

## FUNCTIONS FOR PARAMETRIZATION OF SOLUTIONS OF AN EQUATION IN A FREE MONOID

GENNADY S. MAKANIN AND TATIANA A. MAKANINA

ABSTRACT. In this paper we introduce recursive functions

$$\begin{aligned}\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_s} & \quad (s \geq 0), \\ \mathbf{Th}(x_1, x_2, x_3)_i^{\lambda_1, \dots, \lambda_{2s}} & \quad (i = 1, 2, 3; s \geq 0), \\ \mathbf{Ro}(x_1, x_2, x_3)_i^{\mu_1, \dots, \mu_s} & \quad (i = 1, 2, 3; s \geq 0)\end{aligned}$$

of the word variables  $x_1, x_2, x_3$ , natural number variables  $\lambda_k$  and variables  $\mu_k$  whose values are finite sequences of natural number variables. By means of these functions we give finite expressions for the family of solutions of the equation

$$x_1 x_2 x_3 x_4 = \zeta(x_1, x_2, x_3) x_5,$$

where  $\zeta(x_1, x_2, x_3)$  is an arbitrary word in the alphabet  $x_1, x_2, x_3$ , in a free monoid.

### 1. INTRODUCTION

In 1960 Lyndon [1], [2] considered equations with one unknown in a free group and proved that the family of solutions of such an equation can be represented by a finite number of parametric words. In 1967 Khmelevskii [3] considered equations with three unknowns in a free monoid and proved that the family of solutions of such an equation can be represented by a finite number of parametric words. For a short time after that it was believed that the solutions of all equations in a free group or a free monoid are parametrizable. However in 1971 Khmelevskii [4] pointed out that the solutions of Markov's equation  $x_1 x_3 x_2 = x_2 x_4 x_1$  with four unknowns in a free monoid is not parametrizable by a finite number of parametric words.

Parametrizations of solutions in the Lyndon-Khmelevskii sense (now called primitive parametrization) use variables of two kinds: word variables and natural number variables. We suppose that the idea of finite parametrization of the solutions of the equations in a free group and a free monoid can be saved if we admit an additional kind of variables, namely, variables whose values are the finite sequences of natural number variables. In this paper we intend to demonstrate the possibilities of the parametrization of the solutions of the equations in a free monoid by means of parametrizing functions of variables of the three mentioned kinds on a carefully chosen example of an equation in a free monoid. This equation is not bulky and "contains" many known difficult equations.

---

Received by the editors April 14, 1997.

1991 *Mathematics Subject Classification.* Primary 20M05; Secondary 03D40, 20F10.

©1999 American Mathematical Society

We introduce recursive functions

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_s}, \quad \mathbf{Th}(x_1, x_2, x_3)_i^{\lambda_1, \dots, \lambda_{2s}}, \quad \mathbf{Ro}(x_1, x_2, x_3)_i^{\mu_1, \dots, \mu_s},$$

where  $s \geq 0$ ,  $i = 1, 2, 3$ , of the word variables  $x_1, x_2, x_3$ , natural number variables  $\lambda_k$ , and variables  $\mu_k$  whose values are finite sequences of natural number variables. We consider here an equation of the form

$$x_1 x_2 x_3 x_4 = \zeta(x_1, x_2, x_3) x_5,$$

where  $\zeta(x_1, x_2, x_3)$  is an arbitrary word in the alphabet  $x_1, x_2, x_3$  in a free monoid. We shall give concrete expression from parametric words and parametrizing functions  $\mathbf{Fi}$ ,  $\mathbf{Th}$ ,  $\mathbf{Ro}$ , which describe the family of solutions of the equation  $x_1 x_2 x_3 x_4 = \zeta(x_1, x_2, x_3) x_5$  in a free monoid.

We use the preprints [5] and [6].

## 2. DEFINITIONS AND NOTATION

Let  $\Pi$  be a free monoid (a free semigroup with unit) with a countable alphabet of generators

$$(1) \quad a_1, a_2, \dots, a_k, \dots$$

Let

$$(2) \quad x_1, x_2, \dots, x_n, \dots$$

be a countable alphabet of word variables.

Let

$$(3) \quad \lambda_1, \lambda_2, \dots, \lambda_t, \dots$$

be a countable alphabet of natural number variables (also called natural parameters).

Let

$$(4) \quad \mu_1, \mu_2, \dots, \mu_u, \dots$$

be a countable alphabet of variables (called second parameters) whose values are finite sequences of natural parameters.

Let

$$(5) \quad \nu_1, \nu_2, \dots, \nu_v, \dots$$

be a countable alphabet of variables whose values are finite sequences of second parameters.

Define inductively a *primitive parametric word* as follows: Any word on the alphabet (2) is a primitive word. If  $P$  is a primitive parametric word and  $\lambda$  is a natural parameter, then  $(P)^\lambda$  is a primitive parametric word. If  $P$  and  $Q$  are two primitive parametric words, then  $PQ$  is a primitive parametric word.

We denote by  $\mathbf{L}$  the set of linear polynomials of the form  $k_0 + \sum_{i=1}^r k_i \lambda_i$ , where  $r, k_0, k_1, \dots, k_r$  are natural numbers, and  $\lambda_1, \dots, \lambda_r$  are natural parameters.

A *primitive parametric transformation* is defined by the application

$$\begin{cases} x_i \rightarrow W_i(x_1, \dots, x_n, \lambda_1, \dots, \lambda_q) & (i = 1, \dots, n), \\ \lambda_i \rightarrow L_i(\lambda_1, \dots, \lambda_q) & (i = 1, \dots, q), \end{cases}$$

where every  $W_i$  is a primitive word, and  $L_i \in \mathbf{L}$ . The components of the form  $x_i \rightarrow x_i$  and  $\lambda_i \rightarrow \lambda_i$  are often omitted.

Now we inductively define a parametric word (transformation). Any primitive parametric word (transformation) is a parametric word (transformation).

Define the function  $\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_s}$  for  $s \geq 0$ , where  $x_1$  and  $x_2$  are two word variables, and  $\lambda_1, \dots, \lambda_s$  are natural parameters, inductively as follows:

$$\mathbf{Fi}(x_1, x_2) = 1,$$

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_s} = (\mathbf{Fi}(x_2, x_1)^{\lambda_2, \dots, \lambda_s} x_2)^{\lambda_1} \mathbf{Fi}(x_1, x_2)^{\lambda_3, \dots, \lambda_s} \quad (s \geq 1).$$

In particular,

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1} = (x_2)^{\lambda_1},$$

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1, \lambda_2} = ((x_1)^{\lambda_2} x_2)^{\lambda_1},$$

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1, \lambda_2, \lambda_3} = (((x_2)^{\lambda_3} x_1)^{\lambda_2} x_2)^{\lambda_1} (x_2)^{\lambda_3},$$

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = (((((x_1)^{\lambda_4} x_2)^{\lambda_3} x_1)^{\lambda_2} (x_1)^{\lambda_4} x_2)^{\lambda_1} ((x_1)^{\lambda_4} x_2)^{\lambda_3}.$$

The empty sequence is denoted by  $\emptyset$ . Let  $\mu$  be a variable whose values are finite sequences of natural parameters. The variable  $\mu|$  is connected with a variable  $\mu$  as follows: If  $\mu = \lambda_1, \lambda_2, \dots, \lambda_s$ , where  $s \geq 1$ , then  $\mu| = \lambda_2, \dots, \lambda_s$ . If  $\mu = \emptyset$ , then  $\mu| = \emptyset$ .

A transformation is defined by the application

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 P, x_2 Q)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 Q, x_1 P)^{\mu|} x_2, \end{cases}$$

where  $P, Q$  are parametric words on the alphabet  $x_3, \dots, x_n$  and  $\mu$  is a variable whose values are finite sequences of natural parameters, is a *parametric transformation (by the function  $\mathbf{Fi}$ )*.

We next define by a joint induction the functions

$$\mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s}},$$

$$\mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_1, \dots, \lambda_{2s}},$$

$$\mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s}},$$

for  $s \geq 0$ , where  $x_1, x_2, x_3$  are three word variables, and  $\lambda_1, \dots, \lambda_{2s}$  are natural parameters. Specifically, we set

$$\mathbf{Th}(x_1, x_2, x_3)_i = 1 \quad (i = 1, 2, 3);$$

$$\mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s}} = (\mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s}} x_3)^{\lambda_1} \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s}};$$

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_1, \dots, \lambda_{2s}} \\ &= (\mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s}} x_2 \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_3, \dots, \lambda_{2s}} x_3 \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s}} x_1 \\ & \quad \cdot \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s}} x_1)^{\lambda_2} \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s}}; \end{aligned}$$

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s}} \\ &= \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s}} x_1 \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s}} x_2 \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_3, \dots, \lambda_{2s}}. \end{aligned}$$

Define inductively the auxiliary function

$$\mathbf{Oc}(x_1, x_2, x_3)^{\lambda_1, \dots, \lambda_{2s}}$$

for  $s \geq 0$ , where  $x_1, x_2, x_3$  are three word variables, and  $\lambda_1, \dots, \lambda_{2s}$  are natural parameters, by setting

$$\mathbf{Oc}(x_1, x_2, x_3) = 1;$$

$$\begin{aligned} \mathbf{Oc}(x_1, x_2, x_3)^{\lambda_1, \dots, \lambda_{2s}} &= \mathbf{Oc}(x_1, x_2, x_3)^{\lambda_3, \dots, \lambda_{2s}} (\mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_3, \dots, \lambda_{2s}} x_3 \\ &\quad \cdot \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s}} x_1 \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s}} x_2)^{\lambda_1}. \end{aligned}$$

A transformation defined by the application

$$\begin{cases} x_i \rightarrow \mathbf{Th}(x_1, x_2, x_3)_i^\xi x_i, & i = 1, 2, 3, \\ x_4 \rightarrow x_4 \mathbf{Oc}(x_1, x_2, x_3)^\xi, \end{cases}$$

where  $\xi$  is a variable whose values are even sequences of natural parameters, is called a *parametric transformation (by the function  $\mathbf{Th}$ )*.

Define by a joint induction the functions

$$\mathbf{Ro}(x_1, x_2, x_3)_1^{\mu_1, \dots, \mu_t}, \quad \mathbf{Ro}(x_1, x_2, x_3)_2^{\mu_1, \dots, \mu_t}, \quad \mathbf{Ro}(x_1, x_2, x_3)_3^{\mu_1, \dots, \mu_t}$$

for  $t \geq 0$ , where  $x_1, x_2, x_3$  are three word variables, and  $\mu_1, \dots, \mu_t$  are variables whose values are finite sequences of natural parameters, as follows:

$$\mathbf{Ro}(x_1, x_2, x_3)_i = 1 \quad (i = 1, 2, 3);$$

$$\begin{aligned} \mathbf{Ro}(x_1, x_2, x_3)_1^{\mu_1, \dots, \mu_t} &= \mathbf{Fi}(\mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3 \mathbf{Ro}(x_2, x_1, x_3)_1^{\mu_2, \dots, \mu_t} x_2, \\ &\quad \cdot \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t} x_1 \mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3 \\ &\quad \cdot \mathbf{Ro}(x_2, x_1, x_3)_1^{\mu_2, \dots, \mu_t} x_2 (\mathbf{Ro}(x_1, x_2, x_3)_3^{\mu_1, \dots, \mu_t} x_3)^2)^{\mu_1} \\ &\quad \cdot \mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3 \mathbf{Ro}(x_2, x_1, x_3)_1^{\mu_2, \dots, \mu_t} x_2 \\ &\quad \cdot \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t}; \end{aligned}$$

$$\begin{aligned} \mathbf{Ro}(x_1, x_2, x_3)_2^{\mu_1, \dots, \mu_t} &= \mathbf{Fi}(\mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3 \mathbf{Ro}(x_2, x_1, x_3)_1^{\mu_2, \dots, \mu_t} x_2 \\ &\quad \cdot (\mathbf{Ro}(x_1, x_2, x_3)_3^{\mu_1, \dots, \mu_t} x_3)^2, \mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3 \\ &\quad \cdot \mathbf{Ro}(x_2, x_1, x_3)_1^{\mu_2, \dots, \mu_t} x_2 \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t} x_1)^{\mu_1} \\ &\quad \cdot \mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3 \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t}; \end{aligned}$$

$$\mathbf{Ro}(x_1, x_2, x_3)_3^{\mu_1, \dots, \mu_t} = \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t} x_1 \mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t}.$$

Define inductively the auxiliary function

$$\mathbf{Re}(x_1, x_2, x_3)^{\mu_1, \dots, \mu_t}$$

for  $t \geq 0$ , where  $x_1, x_2, x_3$  are three word variables, and  $\mu_1, \dots, \mu_t$  are variables for sequences of natural parameters:

$$\mathbf{Re}(x_1, x_2, x_3) = 1;$$

$$\mathbf{Re}(x_1, x_2, x_3)^{\mu_1, \dots, \mu_t} = \mathbf{Re}(x_2, x_1, x_3)^{\mu_2, \dots, \mu_t} \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t} x_1.$$

A transformation defined by the application

$$\begin{cases} x_i \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_i^\nu x_i, & i = 1, 2, 3, \\ x_4 \rightarrow x_4 \mathbf{Re}(x_1, x_2, x_3)^\nu, \end{cases}$$

where  $\nu$  is a variable whose values are finite sequences of second parameters, is a *parametric transformation* (by the function  $\mathbf{Ro}$ ).

