CANTOR SETS AND NUMBERS
WITH RESTRICTED PARTIAL QUOTIENTS

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Abstract. For \( j = 1, \ldots, k \) let \( C_j \) be a Cantor set constructed from the interval \( I_j \), and let \( \epsilon_j = \pm 1 \). We derive conditions under which
\[
\epsilon_1 C_1 + \cdots + \epsilon_k C_k = \epsilon_1 I_1 + \cdots + \epsilon_k I_k \quad \text{and} \quad C_{\epsilon_1} \cdots C_{\epsilon_k} = I_{\epsilon_1} \cdots I_{\epsilon_k}.
\]
When these conditions do not hold, we derive a lower bound for the Hausdorff dimension of the above sum and product. We use these results to make corresponding statements about the sum and product of sets \( F(B_j) \), where \( B_j \) is a set of positive integers and \( F(B_j) \) is the set of real numbers \( x \) such that all partial quotients of \( x \), except possibly the first, are members of \( B_j \).

1. Introduction

Let \( x \) be a real number. We say that \( x \) is \textit{badly approximable} if there exists a positive integer \( n \) such that for every rational number \( p/q \),
\[
\left| x - \frac{p}{q} \right| > \frac{1}{nq^2}.
\]
It can be shown that this set is of Lebesgue measure zero; however, it is still quite large. In 1947 Marshall Hall [5] showed that every real number can be expressed as the sum of two badly approximable numbers. In particular, for a positive integer \( m \) let \( F(m) \) denote the set of numbers
\[
F(m) = \{ [t, a_1, a_2, \ldots] : t \in \mathbb{Z}, 1 \leq a_i \leq m \text{ for } i \geq 1 \}
\]
where by \([a_0, a_1, a_2, \ldots]\) we denote the \textit{continued fraction}
\[
a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}
\]
with partial quotients \( a_0, a_1, a_2 \) and so on. It can be shown that for every \( x \in F(4) \) and every \( p/q \in \mathbb{Q} \),
\[
\left| x - \frac{p}{q} \right| > \frac{1}{6q^2}.
\]
so that \( F(4) \) is a set of badly approximable numbers. Hall proved that
\[
F(4) + F(4) = \mathbb{R},
\]
where we define the sum of two sets of real numbers \( A \) and \( B \) by
\[
A + B = \{ a + b : a \in A \text{ and } b \in B \}.
\]
In 1973 Bohuslav Diviš [4] showed that one could not do much better than Hall’s result, namely that
\[
F(3) + F(3) \neq \mathbb{R}.
\]
In 1975 James Hlavka [6] generalized Hall’s results to the case of different sets \( F(m) \) and \( F(n) \). He proved that \( F(m) + F(n) = \mathbb{R} \) holds for \((m, n) = (2, 7)\) or \((m, n) = (3, 4)\), but does not hold for \((m, n) = (2, 4)\).

Now, if (1) holds, then the same equation holds with \( m \) and \( n \) replaced by \( m' \) and \( n' \) respectively, where \( m' \geq m \) and \( n' \geq n \). Further, if either \( n \) or \( m \) is equal to one then trivially (1) does not hold, since \( F(1) \) consists of the points \( \{ [t, 1, 1, 1, \ldots] : t \in \mathbb{Z} \} \).

Hence the only cases of interest left are \((m, n) = (2, 5)\) and \((m, n) = (2, 6)\). Hlavka conjectured that in these two cases (1) would not hold. In work to appear [1] we show that in both cases Hlavka’s conjecture is false.

We can also examine the difference of two sets \( F(m) \) and \( F(n) \). If \( A \) is a set of real numbers we define \(-A\) by
\[
-A = \{ -a : a \in A \}
\]
and denote \( A + (-B) \) by \( A - B \). We have the following result.

**Theorem 1.1.** Let \( m \) and \( n \) be integers. The equations
\[
F(m) + F(n) = \mathbb{R} \quad \text{and} \quad F(m) - F(n) = \mathbb{R}
\]
hold if \((m, n) = (2, 5)\) or \((m, n) = (3, 4)\). Neither of the above equations hold if \((m, n) = (2, 4)\). Additionally,
\[
F(3) + F(3) \neq \mathbb{R} \quad \text{and} \quad F(3) - F(3) = \mathbb{R}.
\]

In 1971 Tom Cusick [2] examined the complementary case of sums of real numbers whose continued fraction expansion contains only large partial quotients. For each positive integer \( l \) we define the set \( G(l) \) by
\[
G(l) = \{ [t, a_1, a_2, \ldots] : t \in \mathbb{Z} \text{ and } a_i \geq l \text{ for } i \geq 1 \}
\]
\[
\cup \{ [t, a_1, a_2, \ldots, a_k] : t, k \in \mathbb{Z}, k \geq 0 \text{ and } a_i \geq l \text{ for } 1 \leq i \leq k \}.
\]
Cusick proved that
\[
G(2) + G(2) = \mathbb{R}.
\]

The above results are special cases of the following general problem. Let \( B \) be a set of positive integers. If \( B \) is a finite set, we let \( F(B) \) denote the set of real numbers which have an infinite continued fraction expansion with all partial quotients, except possibly the first, members of \( B \). For \( B \) infinite, we define \( F(B) \) similarly, but also allow numbers with finite continued fraction expansions. Thus if we define
\[
L_m = \{ 1, 2, \ldots, m \} \quad \text{and} \quad U_l = \{ l, l+1, \ldots \}
\]
Table 1. Values of $\tau(B)$ for certain $B$

<table>
<thead>
<tr>
<th>$B$</th>
<th>$\tau(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_2$</td>
<td>$(-1 + \sqrt{3})/2 = 0.366\ldots$</td>
</tr>
<tr>
<td>$L_3$</td>
<td>$(-6 + 4\sqrt{21})/15 = 0.822\ldots$</td>
</tr>
<tr>
<td>$L_4$</td>
<td>$(-3 + 15\sqrt{2})/14 = 1.300\ldots$</td>
</tr>
<tr>
<td>$L_5$</td>
<td>$\sqrt{5}/5 = 1.788\ldots$</td>
</tr>
<tr>
<td>$L_6$</td>
<td>$(15 + 35\sqrt{15})/66 = 2.281\ldots$</td>
</tr>
<tr>
<td>$L_7$</td>
<td>$(42 + 24\sqrt{77})/91 = 2.775\ldots$</td>
</tr>
<tr>
<td>$U_l$</td>
<td>$1/(l-1)$</td>
</tr>
</tbody>
</table>

for positive integers $m$ and $l$, then $F(m) = F(L_m)$ and $G(l) = F(U_l)$. For sets of positive integers $B_1$ and $B_2$, we wish to know when

$$F(B_1) + F(B_2) = R$$

and when

$$F(B_1) - F(B_2) = R.$$

We shall derive conditions on the sets $B_1$ and $B_2$ such that (2) and (3) follow. Let $B = \{b_1, b_2, \ldots\}$ be a set of positive integers with $|B| > 0$ and $b_1 < b_2 < \ldots$. If $|B| = 1$, then we put $\tau(B) = 0$. Otherwise, we set

$$l = l(B) = \min B \quad \text{and} \quad \Delta_i = \Delta_i(B) = b_{i+1} - b_i.$$

If $B$ is a finite set with $|B| > 1$, then we put

$$m = m(B) = \max B, \quad \delta = \delta(B) = \frac{-lm + \sqrt{l^2m^2 + 4lm}}{2},$$

and

$$\tau(B) = \min_{i < |B|} \min \left\{ \frac{\delta(m-l)}{\Delta_i l m - \delta(m-l)} \cdot \frac{b_{i+1} + 1}{b_i l m + \delta l}, \frac{(m-b_{i+1})l m + \delta(m-l)}{\Delta_i l m - \delta(m-l)} \cdot \frac{b_i l + \delta}{l m + \delta} \right\}.$$
Theorem 1.2. Let $B_1$ and $B_2$ be sets of positive integers, and take $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

1. If $\tau(B_1)\tau(B_2) \geq 1$, then $\epsilon_1 F(B_1) + \epsilon_2 F(B_2) = R$.

2. If $\tau(B_1)\tau(B_2) < 1$, then

$$\dim_H(\epsilon_1 F(B_1) + \epsilon_2 F(B_2)) \geq \frac{\log 2}{\log \left(2 + \frac{1 - \tau(B_1)\tau(B_2)}{\tau(B_1) + \tau(B_2) + 2\tau(B_1)\tau(B_2)}\right)}.$$

For example, we have

$$\dim_H(F(2) + F(2)) \geq 0.658 \ldots .$$

The positive results in Theorem 1.1 follow in part from Theorem 1.2. In addition we mention the following corollaries.

Corollary 1.3. If $l$ and $m$ are positive integers with $m \geq 2l$, then

$$F(m) + G(l) = R.$$

Corollary 1.4. Let $B_o$ denote the set of positive odd integers. Then

$$F(B_o) + F(B_o) = R.$$

Furthermore, if $B$ is a finite set of odd positive integers, then

$$F(B) + F(B) \neq R.$$

Note that if $B(m)$ is the set of positive odd integers less than $m$, then $\tau(B(m))$ approaches one as $m$ tends to infinity. Thus part 1 of Theorem 1.2 is tight in the sense that we cannot replace 1 by any smaller number.

Diviš [4] and Hlavka [6] also developed techniques that allowed them to examine the sum of more than two $F(m)$’s. Diviš showed that

$$F(3) + F(3) + F(3) = R \quad \text{and} \quad F(2) + F(2) + F(2) + F(2) = R,$$

while

$$F(2) + F(2) + F(2) \neq R.$$

Hlavka proved that

$$F(l) + F(m) + F(n) = R$$

holds if $(l, m, n)$ equals $(2, 2, 4)$ or $(2, 3, 3)$ but does not hold for $(l, m, n)$ equal to $(2, 2, 3)$. Together with the work on sums of two $F(m)$’s, these results allow us to determine those finite sets of positive integers $\{m_1, \ldots, m_k\}$ for which

$$\sum_{j=1}^{k} F(m_j) = R.$$

In the case of sums of integers with large partial quotients, Tom Cusick and Robert Lee [3] showed in 1971 that

$$lG(l) = R$$

for every positive integer $l$. We shall extend this to the case where the summands are unequal.

Theorem 1.5. If $l_1, l_2, \ldots, l_k$ are positive integer with

$$\sum_{j=1}^{k} \frac{1}{l_j} \geq 1,$$
then
\[ G(l_1) + \cdots + G(l_k) = \mathbb{R}. \]

Note that if \( l \) is a positive integer and we set \( k = l \) and \( l_1 = l_2 = \cdots = l_k = l \), then we recover the result of Cusick and Lee.

For a non-empty set of positive integers \( B \) we define \( \gamma(B) \) by
\[ \gamma(B) = \frac{\tau(B)}{\tau(B) + 1}. \]

Theorem 1.5 is a consequence of the following general theorem.

**Theorem 1.6.** Let \( k \) be a positive integer and \( B_1, \ldots, B_k \) non-empty sets of positive integers. Let \( \epsilon_j \in \{1, -1\} \) for \( j = 1, \ldots, k \). If
\[ \sum_{j=1}^{k} \gamma(B_j) \geq 1, \]
then
\[ \epsilon_1 F(B_1) + \cdots + \epsilon_k F(B_k) = \mathbb{R}. \] (5)

Otherwise
\[ \dim_H(\epsilon_1 F(B_1) + \cdots + \epsilon_k F(B_k)) \geq \frac{\log 2}{\log \left(1 + \frac{1}{\gamma(B_1) + \cdots + \gamma(B_k)}\right)}. \] (6)

Hall [5] and Cusick [2] also examined products of numbers with bounded partial quotients. For sets \( A \) and \( B \) of real numbers, we define the product of \( A \) and \( B \) by
\[ AB = A \cdot B = \{ab; a \in A \text{ and } b \in B\} \]
and \( A^{-1} \) by
\[ A^{-1} = \{1/a; a \in A \text{ and } a \neq 0\}. \]

We also denote by \( A/B \) the set \( A \cdot (B^{-1}) \). Hall proved that
\[ [1, \infty) \subseteq F(4) \cdot F(4), \] (7)
while Cusick established that
\[ [1, \infty) \subseteq G(2) \cdot G(2). \] (8)

We shall derive the following multiplicative analogue of Theorem 1.6.

**Theorem 1.7.** Let \( k \) be a positive integer. For \( j = 1, \ldots, k \) let \( B_j \) be a set of positive integers and let \( \epsilon_j \in \{1, -1\} \). Set
\[ S_\gamma = \gamma(B_1) + \cdots + \gamma(B_k), \]
\[ S_\epsilon = \epsilon_1 + \cdots + \epsilon_k \]
and
\[ F = F(B_1)^{\epsilon_1} F(B_2)^{\epsilon_2} \cdots F(B_k)^{\epsilon_k}. \]

1. If \( S_\gamma > 1 \) and \( S_\epsilon = k \), then there exists a positive real number \( c_1 \) such that
\[ F \supseteq (-\infty, -c_1] \cup [c_1, \infty). \]
2. If $S_\gamma > 1$ and $|S_\epsilon| < k$, then
\[ F \supseteq (\infty, 0) \cup (0, \infty). \]

3. If $S_\gamma > 1$ and there exists $r$ such that $|B_r| = \infty$ and $\epsilon_r = 1$, then
\[ F = \mathbb{R}. \]

4. If $S_\gamma = 1$ and $S_\epsilon = k$, then there exists a positive real number $c_2$ such that
\[ F \supseteq (c_2, \infty). \]

5. If $S_\gamma = 1$ and $|S_\epsilon| < k$, then $F$ omits at most $k$ points of $(0, \infty)$.

6. If $S_\gamma = 1$ and there exists $r$ such that $|B_r| = \infty$, $\epsilon_r = 1$ and $\Delta_i(B_r)$ is constant, then $F$ omits at most $2k$ real numbers.

