(g) = \underbrace{\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}}}_m \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_K \cdots d\rho_K.$$

In other words, the functional T_K^m is a *measure* in this case. In particular, it follows that if K is a generalized zonoid, then $T_K^m(g)$ is well-defined for all $g \in C(G(n, m))$. More generally,

$$(16) \quad T_{K_1, \dots, K_m}(g) = \underbrace{\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}}}_m \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_{K_1} \cdots d\rho_{K_m},$$

for generalized zonoids K_1, \dots, K_m .

Suppose that Z is a zonotope with center at the origin, that is, a Minkowski sum $Z = \bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_s$ of line segments in \mathbb{R}^n . Recall that \bar{w} denotes the closed line segment with endpoints at w and $-w$. Since the support function of a line segment \bar{w} is given by $h_{\bar{w}}(u) = |u \cdot w|$, it follows that ρ_Z is a finite measure with masses $|w_i|$ concentrated at points $\pm w_i/|w_i|$ in \mathbb{S}^{n-1} . Denote by Z_{i_1, \dots, i_m} the (possibly degenerate) m -face of Z corresponding to the summands $\bar{w}_{i_1}, \dots, \bar{w}_{i_m}$. Equation (15) implies that

$$\begin{aligned} T_Z^m(g) &= \underbrace{\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}}}_m \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_Z \cdots d\rho_Z \\ &= \sum_{i_1=1}^s \cdots \sum_{i_m=1}^s \tilde{g}(w_{i_1}, \dots, w_{i_m}) D(w_{i_1}/|w_{i_1}|, \dots, w_{i_m}/|w_{i_m}|) |2w_{i_1}| \cdots |2w_{i_m}|, \end{aligned}$$

so that

$$(17) \quad T_Z^m(g) = \sum_{i_1=1}^s \cdots \sum_{i_m=1}^s \tilde{g}(w_{i_1}, \dots, w_{i_m}) V_m(Z_{i_1, \dots, i_m}).$$

Recall that the m -volume V_m of a Minkowski linear combination of convex bodies in \mathbb{R}^m expands into a sum of mixed m -volumes according to (5). In particular,

$$V_m(Z_{i_1, \dots, i_m}) = V_m(\bar{w}_{i_1} + \dots + \bar{w}_{i_m}) = m! V_m(\bar{w}_{i_1}, \dots, \bar{w}_{i_m}).$$

The identity (17) now becomes

$$(18) \quad T_Z^m(g) = \sum_{i_1=1}^s \cdots \sum_{i_m=1}^s m! \tilde{g}(w_{i_1}, \dots, w_{i_m}) V_m(\bar{w}_{i_1}, \dots, \bar{w}_{i_m}).$$

The following proposition is a special case of a theorem of Weil [35].

Proposition 4.1. *Let $E \in G(n, m)$. If $K \subseteq E$ is a zonoid and $g \in C(G(n, m))$, then*

$$T_K^m(g) = g(E) V_m(K).$$

Proof. Suppose that $Z \subseteq E$ is a zonotope, given by the Minkowski sum $Z = \bar{w}_1 + \bar{w}_2 + \dots + \bar{w}_s$, for some $w_1, \dots, w_s \in E$. Then (17) implies that

$$\begin{aligned} T_Z^m(g) &= \sum_{i_1=1}^s \cdots \sum_{i_m=1}^s \tilde{g}(w_{i_1}, \dots, w_{i_m}) V_m(Z_{i_1, \dots, i_m}) \\ &= \sum_{1 \leq i_1 < \dots < i_m \leq s} m! \tilde{g}(w_{i_1}, \dots, w_{i_m}) V_m(Z_{i_1, \dots, i_m}) \\ &= g(E) V_m(Z), \end{aligned}$$

since the definition (12) implies that $m! \tilde{g}(w_{i_1}, \dots, w_{i_m}) = g(E)$ if the vectors $\{w_{i_1}, \dots, w_{i_m}\}$ are independent and zero otherwise. Recall that any zonoid in E can be approximated by zonotopes in E . The proposition then follows from the continuity of T_K^m and of the intrinsic volume V_m . \square

Proposition 4.1 can be generalized to include all symmetric compact convex sets inside an m -dimensional subspace E . Theorem 3.1 and Proposition 4.1, along with the identities (17) and (18), suggest the following theorem.