A composition of parametric transformations is a parametric transformation. Any word in the right side of a parametric transformations is a parametric word.

A *coefficient transformation* is defined by the application

$$(6) \quad \begin{cases} x_i \rightarrow X_i & (i = 1, \dots, n), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, t), \\ \mu_i \rightarrow M_i & (i = 1, \dots, u), \\ \nu_i \rightarrow N_i & (i = 1, \dots, r), \end{cases}$$

where every  $X_i$  is a word in the alphabet (1), every  $\Lambda_i$  is a natural number, every  $M_i$  is a finite sequence of  $(\lambda_1, \dots, \lambda_t)$ , and every  $N_i$  is a finite sequence of  $(\mu_1, \dots, \mu_u)$ .

A *coefficient transformation*

$$\begin{cases} x_i \rightarrow X_i & (i = 1, \dots, n), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, l), \\ \mu_i \rightarrow M_i & (i = 1, \dots, m), \\ \nu_i \rightarrow N_i & (i = 1, \dots, p), \end{cases}$$

is called an *extension* of the coefficient transformation (6) if  $l \geq t$ ,  $m \geq u$ , and  $p \geq r$ .

A *parametric equation* in a free monoid is given by an equality of parametric words

$$(7) \quad \Phi(x_1, \dots, x_n, \lambda_1, \dots, \lambda_t) = \Psi(x_1, \dots, x_n, \lambda_1, \dots, \lambda_t).$$

If  $\Phi$  and  $\Psi$  are empty words, the equation (7) is called the *trivial* equation, denoted by 1.

A parametric transformation (a coefficient transformation) is called a *parametric solution* (a *solution*) of the equation (7) if the result of the application of this transformation to (7) is the trivial equation.

We will say that the parametric transformation  $T$  *contains* the coefficient transformation  $C$  by means of the auxiliary transformation  $I$ , if  $TI = C$ .

A finite list of parametric solutions of the equation  $E$  will be called a *general solution* of  $E$ , if every solution of  $E$  is contained in some parametric solution of this list. The general solution of  $E$  will be denoted  $\langle E \rangle$ .

The length of a word  $A$  in the alphabet (1) is denoted by  $|A|$ . The empty word is denoted by 1. The length of a finite sequence  $B$  is denoted by  $|B|$ .

A *condition on natural parameters* has the form

$$(8) \quad L_1(\lambda_1, \dots, \lambda_q) <, \leq, = L_2(\lambda_1, \dots, \lambda_q),$$

where  $L_1, L_2$  are integer polynomials. A coefficient transformation (6) is called a *solution* of the equation  $E$  with condition (8) on natural parameters, if (6) is a

solution of  $E$  that satisfies

$$L_1(\Lambda_1, \dots, \Lambda_q) <, \leq, = L_2(\Lambda_1, \dots, \Lambda_q).$$

A condition on the length of a solution has the form

$$(9) \quad 0, \partial(P(x_1, \dots, x_n, \lambda_1, \dots, \lambda_q)) <, \leq, = \partial(Q(x_1, \dots, x_n, \lambda_1, \dots, \lambda_q)),$$

where  $P, Q$  are two primitive parametric words. A coefficient transformation (6) is called a *solution* of the equation  $E$  with a condition (9) on the lengths of solutions, if (6) is a solution of  $E$  that satisfies

$$0, |P(X_1, \dots, X_n, \Lambda_1, \dots, \Lambda_q)| <, \leq, = |Q(X_1, \dots, X_n, \Lambda_1, \dots, \Lambda_q)|.$$

A condition on the length of a sequence of variables has the form

$$(10) \quad N_i <, \leq, = \partial(\mu_i),$$

where  $N_i$  is a natural number and  $\mu_i$  is a second parameter. A coefficient transformation (6) is called a *solution* of the equation  $E$  with a condition (10) on the length of sequences, if (6) is a solution of  $E$  that satisfies

$$N_i <, \leq, = |M_i|.$$

A parametric transformation  $T$  is called a *parametric solution* of the equation  $E$  with a condition  $R$ , if  $T$  is a parametric solution of  $E$  and the result of the application of  $T$  to  $R$  is a true proposition for any values of the variables.

Let  $E$  be an equation with conditions and let  $R_1, \dots, R_m$  be the list of new conditions. By  $(E, R_i)$  we denote the equation  $E$  with additional condition  $R_i$ . The equation  $E$  is said to be *divided into a collection of equations*  $(E, R_1), \dots, (E, R_m)$ , if every solution  $S$  of  $E$  is a solution of some  $(E, R_i)$ . An equation  $(E, R_1)$  *contains* an equation  $(E, R_2)$  (and we write  $(E, R_1) \supseteq (E, R_2)$ ), if every solution of  $(E, R_2)$  is a solution of  $(E, R_1)$ .

We say that *the equation  $E_1$  is reduced by the parametric transformation  $T$  to the equation  $E_2$* , if  $E_1 T = E_2'$ , where  $E_2' \supseteq E_2$ , and for every solution  $S_1$  of  $E_1$  there exists a solution  $S_2$  of  $E_2$  such that  $S_1 = T S_2^*$  for some extension  $S_2^*$  of  $S_2$ . We say that  $S_2$  is the *image* of  $S_1$  via the transformation  $T$ . We need the extension  $S_2^*$ , because  $T$  could have some variables that are not in  $E_2$ .

**Lemma 1.** *Let  $E_1$  be reduced by  $T$  to  $E_2$ . Let  $S_1$  be a solution of  $E_1$  and  $S_2$  its image via  $T$ . If the parametric solution  $Q_2$  of  $E_2$  contains a solution  $S_2$  of  $E_2$ , then the parametric solution  $TQ_2$  of  $E_1$  contains the solution  $S_1$  of  $E_1$ .*

**Theorem 1.** *Let the equation  $E_1$  be reduced by the parametric transformation  $T$  to the equation  $E_2$ . If the general solution  $\langle E_2 \rangle$  of  $E_2$  is  $Q_1, \dots, Q_r$ , then the general solution  $\langle E_1 \rangle$  of  $E_1$  is  $TQ_1, \dots, TQ_r$ .*

**Theorem 2.** *Let the equation  $E$  be divided into a collection of equations with conditions  $(E, R_1), \dots, (E, R_m)$ . If the general solution  $\langle (E, R_i) \rangle$  of  $(E, R_i)$  is  $Q_{i,1}, \dots, Q_{i,r_i}$  ( $i = 1, \dots, m$ ), then the general solution  $\langle E \rangle$  of  $E$  is  $Q_{1,1}, \dots, Q_{1,r_1}, \dots, Q_{m,1}, \dots, Q_{m,r_m}$ .*

Let

$$(11) \quad K_\alpha(\lambda_1, \dots, \lambda_q) <, \leq, = M_\alpha(\lambda_1, \dots, \lambda_q) \quad (\alpha = 1, \dots, t),$$

where  $K_\alpha, M_\alpha \in \mathbf{L}$ , be a system of linear Diophantine equations and inequations. A transformation

$$\lambda_i \rightarrow L_i, \quad L_i \in \mathbf{L} \quad (i = 1, \dots, q),$$

is called a *parametric solution* of the system (11), if

$$K_\alpha(L_1, \dots, L_q) <, \leq, = M_\alpha(L_1, \dots, L_q) \quad (\alpha = 1, \dots, t)$$

for any values of the variables.

**Theorem 3.** *The family of solutions of the system (11) is described by a finite list of parametric solution (see [7]).*

### 3. PRELIMINARIES

The following seven propositions belong to folklore (see [4], [8], [9]). Observe that a boldface  $\mathbf{n}$  means “the equation in Proposition  $\mathbf{n}$ ”.

**Proposition 1.** *The general solution of the equation*

$$\mathbf{1} \quad x_1 x_2 = x_2 x_1$$

*is described by the transformation*

$$\begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \end{cases}$$

*where  $\alpha, \beta$  are natural parameters.*

**Proposition 2.** *The general solution of the equation*

$$\mathbf{2} \quad x_1 x_2 x_3 = x_3 x_1 x_2$$

*is described by the transformations*

$$\begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow 1, \\ x_3 \rightarrow x_3, \end{cases} \quad \begin{cases} x_1 \rightarrow (x_1 x_2)^\alpha x_1, \\ x_2 \rightarrow (x_2 x_1)^\beta x_2, \\ x_3 \rightarrow (x_1 x_2)^\gamma, \end{cases}$$

*where  $\alpha, \beta, \gamma$  are natural parameters.*

**Proposition 3.** *The general solution of the equation*

$$\mathbf{3} \quad x_1 x_2 x_3 = x_2^\alpha x_1,$$

*where  $\alpha$  is a natural parameter, is described by the transformations*

$$\begin{cases} x_1 \rightarrow x_1^\beta, \\ x_2 \rightarrow x_1^\gamma, \\ x_3 \rightarrow x_1^\delta, \end{cases}$$

*where  $\beta, \gamma, \delta$  are natural parameters.*

**Proposition 4.** *The general solution of the equation*

$$\mathbf{4} \quad x_1 x_3 = x_2 x_1$$

*is described by the transformations*

$$\begin{cases} x_1 \rightarrow x_1, \\ x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow (x_1 x_2)^\alpha x_1, \\ x_2 \rightarrow x_1 x_2, \\ x_3 \rightarrow x_2 x_1, \end{cases}$$

*where  $\alpha$  is a natural parameter.*

**Proposition 5.** *The general solution of the equation*

$$5 \quad x_1x_3 = x_2^\alpha x_1,$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \rightarrow x_1, \\ x_2 \rightarrow x_2, \\ x_3 \rightarrow 1, \\ \alpha \rightarrow 0, \end{cases} \quad \begin{cases} x_1 \rightarrow x_1, \\ x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ \alpha \rightarrow \alpha, \end{cases} \quad \begin{cases} x_1 \rightarrow (x_1x_2)^\beta x_1, \\ x_2 \rightarrow x_1x_2, \\ x_3 \rightarrow (x_2x_1)^\alpha, \\ \alpha \rightarrow \alpha, \end{cases}$$

where  $\beta$  is a natural parameter.

**Proposition 6.** *The general solution of the equation*

$$6 \quad x_1x_2x_3 = x_3x_4$$

is described by the transformations

$$\begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow 1, \\ x_3 \rightarrow x_3, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow x_1, \\ x_2 \rightarrow x_3x_2, \\ x_3 \rightarrow (x_1x_3x_2)^\alpha x_1x_3, \\ x_4 \rightarrow x_2x_1x_3, \end{cases} \quad \begin{cases} x_1 \rightarrow x_3x_1, \\ x_2 \rightarrow x_2, \\ x_3 \rightarrow (x_3x_1x_2)^\alpha x_3, \\ x_4 \rightarrow x_1x_2x_3, \end{cases}$$

where  $\alpha$  is a natural parameter.

