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CANTOR SETS AND NUMBERS WITH RESTRICTED PARTIAL QUOTIENTS

S. ASTELS

ABSTRACT. For $j = 1, \dots, k$ let C_j be a Cantor set constructed from the interval I_j, and let $\epsilon_j = \pm 1$. We derive conditions under which

$$\epsilon_1 C_1 + \cdots + \epsilon_k C_k = \epsilon_1 I_1 + \cdots + \epsilon_k I_k \quad \text{and} \quad C_1^{\epsilon_1} \cdots C_k^{\epsilon_k} = I_1^{\epsilon_1} \cdots I_k^{\epsilon_k}.$$

When these conditions do not hold, we derive a lower bound for the Hausdorff dimension of the above sum and product. We use these results to make corresponding statements about the sum and product of sets $F(B_j)$, where B_j is a set of positive integers and $F(B_j)$ is the set of real numbers x such that all partial quotients of x, except possibly the first, are members of B_j.

1. Introduction

Let x be a real number. We say that x is *badly approximable* if there exists a positive integer n such that for every rational number p/q,

$$\left| x - \frac{p}{q} \right| > \frac{1}{nq^2}.$$

It can be shown that this set is of Lebesgue measure zero; however, it is still quite large. In 1947 Marshall Hall [5] showed that every real number can be expressed as the sum of two badly approximable numbers. In particular, for a positive integer m let $F(m)$ denote the set of numbers

$$F(m) = \{[t, a_1, a_2, \dots]\,;\, t \in \mathbb{Z},\, 1 \le a_i \le m \text{ for } i \ge 1\}$$

where by $[a_0, a_1, a_2, \dots]$ we denote the *continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

with *partial quotients* a_0, a_1, a_2 and so on. It can be shown that for every $x \in F(4)$ and every $p/q \in \mathbb{Q}$,

$$\left| x - \frac{p}{q} \right| > \frac{1}{6q^2}$$

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so that $F(4)$ is a set of badly approximable numbers. Hall proved that

$$F(4) + F(4) = \mathbb{R},$$

where we define the sum of two sets of real numbers A and B by

$$A + B = \{a + b\colon a \in A \text{ and } b \in B\}.$$

In 1973 Bohuslav Diviš [4] showed that one could not do much better than Hall's result, namely that

$$F(3) + F(3) \neq \mathbb{R}.$$

In 1975 James Hlavka [6] generalized Hall's results to the case of different sets $F(m)$ and $F(n)$. He proved that

$$F(m) + F(n) = \mathbb{R} \tag{1}$$

holds for (m, n) equal to (2,7) or (3,4), but does not hold for (m, n) equal to (2,4). Now, if (1) holds, then the same equation holds with m and n replaced by m' and n' respectively, where $m' \geq m$ and $n' \geq n$. Further, if either n or m is equal to one then trivially (1) does not hold, since $F(1)$ consists of the points $\{[t, 1, 1, 1 \dots]\,;\, t \in \mathbb{Z}\}$. Hence the only cases of interest left are $(m, n) = (2, 5)$ and $(m, n) = (2, 6)$. Hlavka conjectured that in these two cases (1) would not hold. In work to appear [1] we show that in both cases Hlavka's conjecture is false.

We can also examine the difference of two sets $F(m)$ and $F(n)$. If A is a set of real numbers we define $-A$ by

$$-A = \{-a;\, a \in A\}$$

and denote $A + (-B)$ by $A - B$. We have the following result.

Theorem 1.1. *Let m and n be integers. The equations*

$$F(m) + F(n) = \mathbb{R} \quad \textit{and} \quad F(m) - F(n) = \mathbb{R}$$

hold if (m, n) equals $(2, 5)$ or $(3, 4)$. Neither of the above equations hold if (m, n) equals $(2, 4)$. Additionally,

$$F(3) + F(3) \neq \mathbb{R} \quad \textit{and} \quad F(3) - F(3) = \mathbb{R}.$$

In 1971 Tom Cusick [2] examined the complementary case of sums of real numbers whose continued fraction expansion contains only large partial quotients. For each positive integer l we define the set $G(l)$ by

$$\begin{aligned} G(l) = &\{[t, a_1, a_2, \dots]\,;\, t \in \mathbb{Z} \text{ and } a_i \geq l \text{ for } i \geq 1\} \\ &\cup \{[t, a_1, a_2, \dots, a_k]\,;\, t, k \in \mathbb{Z},\ k \geq 0 \text{ and } a_i \geq l \text{ for } 1 \leq i \leq k\}\,. \end{aligned}$$

Cusick proved that

$$G(2) + G(2) = \mathbb{R}.$$

The above results are special cases of the following general problem. Let B be a set of positive integers. If B is a finite set, we let $F(B)$ denote the set of real numbers which have an infinite continued fraction expansion with all partial quotients, except possibly the first, members of B. For B infinite, we define $F(B)$ similarly, but also allow numbers with finite continued fraction expansions. Thus if we define

$$L_m = \{1, 2, \dots, m\} \quad \text{and} \quad U_l = \{l, l + 1, \dots\}$$

TABLE 1. Values of $\tau(B)$ for certain B

B	$\tau(B)$
L_2	$(-1+\sqrt{3})/2 = 0.366\ldots$
L_3	$(-6+4\sqrt{21})/15 = 0.822\ldots$
L_4	$(-3+15\sqrt{2})/14 = 1.300\ldots$
L_5	$4\sqrt{5}/5 = 1.788\ldots$
L_6	$(15+35\sqrt{15})/66 = 2.281\ldots$
L_7	$(42+24\sqrt{77})/91 = 2.775\ldots$
U_l	$1/(l-1)$

for positive integers m and l, then $F(m) = F(L_m)$ and $G(l) = F(U_l)$. For sets of positive integers B_1 and B_2, we wish to know when

$$F(B_1) + F(B_2) = \mathbb{R} \tag{2}$$

and when

$$F(B_1) - F(B_2) = \mathbb{R}. \tag{3}$$

We shall derive conditions on the sets B_1 and B_2 such that (2) and (3) follow. Let $B = \{b_1, b_2, \ldots\}$ be a set of positive integers with $|B| > 0$ and $b_1 < b_2 < \ldots$. If $|B| = 1$, then we put $\tau(B) = 0$. Otherwise, we set

$$l = l(B) = \min B \quad \text{and} \quad \Delta_i = \Delta_i(B) = b_{i+1} - b_i.$$

If B is a finite set with $|B| > 1$, then we put

$$m = m(B) = \max B, \quad \delta = \delta(B) = \frac{-lm + \sqrt{l^2m^2 + 4lm}}{2},$$

and

$$\tau(B) = \min_{i<|B|} \min \left\{ \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_{i+1}lm + m + \delta l}{b_i lm + m + \delta l}, \frac{(m - b_{i+1})lm + \delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_i l + \delta}{lm + \delta} \right\}.$$

If B is an infinite set, then we put

$$\tau(B) = \inf_i \min \left\{ \frac{1}{\Delta_i l - 1} \cdot \frac{b_{i+1} l + 1}{b_i l + 1}, \frac{b_i l + 1}{\Delta_i l - 1} \right\}. \tag{4}$$

If we let $\Delta = \max_i \Delta_i$, then if $1 < |B| < \infty$ we have

$$\tau(B) \geq \frac{\delta(m-l)}{\Delta lm - \delta(m-l)} \cdot \frac{lm + \delta - l\Delta}{lm + \delta},$$

and if B is infinite then

$$\tau(B) \geq \frac{1}{\Delta l - 1}.$$

It is a simple matter to calculate $\tau(B)$ for various sets B (see Table 1).

We denote the Hausdorff dimension of a set S by $\dim_H S$. To help determine whether (2) and (3) hold we will prove the following theorem.

Theorem 1.2. *Let B_1 and B_2 be sets of positive integers, and take $\epsilon_1, \epsilon_2 \in \{1, -1\}$.*

1. *If $\tau(B_1)\tau(B_2) \geq 1$, then $\epsilon_1 F(B_1) + \epsilon_2 F(B_2) = \mathbb{R}$.*
2. *If $\tau(B_1)\tau(B_2) < 1$, then*

$$\dim_H(\epsilon_1 F(B_1) + \epsilon_2 F(B_2)) \geq \frac{\log 2}{\log\left(2 + \frac{1-\tau(B_1)\tau(B_2)}{\tau(B_1)+\tau(B_2)+2\tau(B_1)\tau(B_2)}\right)}.$$

For example, we have

$$\dim_H(F(2) + F(2)) \geq 0.658\ldots .$$

The positive results in Theorem 1.1 follow in part from Theorem 1.2. In addition we mention the following corollaries.

Corollary 1.3. *If l and m are positive integers with $m \geq 2l$, then*

$$F(m) + G(l) = \mathbb{R}.$$

Corollary 1.4. *Let B_o denote the set of positive odd integers. Then*

$$F(B_o) + F(B_o) = \mathbb{R}.$$

Furthermore, if B is a finite set of odd positive integers, then

$$F(B) + F(B) \neq \mathbb{R}.$$

Note that if $B(m)$ is the set of positive odd integers less than m, then $\tau(B(m))$ approaches one as m tends to infinity. Thus part 1 of Theorem 1.2 is tight in the sense that we cannot replace 1 by any smaller number.

Diviš [4] and Hlavka [6] also developed techniques that allowed them to examine the sum of more than two $F(m)$'s. Diviš showed that

$$F(3) + F(3) + F(3) = \mathbb{R} \quad \text{and} \quad F(2) + F(2) + F(2) + F(2) = \mathbb{R}$$

while

$$F(2) + F(2) + F(2) \neq \mathbb{R}.$$

Hlavka proved that

$$F(l) + F(m) + F(n) = \mathbb{R}$$

holds if (l, m, n) equals (2,2,4) or (2,3,3) but does not hold for (l, m, n) equal to $(2, 2, 3)$. Together with the work on sums of two $F(m)$'s, these results allow us to determine those finite sets of positive integers $\{m_1, \ldots, m_k\}$ for which

$$\sum_{j=1}^{k} F(m_j) = \mathbb{R}.$$

In the case of sums of integers with large partial quotients, Tom Cusick and Robert Lee [3] showed in 1971 that

$$lG(l) = \mathbb{R}$$

for every positive integer l. We shall extend this to the case where the summands are unequal.

Theorem 1.5. *If k and $l_1, l_2, \ldots, l_k$ are positive integer with*

$$\sum_{j=1}^{k} \frac{1}{l_j} \geq 1,$$

then

$$G(l_1) + \cdots + G(l_k) = \mathbb{R}.$$

Note that if l is a positive integer and we set $k = l$ and $l_1 = l_2 = \cdots = l_k = l$, then we recover the result of Cusick and Lee.

For a non-empty set of positive integers B we define $\gamma(B)$ by

$$\gamma(B) = \frac{\tau(B)}{\tau(B) + 1}.$$

Theorem 1.5 is a consequence of the following general theorem.

Theorem 1.6. *Let k be a positive integer and $B_1, \ldots, B_k$ non-empty sets of positive integers. Let $\epsilon_j \in \{1, -1\}$ for $j = 1, \ldots, k$. If*

$$\sum_{j=1}^{k} \gamma(B_j) \geq 1,$$

then

$$\epsilon_1 F(B_1) + \cdots + \epsilon_k F(B_k) = \mathbb{R}. \tag{5}$$

Otherwise

$$\dim_H(\epsilon_1 F(B_1) + \cdots + \epsilon_k F(B_k)) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\gamma(B_1) + \cdots + \gamma(B_k)}\right)}. \tag{6}$$

Hall [5] and Cusick [2] also examined products of numbers with bounded partial quotients. For sets A and B of real numbers, we define the product of A and B by

$$AB = A \cdot B = \{ab;\ a \in A \text{ and } b \in B\}$$

and A^{-1} by

$$A^{-1} = \{1/a;\ a \in A \text{ and } a \neq 0\}.$$

We also denote by A/B the set $A \cdot (B^{-1})$. Hall proved that

$$[1, \infty) \subseteq F(4) \cdot F(4), \tag{7}$$

while Cusick established that

$$[1, \infty) \subseteq G(2) \cdot G(2). \tag{8}$$

We shall derive the following multiplicative analogue of Theorem 1.6.

Theorem 1.7. *Let k be a positive integer. For $j = 1, \ldots, k$ let B_j be a set of positive integers and let $\epsilon_j \in \{1, -1\}$. Set*

$$S_\gamma = \gamma(B_1) + \cdots + \gamma(B_k),$$
$$S_\epsilon = \epsilon_1 + \cdots + \epsilon_k$$

and

$$F = F(B_1)^{\epsilon_1} F(B_2)^{\epsilon_2} \cdots F(B_k)^{\epsilon_k}.$$

1. *If $S_\gamma > 1$ and $S_\epsilon = k$, then there exists a positive real number c_1 such that*

$$F \supseteq (-\infty, -c_1] \cup [c_1, \infty).$$

2. *If* $S_\gamma > 1$ *and* $|S_\epsilon| < k$, *then*
$$F \supseteq (-\infty, 0) \cup (0, \infty).$$
3. *If* $S_\gamma > 1$ *and there exists* r *such that* $|B_r| = \infty$ *and* $\epsilon_r = 1$, *then*
$$F = \mathbb{R}.$$
4. *If* $S_\gamma = 1$ *and* $S_\epsilon = k$, *then there exists a positive real number* c_2 *such that*
$$F \supseteq [c_2, \infty).$$
5. *If* $S_\gamma = 1$ *and* $|S_\epsilon| < k$, *then* F *omits at most* k *points of* $(0, \infty)$.
6. *If* $S_\gamma = 1$ *and there exists* r *such that* $|B_r| = \infty$, $\epsilon_r = 1$ *and* $\Delta_i(B_r)$ *is constant, then* F *omits at most* $2k$ *real numbers.*
7. *If* $S_\gamma < 1$, *then*
$$\dim_H F \geq \frac{\log 2}{\log\left(1 + \frac{1}{S_\gamma}\right)}.$$

For particular choices of B_j we can calculate c_1 and c_2 in the above theorem explicitly. If we denote by $\langle a_1, a_2, \dots \rangle$ the continued fraction $[0, a_1, a_2, \dots]$, then we have the following improvement to (7).