7. If $S_\gamma < 1$, then
\[ \dim_H F \geq \frac{\log 2}{\log \left(1 + \frac{1}{S_\gamma}\right)}. \]

For particular choices of $B_j$, we can calculate $c_1$ and $c_2$ in the above theorem explicitly. If we denote by $\langle a_1, a_2, \ldots \rangle$ the continued fraction $[0, a_1, a_2, \ldots]$, then we have the following improvement to (7).

**Theorem 1.8.** Define $\alpha_1$ and $\alpha_2$ by
\[ \alpha_1 = (1 - \langle 1, 3 \rangle)(4, 1) = 0.0432 \ldots \]
and
\[ \alpha_2 = (1 - \langle 1, 3 \rangle)(1 - \langle 1, 4 \rangle) = 0.0358 \ldots. \]

Then
\[ F(3) \cdot F(4) = (\infty, -\alpha_1] \cup [\alpha_2, \infty). \]

We may also strengthen and generalize (8).

**Theorem 1.9.** If $k$ and $l_1, l_2, \ldots, l_k$ are positive integers with
\[ \sum_{j=1}^{k} \frac{1}{l_j} \geq 1, \]
then
\[ G(l_1) \cdots G(l_k) = \mathbb{R}. \]

As in the works of Hall, Cusick and Lee, Diviš, and Hlavka, our results hinge on the study of certain Cantor sets. For any set of positive integers $B$ we define the set $C(B)$ by
\[ C(B) = \{ \langle a_1, a_2, \ldots \rangle ; a_i \in B \text{ for every } i \}, \]
where numbers with a finite continued fraction expansion are included in $C(B)$ if and only if $B$ is an infinite set. We shall show that the sets $C(B)$ may be viewed as Cantor sets. We then derive results on sums and products of Cantor sets to prove our results.
2. Cantor Sets

Let $T$ be a connected directed graph. We say that $T$ is a tree if every vertex $V$ of $T$ has at most one edge terminating at $V$, and one vertex $V_R$ has no edges terminating at $V_R$. We call $V_R$ the root of $T$. If there is an edge connecting $V_1$ to $V_2$, then we say that $V_2$ is a subvertex of $V_1$. A vertex with no subvertices is called a leaf. A tree where each vertex has at most $t$ subvertices is called a tree of valence $t$. We will show that our Cantor sets can be represented by trees of valence 2.

Let $A$ be a closed interval of the real line and let $O \subseteq A$ be an open interval. Then

$$A = A^0 \cup O \cup A^1$$

for some closed intervals $A^0$ and $A^1$. We set

$$C^0 = A \quad \text{and} \quad C^1 = A^0 \cup A^1.$$ 

If $O^0$ and $O^1$ are open intervals contained in $A^0$ and $A^1$ respectively, then we have

$$A^0 = A^{00} \cup O^0 \cup A^{01} \quad \text{and} \quad A^1 = A^{10} \cup O^1 \cup A^{11}$$

for some closed intervals $A^{00}$, $A^{01}$, $A^{10}$, and $A^{11}$. We set

$$C^2 = A^{00} \cup A^{01} \cup A^{10} \cup A^{11}.$$ 

We continue this process, forming $C^{j+1}$ from $C^j$ by removing an open interval from each closed interval in the union which comprises $C^j$. We form a tree $D$ with root $A$ as follows. Let the vertices of the tree be the closed intervals $A^w$, for $w$ a finite binary word, and form directed edges joining $A^w$ to $A^{w0}$ and $A^{w1}$. If we define $C$ by

$$C = \bigcap_{j=0}^{\infty} C^j,$$

then we call $D$ a derivation of $C$, and call $C$ the Cantor set derived from $A$ by $D$. The $A^w$’s are called the bridges of $D$. If $A^w$ is a bridge of $D$, then we say that $A^w$ splits as

$$A^w = A^{w0} \cup O^w \cup A^{w1}.$$ 

We extend our definitions of Cantor sets and derivations by allowing the derivation to contain vertices which do not split. Let $A^w$ be such a vertex. We place under $A^w$ the vertex $A^{w0}$, where $A^{w0} = A^w$ as intervals. Thus our derivation may contain infinite stalks, and will be of valence 2. We also allow bridges to split as $A = A^0 \cup O \cup A^1$, where $O = \emptyset$ and $A^0 \cap A^1$ consists of only one point.

Note that the derivation $D$ of a Cantor set $C$ is not uniquely determined by $C$; for example, if we change the order in which the open intervals are removed then we get a different derivation but the same Cantor set.

We denote the length of an interval $I$ by $|I|$. We say that a derivation $D$ is ordered if for any bridges $A$ and $B$ of $D$ with $A = A^0 \cup O \cup A^1$, $B = B^0 \cup O_B \cup B^1$ and $B \subseteq A$ we have $|O| \geq |O_B|$. We define the $t$th level of $D$ to be the set of all vertices $A^w$ in $D$ where $w$ is a binary word of length $t$.

Cantor sets arise in the study of real numbers whose partial quotients are members of a given set. Let $B = \{b_1, b_2, \ldots, b_t\}$ be a finite set of positive integers with
If, for $t \geq 2$ and $b_1 < \cdots < b_t$, we set $l = l(B) = \min B = b_1$, $m = m(B) = \max B = b_t$, $C(B) = \mathcal{F}(B) \cap [0,1] = \{(a_1,a_2,\ldots) : a_i \in B \text{ for } i = 1,2,\ldots\}$, and let $I(B)$ be the closed interval $I(B) = [(\overline{m},l), (l,\overline{m})]$. We have $C(B) \subseteq I(B)$. We now inductively construct a derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$. For any real $a$ and $b$, we denote by $[a,b]$ and $(a,b)$ the intervals $[a,b] = [\min\{a,b\}, \max\{a,b\}]$ and $(a,b) = (\min\{a,b\}, \max\{a,b\})$.

If, for $i < t$, (9) $A = [(\langle a_1,\ldots,a_r,b_i,\overline{m},l \rangle, \langle a_1,\ldots,a_r,m,\overline{m} \rangle)]$ is a bridge of $\mathcal{D}(B)$ of level $n$, then we form the subvertices of $A$ by setting $A^0 = [(\langle a_1,\ldots,a_r,b_i,\overline{m},l \rangle, \langle a_1,\ldots,a_r,b_i,\overline{m} \rangle)]$, $O = (\langle a_1,\ldots,a_r,b_i,\overline{m},l \rangle, \langle a_1,\ldots,a_r,b_i+1,\overline{m},l \rangle)$ and $A^1 = [(\langle a_1,\ldots,a_r,b_{i+1},\overline{m},l \rangle, \langle a_1,\ldots,a_r,m,\overline{m} \rangle)]$.

In this manner we construct the $(n+1)^{th}$ level of the derivation from the $n^{th}$ level. Note that $A^0$ is of the form (9) with $a_r+1 = b_i$ and $b_i$ replaced by $l$. Similarly $A^1$ is also of the form (9). Since $I(B)$ is of the form (9) with $r = 0$ and $i = 1$, by induction we obtain the canonical derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$.

If $B$ is an infinite set, then we may construct a similar derivation. Assume that $B = \{b_1,\ldots\}$ with $b_i < b_{i+1}$ for $i \geq 1$. If we set $l = l(B) = \min B = b_1$, then we have $C(B) \subseteq I(B)$, where $I(B) = [0,1/l]$ and

$C(B) = \{(a_1,a_2,\ldots) : a_i \in B \text{ for } i \geq 1\}$

$\cup \{(a_1,a_2,\ldots,a_k) : k \in \mathbb{Z}, k \geq 0 \text{ and } a_i \in B \text{ for } 1 \leq i \leq k\}$.

If (10) $A = [(\langle a_1,\ldots,a_r,b_i \rangle, \langle a_1,\ldots,a_r \rangle)]$ is a bridge, then we split $A$ by setting $A^0 = [(\langle a_1,\ldots,a_r,b_i \rangle, \langle a_1,\ldots,a_r,b_i,\overline{l} \rangle)]$, $O = (\langle a_1,\ldots,a_r,b_i,\overline{l} \rangle, \langle a_1,\ldots,a_r,b_{i+1} \rangle)$ and $A^1 = [(\langle a_1,\ldots,a_r,b_{i+1} \rangle, \langle a_1,\ldots,a_r \rangle)]$ where by convention we set $\langle a_1,\ldots,a_r \rangle = 0$ if $r = 0$. As above, we construct the canonical derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$ using this process.
For given sets of integers $B_j$, $j = 1, \ldots, k$, we would like to be able to determine if

$$
\sum_{j=1}^{k} C(B_j) = \sum_{j=1}^{k} I(B_j).
$$

To do this we shall derive criteria on general Cantor sets that guarantee (11) holds. Our conditions will be less stringent than those derived previously. Let $C$ be a Cantor set with derivation $D$, and let $A^w$ be a bridge of $D$. We define the thickness of $A^w$ with respect to $D$, denoted by $\tau_D(A^w)$, to be $+\infty$ if $A^w$ does not split. Otherwise we set

$$
\tau(A^w) = \tau_D(A^w) = \min\left\{ \frac{|A^w_0|}{|O^w_1|}, \frac{|A^w_1|}{|O^w_0|} \right\},
$$

where throughout this paper we adopt the convention that $x/0 = +\infty$ for any $x > 0$.

We define the thickness $\tau(D)$ of the derivation $D$ by

$$
\tau(D) = \inf_{A^w} \tau_D(A^w),
$$

where the infimum is taken over all bridges $A^w$ of $D$. We also define $\tau(C)$, the thickness of the Cantor set $C$, by

$$
\tau(C) = \sup_D \tau(D),
$$

where the supremum is taken over all derivations $D$ of $C$. An equivalent definition of $\tau(C)$ may be found in [9], p. 61. It follows from Lemma 3.1 that $\tau(C) = \tau(D_o)$, where $D_o$ is any ordered derivation of $C$. The following observation is trivial yet crucial in our use of thickness.

**Lemma 2.1.** Let $C$ be a Cantor set. Then $C$ is an interval if and only if $\tau(C) = +\infty$.

**Proof.** Let $C$ be derived from $I$ and take $D$ to be any ordered derivation of $C$ from $I$. By Lemma 3.1 we have $\tau(C) = \tau(D)$. If $C \neq I$ then $I$, the root of $D$, must split in $D$ with a nontrivial gap, so

$$
\tau(D) \leq \tau_D(I) < +\infty.
$$

If, on the other hand, $C = I$, then any bridges of $D$ which split do so with a trivial gap, so that $\tau(D) = +\infty$, as required. \qed

For sums of two Cantor sets we shall prove the following result.

**Theorem 2.2.** For $j = 1, 2$ let $C_j$ be a Cantor set derived from $I_j$, with $O_j$ a gap of maximal size in $C_j$. Assume that

$$
|O_1| \leq |I_2| \quad \text{and} \quad |O_2| \leq |I_1|.
$$

1. If $\tau(C_1)\tau(C_2) \geq 1$, then $C_1 + C_2 = I_1 + I_2$.
2. If $\tau(C_1)\tau(C_2) < 1$, then

$$
\tau(C_1 + C_2) \geq \frac{\tau(C_1) + \tau(C_2) + 2\tau(C_1)\tau(C_2)}{1 - \tau(C_1)\tau(C_2)}.
$$

Part 1 of Theorem 2.2 may be derived from work of Sheldon Newhouse; our approach will give an alternative proof. Newhouse [8] established the following result.
Theorem 2.3. Let \( K_1 \) and \( K_2 \) be Cantor sets derived from \( I_1 \) and \( I_2 \) respectively, with \( \tau(K_1)\tau(K_2) > 1 \). Then either \( I_1 \cap I_2 = \emptyset \), \( K_1 \) is contained in a gap of \( K_2 \), \( K_2 \) is contained in a gap of \( K_1 \) or \( K_1 \cap K_2 \neq \emptyset \).

In fact, if Newhouse’s proof is slightly altered then we may replace the condition “\( \tau(K_1)\tau(K_2) > 1 \)” in Theorem 2.3 by the weaker condition “\( \tau(K_1)\tau(K_2) \geq 1 \).”

To see that Part 1 of Theorem 2.2 follow from this modified version of Theorem 2.3, we assume that \( \tau(C_1)\tau(C_2) \geq 1 \) and let \( k \) be any number in \( I_1 + I_2 \). Upon applying Theorem 2.3 (modified) with \( K_1 = k - C_1 \) and \( K_2 = C_2 \) we find that \( (k - C_1) \cap C_2 \neq \emptyset \) and hence \( k \in C_1 + C_2 \).

If \( C \) is a Cantor set, then we define \( \gamma(C) \) by

\[
\gamma(C) = \frac{\tau(C)}{\tau(C) + 1}.
\]

Theorem 2.2 is a special case of the following theorem.

Theorem 2.4. Let \( k \) be a positive integer and for \( j = 1, 2, \ldots, k \) let \( C_j \) be a Cantor set derived from \( I_j \), with \( O_j \) a gap of maximal size in \( C_j \). Let \( S_\gamma = \gamma(C_1) + \cdots + \gamma(C_k) \).

1. If \( S_\gamma \geq 1 \) then \( C_1 + \cdots + C_k \) contains an interval. Otherwise \( C_1 + \cdots + C_k \) contains a Cantor set of thickness at least \( \frac{S_\gamma}{1 - S_\gamma} \).