**Proposition 7.** *The general solution of the equation*

$$7 \quad x_1x_2x_3 = x_2x_3x_4$$

is described by the transformations

$$\begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow x_2, \\ x_3 \rightarrow x_3, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow x_2x_3x_1, \\ x_2 \rightarrow (x_2x_3x_1)^\alpha x_2, \\ x_3 \rightarrow (x_3x_1x_2)^\beta x_3, \\ x_4 \rightarrow x_1x_2x_3, \end{cases} \quad \begin{cases} x_1 \rightarrow x_3x_2x_1, \\ x_2 \rightarrow (x_3x_2x_1)^\alpha x_3x_2, \\ x_3 \rightarrow (x_1x_3x_2)^\beta x_1x_3, \\ x_4 \rightarrow x_2x_1x_3, \end{cases}$$

where  $\alpha, \beta$  are natural parameters.

**Proposition 8.** *The parametric equation*

$$8 \quad x_1R(x_2, x_3)x_4 = (p(x_2, x_3))^{t+1}x_1Q(x_1, x_2, x_3)x_5$$

with  $\partial(P(x_2, x_3)) > 0$ , where  $P, Q, R$  are parametric words and  $t$  is a natural number, is reduced by the parametric transformation  $T$ :

$$x_1 \rightarrow (P(x_2, x_3, \lambda_1, \dots, \lambda_r))^\alpha x_1,$$

where  $\alpha$  is a natural parameter, to the parametric equation  $E$ :

$$\begin{aligned} & x_1R(x_2, x_3)x_4 \\ & = (P(x_2, x_3, \lambda_1, \dots, \lambda_r))^{t+1}x_1Q(P(x_2, x_3, \lambda_1, \dots, \lambda_r)^\alpha x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_5 \end{aligned}$$

with  $\partial(x_1) < \partial(P(x_2, x_3, \lambda_1, \dots, \lambda_r))$ .

*Proof.* It is easy to verify that the substitution of the transformation  $T$  into  $\mathbf{8}$  transforms  $\mathbf{8}$  to some equation  $E'$  which contains  $E$ . On the other hand, let the transformation  $S_1$ :

$$\begin{cases} x_1 \rightarrow X_1, \\ x_i \rightarrow X_i & (i = 2, \dots, 5), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, r), \end{cases}$$

where the  $X_i$  are words in the alphabet (1) and the  $\Lambda_i$  are natural numbers, be an arbitrary solution of the equation  $\mathbf{8}$ . Since  $|P(X_2, X_3, \Lambda_1, \dots, \Lambda_r)| > 0$ , we have  $X_1 = (P(X_2, X_3, \Lambda_1, \dots, \Lambda_r))^A Y_1$  for some word  $Y_1$  in the alphabet (1) and some natural number  $A$  such that

$$A|P(X_2, X_3, \Lambda_1, \dots, \Lambda_r)| \leq |X_1| < (A+1)|P(X_2, X_3, \Lambda_1, \dots, \Lambda_r)|.$$

After the substitution of the solution  $S_1$  in the equation we easily obtain that the coefficient transformation  $S_2$ :

$$\begin{cases} x_1 \rightarrow Y_1, \\ x_i \rightarrow X_i & (i = 2, \dots, 5), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, r) \end{cases}$$

is a solution of the equation  $E$ . The extension  $S_2^*$ :

$$\begin{cases} x_1 \rightarrow Y_1, \\ x_i \rightarrow X_i & (i = 2, \dots, 5), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, r), \\ \alpha \rightarrow A \end{cases}$$

satisfies  $S_1 = TS_2^*$ . Therefore the solution  $S_2$  is the image of  $S_1$  via  $T$ . Thus the parametric equation  $\mathbf{8}$  is reduced by the parametric transformation  $T$  to the parametric equation  $E$ .

#### 4. THE FUNCTION $\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_s}$

**Theorem  $\mathbf{Fi1}$ .** *The following identities hold:*

$$\begin{aligned} \mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}} &= \mathbf{Fi}(x_1, x_1^{\lambda_{2k}} x_2)^{\lambda_1, \dots, \lambda_{2k-1}} & (k \geq 1), \\ \mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k+1}} &= \mathbf{Fi}(x_2^{\lambda_{2k+1}} x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}} x_2^{\lambda_{2k+1}} & (k \geq 0). \end{aligned}$$

*Proof* (Joint induction on  $k$ ). If  $k = 0$  or  $1$ , the proof is obvious.

Suppose that  $k > 1$ . By definition,  $\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}}$  equals

$$(\mathbf{Fi}(x_2, x_1)^{\lambda_2, \dots, \lambda_{2k}} x_2)^{\lambda_1} \mathbf{Fi}(x_1, x_2)^{\lambda_3, \dots, \lambda_{2k}}.$$

According to the induction proposition (second identity), this last expression equals

$$(\mathbf{Fi}(x_1^{\lambda_{2k}} x_2, x_1)^{\lambda_2, \dots, \lambda_{2k-1}} x_1^{\lambda_{2k}} x_2)^{\lambda_1} \mathbf{Fi}(x_1, x_2)^{\lambda_3, \dots, \lambda_{2k}}.$$

According to the induction proposition (first identity), this in turn is equal to

$$(\mathbf{Fi}(x_1^{\lambda_{2k}} x_2, x_1)^{\lambda_2, \dots, \lambda_{2k-1}} x_1^{\lambda_{2k}} x_2)^{\lambda_1} \mathbf{Fi}(x_1, x_1^{\lambda_{2k}} x_2)^{\lambda_3, \dots, \lambda_{2k-1}},$$

and this is equal (by definition) to

$$\mathbf{Fi}(x_1, x_1^{\lambda_{2k}} x_2)^{\lambda_1, \dots, \lambda_{2k-1}}.$$

By definition,  $\mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k+1}}$  equals

$$(\mathbf{Fi}(x_2, x_1)^{\lambda_2, \dots, \lambda_{2k+1}} x_2)^{\lambda_1} \mathbf{Fi}(x_1, x_2)^{\lambda_3, \dots, \lambda_{2k+1}}.$$

According to the induction proposition (first identity), this last expression equals

$$(\mathbf{Fi}(x_2, x_2^{\lambda_{2k+1}} x_1)^{\lambda_2, \dots, \lambda_{2k}} x_2)^{\lambda_1} \mathbf{Fi}(x_1, x_2)^{\lambda_3, \dots, \lambda_{2k+1}},$$

and by the induction proposition (second identity), this is in turn equal to

$$(\mathbf{Fi}(x_2, x_2^{\lambda_{2k+1}} x_1)^{\lambda_2, \dots, \lambda_{2k}} x_2)^{\lambda_1} \mathbf{Fi}(x_2^{\lambda_{2k+1}} x_1, x_2)^{\lambda_3, \dots, \lambda_{2k}} x_2^{\lambda_{2k+1}}.$$

But this is equal (by definition) to

$$\mathbf{Fi}(x_2^{\lambda_{2k+1}} x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}} x_2^{\lambda_{2k+1}}.$$

Consider an equation of the form  $x_1 P x_2 U = x_2 Q x_1 V$ , where  $P, Q$  are parametric words in the alphabet  $x_3, \dots, x_n$  and  $U, V$  are parametric words. Consider the sequence of parametric transformations

$$(12) \quad \begin{array}{ll} 1. & x_1 \rightarrow (x_2 Q)^{\lambda_1} x_1, \\ 2. & x_2 \rightarrow (x_1 P)^{\lambda_2} x_2, \\ 3. & x_1 \rightarrow (x_2 Q)^{\lambda_3} x_1, \\ 4. & x_2 \rightarrow (x_1 P)^{\lambda_4} x_2, \\ \dots & \dots \\ 2k-1. & x_1 \rightarrow (x_2 Q)^{\lambda_{2k-1}} x_1, \\ 2k. & x_2 \rightarrow (x_1 P)^{\lambda_{2k}} x_2, \\ 2k+1. & x_1 \rightarrow (x_2 Q)^{\lambda_{2k+1}} x_1, \\ \dots & \dots \end{array}$$

where  $\lambda_1, \lambda_2, \dots$  are natural parameters.

**Theorem  $\mathbf{Fi}2$ .** *For every natural  $s$ , the sequence (12) of the first  $s$  parametric transformation can be collected by the following common transformation:*

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 P, x_2 Q)^{\lambda_1, \dots, \lambda_s} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 Q, x_1 P)^{\lambda_2, \dots, \lambda_s} x_2. \end{cases}$$

*Proof.* If  $s = 0$  the proposition obviously holds. Consider two cases.

*Case 1.* Suppose that the sequence of the first  $2k - 1$  transformations can be collected by the common transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 P, x_2 Q)^{\lambda_1, \dots, \lambda_{2k-1}} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 Q, x_1 P)^{\lambda_2, \dots, \lambda_{2k-1}} x_2. \end{cases}$$

Let the  $2k$ th transformation be of the form

$$x_2 \rightarrow x_2 P^{\lambda_{2k}} x_2.$$

Then the sequence of the first  $2k$  transformations can be collected by the common transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 P, (x_1 P)^{\lambda_{2k}} x_2 Q)^{\lambda_1, \dots, \lambda_{2k-1}} x_1, \\ x_2 \rightarrow \mathbf{Fi}((x_1 P)^{\lambda_{2k}} x_2 Q, x_1 P)^{\lambda_2, \dots, \lambda_{2k-1}} (x_1 P)^{\lambda_{2k}} x_2. \end{cases}$$

According to Theorem  $\mathbf{Fi}1$  this transformation coincides with the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 P, x_2 Q)^{\lambda_1, \dots, \lambda_{2k}} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 Q, x_1 P)^{\lambda_2, \dots, \lambda_{2k}} x_2. \end{cases}$$

*Case 2.* Suppose that the sequence of the first  $2k$  transformations can be collected in the common transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1P, x_2Q)^{\lambda_1, \dots, \lambda_{2k}} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2Q, x_1P)^{\lambda_2, \dots, \lambda_{2k}} x_2. \end{cases}$$

Let the  $(2k + 1)$ st transformation be of the form

$$x_1 \rightarrow (x_2Q)^{\lambda_{2k+1}} x_1.$$

Then the sequence of the first  $2k+1$  transformations can be collected by the common transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}((x_2Q)^{\lambda_{2k+1}} x_1P, x_2Q)^{\lambda_1, \dots, \lambda_{2k}} (x_2Q)^{\lambda_{2k+1}} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2Q, (x_2Q)^{\lambda_{2k+1}} x_1P)^{\lambda_2, \dots, \lambda_{2k}} x_2. \end{cases}$$

According to Theorem **Fi1** this transformation coincides with the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1P, x_2Q)^{\lambda_1, \dots, \lambda_{2k+1}} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2Q, x_1P)^{\lambda_2, \dots, \lambda_{2k+1}} x_2. \end{cases}$$

**Theorem **Fi3**.** *The following identities hold:*

$$\begin{aligned} \mathbf{Fi}(x_1, x_2)^{0, \lambda_2, \dots, \lambda_s} &= \mathbf{Fi}(x_1, x_2)^{\lambda_3, \dots, \lambda_s}, \\ \mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_s} &= \mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{r-1} + \lambda_{r+1}, \dots, \lambda_s}, \\ \mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{s-1}, 0} &= \mathbf{Fi}(x_1, x_2)^{\lambda_1, \dots, \lambda_{s-1}}. \end{aligned}$$

*Proof.* Follows from Theorem **Fi2**.