Theorem 1.8. *Define* α_1 *and* α_2 *by*
$$\alpha_1 = (1 - \langle \overline{1,3} \rangle)\langle \overline{4,1} \rangle = 0.0432\dots$$
and
$$\alpha_2 = (1 - \langle \overline{1,3} \rangle)(1 - \langle \overline{1,4} \rangle) = 0.0358\dots .$$
Then
$$F(3) \cdot F(4) = (-\infty, -\alpha_1] \cup [\alpha_2, \infty).$$

We may also strengthen and generalize (8).

Theorem 1.9. *If* k *and* $l_1, l_2, \dots, l_k$ *are positive integers with*
$$\sum_{j=1}^{k} \frac{1}{l_j} \geq 1,$$
then
$$G(l_1) \cdots G(l_k) = \mathbb{R}.$$

As in the works of Hall, Cusick and Lee, Diviš, and Hlavka, our results hinge on the study of certain Cantor sets. For any set of positive integers B we define the set $C(B)$ by
$$C(B) = \{\langle a_1, a_2, \dots \rangle \, ; \, a_i \in B \text{ for every } i\},$$
where numbers with a finite continued fraction expansion are included in $C(B)$ if and only if B is an infinite set. We shall show that the sets $C(B)$ may be viewed as Cantor sets. We then derive results on sums and products of Cantor sets to prove our results.

2. Cantor Sets

Let T be a connected directed graph. We say that T is a *tree* if every vertex V of T has at most one edge terminating at V, and one vertex V_R has no edges terminating at V_R. We call V_R the *root* of T. If there is an edge connecting V_1 to V_2, then we say that V_2 is a *subvertex* of V_1. A vertex with no subvertices is called a *leaf.* A tree where each vertex has at most t subvertices is called a tree of *valence* t. We will show that our Cantor sets can be represented by trees of valence 2.

Let A be a closed interval of the real line and let $O \subseteq A$ be an open interval. Then

$$A = A^0 \cup O \cup A^1$$

for some closed intervals A^0 and A^1. We set

$$C^0 = A \qquad \text{and} \qquad C^1 = A^0 \cup A^1.$$

If O^0 and O^1 are open intervals contained in A^0 and A^1 respectively, then we have

$$A^0 = A^{00} \cup O^0 \cup A^{01} \qquad \text{and} \qquad A^1 = A^{10} \cup O^1 \cup A^{11}$$

for some closed intervals A^{00}, A^{01}, A^{10}, and A^{11}. We set

$$C^2 = A^{00} \cup A^{01} \cup A^{10} \cup A^{11}.$$

We continue this process, forming C^{j+1} from C^j by removing an open interval from each closed interval in the union which comprises C^j. We form a tree $\mathcal{D}$ with root A as follows. Let the vertices of the tree be the closed intervals A^w, for w a finite binary word, and form directed edges joining A^w to A^{w0} and A^{w1}. If we define C by

$$C = \bigcap_{j=0}^{\infty} C^j,$$

then we call $\mathcal{D}$ a *derivation* of C, and call C the *Cantor set derived from A by $\mathcal{D}$*. The A^w's are called the *bridges* of $\mathcal{D}$. If A^w is a bridge of $\mathcal{D}$, then we say that A^w *splits* as

$$A^w = A^{w0} \cup O^w \cup A^{w1}.$$

We extend our definitions of Cantor sets and derivations by allowing the derivation to contain vertices which do not split. Let A^w be such a vertex. We place under A^w the vertex A^{w0}, where $A^{w0} = A^w$ as intervals. Thus our derivation may contain infinite stalks, and will be of valence 2. We also allow bridges to split as $A = A^0 \cup O \cup A^1$, where $O = \emptyset$ and $A^0 \cap A^1$ consists of only one point.

Note that the derivation $\mathcal{D}$ of a Cantor set C is not uniquely determined by C; for example, if we change the order in which the open intervals are removed then we get a different derivation but the same Cantor set.

We denote the length of an interval I by $|I|$. We say that a derivation $\mathcal{D}$ is *ordered* if for any bridges A and B of $\mathcal{D}$ with $A = A^0 \cup O \cup A^1$, $B = B^0 \cup O_2 \cup B^1$ and $B \subseteq A$ we have $|O| \geq |O_2|$. We define the t^{th} *level* of $\mathcal{D}$ to be the set of all vertices A^w in $\mathcal{D}$ where w is a binary word of length t.

Cantor sets arise in the study of real numbers whose partial quotients are members of a given set. Let $B = \{b_1, b_2, \ldots, b_t\}$ be a finite set of positive integers with

$t \geq 2$ and $b_1 < \cdots < b_t$. We set $l = l(B) = \min B = b_1$, $m = m(B) = \max B = b_t$,

$$C(B) = F(B) \cap [0,1] = \{\langle a_1, a_2, \dots\rangle \colon a_i \in B \text{ for } i = 1, 2, \dots\},$$

and let $I(B)$ be the closed interval

$$I(B) = [\langle \overline{m, l}\rangle, \langle \overline{l, m}\rangle].$$

We have $C(B) \subseteq I(B)$. We now inductively construct a derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$. For any real a and b, we denote by $[[a, b]]$ and $((a, b))$ the intervals

$$[[a, b]] = [\min\{a, b\}, \max\{a, b\}]$$

and

$$((a, b)) = (\min\{a, b\}, \max\{a, b\}).$$

If, for $i < t$,

$$A = [[\langle a_1, \dots, a_r, b_i, \overline{m, l}\rangle, \langle a_1, \dots, a_r, m, \overline{l, m}\rangle]] \tag{9}$$

is a bridge of $\mathcal{D}(B)$ of level n, then we form the subvertices of A by setting

$$\begin{aligned} A^0 &= [[\langle a_1, \dots, a_r, b_i, \overline{m, l}\rangle, \langle a_1, \dots, a_r, b_i, \overline{l, m}\rangle]], \\ O &= ((\langle a_1, \dots, a_r, b_i, \overline{l, m}\rangle, \langle a_1, \dots, a_r, b_{i+1}, \overline{m, l}\rangle)) \end{aligned}$$

and

$$A^1 = [[\langle a_1, \dots, a_r, b_{i+1}, \overline{m, l}\rangle, \langle a_1, \dots, a_r, m, \overline{l, m}\rangle]].$$

In this manner we construct the $(n+1)^{th}$ level of the derivation from the n^{th} level. Note that A^0 is of the form (9) with $a_{r+1} = b_i$ and b_i replaced by l. Similarly A^1 is also of the form (9). Since $I(B)$ is of the form (9) with $r = 0$ and $i = 1$, by induction we obtain the *canonical* derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$.

If B is an infinite set, then we may construct a similar derivation. Assume that $B = \{b_1, \dots\}$ with $b_i < b_{i+1}$ for $i \geq 1$. If we set $l = l(B) = \min B = b_1$, then we have $C(B) \subseteq I(B)$, where $I(B) = [0, 1/l]$ and

$$\begin{aligned} C(B) = &\{\langle a_1, a_2, \dots\rangle\,;\, a_i \in B \text{ for } i \geq 1\} \\ &\cup \{\langle a_1, a_2, \dots, a_k\rangle\,;\, k \in \mathbb{Z},\ k \geq 0 \text{ and } a_i \in B \text{ for } 1 \leq i \leq k\}. \end{aligned}$$

If

$$A = [[\langle a_1, \dots, a_r, b_i\rangle, \langle a_1, \dots, a_r\rangle]] \tag{10}$$

is a bridge, then we split A by setting

$$\begin{aligned} A^0 &= [[\langle a_1, \dots, a_r, b_i\rangle, \langle a_1, \dots, a_r, b_i, l\rangle]], \\ O &= ((\langle a_1, \dots, a_r, b_i, l\rangle, \langle a_1, \dots, a_r, b_{i+1}\rangle)) \end{aligned}$$

and

$$A^1 = [[\langle a_1, \dots, a_r, b_{i+1}\rangle, \langle a_1, \dots, a_r\rangle]]$$

where by convention we set $\langle a_1, \dots, a_r\rangle = 0$ if $r = 0$. As above, we construct the canonical derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$ using this process.

For given sets of integers B_j, $j = 1, \ldots, k$, we would like to be able to determine if

$$\sum_{j=1}^{k} C(B_j) = \sum_{j=1}^{k} I(B_j). \tag{11}$$

To do this we shall derive criteria on general Cantor sets that guarantee (11) holds. Our conditions will be less stringent than those derived previously. Let C be a Cantor set with derivation $\mathcal{D}$, and let A^w be a bridge of $\mathcal{D}$. We define the *thickness of A^w with respect to $\mathcal{D}$*, denoted by $\tau_{\mathcal{D}}(A^w)$, to be $+\infty$ if A^w does not split. Otherwise we set

$$\tau(A^w) = \tau_{\mathcal{D}}(A^w) = \min\left\{\frac{|A^{w0}|}{|O^w|}, \frac{|A^{w1}|}{|O^w|}\right\},$$

where throughout this paper we adopt the convention that $x/0 = \infty$ for any $x > 0$. We define the *thickness* $\tau(\mathcal{D})$ of the derivation $\mathcal{D}$ by

$$\tau(\mathcal{D}) = \inf_{A^w} \tau_{\mathcal{D}}(A^w),$$

where the infimum is taken over all bridges A^w of $\mathcal{D}$. We also define $\tau(C)$, the *thickness* of the Cantor set C, by

$$\tau(C) = \sup_{\mathcal{D}} \tau(\mathcal{D}),$$

where the supremum is taken over all derivations $\mathcal{D}$ of C. An equivalent definition of $\tau(C)$ may be found in [9], p. 61. It follows from Lemma 3.1 that $\tau(C) = \tau(\mathcal{D}_o)$, where $\mathcal{D}_o$ is any ordered derivation of C. The following observation is trivial yet crucial in our use of thickness.

Lemma 2.1. *Let C be a Cantor set. Then C is an interval if and only if $\tau(C) = \infty$.*

Proof. Let C be derived from I and take $\mathcal{D}$ to be any ordered derivation of C from I. By Lemma 3.1 we have $\tau(C) = \tau(\mathcal{D})$. If $C \neq I$ then I, the root of $\mathcal{D}$, must split in $\mathcal{D}$ with a nontrivial gap, so

$$\tau(\mathcal{D}) \leq \tau_{\mathcal{D}}(I) < \infty.$$

If, on the other hand, $C = I$, then any bridges of $\mathcal{D}$ which split do so with a trivial gap, so that $\tau(\mathcal{D}) = \infty$, as required. □

For sums of two Cantor sets we shall prove the following result.

Theorem 2.2. *For $j = 1, 2$ let C_j be a Cantor set derived from I_j, with O_j a gap of maximal size in C_j. Assume that*

$$|O_1| \leq |I_2| \quad \textit{and} \quad |O_2| \leq |I_1|.$$

1. *If $\tau(C_1)\tau(C_2) \geq 1$, then $C_1 + C_2 = I_1 + I_2$.*
2. *If $\tau(C_1)\tau(C_2) < 1$, then*

$$\tau(C_1 + C_2) \geq \frac{\tau(C_1) + \tau(C_2) + 2\tau(C_1)\tau(C_2)}{1 - \tau(C_1)\tau(C_2)}.$$

Part 1 of Theorem 2.2 may be derived from work of Sheldon Newhouse; our approach will give an alternative proof. Newhouse [8] established the following result.

Theorem 2.3. *Let K_1 and K_2 be Cantor sets derived from I_1 and I_2 respectively, with $\tau(K_1)\tau(K_2) > 1$. Then either $I_1 \cap I_2 = \emptyset$, K_1 is contained in a gap of K_2, K_2 is contained in a gap of K_1 or $K_1 \cap K_2 \neq \emptyset$.*

In fact, if Newhouse's proof is slightly altered then we may replace the condition "$\tau(K_1)\tau(K_2) > 1$" in Theorem 2.3 by the weaker condition "$\tau(K_1)\tau(K_2) \geq 1$".

To see that Part 1 of Theorem 2.2 follow from this modified version of Theorem 2.3, we assume that $\tau(C_1)\tau(C_2) \geq 1$ and let k be any number in $I_1 + I_2$. Upon applying Theorem 2.3 (modified) with $K_1 = k - C_1$ and $K_2 = C_2$ we find that $(k - C_1) \cap C_2 \neq \emptyset$ and hence $k \in C_1 + C_2$.

If C is a Cantor set, then we define $\gamma(C)$ by

$$\gamma(C) = \frac{\tau(C)}{\tau(C) + 1}.$$

Theorem 2.2 is a special case of the following theorem.

Theorem 2.4. *Let k be a positive integer and for $j = 1, 2, \dots, k$ let C_j be a Cantor set derived from I_j, with O_j a gap of maximal size in C_j. Let $S_\gamma = \gamma(C_1) + \cdots + \gamma(C_k)$.*

1. *If $S_\gamma \geq 1$ then $C_1 + \cdots + C_k$ contains an interval. Otherwise $C_1 + \cdots + C_k$ contains a Cantor set of thickness at least*

$$\frac{S_\gamma}{1 - S_\gamma}.$$

Furthermore,

$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\min\{S_\gamma, 1\}}\right)}.$$

2. *If*

$$|I_{r+1}| \geq |O_j| \qquad \text{for } r = 1, \dots, k-1 \text{ and } j = 1, \dots, r, \tag{12}$$

$$|I_1| + \cdots + |I_r| \geq |O_{r+1}| \qquad \text{for } r = 1, \dots, k-1, \tag{13}$$

and $S_\gamma \geq 1$, then

$$C_1 + \cdots + C_k = I_1 + \cdots + I_k.$$

3. *If (12) and (13) hold and $S_\gamma < 1$, then*

$$\tau(C_1 + \cdots + C_k) \geq \frac{S_\gamma}{1 - S_\gamma}.$$

Theorem 2.4 is best possible in the sense that the condition $S_\gamma \geq 1$ in part 1 or part 2 cannot be replaced by $S_\gamma \geq \eta$ for any $\eta < 1$. Similarly, if we multiply the bound for the thickness or the Hausdorff dimension of the sum by $1 + \delta$ for any $\delta > 0$, then the results do not hold in general.