Theorem 4.2. *Suppose that $\mu \in \mathcal{A}_m^n$, and let g denote the generating function of μ ; that is, $g = \Phi(\mu)$. Then $g \in \mathcal{F}_m^n$, with*

$$(19) \quad \mu(K) = T_K^m(g),$$

and

$$(20) \quad \mu(K_1, \dots, K_m) = T_{K_1, \dots, K_m}(g),$$

for all $K, K_1, \dots, K_m \in \mathcal{K}_c^n$.

Here $\mu(K_1, \dots, K_m)$ denotes the mixed valuation generated by μ , as given by Theorem 1.4.

Theorem 4.2 implies that if $g : G(n, m) \rightarrow \mathbb{R}$ is known to be the generating function of a valuation $\mu \in \mathcal{A}_m^n$, then $g \in \mathcal{F}_m^n$; that is, $T_K^m(g)$ is well-defined for all $K \in \mathcal{K}_c^n$.

Proof. To begin, recall that for generalized zonoids K the value of $T_K^m(g)$ is defined, since T_K^m is a *measure* on the Grassmannian $G(n, m)$.

Let u_1, \dots, u_n be an independent set of unit vectors in \mathbb{R}^n , and consider the restriction of the valuation μ to the lattice $\text{Par}(n)$ of (finite unions of) parallelotopes with edges parallel to u_1, \dots, u_n .

Suppose $P \in \text{Par}(n)$ is the parallelotope given by the Minkowski sum

$$P = \alpha_1 \bar{u}_1 + \cdots + \alpha_n \bar{u}_n$$

of line segments. From (17) we then have

$$T_P^m(g) = \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \tilde{g}(u_{i_1}, \dots, u_{i_m}) V_m(\alpha_{i_1} \bar{u}_{i_1} + \cdots + \alpha_{i_m} \bar{u}_{i_m}).$$

Since V_m is a valuation in each m -dimensional coordinate subspace of \mathbb{R}^n , it follows that the functional $T_P^m(g)$ is a continuous translation invariant valuation on $\text{Par}(n)$, homogeneous of degree m . Moreover, if $P \subseteq u_{i_1} \wedge \cdots \wedge u_{i_m}$, then

$$T_P^m(g) = m! \tilde{g}(u_{i_1}, \dots, u_{i_m}) V_m(P) = g(u_{i_1} \wedge \cdots \wedge u_{i_m}) V_m(P) = \mu(P).$$

It follows from Corollary 2.3 that $T_P^m(g) = \mu(P)$ for all parallelotopes in $\text{Par}(n)$.

Since we could have chosen any basis u_1, \dots, u_n for \mathbb{R}^n with which to pursue this argument, it follows that $T_P^m(g) = \mu(P)$ for *all* parallelotopes in \mathbb{R}^n , regardless of the basis of edges u_1, \dots, u_n . Moreover, it now follows that

$$\mu(P_1, \dots, P_m) = \underbrace{\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}}}_m \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_{P_1} \cdots d\rho_{P_m},$$

provided that P_1, \dots, P_m are parallelotopes *with respect to the same basis*. In particular, if $w_1, \dots, w_m \in \mathbb{R}^n$, then

$$\begin{aligned} \mu(\bar{w}_1, \dots, \bar{w}_m) &= \underbrace{\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}}}_m \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_{\bar{w}_1} \cdots d\rho_{\bar{w}_m} \\ &= 2^m \tilde{g}(w_1, \dots, w_m) D(w_1, \dots, w_m) \\ &= \tilde{g}(w_1, \dots, w_m) V_m(\bar{w}_1 + \cdots + \bar{w}_m) \\ &= m! \tilde{g}(w_1, \dots, w_m) V_m(\bar{w}_1, \dots, \bar{w}_m). \end{aligned}$$

For any zonotope $Z = \bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_s$, we now have

$$\begin{aligned} \mu(Z) &= \mu(\bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_s) \\ &= \sum_{i_1=1}^s \cdots \sum_{i_m=1}^s \mu(\bar{w}_{i_1}, \dots, \bar{w}_{i_m}) \\ &= \sum_{i_1=1}^s \cdots \sum_{i_m=1}^s m! \tilde{g}(w_{i_1}, \dots, w_{i_m}) V_m(\bar{w}_{i_1}, \dots, \bar{w}_{i_m}) \\ &= T_Z^m(g), \end{aligned}$$

by (18). It then follows from continuity that $T_K^m(g) = \mu(K)$ whenever K is a zonoid. Since the set of zonoids in \mathcal{K}^n is closed under Minkowski addition, the identity (20) is also verified for zonoids.