**Proposition 9.** *The parametric equation*

$$\mathbf{9} \quad x_1 x_3^\alpha x_2 U(x_1, x_2, x_3, \lambda_1, \dots, \lambda_r) x_4 = x_2 x_3^\beta x_1 V(x_1, x_2, x_3, \lambda_1, \dots, \lambda_r) x_5$$

with  $\partial(x_1 x_3^\alpha) > 0$ ,  $\partial(x_2 x_3^\beta) > 0$ ,  $\alpha + \beta > 0$ , where  $\alpha, \beta, \lambda_1, \dots, \lambda_r$  are natural parameters and  $U, V$  are parametric words, is reduced by the transformation  $T$ :

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 x_3^\alpha, x_2 x_3^\beta)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 x_3^\beta, x_1 x_3^\alpha)^\mu x_2, \end{cases}$$

where  $\mu$  is a variable for sequences of natural parameters, to the equation  $E$ :

$$\begin{aligned} &x_1 x_3^\alpha x_2 U(\mathbf{Fi}(x_1 x_3^\alpha, x_2 x_3^\beta)^\mu x_1, \mathbf{Fi}(x_2 x_3^\beta, x_1 x_3^\alpha)^\mu x_2, x_3, \lambda_1, \dots, \lambda_r) x_4 \\ &= x_2 x_3^\beta x_1 V(\mathbf{Fi}(x_1 x_3^\alpha, x_2 x_3^\beta)^\mu x_1, \mathbf{Fi}(x_2 x_3^\beta, x_1 x_3^\alpha)^\mu x_2, x_3, \lambda_1, \dots, \lambda_r) x_5 \end{aligned}$$

with  $\partial(x_1) < \partial(x_2 x_3^\beta)$ ,  $\partial(x_2) < \partial(x_1 x_3^\alpha)$ .

*Proof.* It is easy to verify that  $T$  reduces **9** to some equation  $E'$  which contains  $E$ .

On the other hand, let the transformation  $S_1$ :

$$\begin{cases} x_i \rightarrow X_i & (i = 1, \dots, 5), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, r), \\ \alpha \rightarrow A, \\ \beta \rightarrow B, \end{cases}$$

where the  $X_i$  are words in the alphabet (1) and  $\Lambda_i, A, B$  are natural numbers, be an arbitrary solution of the equation **9**.

We prove by induction on  $|X_1X_2X_3|$  that  $S_1$  is contained in the parametric transformation  $T$ .

If  $|X_1| < |X_2X_3^B|$  and  $|X_2| < |X_1X_3^A|$ , then the coefficient transformation  $S_2$  :

$$\begin{cases} x_i \rightarrow X_i & (i = 1, \dots, 5), \\ \lambda_i \rightarrow \Lambda_i & (i = 1, \dots, r), \\ \alpha \rightarrow A, \\ \beta \rightarrow B, \\ \mu \rightarrow \emptyset \end{cases}$$

is a solution of the equation  $E$ , and  $S_1 = TS_2$ .

Let  $|X_1| \geq |X_2X_3^B|$ . Since  $|X_2X_3^B| > 0$ , we have  $X_1 = X_2X_3^BY_1$  for some word  $Y_1$  in the alphabet (1), where  $|Y_1| < |X_1|$ .

It is easy to see that equation **9** with the additional condition  $\partial(x_1) > \partial(x_2x_3^\beta)$  is reduced by the transformation  $t$ :

$$x_1 \rightarrow x_2x_3^\beta x_1$$

to the equation  $E'$ :

$$\begin{aligned} & x_1x_3^\alpha x_2U(x_2x_3^\beta x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_4 \\ & = x_2x_3^\beta x_1V(x_2x_3^\beta x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_5 \end{aligned}$$

with  $\partial(x_1x_3^\alpha) > 0$ ,  $\partial(x_2x_3^\beta) > 0$ ,  $\alpha + \beta > 0$ .

The transformation  $S'$ :

$$\begin{cases} x_1 \rightarrow Y_1, \\ x_i \rightarrow X_i & (i = 2, 3, 4, 5), \\ \alpha \rightarrow A, \\ \beta \rightarrow B \end{cases}$$

is a solution of the equation  $E'$ .

Since  $|Y_1| < |X_1|$ , one can use the inductive proposition to see that  $S'$  is contained in the parametric transformation  $T$ . Hence  $S_1$  is contained in the parameter solution  $tT$ . Using Theorem **Fi2**, one can see that the transformation  $tT$  has the form

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1x_3^\alpha, x_2x_3^\beta)^{1,0,\mu}x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2x_3^\beta, x_1x_3^\alpha)^\mu x_2. \end{cases}$$

Let  $|X_2| \geq |X_1X_3^A|$ . Since  $|X_1X_3^A| > 0$ , we have  $X_2 = X_1X_3^AY_2$  for some word  $Y_2$  in the alphabet (1), where  $|Y_2| < |X_2|$ .

It is easy to see that equation **9** with the additional condition  $\partial(x_2) > \partial(x_1x_3^\alpha)$  is reduced by the transformation  $t$ :  $x_2 \rightarrow x_1x_3^\alpha x_2$  to the equation  $E'$ :

$$\begin{aligned} & x_1x_3^\alpha x_2U(x_1, x_1x_3^\alpha x_2, x_3, \lambda_1, \dots, \lambda_r)x_4 \\ & = x_2x_3^\beta x_1V(x_1, x_1x_3^\alpha x_2, x_3, \lambda_1, \dots, \lambda_r)x_5 \end{aligned}$$

with  $\partial(x_1x_3^\alpha) > 0$ ,  $\partial(x_2x_3^\beta) > 0$ ,  $\alpha + \beta > 0$ .

The transformation  $S'$ :

$$\begin{cases} x_2 \rightarrow Y_2, \\ x_i \rightarrow X_i \quad (i = 1, 3, 4, 5), \\ \alpha \rightarrow A, \\ \beta \rightarrow B \end{cases}$$

is a solution of  $E'$ .

Since  $|Y_2| < |X_2|$ , one can use the inductive proposition to see that  $S'$  is contained in the parametric transformation  $T$ . Hence  $S_1$  is contained in  $tT$ . Using Theorem **Fi2**, one can see that the transformation  $tT$  has form

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1x_3^\alpha, x_2x_3^\beta)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2x_3^\beta, x_1x_3^\alpha)^{1,\mu} x_2. \end{cases}$$

**Proposition 10\***. *The general solution of the equation*

$$\mathbf{10}^* \quad x_1x_3^{\alpha+1}x_2 = x_2x_1x_4 \quad \text{with } \partial(x_1) < \partial(x_2) < \partial(x_1x_3^{\alpha+1}),$$

where  $\alpha$  is a natural parameter, is described by the transformation

$$\begin{cases} x_2 \rightarrow x_1(x_2x_3)^\beta x_2, \\ x_3 \rightarrow x_2x_3, \\ x_1 \rightarrow (x_3(x_2x_3)^{\alpha-\beta})^\lambda x_1, \end{cases}$$

followed by one of the two transformations

$$\begin{cases} x_1 \rightarrow (x_1x_3x_2)^\sigma x_1, \\ x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow x_3(x_2x_1x_3)^{\alpha-\beta-\sigma}(x_1x_3x_2)^\sigma x_1(x_2x_1x_3)^\beta x_2, \\ x_1 \rightarrow (x_3x_1x_2)^\tau x_3x_1, \\ x_2 \rightarrow x_1x_2, \\ x_4 \rightarrow x_2x_3(x_1x_2x_3)^{\alpha-\beta-\tau-1}(x_3x_1x_2)^\tau x_3x_1(x_1x_2x_3)^\beta x_1x_2, \end{cases}$$

where  $\beta, \lambda, \sigma, \tau$  are natural parameters, with  $\beta \leq \alpha$ ,  $\sigma + \beta \leq \alpha$ ,  $\tau + \beta + 1 \leq \alpha$ .

*Proof.* The equation  $\mathbf{10}^*$  can be reduced by the transformation

$$x_2 \rightarrow x_1x_2$$

to the equation  $E_1$ :

$$x_3^{\alpha+1}x_1x_2 = x_2x_1x_4 \quad \text{with } 0 < \partial(x_2) < \partial(x_3^{\alpha+1}).$$

The equation  $E_1$  can be reduced by the transformation

$$\begin{cases} x_2 \rightarrow (x_2x_3)^\beta x_2, \\ x_3 \rightarrow x_2x_3, \end{cases}$$

where  $\beta$  is a natural parameter with  $\beta \leq \alpha$ , to the equation  $E_2$ :

$$x_3(x_2x_3)^{\alpha-\beta}x_1(x_2x_3)^\beta x_2 = x_1x_4 \quad \text{with } \partial(x_3) > 0.$$

According to Proposition 27, the equation  $E_2$  is reduced by the parametric transformation

$$x_1 \rightarrow (x_3(x_2x_3)^{\alpha-\beta})^\lambda x_1,$$

where  $\lambda$  is a natural parameter, to the equation  $E_3$ :

$$E_2 \quad \text{with } \partial(x_1) < \partial(x_3(x_2x_3)^{\alpha-\beta}).$$

The equation  $E_3$  can be divided into the collection of equations

(j)  $E_3$  with  $\partial(x_3x_2)^\sigma \leq \partial(x_1) < \partial((x_3x_2)^\sigma x_3)$ ,  $\sigma \leq \alpha - \beta$ ,

(jj)  $E_3$  with  $\partial((x_3x_2)^\tau x_3) \leq \partial(x_1) < \partial((x_3x_2)^{\tau+1}x_3)$ ,  $\tau \leq \alpha - \beta - 1$ .

The equation (j) can be reduced by the transformation

$$\begin{cases} x_1 \rightarrow (x_1x_3x_2)^\sigma x_1, \\ x_3 \rightarrow x_1x_3 \end{cases}$$

to the equation

$$x_3(x_2x_1x_3)^{\alpha-\beta-\sigma}(x_1x_3x_2)^\sigma x_1(x_2x_1x_3)^\beta x_2 = x_4.$$

The equation (jj) can be reduced by the transformation

$$\begin{cases} x_1 \rightarrow (x_3x_1x_2)^\tau x_3x_1, \\ x_2 \rightarrow x_1x_2 \end{cases}$$

to the equation

$$x_2x_3(x_1x_2x_3)^{\alpha-\beta-\tau-1}(x_3x_1x_2)^\tau x_3x_1(x_1x_2x_3)^\beta x_1x_2 = x_4.$$

**Proposition 10.** *The general solution of the equation*

$$10 \quad x_1x_3^{\alpha+1}x_2 = x_2x_1x_4,$$

where  $\alpha$  is a natural parameter, is described by the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1x_3^{\alpha+1}, x_2)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2, x_1x_3^\alpha)^\mu x_2, \end{cases}$$

followed by one of the three transformations

$$\begin{cases} x_1 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_2 \rightarrow 1, \\ x_4 \rightarrow x_3^{\alpha+1}, \end{cases} \quad \langle \mathbf{10}^* \rangle.$$

*Proof.* This follows directly from Propositions 9 and 10\*.

## 5. THE FUNCTION $\mathbf{Th}(x_1, x_2, x_3)_i^{\lambda_1, \dots, \lambda_{2s}}$

**Theorem  $\mathbf{Th1}$ .** *The following identities hold for  $s \geq 0$ :*

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s+2}} \\ &= \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ & \quad x_1x_2x_3)_1^{\lambda_1, \dots, \lambda_{2s}}(x_1x_2x_3)^{\lambda_{2s+1}}; \end{aligned}$$

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_1, \dots, \lambda_{2s+2}} \\ &= \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ & \quad x_1x_2x_3)_2^{\lambda_1, \dots, \lambda_{2s}}(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}; \end{aligned}$$

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s+2}} \\ &= \mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, \\ & \quad x_1 x_2 x_3)_3^{\lambda_1, \dots, \lambda_{2s}} x_1 x_2; \end{aligned}$$

$$\begin{aligned} & \mathbf{Oc}(x_1, x_2, x_3)^{\lambda_1, \dots, \lambda_{2s+2}} \\ &= (x_3 x_1 x_2)^{\lambda_{2s+1}} \mathbf{Oc}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} \\ & \quad x_2, x_1 x_2 x_3)^{\lambda_1, \dots, \lambda_{2s}}. \end{aligned}$$

*Proof.* We argue by a joint induction on  $s$ .