3. Proof of Theorem 2.4

To prove Theorem 2.4 we require several lemmas.

Lemma 3.1. *Let $\mathcal{D}$ be any derivation of C from I. Then there exists an ordered derivation $\mathcal{D}_o$ of C from I with*

$$\tau(\mathcal{D}) \leq \tau(\mathcal{D}_o).$$

Furthermore, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two ordered derivations of C from I, then

$$\tau(\mathcal{D}_1) = \tau(\mathcal{D}_2).$$

Let $\mathcal{D}$ be a derivation of C from I, and assume that $\mathcal{D}$ is not ordered. Then there exists a bridge A of $\mathcal{D}$ which splits as $A = A^0 \cup O \cup A^1$, with A^0 and A^1 splitting as $A^0 = A^{00} \cup O^0 \cup A^{01}$ and $A^1 = A^{10} \cup O^1 \cup A^{11}$ respectively, such that either $|O^0| > |O|$ or $|O^1| > |O|$. Assume without loss of generality that $|O^0| > |O|$. Consider the derivation $\mathcal{D}_s$ which is identical to $\mathcal{D}$ except that the positions of O^0 and O in the tree have been switched, that is, O^0 is removed before O. If we set $A_s = A$ and

$$A_s^0 = A^{00}, \quad O_s = O^0, \quad A_s^1 = A^{01} \cup O \cup A^1,$$
$$A_s^{10} = A^{01}, \quad O_s^1 = O, \quad A_s^{11} = A^1,$$

then in $\mathcal{D}_s$, $A = A_s$ splits as $A_s = A_s^0 \cup O_s \cup A_s^1$ and A_s^1 splits as $A_s^1 = A_s^{10} \cup O_s^1 \cup A_s^{11}$. We claim that

$$\tau(\mathcal{D}) \leq \tau(\mathcal{D}_s). \tag{14}$$

To prove (14) it suffices to show that

$$\min\left\{\frac{|A^0|}{|O|}, \frac{|A^1|}{|O|}, \frac{|A^{00}|}{|O^0|}, \frac{|A^{01}|}{|O^0|}\right\} \leq \min\left\{\frac{|A_s^0|}{|O_s|}, \frac{|A_s^1|}{|O_s|}, \frac{|A_s^{10}|}{|O_s^1|}, \frac{|A_s^{11}|}{|O_s^1|}\right\}. \tag{15}$$

Now,

$$\frac{|A_s^0|}{|O_s|} = \frac{|A^{00}|}{|O^0|}, \qquad \frac{|A_s^1|}{|O_s|} = \frac{|A^{01} \cup O \cup A^1|}{|O^0|} > \frac{|A^{01}|}{|O^0|},$$
$$\frac{|A_s^{10}|}{|O_s^1|} = \frac{|A^{01}|}{|O|} > \frac{|A^{01}|}{|O^0|}, \qquad \frac{|A_s^{11}|}{|O_s^1|} = \frac{|A^1|}{|O|},$$

since $|O| < |O^1|$, and so (15) holds.

We construct our ordered derivation $\mathcal{D}_o$ as follows. First we modify $\mathcal{D}$ to form a new tree $\mathcal{D}^1$ with the property that the first open interval removed is of maximal size. We form this tree by switching (a finite number of times) the order in which open intervals are removed in $\mathcal{D}$, as outlined above. Next we perform the same process on the bridges of level 1 in $\mathcal{D}^1$, forming a new derivation $\mathcal{D}^2$ which has its first two levels ordered. We continue this procedure inductively, forming $\mathcal{D}^{n+1}$ from $\mathcal{D}^n$ by switching the order in which open intervals are removed until for every bridge A of level n in $\mathcal{D}^{n+1}$, the next open interval removed from A is of maximal size. Our ordered derivation $\mathcal{D}_o$ is the derivation with the same root as $\mathcal{D}$ and for which the n^{th} level of $\mathcal{D}_o$ consists of the same bridges as the n^{th} level of $\mathcal{D}^n$.

We will use (14) to prove the first part of our lemma. For $k \in \mathbb{Z}^+$ let $\mathcal{O}_o^k$ be the set of all gaps between intervals in the k^{th} level of the derivation $\mathcal{D}_o$. Let n_k be the minimal number of levels of $\mathcal{D}$ we must descend before all intervals in $\mathcal{O}_o^k$ have been removed. Further, let D_o^k consist of all bridges occurring in the first k levels of $\mathcal{D}_o$, and let D^{n_k} denote the set of all bridges occurring in the first n_k levels of the derivation $\mathcal{D}$. Then for every $k \in \mathbb{Z}^+$ we have

$$\tau(\mathcal{D}) \leq \min_{A \in D^{n_k}} \tau_{\mathcal{D}}(A) \leq \min_{A \in D_o^k} \tau_{\mathcal{D}_o}(A)$$

by a finite number of applications of (14). Thus

$$\tau(\mathcal{D}_o) = \inf_k \min_{A \in D_o^k} \tau_{\mathcal{D}_o}(A) \geq \tau(\mathcal{D}),$$

as required.

Now assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two ordered derivations of C from I. Let $(t_j)_j$ be the sequence of different lengths of open intervals removed in the derivations, in decreasing order (note that both derivations remove the same set of intervals). If no intervals are removed, then $C = I$ and

$$\tau(\mathcal{D}_1) = \infty = \tau(\mathcal{D}_2).$$

Otherwise for every j let B_j be a bridge of minimal width in $\mathcal{D}_1$ such that, in the notation of section 2, $B_j = A^{wd}$ for some binary word w and $d \in \{0, 1\}$, with $|O^w| = t_j$. Then

$$\tau(\mathcal{D}_1) = \inf_j \frac{|B_j|}{t_j}.$$

However, B_j satisfies the same condition with $\mathcal{D}_1$ replaced by $\mathcal{D}_2$, whence

$$\tau(\mathcal{D}_2) = \inf_j \frac{|B_j|}{t_j},$$

and the lemma follows.

Lemma 3.2. *Let C_1 and C_2 be Cantor sets derived by derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Put*

$$\tau_1 = \tau(\mathcal{D}_1) \quad \text{and} \quad \tau_2 = \tau(\mathcal{D}_2).$$

If both τ_1 and τ_2 are greater than zero and neither C_1 nor C_2 contains an interval, then there exist bridges A and B of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively which split as

$$A = A^0 \cup O_1 \cup A^1 \quad \text{and} \quad B = B^0 \cup O_2 \cup B^1$$

such that

$$|A| \geq \frac{\tau_1}{\tau_1 + 1}(\tau_2 + 1)|O_2| \quad \text{and} \quad |B| \geq \frac{\tau_2}{\tau_2 + 1}(\tau_1 + 1)|O_1|.$$

Proof. Let $S = (A_i)_{i=1}^{\infty}$ be a sequence of bridges of $\mathcal{D}_1$, where if $\mathcal{D}_1$ contains a bridge of width t then $|A_i| = t$ for some i, and $|A_i| > |A_{i+1}|$ for $i \geq 1$. Since C_1 does not contain an interval, all A_i split, and $|A_i|$ tends to zero as i increases. We define the sequence $(B_j)_{j=1}^{\infty}$ from $\mathcal{D}_2$ in a similar manner. If O_1^i and O_2^j are the open intervals removed when A_i and B_j split, then

$$|O_1^i| \leq \frac{|A_i|}{2\tau_1 + 1} \quad \text{and} \quad |O_2^j| \leq \frac{|B_j|}{2\tau_2 + 1} \tag{16}$$

for $i, j \geq 1$. Therefore to prove the lemma it suffices to exhibit A_r and B_s with

$$\frac{\tau_2 + 1}{2\tau_2 + 1}\frac{\tau_1}{\tau_1 + 1} \leq \frac{|A_r|}{|B_s|} \leq \frac{2\tau_1 + 1}{\tau_1 + 1}\frac{\tau_2 + 1}{\tau_2},$$

or, equivalently,

$$\frac{\tau_2}{2\tau_2 + 1}|B_s| \leq \frac{\tau_1 + 1}{\tau_1}\frac{\tau_2}{\tau_2 + 1}|A_r| \leq \frac{2\tau_1 + 1}{\tau_1}|B_s|. \tag{17}$$

By (16), for $j \geq 1$ we have

$$\begin{aligned}\max\{|B_j^0|, |B_j^1|\} &\geq \frac{1}{2}\left(|B_j| - |O_2^j|\right)\\ &\geq \frac{1}{2}\left(|B_j| - \frac{|B_j|}{2\tau_2+1}\right)\\ &= \frac{\tau_2}{2\tau_2+1}|B_j|.\end{aligned}$$

Hence

$$\frac{|B_{j+1}|}{|B_j|} \geq \frac{\tau_2}{2\tau_2+1} \quad \text{for } j \geq 1.$$

Therefore

$$\frac{2\tau_1+1}{\tau_1}|B_j| \geq |B_j| \quad \text{and} \quad \frac{\tau_2}{2\tau_2+1}|B_j| \leq |B_{j+1}|,$$

so to establish (17) it is enough to find r and s such that

$$|B_{s+1}| \leq \frac{\tau_1+1}{\tau_1}\frac{\tau_2}{\tau_2+1}|A_r| \leq |B_s|. \tag{18}$$

Since $\{|A_i|\}_i$ and $\{|B_j|\}_j$ are both sequences which are monotonically decreasing to zero and $\tau_2 \neq 0$, (18) must have a solution (r, s). The lemma follows. □

In the next proof we shall make use of the concept of compatibility of bridges, which is similar to an approach used by Hlavka ([6], Theorem 3).

Lemma 3.3. *For $j = 1, 2$ let C_j be a Cantor set derived from I_j with $\overline{O}_j$ the largest gap in C_j. Let $S_\gamma = \gamma(C_1) + \gamma(C_2)$.*

1. *Let α' and β' be any positive real numbers for which $\alpha'\beta' = \tau(C_1)\tau(C_2)$, and put $\alpha = \min\{1, \alpha'\}$ and $\beta = \min\{1, \beta'\}$. If*

$$\beta|\overline{O}_1| \leq |I_2| \quad \textit{and} \quad \alpha|\overline{O}_2| \leq |I_1|, \tag{19}$$

then

$$\tau(C_1 + C_2) \geq \min\left\{\frac{\tau(C_1)+\beta}{1-\beta}, \frac{\tau(C_2)+\alpha}{1-\alpha}\right\}.$$

2. *If $|\overline{O}_1| \leq |I_2|$, $|\overline{O}_2| \leq |I_1|$ and $S_\gamma \geq 1$, then*

$$C_1 + C_2 = I_1 + I_2.$$

3. *If (19) holds with*

$$\alpha' = \gamma(C_1)(\tau(C_2)+1), \quad \beta' = \gamma(C_2)(\tau(C_1)+1)$$

and $S_\gamma < 1$, then

$$\tau(C_1 + C_2) \geq \frac{S_\gamma}{1 - S_\gamma}.$$

4. *If $S_\gamma \geq 1$ then $C_1 + C_2$ contains an interval. Otherwise $C_1 + C_2$ contains a Cantor set of thickness at least*

$$\frac{S_\gamma}{1-S_\gamma}.$$

Proof. We first prove part 1. Assume that (19) holds, and set

$$\tau = \min\left\{\frac{\tau(C_1)+\beta}{1-\beta}, \frac{\tau(C_2)+\alpha}{1-\alpha}\right\}.$$

We will show that $\tau(C_1+C_2) \geq \tau$. To do so we will construct a tree of valence 2 to represent C_1+C_2. This tree might not be a derivation, since bridges of the tree may overlap. However, we will use this tree to construct a derivation of C_1+C_2 with the required thickness.

We will construct our first tree inductively, by setting the root to be I_1+I_2 and showing how each bridge in the tree splits. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be ordered derivations of C_1 and C_2 respectively. If A and B are bridges of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, we say that A and B are *compatible*, and write $A \sim B$, if

$$|A| \geq \alpha|O_2| \quad \text{and} \quad |B| \geq \beta|O_1|,$$

where A and B split as

$$A = A^0 \cup O_1 \cup A^1$$

and

$$B = B^0 \cup O_2 \cup B^1.$$

If A does not split but B does, then we say $A \sim B$ if $|A| \geq \alpha|O_2|$, and similarly if A splits but B does not, then $A \sim B$ if $|B| \geq \beta|O_1|$. Finally, for all bridges A and B, neither of which split, we put $A \sim B$.

We shall construct a derivation for C_1+C_2 using the derivations of C_1 and C_2. Let A and B be bridges of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively with $A \sim B$, and set $D = A+B$. Assume first that both A and B split. Then

$$\frac{\min\{|A^0|, |A^1|\}}{|O_1|} \cdot \frac{\min\{|B^0|, |B^1|\}}{|O_2|} \geq \tau(C_1)\tau(C_2) \geq \alpha\beta,$$

so

$$\min\{|A^0|, |A^1|\} \min\{|B^0|, |B^1|\} \geq \alpha\beta|O_1||O_2|.$$

Thus either

$$\min\{|A^0|, |A^1|\} \geq \alpha|O_2| \tag{20}$$

or

$$\min\{|B^0|, |B^1|\} \geq \beta|O_1|. \tag{21}$$

Assume that (20) holds, and let O_1^0 and O_1^1 be the open intervals removed in the splitting of A^0 and A^1 respectively. Since the derivations are ordered,

$$\beta|O_1^0| \leq \beta|O_1| \leq |B| \quad \text{and} \quad \beta|O_1^1| \leq \beta|O_1| \leq |B|,$$

as $A \sim B$. By (20) we have

$$\alpha|O_2| \leq |A^0| \quad \text{and} \quad \alpha|O_2| \leq |A^1|,$$

whence

$$A^0 \sim B \quad \text{and} \quad A^1 \sim B.$$

We put

$$D^0 = A^0 + B, \quad D^1 = A^1 + B, \quad O_D = D \setminus (D^0 \cup D^1). \tag{22}$$

We have

$$|D^0| = |A^0| + |B|, \quad |D^1| = |A^1| + |B|$$

and, if O_D is non-empty,

$$|O_D| = |O_1| - |B|.$$

Note that if $\beta = 1$ then O_D is necessarily empty. Thus either there is no gap between D^0 and D^1, or

$$\frac{\min\{|D^0|, |D^1|\}}{|O_D|} = \frac{\min\{|A^0|, |A^1|\} + |B|}{|O_1| - |B|} \geq \frac{\tau(C_1) + \beta}{1 - \beta} \geq \tau. \tag{23}$$

For $d = 0, 1$, to determine the splitting of D^d we repeat the above process with A replaced with A^d.