Now suppose that K is a *generalized zonoid*. That is, suppose $K + L = M$, where L and M are zonoids. Then we have

$$\begin{aligned} \mu(K, K_2, \dots, K_m) &= \mu(M, K_2, \dots, K_m) - \mu(L, K_2, \dots, K_m) \\ &= \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_M d\rho_{K_2} \cdots d\rho_{K_m} \\ &\quad - \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_L d\rho_{K_2} \cdots d\rho_{K_m} \\ &= \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \tilde{g}(u_1, \dots, u_m) D(u_1, \dots, u_m) d\rho_K d\rho_{K_2} \cdots d\rho_{K_m}. \end{aligned}$$

In other words, the equation (20) holds if $K_1 = K$ is a generalized zonoid and K_2, \dots, K_m are zonoids. Repeating this argument in each variable, we infer that (20) holds when K_1, \dots, K_m are generalized zonoids. Setting $K = K_1 = \dots = K_m$ then proves (19) for generalized zonoids.

Recall that the set of generalized zonoids in \mathbb{R}^n is dense in the set of all symmetric compact convex sets. From the continuity of μ it now follows that

$$T_K^m(g) = \lim_{K_i \rightarrow K} T_{K_i}^m(g) = \lim_{K_i \rightarrow K} \mu(K_i) = \mu(K),$$

is *well-defined*, where $\{K_i\}$ is *any* sequence of generalized zonoids converging to the symmetric compact convex set K . In other words, $\mu(K) = T_K^m(g)$ for all $K \in \mathcal{K}_c^n$. Since $T_K^m(g)$ is defined for all $K \in \mathcal{K}_c^n$, it follows that $g \in \mathcal{F}_m^n$. The same continuity argument also verifies the mixed valuation expression (20) for all symmetric K_1, \dots, K_m , thereby completing the proof of Theorem 4.2. \square

For subspaces $E, F \in G(n, m)$, denote by $\langle E, F \rangle$ the cosine of the “angle” between the m -subspaces E and F . The value of $\langle E, F \rangle$ measures the distortion of m -volume upon projecting orthogonally from E to F (or vice versa) and is equal to the cosine of the (smaller) Euclidean angle between the straight lines in the exterior tensor space $\wedge^m(\mathbb{R}^n)$ representing the subspaces E and F in Plücker coordinates. If $\{u_1, \dots, u_m\}$ is an orthonormal basis for E and $\{w_1, \dots, w_m\}$ is an orthonormal basis for F , then

$$\langle E, F \rangle = |\det [(u_i \cdot w_j)]|.$$

The following corollary was originally proven directly by Weil [35], but now follows immediately from Theorem 4.2.

Corollary 4.3. *For all $E \in G(n, m)$ and all $K, K_1, \dots, K_m \in \mathcal{K}_c^n$, the value $T_K^m(\langle E, F \rangle)$ is well-defined. Moreover,*

$$(21) \quad V_m(K|E) = T_K^m(\langle E, F \rangle),$$

and

$$(22) \quad V_m(K_1|E, \dots, K_m|E) = T_{K_1, \dots, K_m}(\langle E, F \rangle).$$

Proof. Since $V_m \in \mathcal{A}_m^n$, so is the valuation given by $K \mapsto V_m(K|E)$, where E is a fixed m -dimensional subspace. Meanwhile if $K \subseteq F$ for some $F \in G(n, m)$, then $V_m(K|E) = \langle E, F \rangle V_m(K)$. In other words, the map $F \mapsto \langle E, F \rangle$ is the generating function for the valuation $V_m(\cdot|E)$ given by Theorem 3.1. It follows from Theorem 4.2 that $T_K^m(\langle E, F \rangle)$ is well-defined. Equations (21) and (22) then also follow from Theorem 4.2. \square

5. AN ALTERNATIVE APPROACH

In view of Theorem 4.2, one is tempted to conjecture that $\mathcal{A}_m^n = \{T_{(\cdot)}^m(g) : g \in \mathcal{F}_m^n\}$, in other words, that the map Φ is a bijection between the set \mathcal{A}_m^n of valuations and the space \mathcal{F}_m^n of functions. While this may be true, there remains to show that the functional $K \mapsto T_K^m(g)$ is indeed a valuation on all of \mathcal{K}^n , for every $g \in \mathcal{F}_m^n$.