Furthermore,

\[
\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log \left(1 + \frac{1}{\min(S_\gamma, 1)}\right)}.
\]

2. If

\[
|I_{r+1}| \geq |O_j| \quad \text{for } r = 1, \ldots, k - 1 \text{ and } j = 1, \ldots, r,
\]

\[
|I_1| + \cdots + |I_r| \geq |O_{r+1}| \quad \text{for } r = 1, \ldots, k - 1,
\]

and \( S_\gamma \geq 1 \), then

\[
C_1 + \cdots + C_k = I_1 + \cdots + I_k.
\]

3. If (12) and (13) hold and \( S_\gamma < 1 \), then

\[
\tau(C_1 + \cdots + C_k) \geq \frac{S_\gamma}{1 - S_\gamma}.
\]

Theorem 2.4 is best possible in the sense that the condition \( S_\gamma \geq 1 \) in part 1 or part 2 cannot be replaced by \( S_\gamma \geq \eta \) for any \( \eta < 1 \). Similarly, if we multiply the bound for the thickness or the Hausdorff dimension of the sum by \( 1 + \delta \) for any \( \delta > 0 \), then the results do not hold in general.

3. Proof of Theorem 2.4

To prove Theorem 2.4 we require several lemmas.

Lemma 3.1. Let \( D \) be any derivation of \( C \) from \( I \). Then there exists an ordered derivation \( D_0 \) of \( C \) from \( I \) with

\[
\tau(D) \leq \tau(D_0).
\]
Furthermore, if \( D_1 \) and \( D_2 \) are two ordered derivations of \( C \) from \( I \), then
\[
\tau(D_1) = \tau(D_2).
\]

Let \( D \) be a derivation of \( C \) from \( I \), and assume that \( D \) is not ordered. Then there exists a bridge \( A \) of \( D \) which splits as \( A = A^0 \cup O \cup A^1 \), with \( A^0 \) and \( A^1 \) splitting as \( A^0 = A^{00} \cup O^0 \cup A^{01} \) and \( A^1 = A^{10} \cup O^1 \cup A^{11} \) respectively, such that either \( |O^0| > |O| \) or \( |O^1| > |O| \). Assume without loss of generality that \( |O^0| > |O| \). Consider the derivation \( D_s \) which is identical to \( D \) except that the positions of \( O^0 \) and \( O \) in the tree have been switched, that is, \( O^0 \) is removed before \( O \). If we set \( A_s = A \) and
\[
\begin{align*}
A^0_s &= A^{00}, \quad O_s = O^0, \quad A^1_s = A^{01} \cup O \cup A^1, \\
A^1_s &= A^{10}, \quad O^1 = O, \quad A^{11}_s = A^1,
\end{align*}
\]
then in \( D_s \), \( A = A_s \) splits as \( A_s = A^0_s \cup O_s \cup A^1_s \) and \( A^1_s \) splits as \( A^1_s = A^{10}_s \cup O^1_s \cup A^{11}_s \).

We claim that
\[
\tau(D) \leq \tau(D_s).
\]
To prove (14) it suffices to show that
\[
\min \left\{ \frac{|A^0|}{|O|}, \frac{|A^1|}{|O|}, \frac{|A^{00}|}{|O^0|}, \frac{|A^{10}|}{|O^1|} \right\} \leq \min \left\{ \frac{|A^0_s|}{|O_s|}, \frac{|A^1_s|}{|O_s|}, \frac{|A^{01}|}{|O^1_s|}, \frac{|A^{11}|}{|O^1_s|} \right\}.
\]
Now,
\[
\begin{align*}
\frac{|A^0_s|}{|O_s|} &= \frac{|A^{00}|}{|O^0|}, \quad \frac{|A^1_s|}{|O_s|} = \frac{|A^{01} \cup O \cup A^1|}{|O^0|}, \quad \frac{|A^{10}|}{|O^1_s|} = \frac{|A^1|}{|O^1|}, \quad \frac{|A^{11}|}{|O^1_s|} = \frac{|A^1|}{|O^1|},
\end{align*}
\]
since \( |O| < |O^1| \), and so (15) holds.

We construct our ordered derivation \( D_o \) as follows. First we modify \( D \) to form a new tree \( D^1 \) with the property that the first open interval removed is of maximal size. We form this tree by switching (a finite number of times) the order in which open intervals are removed in \( D \), as outlined above. Next we perform the same process on the bridges of level 1 in \( D^1 \), forming a new derivation \( D^2 \) which has its first two levels ordered. We continue this procedure inductively, forming \( D^{n+1} \) from \( D^n \) by switching the order in which open intervals are removed until for every bridge \( A \) of level \( n \) in \( D^{n+1} \), the next open interval removed from \( A \) is of maximal size. Our ordered derivation \( D_o \) is the derivation with the same root as \( D \) and for which the \( n \)th level of \( D_o \) consists of the same bridges as the \( n \)th level of \( D^n \).

We will use (14) to prove the first part of our lemma. For \( k \in \mathbb{Z}^+ \) let \( \mathcal{O}_k \) be the set of all gaps between intervals in the \( k \)th level of the derivation \( D_o \). Let \( n_k \) be the minimal number of levels of \( D \) we must descend before all intervals in \( \mathcal{O}_k \) have been removed. Further, let \( D^k \) consist of all bridges occurring in the first \( k \) levels of \( D_o \), and let \( D^{n_k} \) denote the set of all bridges occurring in the first \( n_k \) levels of the derivation \( D \). Then for every \( k \in \mathbb{Z}^+ \) we have
\[
\tau(D) \leq \min_{A \in D^{n_k}} \tau_D(A) \leq \min_{A \in D^k} \tau_{D_o}(A)
\]
by a finite number of applications of (14). Thus
\[
\tau(D_o) = \inf_k \min_{A \in D^k} \tau_{D_o}(A) \geq \tau(D),
\]
as required.
Now assume that $D_1$ and $D_2$ are two ordered derivations of $C$ from $I$. Let $(t_j)_j$ be the sequence of different lengths of open intervals removed in the derivations, in decreasing order (note that both derivations remove the same set of intervals). If no intervals are removed, then $C = I$ and

$$\tau(D_1) = \infty = \tau(D_2).$$

Otherwise for every $j$ let $B_j$ be a bridge of minimal width in $D_1$ such that, in the notation of section 2, $B_j = A^{wd}$ for some binary word $w$ and $d \in \{0, 1\}$, with $|O^w| = t_j$. Then

$$\tau(D_1) = \inf_j \frac{|B_j|}{t_j}.$$

However, $B_j$ satisfies the same condition with $D_1$ replaced by $D_2$, whence

$$\tau(D_2) = \inf_j \frac{|B_j|}{t_j},$$

and the lemma follows.

**Lemma 3.2.** Let $C_1$ and $C_2$ be Cantor sets derived by derivations $D_1$ and $D_2$ respectively. Put

$$\tau_1 = \tau(D_1) \quad \text{and} \quad \tau_2 = \tau(D_2).$$

If both $\tau_1$ and $\tau_2$ are greater than zero and neither $C_1$ nor $C_2$ contains an interval, then there exist bridges $A$ and $B$ of $D_1$ and $D_2$ respectively which split as

$$A = A^0 \cup O_1 \cup A^1 \quad \text{and} \quad B = B^0 \cup O_2 \cup B^1$$

such that

$$|A| \geq \frac{\tau_1}{\tau_1 + 1} (\tau_2 + 1)|O_2| \quad \text{and} \quad |B| \geq \frac{\tau_2}{\tau_2 + 1} (\tau_1 + 1)|O_1|.$$

**Proof.** Let $S = (A_i)_{i=1}^{\infty}$ be a sequence of bridges of $D_1$, where if $D_1$ contains a bridge of width $t$ then $|A_i| = t$ for some $i$, and $|A_i| > |A_{i+1}|$ for $i \geq 1$. Since $C_1$ does not contain an interval, all $A_i$ split, and $|A_i|$ tends to zero as $i$ increases. We define the sequence $(B_j)_{j=1}^{\infty}$ from $D_2$ in a similar manner. If $O^1_1$ and $O^2_2$ are the open intervals removed when $A_i$ and $B_j$ split, then

$$|O^i_1| \leq \frac{|A_i|}{2\tau_1 + 1} \quad \text{and} \quad |O^j_2| \leq \frac{|B_j|}{2\tau_2 + 1}$$

for $i, j \geq 1$. Therefore to prove the lemma it suffices to exhibit $A_r$ and $B_s$ with

$$\frac{\tau_2 + 1}{2\tau_2 + 1} \frac{\tau_1}{\tau_1 + 1} \leq \frac{|A_r|}{|B_s|} \leq \frac{2\tau_1 + 1}{\tau_1 + 1} \frac{\tau_2 + 1}{\tau_2}.$$

or, equivalently,

$$\frac{\tau_2}{2\tau_2 + 1} |B_s| \leq \frac{\tau_1 + 1}{\tau_1} \frac{\tau_2}{\tau_2 + 1} |A_r| \leq \frac{2\tau_1 + 1}{\tau_1} |B_s|.$$
By (16), for $j \geq 1$ we have
\[
\max\{|B_j^0|, |B_j^1|\} \geq \frac{1}{2} \left( |B_j| - |O_j^1| \right) \\
\geq \frac{1}{2} \left( |B_j| - \frac{|B_j|}{2\tau_2 + 1} \right) \\
= \frac{\tau_2}{2\tau_2 + 1}|B_j|.
\]
Hence
\[
|B_{j+1}| \geq \frac{\tau_2}{2\tau_2 + 1} |B_j| \quad \text{for } j \geq 1.
\]
Therefore
\[
\frac{2\tau_1 + 1}{\tau_1} |B_j| \geq |B_j| \quad \text{and} \quad \frac{\tau_2}{2\tau_2 + 1} |B_j| \leq |B_{j+1}|,
\]
so to establish (17) it is enough to find $r$ and $s$ such that
\[
|B_{r+1}| \leq \frac{\tau_1 + 1}{\tau_1} \frac{\tau_2}{2\tau_2 + 1} |A_r| \leq |B_s|.
\]
Since $\{|A_i|\}_i$ and $\{|B_j|\}_j$ are both sequences which are monotonically decreasing to zero and $\tau_2 \neq 0$, (18) must have a solution $(r, s)$. The lemma follows.

In the next proof we shall make use of the concept of compatibility of bridges, which is similar to an approach used by Hlavka ([6], Theorem 3).

**Lemma 3.3.** For $j = 1, 2$ let $C_j$ be a Cantor set derived from $I_j$ with $O_j$ the largest gap in $C_j$. Let $S_\gamma = \gamma(C_1) + \gamma(C_2)$.

1. Let $\alpha'$ and $\beta'$ be any positive real numbers for which $\alpha'/\beta' = \tau(C_1)\tau(C_2)$, and put $\alpha = \min\{1, \alpha'\}$ and $\beta = \min\{1, \beta'\}$. If
\[
\beta|O_1| \leq |I_2| \quad \text{and} \quad \alpha|O_2| \leq |I_1|,
\]
then
\[
\tau(C_1 + C_2) \geq \min\left\{ \frac{\tau(C_1) + \beta}{1 - \beta}, \frac{\tau(C_2) + \alpha}{1 - \alpha} \right\}.
\]
2. If $|O_1| \leq |I_2|$, $|O_2| \leq |I_1|$ and $S_\gamma \geq 1$, then
\[
C_1 + C_2 = I_1 + I_2.
\]
3. If (19) holds with
\[
\alpha' = \gamma(C_1)(\tau(C_2) + 1), \quad \beta' = \gamma(C_2)(\tau(C_1) + 1)
\]
and $S_\gamma < 1$, then
\[
\tau(C_1 + C_2) \geq \frac{S_\gamma}{1 - S_\gamma}.
\]
4. If $S_\gamma \geq 1$ then $C_1 + C_2$ contains an interval. Otherwise $C_1 + C_2$ contains a Cantor set of thickness at least
\[
\frac{S_\gamma}{1 - S_\gamma}.
\]
Proof. We first prove part 1. Assume that (19) holds, and set
\[ \tau = \min \left\{ \frac{\tau(C_1) + \beta}{1 - \beta}, \frac{\tau(C_2) + \alpha}{1 - \alpha} \right\}. \]
We will show that \( \tau(C_1 + C_2) \geq \tau \). To do so we will construct a tree of valence 2 to represent \( C_1 + C_2 \). This tree might not be a derivation, since bridges of the tree may overlap. However, we will use this tree to construct a derivation of \( C_1 + C_2 \) with the required thickness.

We will construct our first tree inductively, by setting the root to be \( I_1 + I_2 \) and showing how each bridge in the tree splits. Let \( C_1 \) and \( C_2 \) be ordered derivations of \( C_1 \) and \( C_2 \) respectively. If \( A \) and \( B \) are bridges of \( D_1 \) and \( D_2 \) respectively, we say that \( A \) and \( B \) are compatible, and write \( A \sim B \), if
\[ |A| \geq \alpha |O_2| \quad \text{and} \quad |B| \geq \beta |O_1|. \]
We will construct a derivation for \( C_1 + C_2 \) using the derivations of \( C_1 \) and \( C_2 \). Let \( A \) and \( B \) be bridges of \( D_1 \) and \( D_2 \) respectively with \( A \sim B \), and set \( D = A + B \).