If we find that (21) holds instead of (20), we perform the same process, except we split B instead of A. Again we may bound $\min\{|D^0|, |D^1|\}/|O_D|$ by τ.

If A splits but B does not, then we define D^0, D^1 and O_D as in (22), and find that either O_D is empty or (23) holds. If B splits but A does not, then we proceed in an analogous manner. Finally, if neither A nor B splits, then we let D be the vertex $A + B$, and place under D an infinite stalk composed of vertices D^w where w is a binary word composed of zeros, and $D^w = D$ as intervals.

Since $I_1 \sim I_2$ we find by induction that we may construct a tree T_S of closed intervals $\{D^w\}$ such that

$$C_1 + C_2 = \bigcap_{m \geq 0} \bigcup_{w} D^w,$$

where the union is taken over all binary words w of length m such that D^w is a vertex of T_S. We further have that if V is a vertex of T_S, then either V does not split or

$$\min\{|V^0|, |V^1|\} \geq \tau |O_V|. \tag{24}$$

Now T_S might not be a derivation of $C_1 + C_2$, since we may have some overlap of intervals associated with vertices. We will however use T_S to construct a derivation for $C_1 + C_2$ with the required thickness. Let $H^0 = I_1 + I_2$, $H^1 = D^0 \cup D^1$ and in general

$$H^m = \bigcup_{w} D^w,$$

where the union is over all binary words of length m with D^w in T_S. For each m, H^m will be the union of a finite number of disjoint closed intervals $\{H_i^m\}$. We next define a tree T_H by taking as vertices all intervals $\{H_i^m\}$ and as edges all lines joining vertices H_i^m to H_j^{m+1}, where $H_j^{m+1} \subseteq H_i^m$ as sets. We will convert T_H into a tree where every vertex has at most two subvertices. Let N be a vertex of T_H. We will construct a finite tree T_N with root N and having as leaves the subvertices of N in T_H such that T_N is of valence 2 and T_N satisfies a condition similar to (24).

Let N have subvertices $N_1, N_2, \ldots, N_t$ in T_H. If $t \leq 2$ then we let T_N be the tree with root N and leaves $N_1, \ldots, N_t$. Otherwise, we have that, as intervals,

$$N = N_1 \cup G_1 \cup N_2 \cup G_2 \cup \cdots \cup G_{t-1} \cup N_t, \tag{25}$$

where $G_1, \dots, G_{t-1}$ are open intervals. For intervals J_1 and J_2 we write $J_1 \to J_2$ if $|J_1| \geq \tau |J_2|$. We start by making the following claim.

Claim 1. Let G_r and G_s be two open intervals in (25) with $r < s$. Let J denote the entire closed interval between G_r and G_s. Then $J \to G_r$ or $J \to G_s$. Further, if G_r is any open interval in (25), then

$$\left(N_r \cup \bigcup_{1 \leq n < r} (N_n \cup G_n) \right) \to G_r \tag{26}$$

and

$$\left(N_t \cup \bigcup_{r < n < t} (N_n \cup G_n) \right) \to G_r \tag{27}$$

Proof of Claim 1. Since J contains points of $C_1 + C_2$ and $(C_1 + C_2) \cap (G_r \cup G_s) = \emptyset$, there exists a vertex $V = V^0 \cup O_V \cup V^1$ of T_S with $V \cap J \neq \emptyset$ and either $G_r \subseteq O_V$ or $G_s \subseteq O_V$. Assume without loss of generality that $G_r \subseteq O_V$. If $V^1 \subseteq J$ then

$$\frac{|J|}{|G_r|} \geq \frac{|V^1|}{|O_V|} \geq \tau,$$

so $J \to G_r$. Otherwise $G_s \subseteq V^1$, and since $(C_1 + C_2) \cap G_s = \emptyset$ there exists a vertex $W = W^0 \cup O_W \cup W^1$ in T_S with $W \subseteq V^1$ and $G_s \subseteq O_W$. In this case $W^0 \subseteq J$, so

$$\frac{|J|}{|G_s|} \geq \frac{|W^0|}{|O_W|} \geq \tau.$$

So $J \to G_s$, and the first part of the claim follows.

To prove the second part of the claim we denote by J_r^0 and J_r^1 the left sides of (26) and (27) respectively. As above, we have a vertex V of T_S with $V^0 \subseteq J_r^0$, $V^1 \subseteq J_r^1$ and $G_r \subseteq O_V$, and the claim follows. □

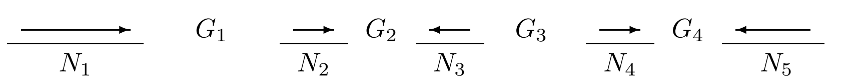
By the claim we have

$$N_1 \to G_1, \qquad N_t \to G_{t-1}$$

and

$$N_j \to G_{j-1} \qquad \text{or} \qquad N_j \to G_j$$

for $j = 2, \dots, t-1$. For example,



Thus there must be some G_{r_1} with

$$N_{r_1} \to G_{r_1} \quad \text{and} \quad N_{r_1+1} \to G_{r_1}. \tag{28}$$

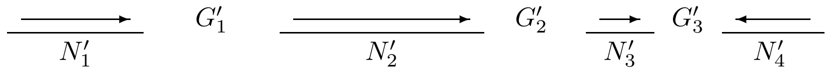
We set $t' = t - 1$,

$$N'_j = \begin{cases} N_j & \text{if } 1 \leq j < r_1, \\ N_{r_1} \cup G_{r_1} \cup N_{r_1+1} & \text{if } j = r_1, \\ N_{j+1} & \text{if } r_1 < j \leq t', \end{cases}$$

and

$$G'_j = \begin{cases} G_j & \text{if } 1 \leq j < r_1, \\ G_{j+1} & \text{if } r_1 \leq j \leq t'. \end{cases}$$

By the claim, $N'_{r_1} \to G'_{r_1 - 1}$ or $N'_{r_1} \to G'_{r_1}$, i.e.



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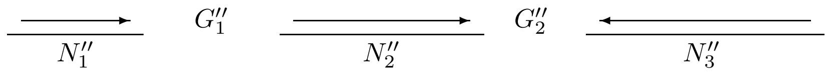
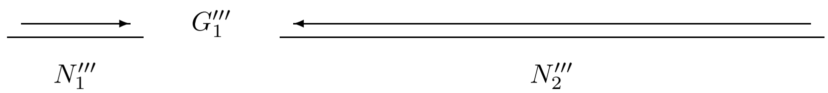
We continue this process until we have only two closed intervals left. In our example, the next step results in

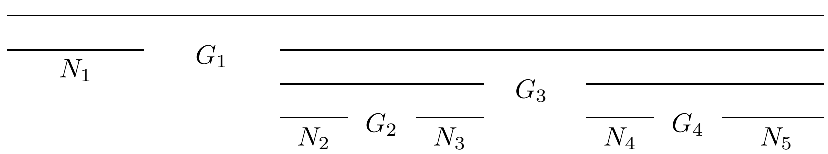


while the last step yields



.

We are now ready to construct our finite tree T_N. Let G_{r_i} be the open interval satisfying (28) at the i^{th} step, for $i = 1, \ldots, t-2$. Further, let $G_{r_{t-1}}$ be the open interval remaining when our process terminates. We form T_N by removing, in order, the open intervals $G_{r_{t-1}}, G_{r_{t-2}}, \ldots, G_{r_1}$:



By our construction, if N^w is a vertex in T_N which splits as

$$N^w = N^{w0} \cup O^w_N \cup N^{w1},$$

then

$$|N^{w0}| \geq \tau |O^w_N| \quad \text{and} \quad |N^{w1}| \geq \tau |O^w_N|.$$

To construct our derivation $\mathcal{D}$ of $C_1 + C_2$, we take as vertices and edges of $\mathcal{D}$ the sets

$$V = \bigcup_{N \in T_S} V(T_N) \quad \text{and} \quad E = \bigcup_{N \in T_S} E(T_N)$$

respectively, where for a tree T we denote the set of vertices of T by $V(T)$ and the set of edges by $E(T)$. We have $\tau(\mathcal{D}) \geq \tau$, and the first part of the lemma follows.

We will use part 1 of the lemma to prove parts 2 and 3. Let

$$\alpha' = \gamma(C_1)(\tau(C_2) + 1) \quad \text{and} \quad \beta' = \gamma(C_2)(\tau(C_1) + 1) \tag{29}$$

and define α and β by

$$\alpha = \min\{1, \alpha'\} \quad \text{and} \quad \beta = \min\{1, \beta'\}. \tag{30}$$

Assume that $|\overline{O}_1| \leq |I_2|$, $|\overline{O}_2| \leq |I_1|$ and $S_\gamma \geq 1$. Then $\tau(C_1)\tau(C_2) \geq 1$, which implies that $\alpha = \beta = 1$. Therefore, by part 1, $\tau(C_1 + C_2) = \infty$, and part 2 follows.

To prove part 3 we first define α', β', α and β by (29) and (30). Note that if $S_\gamma < 1$ then $\alpha = \alpha'$ and $\beta = \beta'$; hence

$$\frac{\tau(C_1)+\beta}{1-\beta} = \frac{\tau(C_1)+\beta'}{1-\beta'} = \frac{S_\gamma}{1-S_\gamma}$$

and

$$\frac{\tau(C_2)+\alpha}{1-\alpha} = \frac{\tau(C_2)+\alpha'}{1-\alpha'} = \frac{S_\gamma}{1-S_\gamma},$$

so by part 1 of the lemma

$$\tau(C_1+C_2) \geq \frac{S_\gamma}{1-S_\gamma},$$

and part 3 follows.

To prove part 4 we first note that if $\tau(C_1) = 0$ or $\tau(C_2) = 0$ then the result follows trivially, whence we may assume $\tau(C_1)$ and $\tau(C_2)$ are both greater than zero. If either C_1 or C_2 contains a bridge that does not split, then $C_1 + C_2$ will contain an interval, hence a set of infinite thickness. Otherwise, by Lemma 3.2 there exist bridges A and B of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, with

$$A = A^0 \cup O_1 \cup A^1 \quad \text{and} \quad B = B^0 \cup O_2 \cup B^1,$$

such that

$$|A| \geq \alpha|O_2| \quad \text{and} \quad |B| \geq \beta|O_1|,$$

where α and β are as defined in (30). By parts 2 and 3 of Lemma 3.3 applied to the Cantor sets

$$C_A = C_1 \cap A \quad \text{and} \quad C_B = C_2 \cap B$$

we have

$$C_A + C_B = A + B$$

if $S_\gamma \geq 1$ and

$$\tau(C_A + C_B) \geq \frac{S_\gamma}{1-S_\gamma}$$

otherwise, and part 4 of the lemma follows. □

To relate thickness to Hausdorff dimension we use the following result.

Lemma 3.4. *If C is a Cantor set, then*

$$\dim_H(C) \geq \frac{\log 2}{\log\left(2 + \frac{1}{\tau(C)}\right)}.$$

Proof. See [9], p. 77. □

Proof of Theorem 2.4. For real numbers γ_1, γ_2 and γ_3 in $[0,1]$ with $\gamma_1 + \gamma_2 < 1$ we put

$$\tau_{12} = \frac{\gamma_1+\gamma_2}{1-\gamma_1-\gamma_2}.$$

Note that,

$$\frac{\tau_{12}}{1+\tau_{12}} = \gamma_1 + \gamma_2$$

so

$$\frac{\tau_{12}}{1+\tau_{12}} + \gamma_3 = \gamma_1 + \gamma_2 + \gamma_3. \tag{31}$$

We first prove part 1. Assume $S_\gamma \geq 1$ and let t be the smallest integer with $\gamma(C_1) + \cdots + \gamma(C_t) \geq 1$. Using Lemma 3.3 (part 4) and (31), we find by induction that $C_1 + \cdots + C_t$ contains an interval, whence $C_1 + \cdots + C_k$ contains an interval.

If $S_\gamma < 1$ then we find by Lemma 3.3 (part 4), (31) and induction that $C_1 + \cdots + C_k$ contains a Cantor set of thickness at least $S_\gamma/(1 - S_\gamma)$, so by Lemma 3.4,

$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log(1 + \frac{1}{S_\gamma})}$$

and part 1 of the theorem follows.

To prove parts 2 and 3 we first note that by (12) and (13) the sets $I_1 + \cdots + I_r$ and I_{r+1} satisfy (19) with $\alpha = \beta = 1$, for $r = 1, \ldots, k-1$. We find by induction, Lemma 3.3 (part 2) and (31) that if $S_\gamma \geq 1$, then

$$C_1 + \cdots + C_k = I_1 + \cdots + I_k,$$

and part 2 of the theorem follows. Similarly, if $S_\gamma < 1$, then by induction, Lemma 3.3 (part 3) and (31) we have

$$\tau(C_1 + \cdots + C_k) \geq \frac{S_\gamma}{1 - S_\gamma},$$

and the theorem follows. □

4. Bounds on the Thickness of $C(B)$

To apply Theorems 2.2 and 2.4 to the cases where the Cantor sets are of the form $C(B_j)$ for some $B_j \subseteq \mathbb{Z}^+$, we need only calculate the thicknesses of the Cantor sets in question.

For $n \geq 0$ we define the n^{th} *convergent* to the continued fraction $[a_0, a_1, \ldots]$ to be the rational number

$$\frac{p_n}{q_n} = [a_0, \ldots, a_n]$$

where p_n and q_n are taken to be coprime. We also define p_n and q_n for $n = -2$ or $n = -1$ by

$$p_{-2} = q_{-1} = 0 \quad \text{and} \quad p_{-1} = q_{-2} = 1.$$

By elementary properties of continued fractions we have

$$p_n = a_n p_{n-1} + p_{n-2}$$

and

$$q_n = a_n q_{n-1} + q_{n-2} \tag{32}$$

for $n \geq 0$. We also have the following result.