*Third identity.* By definition,

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s+2}} \\ &= \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s+2}} x_1 \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s+2}} x_2 \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_3, \dots, \lambda_{2s+2}}. \end{aligned}$$

According to the induction proposition, it is equal to

$$\begin{aligned} & \mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, \\ & \quad x_1 x_2 x_3)_1^{\lambda_3, \dots, \lambda_{2s}} (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1 \\ & \cdot \mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, x_1 x_2 x_3)_2^{\lambda_3, \dots, \lambda_{2s}} \\ & \quad \cdot (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2 \\ & \cdot \mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, x_1 x_2 x_3)_3^{\lambda_3, \dots, \lambda_{2s}} x_1 x_2. \end{aligned}$$

This is equal by definition to

$$\mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, x_1 x_2 x_3)_3^{\lambda_1, \dots, \lambda_{2s}} x_1 x_2.$$

*First identity.* By definition,

$$\mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s+2}} = (\mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s+2}} x_3)^{\lambda_1} \mathbf{Th}(x_1, x_2, x_3)^{\lambda_3, \dots, \lambda_{2s}}.$$

According to the induction proposition and the third identity this is equal to

$$\begin{aligned} & (\mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, x_1 x_2 x_3)_3^{\lambda_1, \dots, \lambda_{2s}} x_1 x_2 x_3)^{\lambda_1} \\ & \cdot \mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, x_1 x_2 x_3)_1^{\lambda_3, \dots, \lambda_{2s}} \\ & \quad \cdot (x_1 x_2 x_3)^{\lambda_{2s+1}}. \end{aligned}$$

This is equal by definition to

$$\begin{aligned} & \mathbf{Th}((x_1 x_2 x_3)^{\lambda_{2s+1}} x_1, (x_2 x_3 x_1 (x_1 x_2 x_3)^{\lambda_{2s+1}} x_1)^{\lambda_{2s+2}} x_2, x_1 x_2 x_3)_1^{\lambda_1, \dots, \lambda_{2s}} \\ & \quad \cdot (x_1 x_2 x_3)^{\lambda_{2s+1}}. \end{aligned}$$

*Second identity.* By definition

$$\begin{aligned} & \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_1, \dots, \lambda_{2s+2}} \\ &= (\mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s+2}} x_2 \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_3, \dots, \lambda_{2s+2}} x_3 \\ & \cdot \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s+2}} x_1 \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s+2}} x_1)^{\lambda_2} \\ & \cdot \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s+2}}. \end{aligned}$$

According to the induction proposition and the first identity, this is equal to

$$\begin{aligned}
& (\mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_2^{\lambda_3, \dots, \lambda_{2s}} \\
& \quad \cdot (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2 \\
& \cdot \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_3^{\lambda_3, \dots, \lambda_{2s}} \\
& \quad \cdot x_1x_2x_3 \\
& \cdot \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_1^{\lambda_3, \dots, \lambda_{2s}} \\
& \quad \cdot (x_1x_2x_3)^{\lambda_{2s+1}}x_1 \\
& \cdot \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_1^{\lambda_1, \dots, \lambda_{2s}} \\
& \quad \cdot (x_1x_2x_3)^{\lambda_{2s+1}}x_1^{\lambda_2} \\
& \cdot \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_2^{\lambda_3, \dots, \lambda_{2s}} \\
& \quad \cdot (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}.
\end{aligned}$$

This is equal by definition to

$$\begin{aligned}
& \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_2^{\lambda_1, \dots, \lambda_{2s}} \\
& \quad \cdot (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}.
\end{aligned}$$

*Fourth identity.* By definition,

$$\begin{aligned}
& \mathbf{Oc}((x_1, x_2, x_3)^{\lambda_1, \dots, \lambda_{2s+2}} \\
& = \mathbf{Oc}(x_1, x_2, x_3)^{\lambda_3, \dots, \lambda_{2s+2}} (\mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_3, \dots, \lambda_{2s+2}} x_3 \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_3, \dots, \lambda_{2s+2}} x_1 \\
& \quad \cdot \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_3, \dots, \lambda_{2s+2}} x_2)^{\lambda_1}.
\end{aligned}$$

According to the induction proposition and the first, second and third identities, this is equal to

$$\begin{aligned}
& (x_3x_1x_2)^{\lambda_{2s+1}} \mathbf{Oc}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_3^{\lambda_3, \dots, \lambda_{2s}} \\
& \cdot (\mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_3^{\lambda_3, \dots, \lambda_{2s}} x_1x_2x_3 \\
& \quad \cdot \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_1^{\lambda_3, \dots, \lambda_{2s}} \\
& \quad \cdot (x_1x_2x_3)^{\lambda_{2s+1}}x_1 \\
& \quad \cdot \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_2^{\lambda_3, \dots, \lambda_s} \\
& \quad \cdot (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2)^{\lambda_1}.
\end{aligned}$$

This is equal by definition to

$$\begin{aligned}
& (x_3x_1x_2)^{\lambda_{2s+1}} \mathbf{Oc}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\
& \quad x_1x_2x_3)^{\lambda_1, \dots, \lambda_{2s}}.
\end{aligned}$$

Consider the equation

$$x_1x_2x_3x_4 = x_3x_1^2x_2$$

and the sequence of joint parametric transformations (13):

$$(13.1) \quad \begin{cases} x_1 \rightarrow (x_1x_2x_3)^{\lambda_1}x_1, \\ x_2 \rightarrow (x_2x_3x_1(x_1x_2x_3)^{\lambda_1}x_1)^{\lambda_2}x_2, \\ x_3 \rightarrow x_1x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^{\lambda_1}; \end{cases}$$

$$(13.2) \quad \begin{cases} x_1 \rightarrow (x_1x_2x_3)^{\lambda_3}x_1, \\ x_2 \rightarrow (x_2x_3x_1(x_1x_2x_3)^{\lambda_3}x_1)^{\lambda_4}x_2, \\ x_3 \rightarrow x_1x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^{\lambda_3}; \end{cases}$$

.....

$$(13.s) \quad \begin{cases} x_1 \rightarrow (x_1x_2x_3)^{\lambda_{2s-1}}x_1, \\ x_2 \rightarrow (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s-1}}x_1)^{\lambda_{2s}}x_2, \\ x_3 \rightarrow x_1x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^{\lambda_{2s-1}}; \end{cases}$$

$$(13.s+1) \quad \begin{cases} x_1 \rightarrow (x_1x_2x_3)^{\lambda_{2s+1}}x_1, \\ x_2 \rightarrow (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ x_3 \rightarrow x_1x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^{\lambda_{2s+1}}; \end{cases}$$

.....

**Theorem Th2.** For every natural  $s$  the sequence of the  $s$  joint parametric transformations (13) can be collected by the following common transformation:

$$(14) \quad \begin{cases} x_1 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s}}x_1, \\ x_2 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_1, \dots, \lambda_{2s}}x_2, \\ x_3 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s}}x_3, \\ x_4 \rightarrow x_4 \mathbf{Oc}(x_1, x_2, x_3)^{\lambda_1, \dots, \lambda_{2s}}. \end{cases}$$

*Proof.* Suppose the sequence of the first  $s$  transformations can be collected by the common transformation (14), and let the  $(s+1)$ st transformation be of the form (13.s+1). Then the sequence of the first  $s+1$  transformations can be collected by the common transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_1^{\lambda_1, \dots, \lambda_{2s}} \\ \quad \cdot (x_1x_2x_3)^{\lambda_{2s+1}}x_1, \\ x_2 \rightarrow \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_2^{\lambda_1, \dots, \lambda_{2s}} \\ \quad \cdot (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ x_3 \rightarrow \mathbf{Th}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_3^{\lambda_1, \dots, \lambda_{2s}} \\ \quad \cdot x_1x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^{\lambda_{2s+1}} \\ \quad \cdot \mathbf{Oc}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)^{\lambda_1, \dots, \lambda_{2s}}. \end{cases}$$

According to Theorem **Th**<sub>1</sub> this transformation coincides with the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_1^{\lambda_1, \dots, \lambda_{2s+2}} x_1, \\ x_2 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_2^{\lambda_1, \dots, \lambda_{2s+2}} x_2, \\ x_3 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_3^{\lambda_1, \dots, \lambda_{2s+2}} x_3, \\ x_4 \rightarrow x_4 \mathbf{Oc}(x_1, x_2, x_3)^{\lambda_1, \dots, \lambda_{2s+2}}. \end{cases}$$

**Proposition 11.** *The general solution of the equation*

$$\mathbf{11} \quad x_2 x_1 x_3 x_4 = x_3 x_1^2 (x_3 x_1)^\alpha x_2$$

with  $\partial(x_3) \leq \partial(x_2) < \partial(x_3 x_1^2 (x_3 x_1)^\alpha)$ , where  $\alpha$  is a natural parameter, is described by the transformations

$$\begin{aligned} \mathbf{t} \quad \begin{cases} x_1 \rightarrow x_1^{\eta+\theta}, \\ x_2 \rightarrow x_3 x_1^\theta, \\ x_4 \rightarrow x_4 (x_1^{\eta+\theta} x_3)^\alpha x_1^\theta, \end{cases} & \quad \mathbf{tt} \quad \begin{cases} x_2 \rightarrow x_3 x_1 (x_1 x_3)^\gamma x_2, \\ x_1 \rightarrow x_2 x_1, \\ x_4 \rightarrow (x_1 x_3 x_2)^{\alpha-\gamma} x_1 x_2 (x_1 x_3 x_2)^\gamma, \end{cases} \\ \langle \mathbf{5} \rangle & \quad \langle \mathbf{2} \rangle \\ \\ \mathbf{t} \mathbf{t} \mathbf{t} \quad \begin{cases} x_2 \rightarrow x_3 x_1 (x_1 x_3)^\delta x_1 x_2, \\ x_3 \rightarrow x_2 x_3, \\ x_4 \rightarrow (x_3 x_1 x_2)^{\alpha-\delta-1} x_3 x_1^2 x_2 (x_3 x_1 x_2)^\delta, \end{cases} & \\ \langle \mathbf{2} \rangle & \end{aligned}$$

where  $\eta, \theta, \gamma, \delta$  are natural parameters with  $\gamma \leq \alpha$  and  $\delta < \alpha$ .

*Proof.* It is easy to see that **11** can be divided into the following collection of equations:

- (j) **11** with  $\partial(x_3) \leq \partial(x_2) < \partial(x_3 x_1)$ ,
- (jj) **11** with  $\partial(x_3 x_1 (x_1 x_3)^\gamma) \leq \partial(x_2) < \partial(x_3 x_1 (x_1 x_3)^\gamma x_1)$ ,
- (jjj) **11** with  $\partial(x_3 x_1 (x_1 x_3)^\delta x_1) \leq \partial(x_2) < \partial(x_3 x_1 (x_1 x_3)^{\delta+1})$ ,

where  $\gamma, \delta$  are natural parameters with  $\gamma \leq \alpha$  and  $\delta < \alpha$ .

It is obvious that (j) is reduced by the parametric transformation

$$\begin{cases} x_2 \rightarrow x_3 x_2, \\ x_1 \rightarrow x_2 x_1, \\ x_4 \rightarrow x_4 x_2 (x_1 x_3 x_2)^\alpha \end{cases}$$

to the equation  $x_2 x_1 x_3 x_4 = x_1 x_2 x_1 x_3$  with  $\partial(x_2) < \partial(x_1)$ . The last equation falls into the system

$$\begin{cases} x_2 x_1 = x_1 x_2, \\ x_3 x_4 = x_1 x_3 \end{cases}$$

with  $\partial(x_2) < \partial(x_1)$ . By Proposition 1 this system is reduced by the parametric transformation

$$\begin{cases} x_1 = x_1^\eta, \\ x_2 = x_1^\theta \end{cases}$$

to the equation  $x_3 x_4 = x_1^\eta x_3$  with  $\theta > \eta$ , that is, to the equation **5**.

The equation (jj) is reduced by the parametric transformation **tt** to the equation **2**.

The equation (jjj) is reduced by the parametric transformation ttt to the equation **2**.

**Proposition 12\***. *The parametric equation*

$$\mathbf{12}^* \quad x_2x_1x_3x_4 = x_3x_1^2(x_3x_1)^\alpha x_2$$

with  $\partial(x_2) < \partial(x_3)$ , where  $\alpha$  is a natural parameter, is reduced by the transformation

$$\begin{cases} x_3 \rightarrow x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^\alpha \end{cases}$$

to the parametric equation **12**.

*Proof.* This is obvious.