Another question also arises: what exactly is \mathcal{F}_m^n ? What conditions must we place on a function $g \in C(G(n, m))$ so that $T_K^m(g)$ is well-defined for all $K \in \mathcal{K}_c^n$? It is not clear that $T_K^m(g)$ is defined for arbitrary symmetric K even when g is *smooth*.

In order to investigate further the space \mathcal{F}_m^n , we continue with yet another interpretation of the generating distribution of a compact convex set K .

Denote by \mathcal{E}_m^n the set of all functions $f : G(n, m) \rightarrow \mathbb{R}$ such that

$$f(E) = \int_{G(n, m)} \langle E, F \rangle d\rho_f(F),$$

for some (signed) measure ρ_f on $G(n, m)$. In this case we call ρ_f a generating measure for f .

Recall that if K is a compact convex subset of \mathbb{R}^n and E is an m -dimensional subspace of \mathbb{R}^n , then the orthogonal projection $K|E$ is a compact convex subset of E . Moreover, the support function $h_{K|E}$ of $K|E$ is given by the restriction of h_K to E . For fixed $K \in \mathcal{K}^n$ the function $V_m(K|E)$ is called the m -th projection function of K .

For all $K \in \mathcal{K}^n$ define the m -th generating distribution \tilde{T}_K^m on \mathcal{E}_m^n by

$$\tilde{T}_K^m(f) = \int_{G(n, m)} V_m(K|E) d\rho_f.$$

We shall see that, although the generating measure ρ_f is not necessarily unique (there may be other measures that generate the function f), the value of $\tilde{T}_K^m(f)$ is well-defined independently of the choice of admissible ρ_f .

Note also that unlike T_K and T_K^m , the functional \tilde{T}_K^m is defined not only for symmetric K , but for *all* compact convex K . Evidently \tilde{T}_K^m is a continuous translation invariant even valuation in the parameter K , homogeneous of degree m . This follows immediately from the fact that $V_m(K|E)$ is such a valuation in the parameter K for each fixed E . For $f \in \mathcal{E}_m^n$, denote $\mu_f(K) = \tilde{T}_K^m(f)$.

Theorem 5.1. *For $f \in \mathcal{E}_m^n$, the valuation μ_f is well-defined. Moreover, the generating function of the valuation μ_f is the function f itself. In other words, $\Phi(\mu_f) = f$.*

Proof. Suppose $f \in \mathcal{E}_m^n$, and let ρ_f be a generating measure for f . If $K \subseteq F$ for some $F \in G(n, m)$, then $V_m(K|E) = \langle E, F \rangle V_m(K)$. Therefore

$$\begin{aligned} \mu_f(K) &= \tilde{T}_K^m(f) \\ &= \int_{G(n, m)} V_m(K|E) d\rho_f \\ &= \int_{G(n, m)} \langle E, F \rangle V_m(K) d\rho_f \\ &= V_m(K)f(E). \end{aligned}$$

It then follows from Theorem 3.1 that f is the generating function of μ_f .

Next, suppose $\bar{\rho}_f$ is another generating measure for f . Let $\bar{\mu}_f$ denote the valuation obtained by replacing ρ_f with $\bar{\rho}_f$. The same argument implies that f is the generating function of $\bar{\mu}_f$. In other words, the two valuations μ_f and $\bar{\mu}_f$ have the same generating function. Theorem 3.1 then implies that $\bar{\mu}_f = \mu_f$. In other words, the valuation μ_f is well-defined, independently of the choice of generating measure ρ for a given function $f \in \mathcal{E}_m^n$. \square

As one might hope, the two functionals T_K^m and \tilde{T}_K^m coincide on \mathcal{E}_m^n .

Theorem 5.2. *For $g \in \mathcal{E}_m^n$ and $K \in \mathcal{K}_c^n$,*

$$\tilde{T}_K^m(g) = T_K^m(g).$$

In particular, $\mathcal{E}_m^n \subseteq \mathcal{F}_m^n$.