Lemma 4.1. *For a fixed $r \geq 0$ and $1 \leq i \leq 4$ assume that $G_i = [a_0, a_1, \ldots, a_r, g_i]$ for some real $g_i > 0$. For $0 \leq n \leq r$ let p_n/q_n be the n^{th} convergent to $[a_0, \ldots, a_r]$, and put $Q = q_{r-1}/q_r$. Then*

1.

$$|G_1 - G_2| = \frac{|g_1 - g_2|}{q_r^2(g_1+Q)(g_2+Q)}$$

and

2.

$$\left|\frac{G_1 - G_2}{G_3 - G_4}\right| = \left|\frac{g_1 - g_2}{g_3 - g_4}\right| \cdot \frac{(g_3+Q)(g_4+Q)}{(g_1+Q)(g_2+Q)}.$$

Proof. See [6], Lemmas 4 and 5. □

Lemma 4.2. *Let $t \geq 2$ be an integer and $B = \{b_1, b_2, \ldots, b_t\}$ a finite set of positive integers with $b_i < b_{i+1}$ for $i = 1, 2, \ldots, t-1$. Let $l = b_1$ and $m = b_t$, and set $\Delta_i = b_{i+1} - b_i$ for $i = 1, 2, \ldots, t-1$. Put*

$$\delta = \frac{-lm + \sqrt{l^2m^2 + 4lm}}{2}.$$

Then

$$\tau(\mathcal{D}(B)) = \min_{1 \leq i < t} \min \left\{ \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_{i+1}lm + m + \delta l}{b_i lm + m + \delta l}, \right. \\ \left. \frac{(m - b_{i+1})lm + \delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_i l + \delta}{lm + \delta} \right\}.$$

Proof. Assume that our bridge is of the form (9). To compute a lower bound for $|A^0|/|O|$ we use part 2 of Lemma 4.1 with

$$g_1 = [b_i, \overline{l, m}], \quad g_2 = [b_i, \overline{m, l}], \quad g_3 = [b_{i+1}, \overline{m, l}], \quad g_4 = [b_i, \overline{l, m}]$$

to find that

$$\begin{aligned} \frac{|A^0|}{|O|} &= \frac{(b_i + \langle \overline{l, m} \rangle) - (b_i + \langle \overline{m, l} \rangle)}{(b_{i+1} + \langle \overline{m, l} \rangle) - (b_i + \langle \overline{l, m} \rangle)} \cdot \frac{b_{i+1} + \langle \overline{m, l} \rangle + Q}{b_i + \langle \overline{m, l} \rangle + Q} \\ &= \frac{\langle \overline{l, m} \rangle - \langle \overline{m, l} \rangle}{\Delta_i + \langle \overline{m, l} \rangle - \langle \overline{l, m} \rangle} \cdot \frac{b_{i+1} + \langle \overline{m, l} \rangle + Q}{b_i + \langle \overline{m, l} \rangle + Q} \\ &= \frac{\delta/l - \delta/m}{\Delta_i + \delta/m - \delta/l} \cdot \frac{b_{i+1} + \delta/m + Q}{b_i + \delta/m + Q} \\ &= \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{(b_{i+1}+Q)m + \delta}{(b_i + Q)m + \delta}. \qquad (33) \end{aligned}$$

Similarly we use part 2 of Lemma 4.1 with

$$g_1 = [m, \overline{l, m}], \quad g_2 = [b_{i+1}, \overline{m, l}], \quad g_3 = [b_{i+1}, \overline{m, l}], \quad g_4 = [b_i, \overline{l, m}]$$

and find that

$$\frac{|A^1|}{|O|} = \frac{(m - b_{i+1})lm + \delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{(b_i+Q)l + \delta}{(m+Q)l + \delta}.$$

Thus

$$\tau(\mathcal{D}(B)) = \inf_{Q} \min_{1\le i<t} \min\left\{ \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{(b_{i+1}+Q)m+\delta}{(b_i+Q)m+\delta}, \right.$$
$$\left. \frac{(m-b_{i+1})lm+\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{(b_i+Q)l+\delta}{(m+Q)l+\delta} \right\}$$
$$= \min_{1\le i<t} \min\left\{ \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_{i+1}lm+m+\delta l}{b_i lm+m+\delta l}, \right.$$
$$\left. \frac{(m-b_{i+1})lm+\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_i l+\delta}{lm+\delta} \right\}$$

since $0 \le Q \le 1/l$, and

$$\frac{q_{-1}}{q_0} = 0 \quad \text{and} \quad \frac{q_0}{q_1} = \frac{1}{l}$$

if $a_1 = l$. □

A similar but simpler result holds in the infinite case.

Lemma 4.3. *Let $B = \{b_1, b_2, \dots\}$ be an infinite set of integers with $b_i < b_{i+1}$ for $i \ge 1$. Let $l = b_1$ and set $\Delta_i = b_{i+1} - b_i$ for $i \ge 1$. Then*

$$\tau(\mathcal{D}(B)) = \inf_{i\ge 1} \min\left\{ \frac{1}{\Delta_i l - 1} \cdot \frac{b_{i+1}l+1}{b_i l+1}, \frac{b_i l+1}{\Delta_i l - 1} \right\}.$$

Proof. We use the same strategy as in the proof of Lemma 4.2. If A is a bridge of the form (10), then by part 2 of Lemma 4.1, with

$$g_1 = [b_i, l], \quad g_2 = b_i, \quad g_3 = b_{i+1}, \quad g_4 = [b_i, l],$$

we find that

$$\frac{|A^0|}{|O|} = \frac{1}{\Delta_i l - 1} \cdot \frac{b_{i+1}+Q}{b_i+Q}. \tag{34}$$

Now if $r = 0$ then

$$\frac{|A^1|}{|O|} = \frac{b_i l+1}{\Delta_i l - 1},$$

while if $r > 0$ we apply part 2 of Lemma 4.1 with

$$g_1 = [a_r, b_{i+1}], \quad g_2 = a_r, \quad g_3 = [a_r, b_i, l] \quad \text{and} \quad g_4 = [a_r, b_{i+1}]$$

and conclude that

$$\begin{aligned}\frac{|A^1|}{|O|} &= \frac{\langle b_{i+1}\rangle}{\langle b_i, l\rangle - \langle b_{i+1}\rangle} \cdot \frac{[a_r, b_i, l] + \frac{q_{r-2}}{q_{r-1}}}{a_r + \frac{q_{r-2}}{q_{r-1}}} \\ &= \frac{b_i l+1}{\Delta_i l - 1} \cdot \frac{a_r q_{r-1} + q_{r-2} + q_{r-1}\langle b_i, l\rangle}{a_r q_{r-1} + q_{r-2}} \\ &= \frac{b_i l+1}{\Delta_i l - 1} \cdot \frac{q_r + q_{r-1}\langle b_i, l\rangle}{q_r} \\ &= \frac{(b_i+Q)l+1}{\Delta_i l - 1} \ge \frac{b_i l+1}{\Delta_i l - 1}\end{aligned}$$

by (32) and since $Q \geq 0$. Therefore

$$\tau(\mathcal{D}(B)) = \inf_Q \inf_i \min\left\{\frac{1}{\Delta_i l - 1}\cdot\frac{b_{i+1}+Q}{b_i+Q}, \frac{b_i l+1}{\Delta_i l - 1}\right\}, \tag{35}$$

and the lemma follows upon minimizing (35), since as in the proof of Lemma 4.2 we have $0 \leq Q \leq 1/l$ with $Q = 0$ if $r = 0$ and $Q = 1/l$ if $r = 1$ and $a_1 = l$. □

Note that $\tau(\mathcal{D}(B))$ equals $\tau(B)$ (as defined in the first section).

Lemma 4.4. *Let B be a set of positive integers with $|B| > 1$. If $\Delta_i(B) = \Delta$ is constant, then $\mathcal{D}(B)$ is ordered, and so*

$$\tau(C(B)) = \tau(\mathcal{D}(B)) = \tau(B).$$

Proof. Assume first that B is finite. In the notation of Lemma 4.2 put

$$O\left((a_1,\ldots,a_r),b_i\right) = \left((\langle a_1,\ldots,a_r,b_i,\overline{l,m}\rangle, \langle a_1,\ldots,a_r,b_{i+1},\overline{m,l}\rangle)\right)$$

for $b_i < m$. Then by part 1 of Lemma 4.1 with

$$g_1 = [b_{i+1},\overline{m,l}] \quad \text{and} \quad g_2 = [b_i,\overline{l,m}]$$

we have

$$|O\left((a_1,\ldots,a_r),b_i\right)| = \frac{\Delta+\delta/m-\delta/l}{q_r^2(b_{i+1}+\delta/m+Q)(b_i+\delta/l+Q)}.$$

Thus

$$|O\left((a_1,\ldots,a_r),b_i\right)| > |O\left((a_1,\ldots,a_r),b_j\right)|$$

for $j > i$, and

$$|O\left((a_1,\ldots,a_r),b_i\right)| > |O\left((a_1,\ldots,a_r,b_i),b_j\right)|$$

for $b_j \in B$, so $\mathcal{D}(B)$ is an ordered derivation. By Lemma 3.1 we have $\tau(C(B)) = \tau(\mathcal{D}(B))$, and Lemma 4.4 follows for B finite.

If B is infinite then we use an analogous approach, where in this case we define

$$O\left((a_1,\ldots,a_r),b_i\right) = \left((\langle a_1,\ldots,a_r,b_i,l\rangle, \langle a_1,\ldots,a_r,b_{i+1}\rangle)\right).$$

□

5. Proofs of Results in the Additive Case

For n an integer and B a set of positive integers with $|B| > 1$ we define $C(n;B)$ by

$$C(n;B) = n + C(B).$$

Using the derivation $\mathcal{D}(B)$ of $C(B)$, we may construct the canonical derivation $n+\mathcal{D}(B)$ of $C(n;B)$ from $n+I(B)$ by translating every interval in $\mathcal{D}$ by n. Similarly we may construct the canonical derivation $n-\mathcal{D}(B)$ of $n-C(B)$ from $n-I(B)$.

Proof of Theorem 1.6. Put

$$S_\gamma = \sum_{j=1}^{k} \gamma(B_j).$$

Assume first that $S_\gamma \geq 1$, and for $N \geq 1$ and $j = 1, \dots, k$ set

$$C_j^N = \bigcup_{n=-N}^{N} C(n; B_j)$$

and

$$I_j^N = [-N + \min C(B_j), N + \max C(B_j)].$$

For $j = 1, \dots, k$ we construct a derivation $\mathcal{D}_j^N$ of C_j^N from I_j^N as follows. Assume that $|I(B_j)| < 1$. Remove from I_j^N the interval $(n + \max C(B_j), n + 1 + \min C(B_j))$ for $n = -N, \dots, N-1$, so that if $A_j = I_j^N$ then

$$\begin{aligned} A_j^0 &= -N + I(B_j), & A_j^1 &= [-N + 1 + \min C(B_j), N + \max C(B_j)], \\ A_j^{10} &= -N + 1 + I(B_j), & A_j^{11} &= [-N + 2 + \min C(B_j), N + \max C(B_j)] \end{aligned}$$

and, ultimately,

$$A_j^{1\cdots 10} = N - 1 + I(B_j) \quad \text{and} \quad A_j^{1\cdots 11} = N + I(B_j). \tag{36}$$

We complete $\mathcal{D}_j^N$ by using the derivations $n + \mathcal{D}(B_j)$, $n = -N, \dots, N$, to split A_j^0, $A_j^{10}, \dots, A_j^{1\cdots 10}$ and $A_j^{1\cdots 11}$. Note that if B_j is finite, then

$$\begin{aligned} \frac{|n + I_j|}{|(n + \max C(B_j), n + 1 + \min C(B_j))|} &= \frac{\langle \overline{l, m} \rangle - \langle \overline{m, l} \rangle}{1 + \langle \overline{m, l} \rangle - \langle \overline{l, m} \rangle} \\ &= \frac{\delta(m - l)}{ml - \delta(m - l)} > \tau(B_j), \end{aligned} \tag{37}$$

and that if B_j is infinite, then

$$\frac{|n + I_j|}{|(n + \max C(B_j), n + 1 + \min C(B_j))|} = \frac{1}{l - 1} \geq \tau(B_j), \tag{38}$$

so in either case $\tau(\mathcal{D}_j^N) \geq \tau(B_j)$.

If $|I_j| = 1$, then we form $\mathcal{D}_j^N$ by removing the gaps in C_j^N in order of descending width, so that again we have $\tau(\mathcal{D}_j^N) \geq \tau(B_j)$.

For $j = 1, \dots, k$, I_j^N has width greater than 2, and all gaps in C_j^N are of width less than 1, whence (12) and (13) hold. Since $S_\gamma \geq 1$ and for a Cantor set C we have $\tau(-C) = \tau(C)$, by part 2 of Theorem 2.4 we get

$$\begin{aligned} \epsilon_1 C_1^N + \cdots + \epsilon_k C_k^N &= \epsilon_1 I_1^N + \cdots + \epsilon_k I_k^N \\ &\supseteq [-k(N-1), k(N-1)], \end{aligned}$$

and (5) follows upon letting N tend to infinity.