**Proposition 12**. *The general solution of the equation*

$$\mathbf{12} \quad x_1x_2x_3x_4 = x_3x_1^2x_2 \quad \text{with } \partial(x_3) > 0$$

is described by the transformation

$$\begin{aligned} \mathbf{T} & \begin{cases} x_1 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_1^\xi x_1, \\ x_2 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_2^\xi x_2, \\ x_3 \rightarrow \mathbf{Th}(x_1, x_2, x_3)_3^\xi x_3, \\ x_4 \rightarrow x_4 \mathbf{Oc}(x_1, x_2, x_3)^\xi, \end{cases} \\ \mathbf{t} & \begin{cases} x_1 \rightarrow (x_1x_3)^\sigma x_1, \\ x_2 \rightarrow (x_3x_1^2(x_3x_1)^\rho)^\rho x_2, \\ x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow x_4, \end{cases} \\ & \langle \mathbf{11} \rangle \end{aligned}$$

where  $\xi$  is a variable whose values are even sequences of natural parameters, and  $\sigma, \rho$  are natural parameters.

*Proof.* It is easy to apply the transformation  $\mathbf{Tt}\langle \mathbf{11} \rangle$  to the equation **12** and to verify, by using the definition of the functions  $\mathbf{Th}(x_1, x_2, x_3)_i^{\lambda_1, \dots, \lambda_{2s}}$ , that it is a parametric solution of **12**.

According to Proposition 8 the equation **12** is reduced by the parametric transformation  $x_1 \rightarrow x_3^\sigma x_1$ , where  $\sigma$  is natural parameter, to the equation  $E_1$ :

$$x_1x_2x_3x_4 = x_3x_1x_3^\sigma x_1x_2 \quad \text{with } \partial(x_1) \leq \partial(x_3).$$

According to condition  $\partial(x_1) < \partial(x_3)$  the equation  $E_1$  is reduced by the parametric transformation  $x_3 \rightarrow x_1x_3$  to the equation  $E_2$ :

$$x_2x_1x_3x_4 = x_3x_1(x_1x_3)^\sigma x_1x_2.$$

According to Proposition 8 the equation  $E_2$  is reduced by the parametric transformation

$$x_2 \rightarrow (x_3x_1(x_1x_3)^\sigma x_1)^\rho x_2,$$

where  $\rho$  is a natural parameter, to the equation  $E_3$ :

$$x_2x_1x_3x_4 = x_3x_1^2(x_3x_1)^\sigma x_2 \quad \text{with } \partial(x_2) < \partial(x_3x_1^2(x_3x_1)^\sigma).$$

It is easy to see that  $E_3$  can be divided into the following collection of equations:

(j)  $E_3$  with  $\partial(x_3) \leq \partial(x_2)$ ,

(jj)  $E_3$  with  $\partial(x_3) > \partial(x_2)$ .

The equation (j) is equation **11**.

According to Proposition 12\* the equation (jj) is reduced by the transformation

$$\begin{cases} x_3 \rightarrow x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^\sigma \end{cases}$$

to the equation **12**.

The sequence of transformations

$$\begin{aligned} x_1 &\rightarrow x_3^\sigma x_1, \\ x_3 &\rightarrow x_1x_3, \\ x_2 &\rightarrow (x_3x_1(x_1x_3)^\sigma x_1)^\rho x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4(x_3x_1x_2)^\alpha \end{aligned}$$

can be collected by the following common transformation  $r$ :

$$\begin{cases} x_1 \rightarrow (x_1x_2x_3)^\sigma x_1, \\ x_2 \rightarrow (x_2x_3x_1(x_1x_2x_3)^\sigma x_1)^\rho x_2, \\ x_3 \rightarrow x_1x_2x_3, \\ x_4 \rightarrow x_4(x_3x_1x_2)^\alpha. \end{cases}$$

Using Theorem **Th2**, one can see that transformation  $rT$  can be obtained from  $T$  by replacing the parameter  $\xi$  by the sequence  $\sigma, \rho, \xi$ .

## 6. SUPPLEMENT

**Proposition 13.** *The general solution of the equation*

$$\mathbf{13} \quad x_1x_2x_3x_4 = x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\kappa+1}x_1$$

with  $\partial(x_1) \leq \partial(x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\kappa+1})$ , where  $\lambda$  and  $\kappa$  are natural parameters, is described by the transformations

$$\mathbf{t1} \quad \begin{cases} x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow x_4x_3(x_2x_1x_3)^\lambda x_2((x_2x_1x_3)^{\lambda+1}x_2)^\kappa x_1, \end{cases} \quad \langle \mathbf{12} \rangle,$$

$$\mathbf{t2} \quad \begin{cases} x_1 \rightarrow x_3x_1^\eta, \\ x_2 \rightarrow x_1^{\eta+\theta}, \\ x_4 \rightarrow x_4(x_1^{\eta+\theta}x_3)^\lambda x_1^{\eta+\theta}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^\kappa x_3x_1^\eta, \end{cases} \quad \langle \mathbf{5} \rangle,$$

$$\mathbf{t3} \quad \begin{cases} x_1 \rightarrow x_3x_1x_2(x_1x_2(x_3x_1x_2)^{\lambda+1})^\tau (x_1x_2x_3)^\sigma x_1, \\ x_2 \rightarrow x_1x_2, \\ x_4 \rightarrow x_2(x_3x_1x_2)^{\lambda-\sigma} (x_1x_2(x_3x_1x_2)^{\lambda+1})^{\kappa-\tau} x_3x_1x_2 \\ \quad \cdot (x_1x_2(x_3x_1x_2)^{\lambda+1})^\tau (x_1x_2x_3)^\sigma x_1, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$t4 \quad \begin{cases} x_1 \rightarrow x_3 x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^\tau (x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^\eta, \\ x_2 \rightarrow x_1^{\eta+\theta}, \\ x_4 \rightarrow x_4 x_1^{\eta+\theta} (x_3 x_1^{\eta+\theta})^\lambda ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^{\kappa-\tau-1} x_3 \\ \quad \cdot x_1^{\eta+\theta} (x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^\tau (x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^\eta, \end{cases}$$

**(5)**,

$$t5 \quad \begin{cases} x_1 \rightarrow x_1 x_3 (x_2 x_1)^{\beta+1} x_2 (((x_2 x_1)^{\beta+1} x_2 x_1 x_3)^{\lambda+1} (x_2 x_1)^{\beta+1} x_2)^\kappa \\ \quad \cdot ((x_2 x_1)^{\beta+1} x_2 x_1 x_3)^{\lambda+1} x_2 x_1, \\ x_2 \rightarrow (x_2 x_1)^{\beta+1} x_2, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow (x_2 x_1)^\beta x_2 (((x_2 x_1)^{\beta+1} x_2 x_1 x_3)^{\lambda+1} (x_2 x_1)^{\beta+1} x_2)^\kappa \\ \quad \cdot ((x_2 x_1)^{\beta+1} x_2 x_1 x_3)^{\lambda+1} x_2 x_1, \end{cases}$$

**(2)**,

$$t6 \quad \begin{cases} x_1 \rightarrow x_3 (x_2 x_3 x_1)^{\beta+1} x_2 (((x_2 x_3 x_1)^{\beta+1} x_2 x_3)^{\lambda+1} (x_2 x_3 x_1)^{\beta+1} x_2)^\kappa \\ \quad \cdot ((x_2 x_3 x_1)^{\beta+1} x_2 x_3)^{\lambda+1} x_2 x_3 x_1, \\ x_2 \rightarrow (x_2 x_3 x_1)^{\beta+1} x_2, \\ x_4 \rightarrow (x_2 x_3 x_1)^\beta x_2 (((x_2 x_3 x_1)^{\beta+1} x_2 x_3)^{\lambda+1} (x_2 x_3 x_1)^{\beta+1} x_2)^\kappa \\ \quad \cdot ((x_2 x_3 x_1)^{\beta+1} x_2 x_3)^{\lambda+1} x_2 x_3 x_1, \end{cases}$$

**(2)**,

$$t7 \quad \begin{cases} x_1 \rightarrow x_1 x_3 x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\sigma x_2 x_1, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow x_3 x_2 (x_1 x_3 x_2)^{\lambda-\sigma-1} ((x_2 x_1 x_3)^{\lambda+1} x_2)^{\kappa-\tau} x_1 x_3 x_2 \\ \quad \cdot ((x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\sigma x_2 x_1, \end{cases}$$

**(2)**

$$t8 \quad \begin{cases} x_1 \rightarrow x_1 x_3 x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\lambda x_2 x_1, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow x_4 x_3 (x_2 x_1 x_3)^\lambda x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^{\kappa-\tau-1} x_1 x_3 x_2 \\ \quad \cdot ((x_3 \rightarrow x_1 x_3 x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\lambda x_2 x_1, \end{cases}$$

**(12)**,

$$t9 \quad \begin{cases} x_1 \rightarrow x_1 x_3 x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^\kappa (x_2 x_1 x_3)^\lambda x_2 x_1, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow x_3 x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^\kappa (x_2 x_1 x_3)^\lambda x_2 x_1, \end{cases}$$

**(2)**,

where  $\eta, \theta, \sigma, \tau, \beta$  are natural parameters.

*Proof.* The equation **13** can be divided into the collection of equations

- (j) **13** with  $\partial(x_1) < \partial(x_3)$ ;
- (jj) **13** with  $\partial(x_3) \leq \partial(x_1) \leq \partial(x_3 x_2)$ ;

(jjj) **13** with

$$\begin{aligned} \partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma) &\leq \partial(x_1) \\ &\leq \partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_2), \quad \tau \leq \kappa, \sigma \leq \lambda + 1; \end{aligned}$$

(jjjj) **13** with

$$\begin{aligned} \partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_2) &\leq \partial(x_1) \\ &< \partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^{\sigma+1}), \quad \tau \leq \kappa, \sigma \leq \lambda. \end{aligned}$$

The equation (j) can be reduced by the transformation

$$\begin{aligned} x_3 &\rightarrow x_1x_3, \\ x_4 &\rightarrow x_4x_3(x_2x_1x_3)^\lambda x_2((x_2(x_1x_3x_2)^{\lambda+1})^\kappa x_1 \end{aligned}$$

to the equation  $x_2x_1x_3x_4 = x_3x_2^2x_1$  with  $\partial(x_3) > 0$ , that is, to the equation **12**.

The equation (jj) can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_3x_1, \\ x_2 &\rightarrow x_1x_2, \\ \begin{cases} x_1 &\rightarrow x_1^\eta, \\ x_2 &\rightarrow x_1^\theta, \end{cases} \\ x_4 &\rightarrow x_4(x_1^{\eta+\theta}x_3)^\lambda x_1^{\eta+\theta}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^\kappa x_3x_1^\eta \end{aligned}$$

to the equation  $x_3x_4 = x_1^\theta x_3$ , that is, to the equation **5**.

The equation (jjj) can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_1, \\ x_2 &\rightarrow x_1x_2 \end{aligned}$$

to the equation  $E$ :

$$\begin{aligned} x_1x_2x_3x_4 &= x_2(x_3x_1x_2)^{\lambda+1-\sigma}((x_1x_2x_3)^{\lambda+1}x_1x_2)^{\kappa-\tau}x_3x_1x_2 \\ &\cdot ((x_1x_2x_3)^{\lambda+1}x_1x_2)^\tau(x_1x_2x_3)^\sigma x_1. \end{aligned}$$

The equation  $E$  with  $\lambda + 1 > \sigma$  can be reduced by the transformation

$$\begin{aligned} x_4 &\rightarrow x_2(x_3x_1x_2)^{\lambda-\sigma}((x_1x_2x_3)^{\lambda+1}x_1x_2)^{\kappa-\tau}x_3x_1x_2 \\ &\cdot ((x_1x_2x_3)^{\lambda+1}x_1x_2)^\tau(x_1x_2x_3)^\sigma x_1 \end{aligned}$$

to the equation  $x_1x_2x_3 = x_2x_3x_1$ , that is, to the equation **2**.