Proof. Suppose $g \in \mathcal{E}_m^n$, and define $\mu_g(K) = \tilde{T}_K^m(g)$ for all $K \in \mathcal{K}^n$. By Theorem 5.1, we have $\Phi(\mu_g) = g$. It then follows from Theorem 4.2 that $g \in \mathcal{F}_m^n$, and that

$$T_K^m(g) = \mu_g(K) = \tilde{T}_K^m(g),$$

for all $K \in \mathcal{K}_c^n$. \square

Remark. Note that $f \in \mathcal{E}_m^n$ if and only if the function f can be approximated by linear combinations of absolute inner product functions $F \mapsto \langle E, F \rangle$. Consequently an even valuation μ has generating function $f_\mu \in \mathcal{E}_m^n$ if and only if μ can be approximated by linear combinations of projection functions $K \mapsto V_m(K|E)$. There remains the interesting question of whether $\mathcal{E}_m^n = \mathcal{F}_m^n$; that is, are *all* continuous even valuations (homogeneous of degree m) approximable by linear combinations of projection functions? Or if not, do there exist easily verified conditions on μ that determine whether $f_\mu \in \mathcal{E}_m^n$?

Recent developments in the study of projection functions [7, 8] and generalized (Grassmannian) cosine transforms [6, 11, 12, 13] may lead to interesting new examples of valuation generating functions, which may in turn shed light on the question of how better to characterize the families \mathcal{E}_m^n and \mathcal{F}_m^n .

6. EVEN VALUATIONS REVISITED

McMullen’s spectral theorem for translation invariant valuations, Theorem 1.5, leads to more general forms for Theorems 3.1, 4.2-5.2.

Denote by $\text{Mod}(n)$ the set of all subspaces of \mathbb{R}^n . The set $\text{Mod}(n)$ consists of the disjoint union

$$\text{Mod}(n) = \{0\} \uplus G(n, 1) \uplus G(n, 2) \uplus \cdots \uplus \{\mathbb{R}^n\}.$$

For $f : \text{Mod}(n) \rightarrow \mathbb{R}$, denote by $f|_{G(n,m)}$ the restriction of f to $G(n,m)$.

Theorem 6.1. *There exists an injective map $\Phi : \mathcal{A}^n \rightarrow C(\text{Mod}(n))$, denoted $\Phi(\mu) = f_\mu$, such that*

$$\mu(K) = \sum_{m=0}^n T_K^m(f_\mu|_{G(n,m)})$$

for all $K \in \mathcal{K}_c^n$.

Proof. Suppose $\mu \in \mathcal{A}^n$. By Theorem 1.5, there exist unique valuations $\mu_m \in \mathcal{A}_m^n$, for $m = 0, \dots, n$, such that

$$\mu(K) = \sum_{m=0}^n \mu_m(K),$$

for all $K \in \mathcal{K}^n$. In other words,

$$\mathcal{A}^n = \mathcal{A}_0^n \oplus \mathcal{A}_1^n \oplus \dots \oplus \mathcal{A}_n^n.$$

Since Theorem 3.1 gives an injective linear map $\Phi : \mathcal{A}_m^n \rightarrow C(G(n, m))$ for each $m = 0, 1, \dots, n$, define $\Phi : \mathcal{A}^n \rightarrow C(\text{Mod}(n))$ to be the direct sum of these component maps. Evidently this extension of Φ remains injective.

If $K \in \mathcal{K}_c^n$ with generating distribution T_K , then

$$\mu(K) = \sum_{m=0}^n \mu_m(K) = \sum_{m=0}^n T_K^m(f_\mu|_{G(n, m)}),$$

by Theorem 4.2 applied to each μ_m . □

Once again the valuation μ is determined uniquely by its values on parallelotopes.

Theorem 6.2. *Let $\mu, \nu \in \mathcal{A}^n$, and suppose that $\mu(P) = \nu(P)$ for all parallelotopes P in \mathbb{R}^n . Then $\mu(K) = \nu(K)$ for all $K \in \mathcal{K}^n$.*

Proof. If P is a parallelotope, then

$$\sum_{m=0}^n \mu_m(P)\epsilon^m = \sum_{m=0}^n \mu_m(\epsilon P) = \mu(\epsilon P) = \nu(\epsilon P) = \sum_{m=0}^n \nu_m(P)\epsilon^m,$$

for all $\epsilon > 0$. It follows that $\mu_m(P) = \nu_m(P)$ for all parallelotopes. By Corollary 3.3, each $\mu_m = \nu_m$, so that $\mu = \nu$. □

Remark. It is sufficient for the valuations μ, ν to agree on the dilates of a single (full-dimensional) parallelotope in each orthogonal frame of \mathbb{R}^n to imply that $\mu = \nu$. The proof is similar to that of Theorem 6.2.

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