If $S_\gamma < 1$, then (6) follows from part 1 of Theorem 2.4 with $C_j = C(B_j)$ for $j = 1, \dots, k$. □

Proof of Theorem 1.2. Theorem 1.2 is a special case of Theorem 1.6, since

$$\gamma(B_1) + \gamma(B_2) \geq 1$$

if and only if

$$\tau(B_1)\tau(B_2) \geq 1,$$

and, further,

$$\frac{\gamma(B_1)+\gamma(B_2)}{1-\gamma(B_1)-\gamma(B_2)} = \frac{\tau(B_1)+\tau(B_2)+2\tau(B_1)\tau(B_2)}{1-\tau(B_1)\tau(B_2)}.$$

□

Proof of Theorem 1.1. As shown by Diviš [4], we have

$$F(3)+F(3)\neq\mathbb{R}.$$

Also, Hlavka [6] established that

$$F(2)+F(4)\neq\mathbb{R} \quad \text{and} \quad F(3)+F(4)=\mathbb{R}.$$

Using Theorem 1.2, we find that

$$F(3)-F(4)=\mathbb{R},$$

and from work to appear [1] we have that

$$F(2)+F(5)=F(2)-F(5)=F(3)-F(3)=\mathbb{R}.$$

Since

$$I(L_2)-I(L_4)\subseteq[-0.462\ldots,0.524\ldots],$$

we know that

$$F(2)-F(4)\neq\mathbb{R},$$

and the theorem follows. □

Proof of Corollary 1.3. Note that $\delta(L_m)>m/(m+1)$, whence

$$\tau(L_m) > \frac{m(m-1)}{m(m+1)-m(m-1)}\cdot\frac{(m-1)(m+1)+m}{m(m+1)+m} > \frac{(m-1)^2}{2m}.$$

Since

$$\tau(U_l)=\frac{1}{l-1}\geq\frac{2}{m-2},$$

we have

$$\tau(L_m)\tau(U_l) > \frac{(m-1)^2}{m(m-2)} > 1,$$

and the result follows from part 1 of Theorem 1.2. □

Before proving Corollary 1.4 we need a preliminary lemma.

Lemma 5.1. *If B is a finite set of odd positive integers, then $1\notin 2C(B)$.*

Proof. Let $m=\max B$ and assume that $1\in 2C(B)$. Then $1\in S$, where

$$S=\left[\langle a_1,a_2,\overline{m,1}\rangle,\langle a_1,a_2,\overline{1,m}\rangle\right]+\left[\langle b_1,b_2,\overline{m,1}\rangle,\langle b_1,b_2,\overline{1,m}\rangle\right]$$

for some odd a_1, a_2, b_1 and b_2 between 1 and m inclusive. Now if both a_1 and b_1 are greater than 1 then $1\notin S$, so we may assume without loss of generality that $a_1=1$. Thus

$$S=\left[\frac{a_2+\theta}{a_2+\theta+1}+\frac{b_2+\theta}{b_1(b_2+\theta)+1},\frac{a_2+\rho}{a_2+\rho+1}+\frac{b_2+\rho}{b_1(b_2+\rho)+1}\right],$$

where $\theta = \langle \overline{m, 1} \rangle$ and $\rho = \langle \overline{1, m} \rangle$. Therefore we have

$$1 \geq \frac{a_2 + \theta}{a_2 + \theta + 1} + \frac{b_2 + \theta}{b_1(b_2 + \theta) + 1} \tag{39}$$

and

$$1 \leq \frac{a_2 + \rho}{a_2 + \rho + 1} + \frac{b_2 + \rho}{b_1(b_2 + \rho) + 1}. \tag{40}$$

It can be shown that (39) and (40) are equivalent to

$$a_2 + 1 - b_1 \leq \frac{1}{b_2 + \theta} - \theta \tag{41}$$

and

$$a_2 + 1 - b_1 \geq \frac{1}{b_2 + \rho} - \rho \tag{42}$$

respectively. But for any integer $n \geq 1$ and real $x \in (0, 1)$ we have

$$-1 < \frac{1}{n + x} - x < 1,$$

and so by (41) and (42) we must have $a_2 + 1 - b_1 = 0$. But this is not possible, since both a_2 and b_1 are odd, and the lemma follows. □

Proof of Corollary 1.4. We find that $\tau(B_o) = 1$, and so, by part 1 of Theorem 1.2,

$$F(B_o) + F(B_o) = \mathbb{R}.$$

Now if B is a finite set of positive odd integers, then $0 \notin 2C(B)$ and $2 \notin 2C(B)$. By Lemma 5.1 we have $1 \notin 2C(B)$, whence

$$1 \notin \mathbb{Z} + 2C(B) = F(B) + F(B),$$

and the result follows. □

6. Products and Quotients

As in [5] and [2] we employ the logarithm function to treat products and quotients of Cantor sets. Given a set S of positive numbers, we form the set S^* by putting

$$S^* = \{\log x\,;\, x \in S\}.$$

If C is Cantor set of positive numbers, then C^* will also be a Cantor set. We construct a derivation $\mathcal{D}^*$ of C^* by taking our bridges to be of the form $[\log a, \log b]$, where $[a, b]$ is a bridge of our derivation $\mathcal{D}$ of C. To relate the Hausdorff dimension of S to that of S^* we will use Lemma 6.2.

Lemma 6.1. *Let E be a set of real numbers with $f : E \to \mathbb{R}$ such that for some positive constant c,*

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in E$. Then

$$\dim_H(f(E)) \leq \dim_H E.$$

Proof. See [7], p. 44. □

Lemma 6.2. *Let $S \subseteq [a,b]$ be a set of real numbers with $a > 0$. Then*

$$\dim_H S = \dim_H(S^*).$$

Proof. For every $x, y \in S$,

$$\frac{1}{b}|x-y| \leq |\log x - \log y| \leq \frac{1}{a}|x-y|,$$

and the lemma follows from Lemma 6.1. □

We have the following multiplicative analogue of Theorem 2.4.

Theorem 6.3. *Let k be a positive integer and for $j = 1, 2, \ldots, k$ let C_j be a Cantor set derived from $I_j \subseteq (0, \infty)$, with O_j a gap in C_j chosen so that $|O_j^*|$ is maximal. Put $S_\gamma = \gamma(C_1^*) + \cdots + \gamma(C_k^*)$.*

1. *If $S_\gamma \geq 1$, then $C_1 \cdots C_k$ contains an interval.*
2. *If $S_\gamma < 1$, then*

$$\dim_H(C_1 \cdots C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{S_\gamma}\right)}.$$

3. *If*

$$\begin{aligned} |I_{r+1}^*| &\geq |O_j^*| \qquad && \text{for } r = 1, \ldots, k-1 \text{ and } j = 1, \ldots, r, \\ |I_1^*| + \cdots + |I_r^*| &\geq |O_{r+1}^*| \qquad && \text{for } r = 1, \ldots, k-1, \end{aligned}$$

and $S_\gamma \geq 1$, then

$$C_1 \cdots C_k = I_1 \cdots I_k.$$

Proof. Note that by Lemma 6.2,

$$\dim_H(C_1 \cdots C_k) = \dim_H(C_1^* + \cdots + C_k^*).$$

We apply Theorem 2.4 to the Cantor sets $C_1^*, \ldots, C_k^*$, and the theorem follows. □

It remains to find a lower bound for $\tau(C^*)$. We start by generalizing a lemma of Cusick ([2], Lemma 2).

Lemma 6.4. *Let $E = [a,b] \subseteq (0, \infty)$ be an interval of real numbers. Suppose that $E = E_1 \cup O \cup E_2$, where*

$$E_1 = [a, a+r], \quad O = (a+r, a+r+s) \quad \text{and} \quad E_2 = [a+r+s, a+r+s+t].$$

If $s < a + r$ and $\tau > 0$ is a real number such that

$$\frac{t - \tau s}{s^2} \geq \frac{1}{a+r}\left(1 + \sum_{2 \leq n < \tau+1} \binom{\tau+1}{n}\right),$$

then

$$\frac{|E_2^*|}{|O^*|} \geq \tau.$$

Proof. We have $|E_2^*| \geq \tau |O^*|$ if and only if

$$\log(a+r+s+t) - \log(a+r+s) \geq \tau(\log(a+r+s) - \log(a+r)),$$

which is equivalent to

$$(a+r+s+t)(a+r)^\tau \geq (a+r+s)^{\tau+1},$$

or, alternatively,

$$1 + \frac{s+t}{a+r} \geq \left(1 + \frac{s}{a+r}\right)^{\tau+1}. \tag{43}$$

Using the power series expansion for $(1+x)^y$ for real y and $|x| < 1$, we find that (43) is equivalent to

$$\frac{t - \tau s}{a+r} \geq \sum_{n=2}^{\infty} \binom{\tau+1}{n} \left(\frac{s}{a+r}\right)^n. \tag{44}$$

Let R be the unique positive integer such that $R \leq \tau + 1 < R + 1$, and for $n \geq 2$ let C_n denote the binomial coefficient in (44). If $n > \tau + 1$, then

$$C_n = \binom{\tau+1}{n} = \frac{\tau+1}{n} \cdot \frac{\tau}{n-1} \cdots \frac{\tau+1-R}{n-R} \cdot \frac{\tau-R}{n-R-1} \cdots \frac{\tau+2-n}{1}.$$

Observe that $(|C_n|)_{n \geq \tau+1}$ is a non-increasing sequence. Further, $C_n C_{n+1} \leq 0$ for $n \geq \tau + 1$. Therefore

$$\left| \sum_{\substack{\tau+1 \leq n \\ 2 \leq n}} \binom{\tau+1}{n} \left(\frac{s}{a+r}\right)^n \right| \leq \left(\frac{s}{a+r}\right)^2. \tag{45}$$

The lemma follows from (44) and (45). □

As a corollary we may find a bound for $\tau(C^*)$.

Corollary 6.5. *Let $C \subseteq \mathbb{R}^+$ be a Cantor set derived by $\mathcal{D}$ and let τ be a real number which is at most $\tau(C)$. Assume that for all bridges $A = [a,b]$ of $\mathcal{D}$ with A^e to the right of A^d $(d, e \in \{0,1\})$ and*

$$\frac{|A^e| - \tau|O|}{|O|^2} < \frac{1}{a}\left(1 + \sum_{2 \leq n < \tau+1} \binom{\tau+1}{n}\right) \tag{46}$$

it follows that

$$\frac{|A^{e*}|}{|O^*|} \geq \tau. \tag{47}$$

Then

$$\tau(C^*) \geq \tau.$$

Proof. Let $A = A^0 \cup O \cup A^1$ be a bridge of $\mathcal{D}$ with A^d to the left of A^e, for $(d,e) = (1,0)$ or $(d,e) = (0,1)$. Since the logarithm function has decreasing slope, it follows that

$$\frac{|A^{d*}|}{|O^*|} \geq \frac{|A^d|}{|O|} \geq \tau.$$

If (46) does not hold, then (47) holds by Lemma 6.4, so in any case

$$\min\left\{\frac{|A^{0*}|}{|O^*|}, \frac{|A^{1*}|}{|O^*|}\right\} \geq \tau,$$

as required. □

We may use Corollary 6.5 to find a bound for $\tau((n \pm C(B))^*)$ for n sufficiently large.

Lemma 6.6. *Let B be a set of positive integers with $|B| > 1$.*

1. *There exists $M_1 \in \mathbb{Z}^+$ such that*
$$\tau(C(n;B)^*) \geq \tau(B)$$
for all $n \geq M_1$.
2. *If τ is a real number with $\tau < \tau(B)$, then there exists $M_2 \in \mathbb{Z}^+$ such that*
$$\tau((n - C(B))^*) \geq \tau$$
for all $n \geq M_2$.
3. *If $|B| = \infty$ and $\Delta_i(B) = \Delta$ is constant, then there exists $M_3 \in \mathbb{Z}^+$ such that*
$$\tau((n - C(B))^*) \geq \tau(B)$$
for all $n \geq M_3$.

Proof. For positive real numbers x we define

$$h(x) = 1 + \sum_{2 \leq t < x+1} \binom{x+1}{t}. \tag{48}$$

Since the lemma holds trivially if $\tau(B) = 0$ or $\tau(B) = \infty$, we may assume that $0 < \tau(B) < \infty$. We first prove part 2. Assume that $\tau < \tau(B)$, say $\tau = \tau(B) - \eta$, where $\eta > 0$. Choose an integer M_2 such that

$$M_2 > \frac{h(\tau)}{\eta} + 1.$$

Let $n \geq M_2$ be an integer and let $\mathcal{D}$ be the canonical derivation of $n - C(B)$. If A is any bridge of $\mathcal{D}$, then $A \subseteq [n-1, \infty)$ and

$$\frac{|A^e|}{|O|} - \tau \geq \tau(B) - \tau = \eta$$

for $e = 0, 1$. Thus

$$\frac{|A^e| - \tau|O|}{|O|^2} \geq \frac{\eta}{|O|} > \eta > \frac{h(\tau)}{M_2 - 1} \geq \frac{h(\tau)}{n-1},$$

and so (46) never holds. Therefore, by Corollary 6.5,

$$\tau_{\mathcal{D}^*}(A^*) \geq \tau,$$

and part 2 of the lemma follows.

We next prove part 3. Let M_3 and n be integers with $M_3 \geq \Delta h(\tau(B)) + 1$ and $n \geq M_3$. Let $\mathcal{D}$ be the canonical derivation of $n - C(B)$. To bound the quantity

$$\frac{|A^e| - \tau|O|}{|O|^2}$$

for all bridges A of $\mathcal{D}$ we need only compute the bound for all bridges of $\mathcal{D}(B)$. Let A be a bridge of type (10). From (34) we have

$$\frac{|A^0|}{|O|} - \tau(B) = \frac{1}{\Delta l - 1}\left(\frac{b_{i+1} + Q}{b_i + Q} - 1\right) = \frac{\Delta}{(\Delta l - 1)(b_i + Q)}.$$

By Lemma 4.1 (part 1) with $g_1 = [b_{i+1}]$ and $g_2 = [b_i, l]$ we have

$$|O| = \langle a_1 \ldots, a_r, g_1\rangle - \langle a_1 \ldots, a_r, g_2\rangle = \frac{\Delta - 1/l}{q_r^2(b_{i+1} + Q)(b_i + 1/l + Q)}.$$

Therefore

$$\frac{|A^0| - \tau|O|}{|O|^2} = \frac{\Delta}{(\Delta l - 1)(b_i + Q)} \cdot \frac{q_r^2(b_{i+1} + Q)(b_i + 1/l + Q)}{\Delta - 1/l}$$
$$> \frac{q_r^2 b_{i+1}}{\Delta l} \geq \frac{1}{\Delta} > \frac{h(\tau(B))}{n-1}.$$

Similarly we find that

$$\frac{|A^1| - \tau|O|}{|O|^2} > \frac{q_r^2 b_i^2 b_{i+1}}{\Delta^2} > \frac{1}{\Delta} > \frac{h(\tau(B))}{n-1},$$

and part 3 of the lemma follows from Corollary 6.5.