Assume first that both \( A \) and \( B \) split. Then
\[ \min\{|A^0|, |A^1|\}, \min\{|B^0|, |B^1|\} \geq \tau(C_1) \tau(C_2) \geq \alpha \beta, \]
doing so
\[ \min\{|A^0|, |A^1|\} \min\{|B^0|, |B^1|\} \geq \alpha \beta |O_1| |O_2|. \]
Thus either
\[ \min\{|A^0|, |A^1|\} \geq \alpha |O_2|, \tag{20} \]
or
\[ \min\{|B^0|, |B^1|\} \geq \beta |O_1|. \tag{21} \]
Assume that (20) holds, and let \( O^0_1 \) and \( O^1_1 \) be the open intervals removed in the splitting of \( A^0 \) and \( A^1 \) respectively. Since the derivations are ordered,
\[ \beta |O^0_1| \leq \beta |O_1| \leq |B| \quad \text{and} \quad \beta |O^1_1| \leq \beta |O_1| \leq |B|, \]
as \( A \sim B \). By (20) we have
\[ \alpha |O_2| \leq |A^0| \quad \text{and} \quad \alpha |O_2| \leq |A^1|, \]
whence
\[ A^0 \sim B \quad \text{and} \quad A^1 \sim B. \]
We put
\[(22) \quad D^0 = A^0 + B, \quad D^1 = A^1 + B, \quad O_D = D \setminus (D^0 \cup D^1).\]
We have
\[|D^0| = |A^0| + |B|, \quad |D^1| = |A^1| + |B|\]
and, if \(O_D\) is non-empty,
\[|O_D| = |O_1| - |B|.\]
Note that if \(\beta = 1\) then \(O_D\) is necessarily empty. Thus either there is no gap between \(D^0\) and \(D^1\), or
\[(23) \quad \frac{\min\{|D^0|, |D^1|\}}{|O_D|} = \frac{\min\{|A^0|, |A^1|\} + |B|}{|O_1| - |B|} \geq \frac{\tau(C_1) + \beta}{1 - \beta} \geq \tau.\]
For \(d = 0, 1\), to determine the splitting of \(D^d\) we repeat the above process with \(A\) replaced with \(A^d\).
If we find that (21) holds instead of (20), we perform the same process, except we split \(B\) instead of \(A\). Again we may bound \(\min\{|D^0|, |D^1|\}/|O_D|\) by \(\tau\).
If \(A\) splits but \(B\) does not, then we define \(D^0, D^1\) and \(O_D\) as in (22), and find that either \(O_D\) is empty or (23) holds. If \(B\) splits but \(A\) does not, then we proceed in an analogous manner. Finally, if neither \(A\) nor \(B\) splits, then we let \(D\) be the vertex \(A + B\), and place under \(D\) an infinite stalk composed of vertices \(D^w\) where \(w\) is a binary word composed of zeros, and \(D^w = D\) as intervals.
Since \(I_1 \sim I_2\) we find by induction that we may construct a tree \(T_S\) of closed intervals \(\{D^w\}\) such that
\[C_1 + C_2 = \bigcap_{m \geq 0} \bigcup D^w,\]
where the union is taken over all binary words \(w\) of length \(m\) such that \(D^w\) is a vertex of \(T_S\). We further have that if \(V\) is a vertex of \(T_S\), then either \(V\) does not split or
\[(24) \quad \min\{|V^0|, |V^1|\} \geq \tau|O_V|.
Now \(T_S\) might not be a derivation of \(C_1 + C_2\), since we may have some overlap of intervals associated with vertices. We will however use \(T_S\) to construct a derivation for \(C_1 + C_2\) with the required thickness. Let \(H^0 = I_1 + I_2, H^1 = D^0 \cup D^1\) and in general
\[H^m = \bigcup_{w} D^w,\]
where the union is over all binary words of length \(m\) with \(D^w\) in \(T_S\). For each \(m\), \(H^m\) will be the union of a finite number of disjoint closed intervals \(\{H_i^m\}\). We next define a tree \(T_H\) by taking as vertices all intervals \(\{H_i^m\}\) and as edges all lines joining vertices \(H_i^m\) to \(H_j^{m+1}\), where \(H_j^{m+1} \subseteq H_i^m\) as sets. We will convert \(T_H\) into a tree where every vertex has at most two subvertices. Let \(N\) be a vertex of \(T_H\).
We will construct a finite tree \(T_N\) with root \(N\) and having as leaves the subvertices of \(N\) in \(T_H\) such that \(T_N\) is of valence 2 and \(T_N\) satisfies a condition similar to (24).
Let \(N\) have subvertices \(N_1, N_2, \ldots, N_t\) in \(T_H\). If \(t \leq 2\) then we let \(T_N\) be the tree with root \(N\) and leaves \(N_1, \ldots, N_t\). Otherwise, we have that, as intervals,
\[(25) \quad N = N_1 \cup G_1 \cup N_2 \cup G_2 \cup \cdots \cup G_{t-1} \cup N_t,\]
where $G_1, \ldots, G_{t-1}$ are open intervals. For intervals $J_1$ and $J_2$ we write $J_1 \rightarrow J_2$ if $|J_1| \geq \tau|J_2|$. We start by making the following claim.

**Claim 1.** Let $G_r$ and $G_s$ be two open intervals in (25) with $r < s$. Let $J$ denote the entire closed interval between $G_r$ and $G_s$. Then $J \rightarrow G_r$ or $J \rightarrow G_s$. Further, if $G_r$ is any open interval in (25), then

$$
(26) \quad \left( N_r \cup \bigcup_{1 \leq n < r} (N_n \cup G_n) \right) \rightarrow G_r
$$

and

$$
(27) \quad \left( N_t \cup \bigcup_{r < n < t} (N_n \cup G_n) \right) \rightarrow G_r
$$

**Proof of Claim 1.** Since $J$ contains points of $C_1 + C_2$ and $(C_1 + C_2) \cap (G_r \cup G_s) = \emptyset$, there exists a vertex $V = V_0 \cup O_V \cup V_1$ of $T_S$ with $V \cap J \neq \emptyset$ and either $G_r \subseteq O_V$ or $G_s \subseteq O_V$. Assume without loss of generality that $G_r \subseteq O_V$. If $V_1 \subseteq J$ then $|J| \geq |V_1| \geq \tau$, so $J \rightarrow G_r$. Otherwise $G_s \subseteq V_1$, and since $(C_1 + C_2) \cap G_s = \emptyset$ there exists a vertex $W = W_0 \cup O_W \cup W_1$ in $T_S$ with $W \subseteq V_1$ and $G_s \subseteq O_W$. In this case $W_0 \subseteq J$, so $|J| \geq |W_0| \geq \tau$. So $J \rightarrow G_s$, and the first part of the claim follows.

To prove the second part of the claim we denote by $J^0_r$ and $J^1_r$ the left sides of (26) and (27) respectively. As above, we have a vertex $V$ of $T_S$ with $V \subseteq J^0_r$, $V^1 \subseteq J^1_r$ and $G_r \subseteq O_V$, and the claim follows. 

By the claim we have

$$
N_1 \rightarrow G_1, \quad N_t \rightarrow G_{t-1}
$$

and

$$
N_j \rightarrow G_j \quad \text{or} \quad N_j \rightarrow G_j
$$

for $j = 2, \ldots, t-1$. For example,

$$
N_1 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_5.
$$

Thus there must be some $G_{r_1}$ with

$$
(28) \quad N_{r_1} \rightarrow G_{r_1} \quad \text{and} \quad N_{r_1+1} \rightarrow G_{r_1}.
$$

We set $t' = t - 1$,

$$
N'_j = \begin{cases} 
N_j & \text{if } 1 \leq j < r_1, \\
N_{r_1} \cup G_{r_1} \cup N_{r_1+1} & \text{if } j = r_1, \\
N_{j+1} & \text{if } r_1 < j \leq t',
\end{cases}
$$
and

\[ G'_j = \begin{cases} 
G_j & \text{if } 1 \leq j < r_1, \\
G_{j+1} & \text{if } r_1 \leq j \leq t'. 
\end{cases} \]

By the claim, \( N_{r_1}' \rightarrow G'_{r_1-1} \) or \( N_{r_1}' \rightarrow G'_{r_1} \), i.e.

\[ N_1' \rightarrow G'_1 \rightarrow N_2' \rightarrow G'_2 \rightarrow N_3' \rightarrow G'_3 \rightarrow N_4'. \]

We continue this process until we have only two closed intervals left. In our example, the next step results in

\[ N_1'' \rightarrow G''_1 \rightarrow N_2'' \rightarrow G''_2 \rightarrow N_3'' \rightarrow G''_3. \]

while the last step yields

\[ N_1''' \rightarrow G'''_1 \rightarrow N_2'''' \rightarrow G'''_2. \]

We are now ready to construct our finite tree \( T_N \). Let \( G'_{r_i} \) be the open interval satisfying (28) at the \( i \)th step, for \( i = 1, \ldots, t-2 \). Further, let \( G'_{r_{t-1}} \) be the open interval remaining when our process terminates. We form \( T_N \) by removing, in order, the open intervals \( G'_{r_{t-1}}, G'_{r_{t-2}}, \ldots, G'_{r_1} \):

By our construction, if \( N^w \) is a vertex in \( T_N \) which splits as

\[ N^w = N^{w_0} \cup O^w_N \cup N^{w_1}, \]

then

\[ |N^{w_0}| \geq \tau |O^w_N| \quad \text{and} \quad |N^{w_1}| \geq \tau |O^w_N|. \]

To construct our derivation \( D \) of \( C_1 + C_2 \), we take as vertices and edges of \( D \) the sets

\[ V = \bigcup_{N \in T_S} V(T_N) \quad \text{and} \quad E = \bigcup_{N \in T_S} E(T_N) \]

respectively, where for a tree \( T \) we denote the set of vertices of \( T \) by \( V(T) \) and the set of edges by \( E(T) \). We have \( \tau(D) \geq \tau \), and the first part of the lemma follows.

We will use part 1 of the lemma to prove parts 2 and 3. Let

\[ \alpha' = \gamma(C_1)(\tau(C_2) + 1) \quad \text{and} \quad \beta' = \gamma(C_2)(\tau(C_1) + 1) \]

and define \( \alpha \) and \( \beta \) by

\[ \alpha = \min\{1, \alpha'\} \quad \text{and} \quad \beta = \min\{1, \beta'\}. \]

Assume that \( |\mathcal{O}_1| \leq |\mathcal{I}_2|, |\mathcal{O}_2| \leq |\mathcal{I}_1| \) and \( S_\gamma \geq 1 \). Then \( \tau(C_1)\tau(C_2) \geq 1 \), which implies that \( \alpha = \beta = 1 \). Therefore, by part 1, \( \tau(C_1 + C_2) = \infty \), and part 2 follows.
To prove part 3 we first define $\alpha'$, $\beta'$, $\alpha$ and $\beta$ by (29) and (30). Note that if $S_\gamma < 1$ then $\alpha = \alpha'$ and $\beta = \beta'$; hence
\[
\frac{\tau(C_1) + \beta}{1 - \beta} = \frac{\tau(C_1) + \beta'}{1 - \beta'} = \frac{S_\gamma}{1 - S_\gamma}
\]
and
\[
\frac{\tau(C_2) + \alpha}{1 - \alpha} = \frac{\tau(C_2) + \alpha'}{1 - \alpha'} = \frac{S_\gamma}{1 - S_\gamma},
\]
so by part 1 of the lemma
\[
\tau(C_1 + C_2) \geq \frac{S_\gamma}{1 - S_\gamma},
\]
and part 3 follows.

To prove part 4 we first note that if $\tau(C_1) = 0$ or $\tau(C_2) = 0$ then the result follows trivially, whence we may assume $\tau(C_1)$ and $\tau(C_2)$ are both greater than zero. If either $C_1$ or $C_2$ contains a bridge that does not split, then $C_1 + C_2$ will contain an interval, hence a set of infinite thickness. Otherwise, by Lemma 3.2 there exist bridges $A$ and $B$ of $D_1$ and $D_2$ respectively, with
\[
A = A^0 \cup O_1 \cup A^1 \quad \text{and} \quad B = B^0 \cup O_2 \cup B^1,
\]
such that
\[
|A| \geq \alpha|O_2| \quad \text{and} \quad |B| \geq \beta|O_1|,
\]
where $\alpha$ and $\beta$ are as defined in (30). By parts 2 and 3 of Lemma 3.3 applied to the Cantor sets
\[
C_A = C_1 \cap A \quad \text{and} \quad C_B = C_2 \cap B
\]
we have
\[
C_A + C_B = A + B
\]
if $S_\gamma \geq 1$ and
\[
\tau(C_A + C_B) \geq \frac{S_\gamma}{1 - S_\gamma}
\]
otherwise, and part 4 of the lemma follows. \hfill \Box

To relate thickness to Hausdorff dimension we use the following result.

**Lemma 3.4.** If $C$ is a Cantor set, then
\[
\dim_H(C) \geq \frac{\log 2}{\log \left(2 + \frac{1}{\tau(C)}\right)}.
\]

*Proof.* See [9], p. 77. \hfill \Box

*Proof of Theorem 2.4.* For real numbers $\gamma_1$, $\gamma_2$ and $\gamma_3$ in $[0, 1]$ with $\gamma_1 + \gamma_2 < 1$ we put
\[
\tau_{12} = \frac{\gamma_1 + \gamma_2}{1 - \gamma_1 - \gamma_2}.
\]
Note that,
\[ \frac{\tau_12}{1 + \tau_12} = \gamma_1 + \gamma_2 \]
so
\[ \frac{\tau_12}{1 + \tau_12} + \gamma_3 = \gamma_1 + \gamma_2 + \gamma_3. \]

We first prove part 1. Assume \( S_\gamma \geq 1 \) and let \( t \) be the smallest integer with \( \gamma(C_1) + \cdots + \gamma(C_t) \geq 1 \). Using Lemma 3.3 (part 4) and (31), we find by induction that \( C_1 + \cdots + C_t \) contains an interval, whence \( C_1 + \cdots + C_k \) contains an interval.

If \( S_\gamma < 1 \) then we find by Lemma 3.3 (part 4), (31) and induction that \( C_1 + \cdots + C_k \) contains a Cantor set of thickness at least \( S_\gamma/(1 - S_\gamma) \), so by Lemma 3.4,
\[ \dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log(1 + \tau_12)} \]
and part 1 of the theorem follows.