The equation  $E$  with  $\lambda + 1 = \sigma$  and  $\tau < \kappa$  can be reduced by the transformation

$$\begin{aligned} \begin{cases} x_1 &\rightarrow x_1^\eta, \\ x_2 &\rightarrow x_1^\theta, \end{cases} \\ x_4 &\rightarrow x_4x_1^{\eta+\theta}(x_3x_1^{\eta+\theta})^\lambda((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\kappa-\tau-1} \\ &\cdot x_3x_1^{\eta+\theta}(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^\tau(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^\eta \end{aligned}$$

to the equation  $x_3x_4 = x_1^\theta x_3$ , that is, to the equation **5**.

The equation  $E$  with  $\lambda + 1 = \sigma$  and  $\tau = \kappa$  can be reduced by the transformation

$$\begin{aligned} x_2 &\rightarrow x_1^\beta x_2, \\ x_1 &\rightarrow x_2 x_1, \\ x_4 &\rightarrow (x_2 x_1)^\beta x_2 (((x_2 x_1)^{\beta+1} x_2 x_3)^{\lambda+1} (x_2 x_1)^{\beta+1} x_2)^\tau (x_2 x_1)^{\beta+1} x_2 x_3)^\sigma x_2 x_1, \end{aligned}$$

then both  $x_1 \rightarrow x_3 x_1$  and  $x_3 \rightarrow x_1 x_3$ , to the equation **2**.

The equation (jjjj) can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_3 x_2 ((x_2 x_3)^{\lambda+1} x_2)^\tau (x_2 x_3)^\sigma x_2 x_1, \\ x_3 &\rightarrow x_1 x_3 \end{aligned}$$

to the equation  $F$ :

$$\begin{aligned} x_2 x_1 x_3 x_4 &= x_3 x_2 (x_1 x_3 x_2)^{\lambda-\sigma} ((x_2 x_1 x_3)^{\lambda+1} x_2)^{\kappa-\tau} x_1 x_3 x_2 \\ &\quad \cdot ((x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\sigma x_2 x_1 \quad \text{with } \partial(x_3) > 0. \end{aligned}$$

The equation  $F$  with  $\lambda > \sigma$  can be reduced by the transformation

$$\begin{aligned} x_4 &= x_3 x_2 (x_1 x_3 x_2)^{\lambda-\sigma-1} ((x_2 x_1 x_3)^{\lambda+1} x_2)^{\kappa-\tau} x_1 x_3 x_2 \\ &\quad \cdot ((x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\sigma x_2 x_1 \end{aligned}$$

to the equation  $x_1 x_2 x_3 = x_3 x_1 x_2$ , that is, to the equation **2** with  $\partial(x_3) > 0$ . On the other hand, t7 is the parametric solution of **13**.

The equation  $F$  with  $\lambda = \sigma$  and  $\kappa > \tau$  can be reduced by the transformation

$$\begin{aligned} x_4 &\rightarrow x_4 x_3 (x_2 x_1 x_3)^\lambda x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^{\kappa-\tau-1} x_1 x_3 x_2 \\ &\quad \cdot ((x_2 x_1 x_3)^{\lambda+1} x_2)^\tau (x_2 x_1 x_3)^\lambda x_2 x_1 \end{aligned}$$

to the equation  $x_2 x_1 x_3 x_4 = x_3 x_2^2 x_1$  with  $\partial(x_3) > 0$ , that is, to the equation **12**.

The equation  $F$  with  $\lambda = \sigma$  and  $\kappa = \tau$  can be reduced by the transformation

$$x_4 \rightarrow x_3 x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^\kappa (x_2 x_1 x_3)^\lambda x_2 x_1$$

to the equation  $x_2 x_1 x_3 = x_3 x_2 x_1$  with  $\partial(x_3) > 0$ , that is, to the equation **2** with  $\partial(x_3) > 0$ . On the other hand, t9 is the parametric solution of **13**.

**Proposition 14.** *The general solution of the equation*

$$\mathbf{14} \quad x_1 x_2 x_3 x_4 = x_3 x_2 (x_2 (x_3 x_2)^{\lambda+1})^{\tau+1} x_1$$

where  $\lambda$  and  $\tau$  are natural parameters, is described by the transformations

$$\mathbf{13} \quad x_1 \rightarrow (x_3 x_2 (x_2 (x_3 x_2)^{\lambda+1})^{\tau+1})^\rho x_1, \quad \begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases}$$

where  $\rho$  is a natural parameter.

*Proof.* This follows directly from Propositions 8 and 13.

**Proposition 15.** *The general solution of the equation*

$$\mathbf{15} \quad x_1 x_2 x_3 x_4 = (x_3 x_1)^{\tau+2} x_2,$$

where  $\tau$  is a natural parameter, is described by the transformations

$$x_3 \rightarrow 1, \quad \begin{cases} x_1 \rightarrow (x_1 x_3)^\alpha x_1, \\ x_3 \rightarrow x_1 x_3, \end{cases} \begin{matrix} \langle \mathbf{5} \rangle, \\ \langle \mathbf{14} \rangle, \end{matrix}$$

where  $\alpha$  is a natural parameter.

*Proof.* The equation **15** can be divided into the collection of the equations **15** with  $\partial(x_3) = 0$  and **15** with  $\partial(x_3) > 0$ . In the second case we use Proposition 8.

**Proposition 16.** *The general solution of the equation*

$$\mathbf{16} \quad x_1 x_2 x_3 x_4 = x_3 x_2^{\tau+2} x_1$$

with  $\partial(x_1) < \partial(x_3 x_2^{\tau+2})$  and  $\partial(x_3) < \partial(x_1 x_2)$ , where  $\tau$  is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \rightarrow x_3 (x_1 x_2)^\nu x_1, \\ x_2 \rightarrow x_1 x_2, \\ x_4 \rightarrow x_4 (x_1 x_2)^\nu x_1, \end{cases} \begin{cases} x_1 \rightarrow x_3 (x_1 x_2)^{\tau+1} x_1, \\ x_2 \rightarrow x_1 x_2, \\ x_4 \rightarrow (x_2 x_1)^{\tau+1}, \end{cases} \begin{matrix} \langle \mathbf{5} \rangle \\ \langle \mathbf{2} \rangle \end{matrix}$$

$$\begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta, \end{cases}$$

$$\begin{cases} x_3 \rightarrow x_1 x_3, \\ x_2 \rightarrow x_3 x_2, \end{cases} \langle \mathbf{15} \rangle,$$

where  $\nu, \eta, \theta$  are natural parameters with  $\nu \leq \tau$ .

*Proof.* The equation **16** can be divided into the collection of equations

(j) **16** with  $\partial(x_3) \leq \partial(x_1)$ ,

(jj) **16** with  $\partial(x_1) \leq \partial(x_3)$ .

The equation (j) can be reduced by the transformation  $x_1 \rightarrow x_3 x_1$  to the equation  $E_1$ :

$$x_1 x_2 x_3 x_4 = x_2^{\tau+2} x_3 x_1 \quad \text{with } \partial(x_1) < \partial(x_2^{\tau+2}).$$

The equation  $E_1$  can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_2^\nu x_1 & (\nu < \tau + 2), \\ x_2 &\rightarrow x_1 x_2 \end{aligned}$$

to the equation  $E_2$ :

$$x_1 x_2 x_3 x_4 = x_2 (x_1 x_2)^{\tau+1-\nu} x_3 (x_1 x_2)^\tau x_1.$$

If  $\tau + 1 > \nu$ , then the equation  $E_2$  falls into the system  $E_3$ :

$$\begin{cases} x_1 x_2 = x_2 x_1, \\ x_3 x_4 = x_2 (x_1 x_2)^{\tau-\nu} x_3 (x_1 x_2)^\nu x_1. \end{cases}$$

The system  $E_3$  can be reduced by the transformation

$$x_4 \rightarrow x_4 (x_1 x_2)^\tau x_1$$

to the system  $E_4$ :

$$\begin{cases} x_1x_2 = x_2x_1, \\ x_3x_4 = x_2(x_1x_2)^{\tau-\nu}x_3. \end{cases}$$

By Proposition 1 the system  $E_4$  is reduced by the parametric transformation

$$\begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta \end{cases}$$

to the equation

$$x_3x_4 = x_1^{\theta+(\nu+\theta)(\tau-\nu)}x_3,$$

that is, to the equation **5**.

If  $\tau + 1 = \nu$ , then the equation  $E_2$  can be reduced by the transformation  $x_4 \rightarrow (x_2x_1)^\nu$  to the equation  $x_1x_2x_3 = x_2x_3x_1$ , that is, to the equation **2**.

The equation (jj) can be reduced by the transformation  $x_3 \rightarrow x_1x_3$  to the equation  $E_5$ :

$$x_2x_1x_3x_4 = x_3x_2^{\tau+2}x_1 \quad \text{with } \partial(x_3) < \partial(x_2).$$

The equation  $E_5$  can be reduced by the transformation  $x_2 \rightarrow x_3x_2$  to the equation

$$x_2x_1x_3x_4 = (x_3x_2)^{\tau+2}x_1,$$

that is the equation **15**.

**Proposition 17.** *The general solution of the equation*

$$\mathbf{17} \quad x_1x_2x_3x_4 = x_3x_2^{\tau+2}x_1,$$

where  $\tau$  is a natural parameter, is described by the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1x_2, x_3x_2^{\tau+2})^\mu x_1, \\ x_3 \rightarrow \mathbf{Fi}(x_3x_2^{\tau+2}, x_1x_2)^\mu x_3, \end{cases}$$

where  $\mu$  is a variable whose values are sequences of natural parameters, followed by one of the three transformations

$$\begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \langle \mathbf{16} \rangle.$$

*Proof.* This follows directly from Propositions 9 and 16.

**Proposition 18.** *The general solution of the equation*

$$\mathbf{18} \quad x_1x_2x_3x_4 = x_2^{\sigma+3}x_3x_1,$$

where  $\sigma$  is a natural parameter, is described by the transformations

$$x_1 \rightarrow x_2^{\sigma+3}x_1, \quad \begin{cases} x_1 \rightarrow (x_1x_2)^{\sigma+2}x_1, \\ x_2 \rightarrow x_1x_2, \\ x_4 \rightarrow (x_2x_1)^{\sigma+2}, \end{cases} \quad \langle \mathbf{2} \rangle, \quad \begin{cases} x_1 \rightarrow (x_1x_2)^\nu x_1, \\ x_2 \rightarrow x_1x_2, \\ x_4 \rightarrow x_4(x_1x_2)^\nu x_1, \end{cases} \quad \begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta, \end{cases} \quad \langle \mathbf{5} \rangle,$$

where  $\nu, \eta, \theta$  are natural parameters with  $\nu \leq \sigma + 1$ .

*Proof.* The equation **18** can be divided into the collection of equations

- (j) **18** with  $\partial(x_2^{\sigma+3}) \leq \partial(x_1)$ ,
- (jj) **18** with  $\partial(x_2^{\sigma+2}) \leq \partial(x_1) < \partial(x_2^{\sigma+3})$ ,
- (jjj) **18** with  $\partial(x_2^\nu) \leq \partial(x_1) < \partial(x_2^{\nu+1})$ ,  $\nu < \sigma + 2$ .

The equation (j) can be reduced by the transformation  $x_1 \rightarrow x_2^{\sigma+3}x_1$  to the equation  $x_1x_2x_3x_4 = x_3x_2^{\sigma+3}x_1$ , that is, to **17** with  $\tau > 0$ .

The equation (jj) can be reduced by the transformation  $x_1 \rightarrow x_2^{\sigma+2}x_1$ ,  $x_2 \rightarrow x_1x_2$ ,  $x_4 \rightarrow (x_2x_1)^{\sigma+2}$  to the equation  $x_1x_2x_3 = x_2x_3x_1$ , that is, to **2**.

The equation (jjj) can be reduced by the transformation  $x_1 \rightarrow x_2^\nu x_1$  with  $\nu < \sigma + 2$ ,  $x_2 \rightarrow x_1x_2$ ,  $x_4 \rightarrow x_4(x_1x_2)^\nu x_1$  to the equation *E*:

$$x_1x_2x_3x_4 = x_2x_1x_2(x_1x_2)^{\sigma+1-\nu}x_3.$$

By Proposition 1 the equation *E* is reduced by the parametric transformation

$$\begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta \end{cases}$$

to the equation

$$x_3x_4 = x_1^{\theta+(\eta+\theta)(\sigma+1-\nu)}x_3,$$

that is, to **5**.