We now prove part 1. Assume first that B is finite, and put

$$T = \frac{\delta(m-l)}{\max\{\Delta_i\} lm - \delta(m-l)}.$$

Let M_1 and n be integers with $M_1 \geq (m+2)^2 h(\tau(B))/T$ and $n \geq M_1$. If A is a bridge of $\mathcal{D}(B)$ of the form (9), then from (33) we have

$$\frac{|A^0|}{|O|} - \tau \geq \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \left(\frac{(b_{i+1}+Q)m + \delta}{(b_i + Q)m + \delta} - \frac{(b_{i+1} + 1/l)m + \delta}{(b_i + 1/l)m + \delta} \right)$$
$$\geq T \cdot \frac{m^2 \Delta_i (1/l - Q)}{((b_i + Q)m + \delta)((b_i + 1/l)m + \delta)}$$
$$\geq T \cdot \frac{m^2 \Delta_i q_{r-2}}{l q_r ((b_i + Q)m + \delta)((b_i + 1/l)m + \delta)},$$

since $q_r \geq l q_{r-1} + q_{r-2}$. From Lemma 4.1 (part 1) with $g_1 = [b_{i+1}, \overline{m, l}]$ and $g_2 = [b_i, \overline{l, m}]$ we have

$$|O| = \frac{\Delta_i + \delta/m - \delta/l}{q_r^2 (b_{i+1} + \delta/m + Q)(b_i + \delta/l + Q)},$$

whence if $r \neq 1$ then

$$\frac{|A^0| - \tau|O|}{|O|^2} > T \frac{q_{r-2} q_r}{l} > \frac{h(\tau(B))}{n}. \tag{49}$$

Similarly we have

$$\frac{|A^1| - \tau|O|}{|O|^2} > T \frac{q_{r-1} q_r b_i b_{i+1}}{\Delta_i (m+2)^2} > T \frac{q_{r-1} q_r b_i}{(m+2)^2} \geq \frac{h(\tau(B))}{n} \tag{50}$$

if $r \neq 0$.

Now if $r = 1$ then in $C(n; B)$ we have A^1 on the right side of A^0, and if $r = 0$ then A^0 is to the right of A^1. Thus by Corollary 6.5, part 1 of the lemma follows for B finite.

If B is infinite then we take M_1 greater than $(\max \Delta_i) l^2 h(\tau(B))$ and use an argument analogous to the above to establish our result. □

7. Proofs of Theorems 1.7, 1.8 and 1.9

Proof of Theorem 1.7. We may assume without loss of generality that $0 < \tau(B_j) < \infty$ for $1 \leq j \leq k$. Assume first that $S_\gamma > 1$. Let η be the positive real number

$$\eta = \frac{S_\gamma - 1}{k}.$$

We will follow an approach similar to that used in the proof of Theorem 1.6. For $j = 1, \dots, k$, by Lemma 6.6 (part 1) there exists a positive integer M_j such that for $n \geq M_j$,

$$\tau(C(n; B_j)^*) \geq \tau(B_j).$$

Let $M = \max M_j$. We define $\tilde{C}_j^N$ and $\tilde{I}_j^N$ for $N \geq M$ by

$$\tilde{C}_j^N = \bigcup_{n=M}^{N} C(n; B_j)$$

and

$$\tilde{I}_j^N = [M + \min C(B_j), N + \max C(B_j)].$$

Our definition of $\tilde{\mathcal{D}}_j^N$ is analogous to that for $\mathcal{D}_j^N$ in the proof of Theorem 1.6. Note that since the logarithm function is not linear, in the notation of (36) we may have

$$\frac{|(A^{1\cdots 11})^*|}{|(O^{1\cdots 1})^*|} < \tau(B_j). \tag{51}$$

However, if we set

$$\eta' = \frac{\eta(\tau(B_j) + 1)^2}{1 + \eta(\tau(B_j) + 1)}$$

and take N sufficiently large, then

$$\frac{|(A^{1\cdots 11})^*|}{|(O^{1\cdots 1})^*|} > \tau(B_j) - \eta'.$$

Thus

$$\tau((\tilde{C}_j^N)^*) > \tau(B_j) - \eta',$$

so

$$\gamma((\tilde{C}_j^N)^*) > \gamma(B_j) - \eta,$$

whence it follows that

$$\gamma((\tilde{C}_1^N)^*) + \cdots + \gamma((\tilde{C}_k^N)^*) > S_\gamma - k\eta = 1.$$

Now

$$\gamma(((\tilde{C}_j^N)^{-1})^*) = \gamma((\tilde{C}_j^N)^*),$$

so if $N > M + 1$ then all the conditions of Theorem 6.3 (part 3) are satisfied, and we find that

$$\begin{aligned}(\tilde{C}_1^N)^{\epsilon_1} \cdots (\tilde{C}_k^N)^{\epsilon_k} &= (\tilde{I}_1^N)^{\epsilon_1} \cdots (\tilde{I}_k^N)^{\epsilon_k} \\ &\supseteq \left[\frac{(M+1)^{S_\epsilon^+}}{N^{S_\epsilon^-}}, \frac{N^{S_\epsilon^+}}{(M+1)^{S_\epsilon^-}}\right],\end{aligned} \tag{52}$$

where $S_\epsilon^+ = |\{j;\, \epsilon_j = 1\}|$ and $S_\epsilon^- = k - S_\epsilon^+$. We let N tend to infinity in (52) and find that if $S_\epsilon = k$, then

$$\big[(M+1)^k, \infty\big) \subseteq F, \tag{53}$$

and if $|S_\epsilon| < k$, then

$$(0, \infty) \subseteq F. \tag{54}$$

To extend these results to the negative axis we consider the set

$$\tilde{C}_1^{N-} = \bigcup_{n=M}^{N} (n - C(B_1))$$

for $N > M+1$. By Lemma 6.6 (part 2) we find that for M and N sufficiently large,

$$\tau((\tilde{C}_1^{N-})^*) > \tau(B_1) - \eta'.$$

As before we have by part 3 of Theorem 6.3 that

$$\left[\frac{(M+1)^{S_\epsilon^+}}{(N-1)^{S_\epsilon^-}}, \frac{(N-1)^{S_\epsilon^+}}{(M+1)^{S_\epsilon^-}}\right] \subseteq (\tilde{C}_1^{N-})^{\epsilon_1}(\tilde{C}_2^{N})^{\epsilon_2}\cdots(\tilde{C}_k^{N})^{\epsilon_k}.$$

However,

$$n - C(B_1) = -(-n + C(B_1)) \subseteq -F(B_1)$$

for every n, whence

$$\left[\frac{(M+1)^{S_\epsilon^+}}{(N-1)^{S_\epsilon^-}}, \frac{(N-1)^{S_\epsilon^+}}{(M+1)^{S_\epsilon^-}}\right] \subseteq -F.$$

Taking the limit as N approaches infinity, we find that

$$(-\infty, -(M+1)^k] \subseteq F \tag{55}$$

if $S_\epsilon = k$, and

$$(-\infty, 0) \subseteq F \tag{56}$$

if $|S_\epsilon| < k$. Part 1 of the theorem follows from (53) and (55), while part 2 is a consequence of (54) and (56).

We now assume that $S_\gamma > 1$ and $|B_r| = \infty$ for some r with $\epsilon_r = 1$. Now,

$$C(B_r) = \bigcup_{b_i \in B_r} \frac{1}{C(b_i; B_r)} = \frac{1}{\bigcup_{b_i \in B_r} C(b_i; B_r)}.$$

This is similar to the case $S_\gamma > 1$ and $|S_\epsilon| < k$, where instead of dividing by the set

$$\bigcup_{n=M}^{N} C(n; B_r)$$

we are dividing by

$$C' = \bigcup_{\substack{M \leq n \leq N \\ n \in B_r}} C(n; B_r).$$

Notice that

$$\frac{|[b_{i+1}, b_{i+1} + 1/l_r]|}{|(b_i + 1/l_r, b_{i+1})|} = \frac{1}{\Delta_i l_r - 1}. \tag{57}$$

Choose M sufficiently large so that $\Delta(M) = \max_{j \geq s} \Delta_j(B_r)$ occurs infinitely often in the sequence $(\Delta_1(B_r), \Delta_2(B_r), \ldots)$, where $s = s(M)$ is the unique positive integer with $b_{s-1} < M \leq b_s$. Then, by (4),

$$\tau(B_r) \leq \frac{1}{\Delta(M) l_r - 1},$$

so by (57) $\tau((C')^*) \geq \tau(B_r) - \eta'$ for N sufficiently large. We proceed in a similar manner as in the proof of part 2 of the theorem and find that

$$(-\infty, 0) \cup (0, \infty) \subseteq F.$$

However, $0 \in F(B_r)$, and part 3 of the theorem follows.

Now assume that $S_\gamma = 1$. As above, it might be the case that for some j we have $\tau(\tilde{C}_j^N) < \tau(B_j)$; however, if we choose N sufficiently large, then by Lemma 6.6, part 1, for each j there will be at most one bridge A_j^w with $\tau((A_j^w)^*) < \tau(B_j)$, namely the bridge

$$A_j^w = [N - 1 + \min C(B_j), N + \max C(B_j)].$$

Therefore by the proof of part 2 of Theorem 2.4 we find that

$$(\tilde{C}_1^N)^{\epsilon_1} \cdots (\tilde{C}_k^N)^{\epsilon_k} \supseteq (\tilde{I}_1^N)^{\epsilon_1} \cdots (\tilde{I}_k^N)^{\epsilon_k} \setminus \bigcup_{j=1}^{k} V_j^N, \tag{58}$$

where, for $j = 1, \ldots, k$, V_j^N is an open interval. Now assume that $N > 1 + 2\tau(B_j)^{-1}$. Then

$$\begin{aligned}\tau(A_j^{w*}) &= \frac{|[N + \min C(B_j), N + \max C(B_j)]^*|}{|(N - 1 + \max C(B_j), N + \min C(B_j))^*|} \\ &\geq \frac{|[N-1, N]^*|}{|(N - 1 - \tau(B_j)^{-1}, N-1)^*|},\end{aligned}$$

since

$$|[N + \min C(B_j), N + \max C(B_j)]| \leq 1$$

and

$$\frac{|[N + \min C(B_j), N + \max C(B_j)]|}{|(N - 1 + \max C(B_j), N + \min C(B_j))|} \geq \tau(B_j)$$

by (37) and (38), and since the derivative of the logarithm function is decreasing. Thus

$$\tau(A_j^{w*}) \geq \frac{\log\left(1 + \frac{1}{N-1}\right)}{\log\left(1 + \frac{\tau(B_j)^{-1}}{N-1-\tau(B_j)^{-1}}\right)} > \frac{\frac{1}{N-1} - \frac{1}{2(N-1)^2}}{\frac{\tau(B_j)^{-1}}{N-1-\tau(B_j)^{-1}}}$$

by the power series expansion of $\log(1+x)$ for $|x| < 1$. Therefore if we put $\beta_j = 1/2 + \tau(B_j)^{-1}$, then

$$\tau(A_j^{w*}) > \tau(B_j)\left(1 - \frac{\beta_j}{(N-1)}\right),$$

so

$$\gamma(C_j^{N*}) > \frac{\tau(B_j)\left(1 - \frac{\beta_j}{(N-1)}\right)}{\tau(B_j)\left(1 - \frac{\beta_j}{(N-1)}\right) + 1} > \gamma(B_j)\left(1 - \frac{\beta_j}{(N-1)}\right).$$

Let $\beta = \max \beta_j$. By Theorem 2.4 (part 3) we have, for N sufficiently large,

$$\tau\left(\epsilon_1(\tilde{C}_1^N)^* + \cdots + \epsilon_k(\tilde{C}_k^N)^*\right) \geq \frac{S_\gamma\left(1 - \frac{\beta}{(N-1)}\right)}{1 - S_\gamma\left(1 - \frac{\beta}{(N-1)}\right)} = \frac{N-1}{\beta} - 1.$$

Therefore

$$|V_j^{N*}| < \frac{\beta k \log N}{N-1-\beta},$$

so $|V_j^N| \to 0$ as $N \to \infty$. We take the limit as N approaches infinity in (58), and parts 4 and 5 of the theorem follow.

If $S_\gamma = 1$ and for some r we have $|B_r| = \infty$, $\epsilon_r = 1$ and $\Delta_i(B_r)$ constant, then we may extend our results to the negative reals. We use Lemma 6.6 (part 3) and an approach similar to that used in the proof of part 3 of the theorem, and part 6 follows.

Finally, if $S_\gamma < 1$, then by Lemma 6.6 (part 1) we have

$$\tau(C(M; B_j)^*) \geq \tau(B_j)$$

for $j = 1, \ldots, k$ and M sufficiently large, whence by Theorem 6.3 (part 2)

$$\dim_H(C(M; B_1)^{\epsilon_1} \cdots C(M; B_k)^{\epsilon_k}) \geq \frac{\log 2}{\log\left(1 + \frac{1}{S_\gamma}\right)},$$

and the theorem follows. ☐

Our methods of proving Theorems 1.7, 1.8 and 1.9 differ from that employed by Hall in [5]. He covers part of the real line by intervals of the form

$$I(n; L_4) \cdot I(n; L_4) \qquad \text{or} \qquad I(n; L_4) \cdot I(n+1; L_4)$$

and then shows that

$$C(n; L_4) \cdot C(n; L_4) = I(n; L_4) \cdot I(n, L_4)$$

and

$$C(n; L_4) \cdot C(n+1; L_4) = I(n; L_4) \cdot I(n+1, L_4).$$

Proof of Theorem 1.8. Note that $\gamma(L_3) + \gamma(L_4) = 1.0165\ldots > 1$. For positive integers m, define $F^+(m)$ and $F^-(m)$ by

$$F^+(m) = \{[n, a_1, a_2, \ldots];\ n \geq 0 \text{ and } 1 \leq a_i \leq m \text{ for } i \geq 1\},$$
$$F^-(m) = \{[n, a_1, a_2, \ldots];\ n < 0 \text{ and } 1 \leq a_i \leq m \text{ for } i \geq 1\}.$$

We will first show that

$$\tau(C(n; L_3)^*) \geq \tau(L_3) \quad \text{and} \quad \tau(C(n; L_4)^*) \geq \tau(L_4) \tag{59}$$

for $n \geq 0$, and that

$$\tau(n - C(L_4))^*) \geq 1.255 \tag{60}$$

for $n \geq 1$. If $C \subseteq (0, \infty)$ is a Cantor set with derivation $\mathcal{D}$ and $E = E_1 \cup O \cup E_2$ is a bridge of $\mathcal{D}$ with E_1 to the left of E_2, then, for any integer $n \geq 0$,

$$\frac{|(n+E_1)^*|}{|(n+O)^*|} > \frac{|n+E_1|}{|n+O|} \geq \tau(\mathcal{D})$$

and

$$\frac{|(n+E_2)^*|}{|(n+O)^*|} > \frac{|E_2^*|}{|O^*|} \geq \tau(\mathcal{D}^*),$$

since the second derivative of the logarithm function is negative and has decreasing magnitude. Therefore

$$\tau((n+C)^*) \geq \min\{\tau(C), \tau(C^*)\}.$$

Thus to prove (59) it suffices to show that $\tau(C(L_3)^*) \geq \tau(L_3)$ and $\tau(C(L_4)^*) \geq \tau(L_4)$. Similarly, to establish (60) we need only show that $\tau(1 - C(L_4))^*) \geq 1.255$.