To prove parts 2 and 3 we first note that by (12) and (13) the sets \( I_1 + \cdots + I_r \) and \( I_{r+1} \) satisfy (19) with \( \alpha = \beta = 1 \), for \( r = 1, \ldots, k - 1 \). We find by induction, Lemma 3.3 (part 2) and (31) that if \( S_\gamma \geq 1 \), then
\[ C_1 + \cdots + C_k = I_1 + \cdots + I_k, \]
and part 2 of the theorem follows. Similarly, if \( S_\gamma < 1 \), then by induction, Lemma 3.3 (part 3) and (31) we have
\[ \tau(C_1 + \cdots + C_k) \geq S_\gamma/(1 - S_\gamma), \]
and the theorem follows.

4. Bounds on the Thickness of \( C(B) \)

To apply Theorems 2.2 and 2.4 to the cases where the Cantor sets are of the form \( C(B_j) \) for some \( B_j \subseteq \mathbb{Z}_+ \), we need only calculate the thicknesses of the Cantor sets in question.

For \( n \geq 0 \) we define the \( n^{th} \) convergent to the continued fraction \([a_0, a_1, \ldots]\) to be the rational number
\[ \frac{p_n}{q_n} = [a_0, \ldots, a_n] \]
where \( p_n \) and \( q_n \) are taken to be coprime. We also define \( p_n \) and \( q_n \) for \( n = -2 \) or \( n = -1 \) by
\[ p_{-2} = q_{-1} = 0 \quad \text{and} \quad p_{-1} = q_{-2} = 1. \]

By elementary properties of continued fractions we have
\[ p_n = a_np_{n-1} + p_{n-2} \]
and
\[ q_n = a_nq_{n-1} + q_{n-2} \]
for \( n \geq 0 \). We also have the following result.

**Lemma 4.1.** For a fixed \( r \geq 0 \) and \( 1 \leq i \leq 4 \) assume that \( G_i = [a_0, a_1, \ldots, a_r, g_i] \) for some real \( g_i > 0 \). For \( 0 \leq n \leq r \) let \( p_n/q_n \) be the \( n^{th} \) convergent to \([a_0, \ldots, a_r] \), and put \( Q = q_{r-1}/q_r \). Then
and find that
\[
\frac{|A_1|}{|O|} = \frac{(m - b_{i+1})l + \delta(m - l)}{\Delta_i lm - \delta(m - l)} \cdot \frac{(b_i + Q)l + \delta}{(m + Q)l + \delta}.
\]
Thus
\[
\tau(D(B)) = \inf_Q \min_{1 \leq i < t} \min \left\{ \frac{\delta(m - l)}{\Delta_lm - \delta(m - l)}, \frac{(b_{i+1} + Q)m + \delta}{(b_i + Q)m + \delta}, \frac{(m - b_{i+1})m + \delta(m - l)}{\Delta_lm - \delta(m - l)}, \frac{(b_i + Q)l + \delta}{(m + Q)l + \delta} \right\}
\]

\[= \min_{1 \leq i < t} \min \left\{ \frac{\delta(m - l)}{\Delta_lm - \delta(m - l)}, \frac{b_{i+1}l + m + \delta l}{b_il + m + \delta l}, \frac{(m - b_{i+1})l + \delta(m - l)}{\Delta_lm - \delta(m - l)}, \frac{b_il + \delta}{l + \delta} \right\}
\]
since \(0 \leq Q \leq 1/l\), and
\[
\frac{q-1}{q_0} = 0 \quad \text{and} \quad \frac{q_0}{q_1} = \frac{1}{l}
\]
if \(a_1 = l\).

\[\square\]

A similar but simpler result holds in the infinite case.

**Lemma 4.3.** Let \(B = \{b_1, b_2, \ldots\} \) be an infinite set of integers with \(b_i < b_{i+1}\) for \(i \geq 1\). Let \(l = b_1\) and set \(\Delta_i = b_{i+1} - b_i\) for \(i \geq 1\). Then
\[
\tau(D(B)) = \inf_{i \geq 1} \min \left\{ \frac{1}{\Delta_il - 1}, \frac{b_{i+1}l + 1}{b_il + 1}, \frac{b_il + 1}{\Delta_il - 1} \right\}
\]

**Proof.** We use the same strategy as in the proof of Lemma 4.2. If \(A\) is a bridge of the form (10), then by part 2 of Lemma 4.1, with
\[
g_1 = [b_i, l], \quad g_2 = b_i, \quad g_3 = b_{i+1}, \quad g_4 = [b_i, l],
\]
we find that
\[
\frac{|A^0|}{|O|} = \frac{1}{\Delta_il - 1} \cdot \frac{b_{i+1} + Q}{b_i + Q}.
\]

Now if \(r = 0\) then
\[
\frac{|A^1|}{|O|} = \frac{b_il + 1}{\Delta_il - 1},
\]
while if \(r > 0\) we apply part 2 of Lemma 4.1 with
\[
g_1 = [a_r, b_{i+1}], \quad g_2 = a_r, \quad g_3 = [a_r, b_i, l] \quad \text{and} \quad g_4 = [a_r, b_{i+1}]
\]
and conclude that
\[
\frac{|A^1|}{|O|} = \frac{\langle b_{i+1} \rangle}{\langle b_i, l \rangle - \langle b_{i+1} \rangle} \cdot \frac{[a_r, b_i, l] + \frac{q_{r-2}}{q_{r-1}}}{a_r + \frac{q_{r-2}}{q_{r-1}}} = \frac{b_il + 1}{\Delta_il - 1} \cdot \frac{a_rq_{r-1} + q_{r-2} + q_{r-1}\langle b_i, l \rangle}{a_rq_{r-1} + q_{r-2}} = \frac{b_il + 1}{\Delta_il - 1} \cdot \frac{q_r + q_{r-1}\langle b_i, l \rangle}{q_r} = \frac{(b_i + Q)l + 1}{\Delta_il - 1} \geq \frac{b_il + 1}{\Delta_il - 1}
\]
by (32) and since \( Q \geq 0 \). Therefore

\[
\tau(D(B)) = \inf_Q \inf_i \min \left\{ \frac{1}{\Delta,i-1} \cdot \frac{b_i+1 + Q}{b_i} \cdot \frac{b_i + 1}{\Delta,i-1} \right\},
\]

and the lemma follows upon minimizing (35), since as in the proof of Lemma 4.2 we have \( 0 \leq Q \leq 1/l \) with \( Q = 0 \) if \( r = 0 \) and \( Q = 1/l \) if \( r = 1 \) and \( a_1 = l \).

Note that \( \tau(D(B)) \) equals \( \tau(B) \) (as defined in the first section).

**Lemma 4.4.** Let \( B \) be a set of positive integers with \( |B| > 1 \). If \( \Delta_i(B) = \Delta \) is constant, then \( D(B) \) is ordered, and so

\[
\tau(C(B)) = \tau(D(B)) = \tau(B).
\]

**Proof.** Assume first that \( B \) is finite. In the notation of Lemma 4.2 put

\[
O((a_1, \ldots, a_r), b_i) = (((a_1, \ldots, a_r, b_i, l, m), (a_1, \ldots, a_r, b_{i+1}, m, l)))
\]

for \( b_i < m \). Then by part 1 of Lemma 4.1 with

\[
g_1 = [b_{i+1}, m, l] \quad \text{and} \quad g_2 = [b_i, l, m]
\]

we have

\[
|O((a_1, \ldots, a_r), b_i)| = \frac{\Delta + \delta/m - \delta/l}{q^2(b_i+1 + \delta/m + Q)(b_i + \delta/l + Q)}.
\]

Thus

\[
|O((a_1, \ldots, a_r), b_i)| > |O((a_1, \ldots, a_r), b_j)|
\]

for \( j > i \), and

\[
|O((a_1, \ldots, a_r), b_i)| > |O((a_1, \ldots, a_r, b_i), b_j)|
\]

for \( b_j \in B \), so \( D(B) \) is an ordered derivation. By Lemma 3.1 we have \( \tau(C(B)) = \tau(D(B)) \), and Lemma 4.4 follows for \( B \) finite.

If \( B \) is infinite then we use an analogous approach, where in this case we define

\[
O((a_1, \ldots, a_r), b_i) = (((a_1, \ldots, a_r, b_i, l), (a_1, \ldots, a_r, b_{i+1}))).
\]

\[\square\]

5. **Proofs of Results in the Additive Case**

For \( n \) an integer and \( B \) a set of positive integers with \( |B| > 1 \) we define \( C(n; B) \) by

\[
C(n; B) = n + C(B).
\]

Using the derivation \( D(B) \) of \( C(B) \), we may construct the canonical derivation \( n + D(B) \) of \( C(n; B) \) from \( n + I(B) \) by translating every interval in \( D \) by \( n \). Similarly we may construct the canonical derivation \( n - D(B) \) of \( n - C(B) \) from \( n - I(B) \).

**Proof of Theorem 1.6.** Put

\[
S_\gamma = \sum_{j=1}^k \gamma(B_j).
\]
Assume first that \( S, \gamma \geq 1 \), and for \( N \geq 1 \) and \( j = 1, \ldots, k \) set
\[
C_j^N = \bigcup_{n=-N}^{N} C(n; B_j)
\]
and
\[
I_j^N = [-N + \min C(B_j), N + \max C(B_j)].
\]
For \( j = 1, \ldots, k \) we construct a derivation \( D_j^N \) of \( C_j^N \) from \( I_j^N \) as follows. Assume that \( |I(B_j)| < 1 \). Remove from \( I_j^N \) the interval \( (n + \max C(B_j), n + 1 + \min C(B_j)) \) for \( n = -N, \ldots, N - 1 \), so that if \( A_j = I_j^N \) then
\[
A_j^0 = -N + I(B_j), \quad A_j^1 = [-N + 1 + \min C(B_j), N + \max C(B_j)],
\]
and, ultimately,
\[
A_j^{1..10} = N - 1 + I(B_j) \quad \text{and} \quad A_j^{1..11} = N + I(B_j).
\]
We complete \( D_j^N \) by using the derivations \( n + D(B_j), n = -N, \ldots, N \), to split \( A_j^0, A_j^{10}, \ldots, A_j^{1..10} \) and \( A_j^{1..11} \). Note that if \( B_j \) is finite, then
\[
\frac{|n + I_j|}{|(n + \max C(B_j), n + 1 + \min C(B_j))|} = \frac{(\overline{l,m}) - (\overline{m,l})}{1 + (\overline{m,l}) - (\overline{l,m})} = \frac{\delta(m-l)}{ml - \delta(m-l)} > \tau(B_j),
\]
and that if \( B_j \) is infinite, then
\[
\frac{|n + I_j|}{|(n + \max C(B_j), n + 1 + \min C(B_j))|} = \frac{1}{l-1} \geq \tau(B_j),
\]
so in either case \( \tau(D_j^N) \geq \tau(B_j) \).

If \( |I_j| = 1 \), then we form \( D_j^N \) by removing the gaps in \( C_j^N \) in order of descending width, so that again we have \( \tau(D_j^N) \geq \tau(B_j) \).

For \( j = 1, \ldots, k \), \( I_j^N \) has width greater than 2, and all gaps in \( C_j^N \) are of width less than 1, whence (12) and (13) hold. Since \( S, \gamma \geq 1 \) and for a Cantor set \( C \) we have \( \tau(-C) = \tau(C) \), by part 2 of Theorem 2.4 we get
\[
\epsilon_1 C_1^N + \cdots + \epsilon_k C_k^N = \epsilon_1 I_1^N + \cdots + \epsilon_k I_k^N \supseteq [-k(N-1), k(N-1)],
\]
and (5) follows upon letting \( N \) tend to infinity.

If \( S, \gamma < 1 \), then (6) follows from part 1 of Theorem 2.4 with \( C_j = C(B_j) \) for \( j = 1, \ldots, k \).

**Proof of Theorem 1.2.** Theorem 1.2 is a special case of Theorem 1.6, since
\[
\gamma(B_1) + \gamma(B_2) \geq 1
\]
if and only if
\[
\tau(B_1) \tau(B_2) \geq 1,
\]
and, further,
\[
\frac{\gamma(B_1) + \gamma(B_2)}{1 - \gamma(B_1) - \gamma(B_2)} = \frac{\tau(B_1) + \tau(B_2) + 2\tau(B_1)\tau(B_2)}{1 - \tau(B_1)\tau(B_2)}.
\]

**Proof of Theorem 1.1.** As shown by Diviš [4], we have
\[
F(3) + F(3) \neq R.
\]
Also, Hlavka [6] established that
\[
F(2) + F(4) \neq R \quad \text{and} \quad F(3) + F(4) = R.
\]
Using Theorem 1.2, we find that
\[
F(3) - F(4) = R,
\]
and from work to appear [1] we have that
\[
F(2) + F(5) = F(2) - F(5) = F(3) - F(3) = R.
\]
Since
\[
I(L_2) - I(L_4) \subseteq [-0.462 \ldots, 0.524 \ldots],
\]
we know that
\[
F(2) - F(4) \neq R,
\]
and the theorem follows. \(\square\)

**Proof of Corollary 1.3.** Note that \(\delta(L_m) > m/(m + 1)\), whence
\[
\tau(L_m) > \frac{m(m - 1)}{m(m + 1) - m(m - 1)} \cdot \frac{(m - 1)(m + 1) + m}{m(m + 1) + m} = \frac{(m - 1)^2}{2m}.
\]
Since
\[
\tau(U_l) = \frac{1}{l - 1} \geq \frac{2}{m - 2},
\]
we have
\[
\tau(L_m)\tau(U_l) > \frac{(m - 1)^2}{m(m - 2)} > 1,
\]
and the result follows from part 1 of Theorem 1.2. \(\square\)

Before proving Corollary 1.4 we need a preliminary lemma.

**Lemma 5.1.** If \(B\) is a finite set of odd positive integers, then \(1 \notin 2C(B)\).

**Proof.** Let \(m = \max B\) and assume that \(1 \in 2C(B)\). Then \(1 \in S\), where
\[
S = [(a_1, a_2, m, \overline{1})], (a_1, a_2, \overline{1}, m)],\]
for some odd \(a_1, a_2, b_1\) and \(b_2\) between 1 and \(m\) inclusive. Now if both \(a_1\) and \(b_1\) are greater than 1 then 1 \(\notin S\), so we may assume without loss of generality that \(a_1 = 1\). Thus
\[
S = \left[\frac{a_2 + \theta}{a_2 + \theta + 1}, \frac{b_2 + \theta}{b_2 + (b_2 + \theta + 1)}, \frac{a_2 + \rho}{a_2 + \rho + 1}, \frac{b_2 + \rho}{b_2 + (b_2 + \rho + 1)}\right],
\]
where \( \theta = (m, 1) \) and \( \rho = (1, m) \). Therefore we have

\[
1 \geq \frac{a_2 + \theta}{a_2 + \theta + 1} + \frac{b_2 + \theta}{b_1(b_2 + \theta) + 1}
\]

and

\[
1 \leq \frac{a_2 + \rho}{a_2 + \rho + 1} + \frac{b_2 + \rho}{b_1(b_2 + \rho) + 1}.
\]

It can be shown that (39) and (40) are equivalent to

\[
a_2 + 1 - b_1 \leq \frac{1}{b_2 + \theta} - \theta
\]

and

\[
a_2 + 1 - b_1 \geq \frac{1}{b_2 + \rho} - \rho
\]

respectively. But for any integer \( n \geq 1 \) and real \( x \in (0, 1) \) we have

\[-1 < \frac{1}{n + x} - x < 1,
\]

and so by (41) and (42) we must have \( a_2 + 1 - b_1 = 0 \). But this is not possible, since both \( a_2 \) and \( b_1 \) are odd, and the lemma follows.

**Proof of Corollary 1.4.** We find that \( \tau(B_o) = 1 \), and so, by part 1 of Theorem 1.2,

\[F(B_o) + F(B_o) = \mathbb{R}.
\]

Now if \( B \) is a finite set of positive odd integers, then \( 0 \not\in 2C(B) \) and \( 2 \not\in 2C(B) \). By Lemma 5.1 we have \( 1 \not\in 2C(B) \), whence

\[1 \not\in \mathbb{Z} + 2C(B) = F(B) + F(B),
\]

and the result follows.

### 6. Products and Quotients

As in [5] and [2] we employ the logarithm function to treat products and quotients of Cantor sets. Given a set \( S \) of positive numbers, we form the set \( S^* \) by putting

\[S^* = \{\log x : x \in S\}.
\]

If \( C \) is Cantor set of positive numbers, then \( C^* \) will also be a Cantor set. We construct a derivation \( D^* \) of \( C^* \) by taking our bridges to be of the form \( [\log a, \log b] \), where \( [a, b] \) is a bridge of our derivation \( D \) of \( C \). To relate the Hausdorff dimension of \( S \) to that of \( S^* \) we will use Lemma 6.2.

**Lemma 6.1.** Let \( E \) be a set of real numbers with \( f : E \rightarrow \mathbb{R} \) such that for some positive constant \( c \),

\[|f(x) - f(y)| \leq c|x - y|
\]

for all \( x, y \in E \). Then

\[\dim_H(f(E)) \leq \dim_H E.
\]

**Proof.** See [7], p. 44.
Lemma 6.2. Let \( S \subseteq [a, b] \) be a set of real numbers with \( a > 0 \). Then
\[
\dim_H S = \dim_H (S^*) .
\]

Proof. For every \( x, y \in S \),
\[
\frac{1}{b} |x - y| \leq |\log x - \log y| \leq \frac{1}{a} |x - y| ,
\]
and the lemma follows from Lemma 6.1. \( \square \)

We have the following multiplicative analogue of Theorem 2.4.

Theorem 6.3. Let \( k \) be a positive integer and for \( j = 1, 2, \ldots, k \) let \( C_j \) be a Cantor set derived from \( I_j \subseteq (0, \infty) \), with \( O_j \) a gap in \( C_j \) chosen so that \( |O_j^*| \) is maximal.

Put \( S_\gamma = \gamma(C_1^*) + \cdots + \gamma(C_k^*) \).

1. If \( S_\gamma \geq 1 \), then \( C_1 \cdots C_k \) contains an interval.
2. If \( S_\gamma < 1 \), then
\[
\dim_H(C_1 \cdots C_k) \geq \frac{\log 2}{\log \left( 1 + \frac{1}{S_\gamma} \right)} .
\]
3. If
\[
|I_{r+1}^*| \geq |O_r^*| \quad \text{for } r = 1, \ldots, k - 1 \text{ and } j = 1, \ldots, r,
\]
\[
|I_1^*| + \cdots + |I_r^*| \geq |O_{r+1}^*| \quad \text{for } r = 1, \ldots, k - 1,
\]
and \( S_\gamma \geq 1 \), then
\[
C_1 \cdots C_k = I_1 \cdots I_k .
\]

Proof. Note that by Lemma 6.2,
\[
\dim_H(C_1 \cdots C_k) = \dim_H(C_1^* + \cdots + C_k^*) .
\]
We apply Theorem 2.4 to the Cantor sets \( C_1^*, \ldots, C_k^* \), and the theorem follows. \( \square \)

It remains to find a lower bound for \( \tau(C^*) \). We start by generalizing a lemma of Cusick ([2], Lemma 2).

Lemma 6.4. Let \( E = [a, b] \subseteq (0, \infty) \) be an interval of real numbers. Suppose that \( E = E_1 \cup O \cup E_2 \), where
\[
E_1 = [a, a + r] , \quad O = (a + r, a + r + s) \quad \text{and} \quad E_2 = [a + r + s, a + r + s + t] .
\]
If \( s < a + r \) and \( \tau > 0 \) is a real number such that
\[
\frac{t - \tau s}{s^2} \geq \frac{1}{a + r} \left( 1 + \sum_{2 \leq n < \tau + 1} \left( \frac{\tau + 1}{n} \right) \right) ,
\]
then
\[
\frac{|E_2^*|}{|O^*|} \geq \tau .
\]

Proof. We have \( |E_2^*| \geq \tau |O^*| \) if and only if
\[
\log(a + r + s + t) - \log(a + r + s) \geq \tau (\log(a + r + s) - \log(a + r)) ,
\]
which is equivalent to
\[
(a + r + s + t)(a + r)^\tau \geq (a + r + s)^{\tau + 1} .
\]
or, alternatively,
\begin{equation}
1 + \frac{s + t}{a + r} \geq \left(1 + \frac{s}{a + r}\right)^{\tau + 1}.
\end{equation}

Using the power series expansion for \((1 + x)^y\) for real \(y\) and \(|x| < 1\), we find that (43) is equivalent to
\begin{equation}
\frac{t - \tau s}{a + r} \geq \sum_{n=2}^{\infty} \left( \frac{\tau + 1}{n} \right) \left( \frac{s}{a + r} \right)^n.
\end{equation}

Let \(R\) be the unique positive integer such that \(R \leq \tau + 1 < R + 1\), and for \(n \geq 2\) let \(C_n\) denote the binomial coefficient in (44). If \(n > \tau + 1\), then
\(C_n = \left( \frac{\tau + 1}{n} \right) \frac{\tau}{n-1} \cdots \frac{\tau + 1 - R}{n-R} \frac{\tau - R}{n-R-1} \cdots \frac{\tau + 2 - n}{1}\).

Observe that \(\{C_n\}_{n \geq \tau + 1}\) is a non-increasing sequence. Further, \(C_n C_{n+1} \leq 0\) for \(n \geq \tau + 1\). Therefore
\begin{equation}
\left| \sum_{\tau + 1 \leq n \leq \tau + \frac{1}{2}} \left( \frac{\tau + 1}{n} \right) \left( \frac{s}{a + r} \right)^n \right| \leq \left( \frac{s}{a + r} \right)^2.
\end{equation}

The lemma follows from (44) and (45).

As a corollary we may find a bound for \(\tau(C^*)\).

**Corollary 6.5.** Let \(C \subseteq \mathbb{R}^+\) be a Cantor set derived by \(D\) and let \(\tau\) be a real number which is at most \(\tau(C)\). Assume that for all bridges \(A = [a, b]\) of \(D\) with \(A^c\) to the right of \(A^d\) (\(d, e \in \{0, 1\}\)) and
\begin{equation}
\frac{|A^c| - \tau |O|}{|O|^2} < \frac{1}{a} \left( 1 + \sum_{2 \leq n < \tau + 1} \left( \frac{\tau + 1}{n} \right) \right)
\end{equation}

it follows that
\begin{equation}
\frac{|A^c_x|}{|O^+|} \geq \tau.
\end{equation}

Then
\(\tau(C^*) \geq \tau\).

**Proof.** Let \(A = A^0 \cup O \cup A^1\) be a bridge of \(D\) with \(A^d\) to the left of \(A^c\), for \((d, e) = (1, 0)\) or \((d, e) = (0, 1)\). Since the logarithm function has decreasing slope, it follows that
\(\frac{|A^d^*|}{|O^+|} \geq \frac{|A^d|}{|O|} \geq \tau\).

If (46) does not hold, then (47) holds by Lemma 6.4, so in any case
\[\min \left\{ \frac{|A^0_x|}{|O^+|}, \frac{|A^1_x|}{|O^+|} \right\} \geq \tau,
\]
as required.

We may use Corollary 6.5 to find a bound for \(\tau((n \pm C(B))^*)\) for \(n\) sufficiently large.
Lemma 6.6. Let $B$ be a set of positive integers with $|B| > 1$.

1. There exists $M_1 \in \mathbb{Z}^+$ such that
   $$\tau(C(n; B)^*) \geq \tau(B)$$
   for all $n \geq M_1$.

2. If $\tau$ is a real number with $\tau < \tau(B)$, then there exists $M_2 \in \mathbb{Z}^+$ such that
   $$\tau((n - C(B))^*) \geq \tau$$
   for all $n \geq M_2$.

3. If $|B| = \infty$ and $\Delta_i(B) = \Delta$ is constant, then there exists $M_3 \in \mathbb{Z}^+$ such that
   $$\tau((n - C(B))^*) \geq \tau(B)$$
   for all $n \geq M_3$.

Proof. For positive real numbers $x$ we define

$$h(x) = 1 + \sum_{2 \leq t < x+1} \binom{x+1}{t}.$$  \hspace{1cm} (48)

Since the lemma holds trivially if $\tau(B) = 0$ or $\tau(B) = \infty$, we may assume that $0 < \tau(B) < \infty$. We first prove part 2. Assume that $\tau < \tau(B)$, say $\tau = \tau(B) - \eta$, where $\eta > 0$. Choose an integer $M_2$ such that

$$M_2 > \frac{h(\tau)}{\eta} + 1.$$ 

Let $n \geq M_2$ be an integer and let $D$ be the canonical derivation of $n - C(B)$. If $A$ is any bridge of $D$, then $A \subseteq [n - 1, \infty)$ and

$$\frac{|A^e|}{|O|} - \tau \geq \tau(B) - \tau = \eta$$

for $e = 0, 1$. Thus

$$\frac{|A^e| - \tau|O|}{|O|^2} \geq \frac{\eta}{|O|} > \eta > \frac{h(\tau)}{M_2 - 1} \geq \frac{h(\tau)}{n - 1},$$

and so (46) never holds. Therefore, by Corollary 6.5,

$$\tau_{D^*}(A^*) \geq \tau,$$

and part 2 of the lemma follows.

We next prove part 3. Let $M_3$ and $n$ be integers with $M_3 \geq \Delta h(\tau(B)) + 1$ and $n \geq M_3$. Let $D$ be the canonical derivation of $n - C(B)$. To bound the quantity

$$\frac{|A^e| - \tau|O|}{|O|^2}$$

for all bridges $A$ of $D$ we need only compute the bound for all bridges of $D(B)$. Let $A$ be a bridge of type (10). From (34) we have

$$\frac{|A^0|}{|O|} - \tau(B) = \frac{1}{\Delta l - 1} \left( \frac{b_{i+1} + Q}{b_i + Q} - 1 \right) = \frac{\Delta}{(\Delta l - 1)(b_i + Q)}.$$ 

By Lemma 4.1 (part 1) with $g_1 = [b_{i+1}]$ and $g_2 = [b_i, l]$ we have

$$|O| = (a_1, \ldots, a_r, g_1) - (a_1, \ldots, a_r, g_2) = \frac{\Delta - 1/l}{q_l^2(b_{i+1} + Q)(b_i + 1/l + Q)}.$$
Therefore
\[
\frac{|A^0| - \tau |O|}{|O|^2} = \frac{\Delta}{(\Delta l - 1)(b_i + Q)} \frac{q_i^2(b_{i+1} + Q)(b_i + 1/l + Q)}{\Delta - 1/l} > \frac{q_i^2 b_{i+1}}{\Delta l} \geq \frac{1}{\Delta} > h(\tau(B)) \frac{n}{n - 1}.
\]
Similarly we find that
\[
\frac{|A^1| - \tau |O|}{|O|^2} > \frac{q_i^2 b_{i+1}}{\Delta^2} \frac{1}{\Delta} > \frac{h(\tau(B))}{n - 1},
\]
and part 3 of the lemma follows from Corollary 6.5.