We first examine $C(L_3)$. If $r > 1$, $d \in \{0, 1\}$ and A is a bridge of the form (9), then by (49) and (50) we have

$$\frac{|A^d| - \tau|O|}{|O|^2} > T\frac{q_{r-1}q_r}{25}. \tag{61}$$

If we define $h(x)$ as in (48), then by using a Maple program we find that

$$\min_A \tau(A^*) = 0.833\ldots, \tag{62}$$

where the minimum is taken over all bridges A of $C(L_3)$ with

$$q_r q_{r-1} < \frac{25}{T} \cdot \frac{h(\tau(L_3))}{\langle\overline{3,1}\rangle}.$$

By (61), (62) and Corollary 6.5 we have

$$\tau(C(L_3)^*) \geq \tau(L_3) = 0.822\ldots . \tag{63}$$

Similarly we find that

$$\tau(C(L_4)^*) \geq \tau(L_4) = 1.300\ldots$$

and

$$\tau((1 - C(L_4))^*) \geq 1.255\ldots .$$

Therefore (59) and (60) hold. Since $0.822 \times 1.255 > 1$, we find by an approach analogous to that used in the proof of part 1 of Theorem 1.7 that

$$F^+(3) \cdot F^+(4) = [\langle\overline{3,1}\rangle\langle\overline{4,1}\rangle,\ \infty) \tag{64}$$

and

$$F^+(3) \cdot F^-(4) = (-\infty,\ \langle\overline{3,1}\rangle(-1 + \langle\overline{1,4}\rangle)]. \tag{65}$$

Now put

$$\begin{aligned} I_3^- &= [1 - \langle 1, \overline{3,1}\rangle, 1 - \langle 1, \overline{1,3}\rangle] = [0.2087\ldots, 0.4417\ldots], \\ I_4^+ &= [\langle 4, \overline{1,4}\rangle, \langle 3, \overline{4,1}\rangle] = [0.2071\ldots, 0.3118\ldots], \\ I_4^- &= [1 - \langle 1, \overline{4,1}\rangle, 1 - \langle 1, \overline{1,4}\rangle] = [0.1715\ldots, 0.4530\ldots], \end{aligned}$$

$$C_3^- = C(L_3) \cap I_3^-, \quad C_4^+ = C(L_4) \cap I_4^+ \quad \text{and} \quad C_4^- = (1 - C(L_4)) \cap I_4^-.$$

Since $1 - I_3^-$ is a bridge of $\mathcal{D}(L_3)$, we may use $\mathcal{D}(L_3)$ to construct a derivation of C_3^- from I_3^-. To bound $\tau((C_3^-)^*)$ we use the same process that was used to establish (63). Specifically, we find that

$$\tau((C_3^-)^*) \geq \tau(L_3). \tag{66}$$

Similarly we have

$$\tau((C_4^-)^*) \geq \tau(L_4) \quad \text{and} \quad \tau((C_4^+)^*) \geq \tau(L_4). \tag{67}$$

The largest gap in $(C_3^-)^*$, $(C_4^+)^*$ and $(C_4^-)^*$ has width $0.1563\ldots$, $0.0943\ldots$ and $0.1256\ldots$ respectively. Further,

$$|(I_3^-)^*| = 0.7497\ldots, \quad |(I_4^+)^*| = 0.4091\ldots \quad \text{and} \quad |(I_4^-)^*| = 0.9710\ldots,$$

so by (66), (67) and part 3 of Theorem 6.3,

$$C_3^- \cdot C_4^+ = I_3^- \cdot I_4^+ = [(1 - \langle\overline{1,3}\rangle)\langle\overline{4,1}\rangle, (1 - \langle 1, \overline{1,3}\rangle)\langle 3, \overline{4,1}\rangle] \tag{68}$$

and

$$C_3^- \cdot C_4^- = I_3^- \cdot I_4^- = [(1 - \langle\overline{1,3}\rangle)(1 - \langle\overline{1,4}\rangle), (1 - \langle 1, \overline{1,3}\rangle)(1 - \langle 1, \overline{1,4}\rangle)]. \tag{69}$$

Since

$$F(3) \cdot F(4) \subseteq \big(-\infty, (-1 + \langle\overline{1,3}\rangle)\langle\overline{4,1}\rangle\big] \cup \big[(-1 + \langle\overline{1,3}\rangle)(-1 + \langle\overline{1,4}\rangle), \infty\big),$$

the theorem follows from (64), (65), (68) and (69). □

Proof of Theorem 1.9. Our proof will be similar to that of Theorem 1.7. Let S_γ denote the number

$$S_\gamma = \sum_{j=1}^{k} \frac{1}{l_j}.$$

If $S_\gamma > 1$ then the theorem follows from part 3 of Theorem 1.7. Assume that $S_\gamma = 1$. If $l_j = 1$ for any j then the theorem follows trivially, so we may assume that $l_j \geq 2$ for $j = 1, \ldots, k$.

First assume that $k = 2$; then $l_1 = l_2 = 2$. For $M \geq 2$ sufficiently large we have by Lemma 6.6 (part 1) that

$$\tau(C(n; U_2)^*) \geq \tau(U_2) = 1$$

for $n \geq M$. For positive integers $N > M$ we define C^N and I^N by

$$C^N = \left[N + \frac{1}{2}, N + 1\right] \cup \bigcup_{n=M}^{N} C(n; U_2)$$

and

$$I^N = [M, N + 1].$$

We may construct a derivation $\mathcal{D}^N$ of C^N from I^N in a manner similar to that used in the proof of Theorem 1.7, the only difference being that the rightmost interval in every level of the tree contains the closed interval $[N + 1/2, N + 1]$. Because of this we avoid the problems faced in (51), so that

$$\tau((C^N)^*) \geq \tau(U_2) = 1.$$

Thus $(C^N)^{-1}$ and $C(M; U_2)$ satisfy the requirements of Theorem 6.3, part 3, so

$$\frac{C(M; U_2)}{C^N} = \frac{[M, M + 1/2]}{[M, N + 1]} = \left[\frac{M}{N + 1}, 1 + \frac{1}{2M}\right].$$

Therefore

$$\frac{C(M; U_2)}{\bigcup_{n=M}^{N} C(n; U_2)} \supseteq \left[\frac{2M + 1}{2N + 1}, 1\right].$$

But

$$C(0;U_2) = \frac{1}{\bigcup_{n=2}^{\infty} C(n;U_2)},$$

whence

$$C(M;U_2)\cdot C(0;U_2) \supseteq \left[\frac{2M+1}{2N+1}, 1\right].$$

This holds for every $N > M$; thus

$$C(M;U_2)\cdot C(0;U_2) \supseteq (0,1]. \tag{70}$$

By taking reciprocals we have

$$C(M;U_2)^{-1}\cdot C(0;U_2)^{-1} \supseteq [1,\infty).$$

However,

$$C(M;U_2)^{-1} \subseteq C(0;U_2) \quad \text{and} \quad C(0;U_2)^{-1} \subseteq G(2),$$

so

$$C(0;U_2)\cdot G(2) \supseteq [1,\infty). \tag{71}$$

By (70) and (71) we have

$$(0,\infty) \subseteq G(2)\cdot G(2). \tag{72}$$

As in the proof of Theorem 1.7 we may extend our results to the negative real axis, so that

$$(-\infty,0) \subseteq G(2)\cdot G(2). \tag{73}$$

Since $0 \in G(2)$, we have by (72) and (73) that

$$G(2)\cdot G(2) = \mathbb{R}, \tag{74}$$

as required.

Now assume that $k > 2$. To prove the theorem we will use an approach similar to that used to establish (74). Without loss of generality we may assume that

$$l_1 \ge l_2 \ge \cdots \ge l_k.$$

Now, for M sufficiently large, for all $n \ge M$ and all $j = 1,\dots,k$ the largest gap in $C(n;U_{l_j})^*$ is $(O_j^n)^*$, where O_j^n is the largest gap in $C(n;U_{l_j})$, namely

$$O_j^n = ([n,l_j+1],[n,l_j,l_j]).$$

By Lemma 6.6 (part 1) there exists a positive integer $M_1 \ge M$ such that

$$M_1 > \max\left\{l_1, \frac{l_{k-1}^2}{l_k}\right\}, \qquad (l_2 l_3 \cdots l_k) \text{ divides } M_1$$

and

$$\tau(C(n;U_{l_j})^*) \ge \tau(U_{l_j}) \tag{75}$$

for all $n \ge M_1$ and all $j = 1,\dots,k$. For $j = 2,\dots,k$ we set

$$M_j = \frac{l_{j-1}^2}{l_j} M_{j-1}. \tag{76}$$

By our choice of $M_1,\dots,M_k$, we have, by calculation,

$$|(O_j^{M_j})^*| \le |I(M_{j+1};U_{l_{j+1}})^*| \le |I(M_j;U_{l_j})^*| \tag{77}$$

for $j = 1, \ldots, k-1$. For integers $N > 2M_k$ we define C_k^N and I_k^N by

$$C_k^N = \left[N + \frac{1}{l_k}, N+1\right] \cup \bigcup_{n=M_k}^{N} C(n; U_{l_k}) \quad \text{and} \quad I_k^N = [M_k, N+1].$$

Now the largest gap in $(C_k^N)^*$ is either $(O_k^{M_k})^*$ or $(M_k + 1/l_k, M_k + 1)^*$. Since $l_k \le l_{k-1}$ and $k \ge 3$, we have

$$\frac{l_k}{l_1 M_1 (l_1 \cdots l_{k-1})} \le \frac{1}{l_1^2 M_1} < \frac{1}{l_1 M_1}. \tag{78}$$

Also, by (76) it follows that

$$M_j = \frac{l_1 M_1}{l_j^2}(l_1 \cdots l_j) \tag{79}$$

for $j = 1, \ldots, k$, whence with (78) we find that

$$\frac{1}{M_k} \le \frac{1}{l_1 M_1}.$$

Thus

$$\log\left(\frac{M_k + 1}{M_k + 1/l_k}\right) \le \log\left(\frac{M_1 + 1/l_1}{M_1}\right).$$

Equivalently,

$$|(M_k + 1/l_k, M_k + 1)^*| \le |I(M_1; U_{l_1})^*|. \tag{80}$$

As in the proof of the case $k = 2$, we have

$$\tau((C_k^N)^*) \ge \tau(U_{l_k}). \tag{81}$$

By (75), (77), (80), (81) and Theorem 6.3 (part 3),

$$\frac{C(M_1; U_{l_1}) \cdots C(M_{k-1}; U_{l_{k-1}})}{C_k^N} = \left[\frac{M_1 \cdots M_{k-1}}{N+1}, \frac{(M_1 + 1/l_1) \cdots (M_{k-1} + 1/l_{k-1})}{M_k}\right].$$

Thus

$$\frac{C(M_1; U_{l_1}) \cdots C(M_{k-1}; U_{l_{k-1}})}{\bigcup_{n=M_k}^N C(n; U_{l_k})} \supseteq \left[\frac{(M_1 + 1/l_1) \cdots (M_{k-1} + 1/l_{k-1})}{N + 1/l_k}, \frac{M_1 \cdots M_{k-1}}{M_k}\right]. \tag{82}$$

Since $M_1 > l_{k-1}^2 / l_k$ and $k > 2$, we have by (79) that

$$\frac{M_1 \cdots M_{k-1}}{M_k} > 1. \tag{83}$$

Also, since $M_k \ge l_k$,

$$\left(\bigcup_{n=M_k}^{N} C(n; U_{l_k})\right)^{-1} \subseteq C(0; U_{l_k}), \tag{84}$$

so by (82), (83) and (84), upon taking the limit as N approaches infinity, we have

$$(0, 1] \subseteq G(l_1) \cdots G(l_k). \tag{85}$$

Since $M_j \geq l_j$ for $j = 1, \ldots, k-1$ we may take reciprocals in (82) and let N tend to infinity, so that

$$[1, \infty) \subseteq G(l_1) \cdots G(l_k).$$

With (85) we have

$$(0, \infty) \subseteq G(l_1) \cdots G(l_k).$$

As before we may extend our results to the negative reals, finding that

$$(-\infty, 0) \subseteq G(l_1) \cdots G(l_k).$$

Since $0 \in G(l_1)$, our result follows. □

8. Final Remarks

The problem of proving negative results for products seems to be much more difficult than for sums. For example, to prove that $[a, \infty) \not\subseteq F(2) \cdot F(2)$ for some a, it would not suffice to find a single gap modulo one in $C(L_2) \cdot C(L_2)$. Rather, we would have to show that the same gap existed in each $\tilde{C}_1^N \tilde{C}_2^N$.

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Department of Pure Mathematics, The University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: sastels@barrow.uwaterloo.ca