We now prove part 1. Assume first that \(B\) is finite, and put
\[
T = \frac{\delta(m - l)}{\max(\Delta l, m - \delta(m - l))}.
\]
Let \(M_1\) and \(n\) be integers with \(M_1 \geq (m + 2)^2 h(\tau(B))/T\) and \(n \geq M_1\). If \(A\) is a bridge of \(D(B)\) of the form (9), then from (33) we have
\[
\frac{|A^0|}{|O|} - \tau \geq \frac{\delta(m - l)}{\Delta, l, m - \delta(m - l)} \left(\frac{(b_{i+1} + Q)m + \delta}{(b_i + Q)m + \delta} - \frac{(b_{i+1} + 1/l)m + \delta}{(b_i + 1/l)m + \delta}\right) \\
\geq T \cdot \frac{m^2 \Delta_l (1/l - Q)}{((b_i + Q)m + \delta)((b_i + 1/l)m + \delta)} \\
\geq T \cdot \frac{\delta/\Delta \cdot q_{r-2}}{l q_r ((b_i + Q)m + \delta)((b_i + 1/l)m + \delta)},
\]
since \(q_r \geq l q_{r-1} + q_{r-2}\). From Lemma 4.1 (part 1) with \(g_1 = [b_{i+1}, m, l]\) and \(g_2 = [b_i, l, m]\) we have
\[
|O| = \frac{\Delta l + \delta/m - \delta/l}{q_i^2 (b_{i+1} + \delta/m + Q)(b_i + \delta/l + Q)},
\]
whence if \(r \neq 1\) then
\[(49) \quad \frac{|A^0| - \tau |O|}{|O|^2} > T \frac{q_{r-2} q_r}{l} > \frac{h(\tau(B))}{n}.
\]
Similarly we have
\[(50) \quad \frac{|A^1| - \tau |O|}{|O|^2} > T \frac{q_{r-1} q_{r-2} b_i b_{i+1}}{\Delta_i (m + 2)^2} > T \frac{q_{r-1} q_{r-2} b_i}{(m + 2)^2} > \frac{h(\tau(B))}{n},
\]
if \(r \neq 0\).

Now if \(r = 1\) then in \(C(n; B)\) we have \(A^1\) on the right side of \(A^0\), and if \(r = 0\) then \(A^0\) is to the right of \(A^1\). Thus by Corollary 6.5, part 1 of the lemma follows for \(B\) finite.

If \(B\) is infinite then we take \(M_1\) greater than \((\max \Delta_i)^2 h(\tau(B))\) and use an analogous argument to the above to establish our result.

7. Proofs of Theorems 1.7, 1.8 and 1.9

Proof of Theorem 1.7. We may assume without loss of generality that \(0 < \tau(B_j) < \infty\) for \(1 \leq j \leq k\). Assume first that \(S_\gamma > 1\). Let \(\eta\) be the positive real number
\[
\eta = \frac{S_\gamma - 1}{k}.
\]
We will follow an approach similar to that used in the proof of Theorem 1.6. For \( j = 1, \ldots, k \), by Lemma 6.6 (part 1) there exists a positive integer \( M_j \) such that for \( n \geq M_j \),
\[
\tau(C(n; B_j)^*) \geq \tau(B_j).
\]
Let \( M = \max M_j \). We define \( \tilde{C}_j^N \) and \( \tilde{I}_j^N \) for \( N \geq M \) by
\[
\tilde{C}_j^N = \bigcup_{n=M}^N C(n; B_j)
\]
and
\[
\tilde{I}_j^N = [M + \min C(B_j), N + \max C(B_j)].
\]
Our definition of \( \tilde{D}_j^N \) is analogous to that for \( D_j^N \) in the proof of Theorem 1.6. Note that since the logarithm function is not linear, in the notation of (36) we may have
\[
\frac{|[A_1^{\cdots 11}]^*|}{|[O_1^{\cdots 1}]^*|} < \tau(B_j),
\]
(51)
However, if we set
\[
\eta' = \frac{\eta(\tau(B_j) + 1)^2}{1 + \eta(\tau(B_j) + 1)}
\]
and take \( N \) sufficiently large, then
\[
\frac{|[A_1^{\cdots 11}]^*|}{|[O_1^{\cdots 1}]^*|} > \tau(B_j) - \eta'.
\]
Thus
\[
\tau((\tilde{C}_j^N)^*) > \tau(B_j) - \eta',
\]
so
\[
\gamma((\tilde{C}_j^N)^*) > \gamma(B_j) - \eta,
\]
whence it follows that
\[
\gamma((\tilde{C}_1^N)^*) + \cdots + \gamma((\tilde{C}_k^N)^*) > S_\gamma - k\eta = 1.
\]
Now
\[
\gamma(((\tilde{C}_j^N)^{-1})^*) = \gamma((\tilde{C}_j^N)^*),
\]
so if \( N > M + 1 \) then all the conditions of Theorem 6.3 (part 3) are satisfied, and we find that
\[
(\tilde{C}_1^N)^{1\cdot} \cdots (\tilde{C}_k^N)^{k\cdot} = (\tilde{I}_1^N)^{1\cdot} \cdots (\tilde{I}_k^N)^{k\cdot}
\]
(52)
\[
\geq \left[ \begin{array}{c}
(M+1)^{S_\nu^+} \cdot (M+1)^{S_\nu^-}
\end{array} \right],
\]
where \( S_\nu^+ = |\{j; \epsilon_j = 1\}| \) and \( S_\nu^- = k - S_\nu^+ \). We let \( N \) tend to infinity in (52) and find that if \( S_\nu = k \), then
\[
[(M+1)^k, \infty) \subseteq F,
\]
(53)
and if \( |S_\nu| < k \), then
\[
(0, \infty) \subseteq F.
\]
(54)
To extend these results to the negative axis we consider the set
\[ \tilde{C}_1^{N-} = \bigcup_{n=M}^{N} (n - C(B_1)) \]
for \( N > M + 1 \). By Lemma 6.6 (part 2) we find that for \( M \) and \( N \) sufficiently large,
\[ \tau((\tilde{C}_1^{N-})^*) > \tau(B_1) - \eta'. \]
As before we have by part 3 of Theorem 6.3 that
\[
\left( M + 1 \right)^{S^+} \left( N - 1 \right)^{S^+} \left( M + 1 \right)^{S^-} \left( N - 1 \right)^{S^-} \subseteq (\tilde{C}_1^{N-})^{s_1} (\tilde{C}_2^{N})^{s_2} \ldots (\tilde{C}_k^{N})^{s_k}.
\]
However, \( n - C(B_1) = -(-n + C(B_1)) \subseteq -F(B_1) \) for every \( n \), whence
\[
\left( M + 1 \right)^{S^+} \left( N - 1 \right)^{S^+} \left( M + 1 \right)^{S^-} \left( N - 1 \right)^{S^-} \subseteq -F.
\]
Taking the limit as \( N \) approaches infinity, we find that
\[
(-\infty, -(M + 1)^k] \subseteq F
\]
if \( S_\epsilon = k \), and
\[
(-\infty, 0) \subseteq F
\]
if \( |S_\epsilon| < k \). Part 1 of the theorem follows from (53) and (55), while part 2 is a consequence of (54) and (56).

We now assume that \( S_\gamma > 1 \) and \( |B_r| = \infty \) for some \( r \) with \( \epsilon_r = 1 \). Now,
\[ C(B_r) = \bigcup_{b_i \in B_r} \frac{1}{C(b_i; B_r)} = \frac{1}{\bigcup_{b_i \in B_r} C(b_i; B_r)}. \]
This is similar to the case \( S_\gamma > 1 \) and \( |S_\epsilon| < k \), where instead of dividing by the set
\[ \bigcup_{n=M}^{N} C(n; B_r) \]
we are dividing by
\[ C' = \bigcup_{M \leq n \leq N} C(n; B_r). \]
Notice that
\[
\frac{|[b_{i+1}, b_{i+1} + 1/l_r]|}{|[b_i + 1/l_r, b_{i+1} + 1/l_r]|} = \frac{1}{\Delta l_r - 1}.
\]
Choose \( M \) sufficiently large so that \( \Delta(M) = \max_{i \geq s} \Delta_j(B_r) \) occurs infinitely often in the sequence \( \Delta_1(B_r), \Delta_2(B_r), \ldots \), where \( s = s(M) \) is the unique positive integer with \( b_{s-1} < M \leq b_s \). Then, by (4),
\[ \tau(B_r) \leq \frac{1}{\Delta(M)l_r - 1}. \]
Let \( \beta \) so \( \in \) and \( \eta \) for \( N \) sufficiently large. We proceed in a similar manner as in the proof of part 2 of the theorem and find that
\[
(\infty, 0) \cup (0, \infty) \subseteq F.
\]
However, \( 0 \in F(B_e) \), and part 3 of the theorem follows.

Now assume that \( S_1 = 1 \). As above, it might be the case that for some \( j \) we have \( \tau(\hat{C}_j^N) < \tau(B_j) \); however, if we choose \( N \) sufficiently large, then by Lemma 6.6, part 1, for each \( j \) there will be at most one bridge \( A_j^w \) with \( \tau((A_j^w)^*) < \tau(B_j) \), namely the bridge
\[
A_j^w = [N - 1 + \min C(B_j), N + \max C(B_j)].
\]
Therefore by the proof of part 2 of Theorem 2.4 we find that
\[
(\hat{C}_1^N) \cdots (\hat{C}_k^N)^{\epsilon_k} \supseteq (\hat{I}_1^N) \cdots (\hat{I}_k^N)^{\epsilon_k} \setminus \bigcup_{j=1}^k V_j^N,
\]
where, for \( j = 1, \ldots, k, V_j^N \) is an open interval. Now assume that \( N > 1 + 2\tau(B_j)^{-1} \). Then
\[
\tau(A_j^{w*}) = \frac{|[N + \min C(B_j), N + \max C(B_j)]^*|}{|[N - 1 + \max C(B_j), N + \min C(B_j)]^*|} \geq \frac{|N - 1, N^*|}{|[N - 1 - \tau(B_j)^{-1}, N - 1]^*|},
\]
since
\[
|[N + \min C(B_j), N + \max C(B_j)]| \leq 1
\]
and
\[
\frac{|[N + \min C(B_j), N + \max C(B_j)]|}{|[N - 1 + \max C(B_j), N + \min C(B_j)]|} \geq \tau(B_j)
\]
by (37) and (38), and since the derivative of the logarithm function is decreasing. Thus
\[
\tau(A_j^{w*}) \geq \frac{\log \left( 1 + \frac{1}{N-1} \right)}{\log \left( 1 + \frac{1}{\tau(B_j)^{-1}} \right)} \geq \frac{\frac{1}{N - 1} - \frac{1}{2(N-1)^2}}{\frac{\tau(B_j)^{-1}}{N - 1 - \tau(B_j)^{-1}}},
\]
by the power series expansion of \( \log(1 + x) \) for \( |x| < 1 \). Therefore if we put \( \beta_j = 1/2 + \tau(B_j)^{-1} \), then
\[
\tau(A_j^{w*}) > \tau(B_j) \left( 1 - \frac{\beta_j}{N - 1} \right),
\]
so
\[
\gamma(C_j^{N*}) > \frac{\tau(B_j) \left( 1 - \frac{\beta_j}{N - 1} \right)}{\tau(B_j) \left( 1 - \frac{\beta_j}{N - 1} \right) + 1} > \gamma(B_j) \left( 1 - \frac{\beta_j}{N - 1} \right).
\]
Let \( \beta = \max \beta_j \). By Theorem 2.4 (part 3) we have, for \( N \) sufficiently large,
\[
\tau \left( \epsilon_1(\hat{C}_1^N)^* + \cdots + \epsilon_k(\hat{C}_k^N)^* \right) \geq \frac{S_\gamma \left( 1 - \frac{\beta}{N - 1} \right)}{1 - S_\gamma \left( 1 - \frac{\beta}{N - 1} \right)} = \frac{N - 1}{\beta} - 1.
\]
Therefore

$$|V_j^{N*}| < \frac{\beta k \log N}{N - 1 - \beta},$$

so $|V_j^N| \to 0$ as $N \to \infty$. We take the limit as $N$ approaches infinity in (58), and parts 4 and 5 of the theorem follow.

If $S_\gamma = 1$ and for some $r$ we have $|B_r| = \infty$, $\epsilon_r = 1$ and $\Delta_\gamma(B_r)$ constant, then we may extend our results to the negative reals. We use Lemma 6.6 (part 3) and an approach similar to that used in the proof of part 3 of the theorem, and part 6 follows.

Finally, if $S_\gamma < 1$, then by Lemma 6.6 (part 1) we have

$$\tau(C(M; B_j)^* \geq \tau(B_j)$$

for $j = 1, \ldots, k$ and $M$ sufficiently large, whence by Theorem 6.3 (part 2)

$$\dim_H(C(M; B_1)^{\tau_1} \cdots C(M; B_k)^{\tau_k}) \geq \frac{\log 2}{\log \left(1 + \frac{1}{S_\gamma}\right)},$$

and the theorem follows.

\[\square\]

Our methods of proving Theorems 1.7, 1.8 and 1.9 differ from that employed by Hall in [5]. He covers part of the real line by intervals of the form

$$I(n; L_4) \cdot I(n; L_4) \quad \text{or} \quad I(n; L_4) \cdot I(n + 1; L_4)$$

and then shows that

$$C(n; L_4) \cdot C(n; L_4) = I(n; L_4) \cdot I(n, L_4)$$

and

$$C(n; L_4) \cdot C(n + 1; L_4) = I(n; L_4) \cdot I(n + 1, L_4).$$

**Proof of Theorem 1.8.** Note that $\gamma(L_3) + \gamma(L_4) = 1.0165 \ldots > 1$. For positive integers $m$, define $F^+(m)$ and $F^-(m)$ by

$$F^+(m) = \{[n, a_1, a_2, \ldots]; n \geq 0 \text{ and } 1 \leq a_i \leq m \text{ for } i \geq 1\},$$

$$F^-(m) = \{[n, a_1, a_2, \ldots]; n < 0 \text{ and } 1 \leq a_i \leq m \text{ for } i \geq 1\}.$$

We will first show that

$$\tau(C(n; L_3)^*) \geq \tau(L_3) \quad \text{and} \quad \tau(C(n; L_4)^*) \geq \tau(L_4)$$

for $n \geq 0$, and that

$$\tau(n - C(L_4)^*) \geq 1.255$$

for $n \geq 1$. If $C \subseteq (0, \infty)$ is a Cantor set with derivation $D$ and $E = E_1 \cup O \cup E_2$ is a bridge of $D$ with $E_1$ to the left of $E_2$, then, for any integer $n \geq 0$,

$$\frac{|(n + E_1)^*|}{|(n + O)^*|} > \frac{|n + E_1|}{|n + O|} \geq \tau(D)$$

and

$$\frac{|(n + E_2)^*|}{|(n + O)^*|} > \frac{|E_2|}{|O^*|} \geq \tau(D^*),$$
since the second derivative of the logarithm function is negative and has decreasing magnitude. Therefore
\[ \tau((u + C)^*) \geq \min\{\tau(C), \tau(C^*)\}. \]

Thus to prove (59) it suffices to show that \( \tau(C(L_3)^*) \geq \tau(L_3) \) and \( \tau(C(L_4)^*) \geq \tau(L_4) \). Similarly, to establish (60) we need only show that \( \tau(1 - C(L_4)^*) \geq 1.255 \).

We first examine \( C(L_3) \). If \( r > 1, d \in \{0, 1\} \) and \( A \) is a bridge of the form (9), then by (49) and (50) we have
\[
\frac{|A^d| - \tau(O)}{|O|^2} > T\frac{q_r - 1}{25}.
\]

If we define \( h(x) \) as in (48), then by using a Maple program we find that
\[
\min_A \tau(A^*) = 0.833 \ldots,
\]
where the minimum is taken over all bridges \( A \) of \( C(L_3) \) with
\[
q_r q_r - 1 < \frac{25}{T} \cdot \frac{h(\tau(L_3))}{(3, 4)}.
\]

By (61), (62) and Corollary 6.5 we have
\[
\tau(C(L_3)^*) \geq \tau(L_3) = 0.822 \ldots .
\]

Similarly we find that
\[
\tau(C(L_4)^*) \geq \tau(L_4) = 1.300 \ldots
\]

and
\[
\tau((1 - C(L_4)^*)^*) \geq 1.255 \ldots .
\]

Therefore (59) and (60) hold. Since \( 0.822 \times 1.255 > 1 \), we find by an approach analogous to that used in the proof of part 1 of Theorem 1.7 that
\[
F^+(3) \cdot F^+(4) = ([3, 4] \langle 4, 1 \rangle, \infty)
\]
and
\[
F^+(3) \cdot F^-(4) = (-\infty, (3, 1] (-1 + [1, 4])].
\]

Now put
\[
I_3^- = [1 - (1, 3), 1 - (1, 4)] = [0.2087 \ldots , 0.4417 \ldots ],
\]
\[
I_4^+ = [\langle 4, 1, 4 \rangle, \langle 3, 4, 1 \rangle] = [0.2971 \ldots , 0.3118 \ldots ],
\]
\[
I_4^- = [1 - (1, 4), 1 - (1, 3)] = [0.1715 \ldots , 0.4530 \ldots ],
\]
\[
C_3^- = C(L_3) \cap I_3^-, \ C_4^+ = C(L_4) \cap I_4^+ \quad \text{and} \quad C_4^- = (1 - C(L_4)) \cap I_4^- .
\]

Since \( 1 - I_3^- \) is a bridge of \( D(L_3) \), we may use \( D(L_3) \) to construct a derivation of \( C_3^- \) from \( I_3^- \). To bound \( \tau(C_3^-) \) we use the same process that was used to establish (63). Specifically, we find that
\[
\tau((C_3^-)^*) \geq \tau(L_3). \quad (66)
\]

Similarly we have
\[
\tau((C_4^-)^*) \geq \tau(L_4) \quad \text{and} \quad \tau((C_4^+)^*) \geq \tau(L_4). \quad (67)
\]
The largest gap in \((C^-_3)^*\), \((C^+_4)^*\) and \((C^-_4)^*\) has width 0.1563\ldots, 0.0943\ldots and 0.1256\ldots respectively. Further, 
\[ |(I^-_3)^*| = 0.7497\ldots, \quad |(I^+_4)^*| = 0.4091\ldots \quad \text{and} \quad |(I^-_4)^*| = 0.9710\ldots, \]
so by (66), (67) and part 3 of Theorem 6.3, 
\[ C^-_3 \cdot C^+_4 = I^-_3 \cdot I^+_4 = [(1 - \langle 1, 3 \rangle)(4, 1), (1 - \langle 1, 1 \rangle)(3, 4, 1)] \]
and 
\[ C^-_3 \cdot C^-_4 = I^-_3 \cdot I^-_4 = [(1 - \langle 1, 3 \rangle)(1 - \langle 1, 4 \rangle), (1 - \langle 1, 1 \rangle)(1 - \langle 1, 1 \rangle)]. \]
Since 
\[ F(3) \cdot F(4) \subseteq (-\infty, (-1 + \langle 1, 3 \rangle)(4, 1)] \cup [(1 + \langle 1, 3 \rangle)(-1 + \langle 1, 4 \rangle), \infty), \]
the theorem follows from (64), (65), (68) and (69).

Proof of Theorem 1.9. Our proof will be similar to that of Theorem 1.7. Let 
\[ S_\gamma = \sum_{j=1}^{k} \frac{1}{l_j}. \]
If \( S_\gamma > 1 \) then the theorem follows from part 3 of Theorem 1.7. Assume that 
\( S_\gamma = 1 \). If \( l_j = 1 \) for any \( j \) then the theorem follows trivially, so we may assume 
that \( l_j \geq 2 \) for \( j = 1, \ldots, k \).
First assume that \( k = 2 \); then \( l_1 = l_2 = 2 \). For \( M \geq 2 \) sufficiently large we have 
by Lemma 6.6 (part 1) that 
\[ \tau(C(n; U_2)^*) \geq \tau(U_2) = 1 \]
for \( n \geq M \). For positive integers \( N > M \) we define \( C^N \) and \( I^N \) by 
\[ C^N = \left[ N + \frac{1}{2}, N + 1 \right] \cup \bigcup_{n=M}^{N} C(n; U_2) \]
and 
\[ I^N = [M, N + 1]. \]
We may construct a derivation \( D^N \) of \( C^N \) from \( I^N \) in a manner similar to that used 
in the proof of Theorem 1.7, the only difference being that the rightmost interval 
in every level of the tree contains the closed interval \([N + 1/2, N + 1]\). Because of 
this we avoid the problems faced in (51), so that 
\[ \tau((C^N)^*) \geq \tau(U_2) = 1. \]
Thus \( (C^N)^{-1} \) and \( C(M; U_2) \) satisfy the requirements of Theorem 6.3, part 3, so 
\[ C(M; U_2) \supseteq \left[ \frac{2M + 1}{2N + 1}, 1 \right]. \]
But
\[ C(0; U_2) = \frac{1}{\bigcup_{n=2}^{\infty} C(n; U_2)}, \]
whence
\[ C(M; U_2) \cdot C(0; U_2) \supseteq \left[ \frac{2M + 1}{2N + 1}, 1 \right]. \]
This holds for every \( N > M \); thus
\[ C(M; U_2) \cdot C(0; U_2) \supseteq (0, 1]. \quad (70) \]
By taking reciprocals we have
\[ C(M; U_2)^{-1} \cdot C(0; U_2)^{-1} \supseteq [1, \infty). \]
However,
\[ C(M; U_2)^{-1} \subseteq C(0; U_2) \text{ and } C(0; U_2)^{-1} \subseteq G(2), \]
so
\[ (0, \infty) \subseteq G(2) \cdot G(2). \quad (71) \]
By (70) and (71) we have
\[ (0, \infty) \subseteq G(2) \cdot G(2). \quad (72) \]
As in the proof of Theorem 1.7 we may extend our results to the negative real axis, so that
\[ (-\infty, 0) \subseteq G(2) \cdot G(2). \quad (73) \]
Since \( 0 \in G(2) \), we have by (72) and (73) that
\[ G(2) \cdot G(2) = \mathbb{R}, \quad (74) \]
as required.

Now assume that \( k > 2 \). To prove the theorem we will use an approach similar to that used to establish (74). Without loss of generality we may assume that
\[ l_1 \geq l_2 \geq \cdots \geq l_k. \]
Now, for \( M \) sufficiently large, for all \( n \geq M \) and all \( j = 1, \ldots, k \) the largest gap in \( C(n; U_{l_j})^* \) is \( (O_{l_j}^a)^* \), where \( O_{l_j}^a \) is the largest gap in \( C(n; U_{l_j}) \), namely
\[ O_{l_j}^a = ([n, l_j + 1], [n, l_j, l_j]). \]
By Lemma 6.6 (part 1) there exists a positive integer \( M_1 \geq M \) such that
\[ M_1 > \max \left\{ l_1, \frac{l_2^2 - 1}{l_2} \right\}, \quad (l_2 l_3 \cdots l_k) \text{ divides } M_1 \]
and
\[ \tau(C(n; U_{l_j})^*) \geq \tau(U_{l_j}) \]
for all \( n \geq M_1 \) and all \( j = 1, \ldots, k \). For \( j = 2, \ldots, k \) we set
\[ M_j = \frac{l_j^2 - 1}{l_j} M_{j-1}. \quad (76) \]
By our choice of \( M_1, \ldots, M_k \), we have, by calculation,
\[ |(O_{l_j}^{M_j})^*| \leq |I(M_{j+1}; U_{l_{j+1}})^*| \leq |I(M_{j}; U_{l_{j}})^*| \]
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for \( j = 1, \ldots, k - 1 \). For integers \( N > 2M_k \) we define \( C_k^N \) and \( I_k^N \) by

\[
C_k^N = \left[ N + \frac{1}{l_k}, N + 1 \right] \cup \bigcup_{n=M_k}^{N} C(n; U_{l_k}) \quad \text{and} \quad I_k^N = [M_k, N + 1].
\]

Now the largest gap in \((C_k^N)^*\) is either \((O_k^{M_k})^*\) or \((M_k + 1/l_k, M_k + 1)^*\). Since \( l_k \leq l_{k-1} \) and \( k \geq 3 \), we have

\[
\frac{l_k}{l_1 M_1 (l_1 \cdots l_{k-1})} \leq \frac{1}{l_1^2 M_1} < \frac{1}{l_1 M_1}.
\]

Also, by (76) it follows that

\[
M_j = \frac{l_1 M_1}{l_j^2} (l_1 \cdots l_j)
\]

for \( j = 1, \ldots, k \), whence with (78) we find that

\[
\frac{1}{M_k} \leq \frac{1}{l_1 M_1}.
\]

Thus

\[
\log \left( \frac{M_k + 1}{M_k + 1/l_k} \right) \leq \log \left( \frac{M_1 + 1/l_1}{M_1} \right).
\]

Equivalently,

\[
|(M_k + 1/l_k, M_k + 1)^*| \leq |I(M_1; U_{l_1})^*|.
\]

As in the proof of the case \( k = 2 \), we have

\[
\tau((C_k^N)^*) \geq \tau(U_{l_k}).
\]

By (75), (77), (80), (81) and Theorem 6.3 (part 3),

\[
\frac{C(M_1; U_{l_1}) \cdots C(M_{k-1}; U_{l_{k-1}})}{C_k^N} = \left[ \frac{M_1 \cdots M_{k-1}}{N + 1}, \frac{(M_1 + 1/l_1) \cdots (M_{k-1} + 1/l_{k-1})}{M_k} \right].
\]

Thus

\[
\frac{C(M_1; U_{l_1}) \cdots C(M_{k-1}; U_{l_{k-1}})}{\bigcup_{n=M_k}^{N} C(n; U_{l_k})} \supseteq \left[ \frac{(M_1 + 1/l_1) \cdots (M_{k-1} + 1/l_{k-1})}{N + 1/l_k}, \frac{M_1 \cdots M_{k-1}}{M_k} \right].
\]

Since \( M_1 > l_{k-1}^2/l_k \) and \( k > 2 \), we have by (79) that

\[
\frac{M_1 \cdots M_{k-1}}{M_k} > 1.
\]

Also, since \( M_k \geq l_k \),

\[
\left( \bigcup_{n=M_k}^{N} C(n; U_{l_k}) \right)^{-1} \subseteq C(0; U_{l_k}),
\]

so by (82), (83) and (84), upon taking the limit as \( N \) approaches infinity, we have

\[
(0, 1) \subseteq G(l_1) \cdots G(l_k).
\]
Since \( M_j \geq l_j \) for \( j = 1, \ldots, k - 1 \) we may take reciprocals in (82) and let \( N \) tend to infinity, so that

\[
[1, \infty) \subseteq G(l_1) \cdots G(l_k).
\]

With (85) we have

\[
(0, \infty) \subseteq G(l_1) \cdots G(l_k).
\]

As before we may extend our results to the negative reals, finding that

\[
(-\infty, 0) \subseteq G(l_1) \cdots G(l_k).
\]

Since \( 0 \in G(l_1) \), our result follows.

8. Final Remarks

The problem of proving negative results for products seems to be much more difficult than for sums. For example, to prove that \([a, \infty) \not\subseteq F(2) \cdot F(2)\) for some \( a \), it would not suffice to find a single gap modulo one in \( C(L_2) \cdot C(L_2) \). Rather, we would have to show that the same gap existed in each \( \tilde{C}_N \tilde{C}_N \).

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References