**Proposition 19.** *The general solution of the equation*

$$\mathbf{19} \quad x_1x_2x_3x_4 = x_3x_1^{\sigma+3}x_2 \quad \text{with } \partial(x_3) < \partial(x_1x_2),$$

where  $\sigma$  is a natural parameter, is described by the transformations

$$\langle \mathbf{15} \rangle, \quad \begin{cases} x_3 \rightarrow x_1x_3, \\ x_2 \rightarrow x_3x_2, \end{cases} \quad \langle \mathbf{18} \rangle.$$

*Proof.* The equation **19** can be divided into the collection of equations

- (j) **19** with  $\partial(x_3) \leq \partial(x_1)$ ,
- (jj) **19** with  $\partial(x_3) \geq \partial(x_1)$ .

The equation (j) can be reduced by the transformation  $x_1 \rightarrow x_3x_1$  to the equation  $x_1x_2x_3x_4 = (x_3x_1)^{\sigma+3}x_2$ , that is, to **15** with  $\tau > 0$ .

The equation (jj) can be reduced by the transformation  $x_3 \rightarrow x_1x_3$ ,  $x_2 \rightarrow x_3x_2$  to the equation  $x_2x_1x_3x_4 = x_1^{\sigma+3}x_3x_2$ , that is, to **18**.

**Proposition 20.** *The general solution of the equation*

$$\mathbf{20} \quad x_1x_2x_3x_4 = x_3x_1^{\tau+3}x_2,$$

where  $\tau$  is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad x_3 \rightarrow (x_1x_2)^\alpha x_3, \quad \langle \mathbf{19} \rangle,$$

where  $\alpha$  is a natural parameter.

*Proof.* The equation **20** can be divided into the collection of equations **20** with  $\partial(x_1x_2) = 0$  and **20** with  $\partial(x_1x_2) > 0$ . In the second case we use Proposition 8.

7. THE FUNCTION  $\mathbf{Ro}(x_1, x_2, x_3)_i^{\mu_1, \dots, \mu_t}$ 

Consider the equation

$$x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1$$

and a sequence of transformations (15):

$$(15.1) \quad \begin{cases} x_1 \rightarrow \mathbf{Fi}(x_3 x_2 x_1, x_3 x_2 (x_1 x_3)^2)^{\mu_1} x_3 x_2 x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_3 x_2 (x_1 x_3)^2, x_3 x_2 x_1)^{\mu_1} x_3 x_2, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow x_4 x_1, \end{cases}$$

$$(15.2) \quad \begin{cases} x_1 \rightarrow \mathbf{Fi}(x_3 x_1 (x_2 x_3)^2, x_3 x_1 x_2)^{\mu_2} x_3 x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_3 x_1 x_2, x_3 x_1 (x_2 x_3)^2)^{\mu_2} x_3 x_1 x_2, \\ x_3 \rightarrow x_2 x_3, \\ x_4 \rightarrow x_4 x_2, \end{cases}$$

$$(15.2k+1) \quad \begin{cases} x_1 \rightarrow \mathbf{Fi}(x_3 x_2 x_1, x_3 x_2 (x_1 x_3)^2)^{\mu_{2k+1}} x_3 x_2 x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_3 x_2 (x_1 x_3)^2, x_3 x_2 x_1)^{\mu_{2k+1}} x_3 x_2, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow x_4 x_1, \end{cases}$$

$$(15.2k+2) \quad \begin{cases} x_1 \rightarrow \mathbf{Fi}(x_3 x_1 (x_2 x_3)^2, x_3 x_1 x_2)^{\mu_{2k+2}} x_3 x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_3 x_1 x_2, x_3 x_1 (x_2 x_3)^2)^{\mu_{2k+2}} x_3 x_1 x_2, \\ x_3 \rightarrow x_2 x_3, \\ x_4 \rightarrow x_4 x_2, \end{cases}$$

**Theorem  $\mathbf{Ro}1$ .** For every natural  $t$  the sequence of the  $t$  joint parametric transformations (15) can be collected by the following common transformation:

$$(16) \quad \begin{cases} x_1 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_1^{\mu_1, \dots, \mu_t} x_1, \\ x_2 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_2^{\mu_1, \dots, \mu_t} x_2, \\ x_3 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_3^{\mu_1, \dots, \mu_t} x_3, \\ x_4 \rightarrow x_4 \mathbf{Re}(x_1, x_2, x_3)^{\mu_1, \dots, \mu_t}. \end{cases}$$

*Proof.* If  $t = 0$ , it is obvious. Let  $t = 1$ . By the definition of  $\mathbf{Ro}(x_1, x_2, x_3)_i^{\mu_1, \dots, \mu_t}$  we have

$$\mathbf{Ro}(x_1, x_2, x_3)_1^{\mu_1} = \mathbf{Fi}(x_3 x_2 x_1, x_3 x_2 (x_1 x_3)^2)^{\lambda_1, \dots, \lambda_s} x_3 x_2,$$

$$\mathbf{Ro}(x_1, x_2, x_3)_2^{\mu_1} = \mathbf{Fi}(x_3 x_2 (x_1 x_3)^2, x_3 x_2 x_1)^{\lambda_2, \dots, \lambda_s} x_3,$$

$$\mathbf{Ro}(x_1, x_2, x_3)_3^{\mu_1} = x_1,$$

$$\mathbf{Re}(x_1, x_2, x_3)^{\mu_1} = x_1,$$

and the transformation (16) coincides with (15.1).

Let  $t > 1$ . By the inductive proposition the sequence of transformations (15.2), ..., (15. $t$ ) can be collected into the transformation

$$(17) \quad \begin{cases} x_1 \rightarrow \mathbf{Ro}(x_2, x_1, x_3)_2^{\mu_2, \dots, \mu_t} x_1, \\ x_2 \rightarrow \mathbf{Ro}(x_2, x_1, x_3)_1^{\mu_2, \dots, \mu_t} x_2, \\ x_3 \rightarrow \mathbf{Ro}(x_2, x_1, x_3)_3^{\mu_2, \dots, \mu_t} x_3, \\ x_4 \rightarrow x_4 \mathbf{Re}(x_2, x_1, x_3)^{\mu_2, \dots, \mu_t}. \end{cases}$$

Substituting (17) into (15.1), we obtain (16).

**Proposition 21.** *The general solution of the equation*

$$\mathbf{21} \quad x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1$$

is described by the transformations

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1, x_2 x_3^2)^{\mu} x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 x_3^2, x_1)^{\mu} x_2, \end{cases}$$

where  $\mu$  is a variable for sequences of natural parameters, followed by one of the three transformations

$$\begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow 1, \\ x_4 \rightarrow x_3, \end{cases} \quad \langle \mathbf{22} \rangle.$$

*Proof.* This follows directly from Propositions 9 and 22.

**Proposition 22.** *The general solution of the equation*

$$\mathbf{22} \quad x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1 \quad \text{with } \partial(x_2) < \partial(x_1) < \partial(x_2 x_3^2)$$

is described by the transformations  $\langle \mathbf{23} \rangle$ ,  $\langle \mathbf{24} \rangle$ ,  $\langle \mathbf{25} \rangle$ .

*Proof.* The equation  $\mathbf{22}$  can be divided into the collection of equations:

- (j)  $\mathbf{22}$  with  $\partial(x_1) \leq \partial(x_3)$ ,
- (jj)  $\mathbf{22}$  with  $\partial(x_2 x_3) \leq \partial(x_1)$ ,
- (jjj)  $\mathbf{22}$  with  $\partial(x_3) \leq \partial(x_1) \leq \partial(x_2 x_3)$ .

By definition (j) is  $\mathbf{23}$ , (jj) is  $\mathbf{24}$ , (jjj) is  $\mathbf{25}$ .

**Proposition 23.** *The general solution of the equation*

$$\mathbf{23} \quad x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1,$$

with  $\partial(x_2) < \partial(x_1) < \partial(x_2 x_3^2)$  and  $\partial(x_1) \leq \partial(x_3)$ , is described by the transformation

$$t \quad \begin{cases} x_1 \rightarrow x_2 x_1, \\ x_3 \rightarrow x_1 x_2 x_3, \\ x_4 \rightarrow x_3 x_2 x_1, \end{cases} \quad \langle \mathbf{2} \rangle.$$

*Proof.* The equation  $\mathbf{23}$  can be reduced by the transformation  $x_1 \rightarrow x_2 x_1$  to the equation  $E_1$ :

$$x_1 x_2 x_3 x_4 = x_3^2 x_2 x_1 \quad \text{with } 0 < \partial(x_1) < \partial(x_3^2), \partial(x_2 x_1) \leq \partial(x_3).$$

The equation  $E_1$  can be reduced by the transformation  $x_3 \rightarrow x_1x_2x_3$  to the equation  $E_2$ :

$$x_1x_2x_3x_4 = x_3x_1x_2x_3x_2x_1 \quad \text{with } \partial(x_1) > 0.$$

The equation  $E_2$  can be reduced by the transformation  $x_4 \rightarrow x_3x_2x_1$  to the equation  $x_1x_2x_3 = x_3x_1x_2$  with  $\partial(x_1) > 0$ , that is, to the equation **2** with  $\partial(x_1) > 0$ .

On the other hand, the transformation  $t$  is a parametric solution of **23**.

**Proposition 24.** *The general solution of the equation*

$$\mathbf{24} \quad x_1x_2x_3x_4 = x_2x_3^2x_1,$$

with  $\partial(x_2) < \partial(x_1) < \partial(x_2x_3^2)$  and  $\partial(x_2x_3) \leq \partial(x_1)$ , is described by the transformation

$$t \quad \begin{cases} x_1 \rightarrow x_2x_3x_1, \\ x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow x_3x_1, \end{cases} \quad \langle \mathbf{2} \rangle.$$

*Proof.* The equation **24** can be reduced by the transformation  $x_1 \rightarrow x_2x_3x_1$  to the equation  $E_1$ :

$$x_1x_2x_3x_4 = x_3x_2x_3x_1 \quad \text{with } 0 < \partial(x_1) < \partial(x_3).$$

The equation  $E_1$  can be reduced by the transformation  $x_3 \rightarrow x_1x_3$  to the equation  $E_2$ :

$$x_2x_1x_3x_4 = x_3x_2x_1x_3x_1$$

with  $\partial(x_1) > 0$ ,  $\partial(x_3) > 0$ . The equation  $E_2$  can be reduced by the transformation  $x_4 \rightarrow x_3x_1$  to the equation  $x_2x_1x_3 = x_3x_2x_1$  with  $\partial(x_1) > 0$ ,  $\partial(x_3) > 0$ , that is, to the equation **2** with  $\partial(x_1) > 0$ ,  $\partial(x_3) > 0$ .

On the other hand, the transformation  $t$  is a parametric solution of **24**.

**Proposition 25.** *The general solution of the equation*

$$\mathbf{25} \quad x_1x_2x_3x_4 = x_2x_3^2x_1,$$

with  $\partial(x_2) < \partial(x_1) < \partial(x_2x_3^2)$  and  $\partial(x_3) \leq \partial(x_1) \leq \partial(x_2x_3)$ , is described by the transformation

$$t \quad \begin{cases} x_1 \rightarrow x_2x_1, \\ x_3 \rightarrow x_1x_3, \\ x_2 \rightarrow x_3x_2, \\ x_4 \rightarrow x_4x_1, \end{cases} \quad \begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1, \end{cases} \quad \langle \mathbf{21} \rangle.$$

*Proof.* The equation **25** can be reduced by the transformation  $x_1 \rightarrow x_2x_1$  to the equation  $E_1$ :

$$x_1x_2x_3x_4 = x_3^2x_2x_1$$

with  $0 < \partial(x_1) < \partial(x_3^2)$  and  $\partial(x_3) \leq \partial(x_1x_2)$ . The equation  $E_1$  can be reduced by the transformation  $x_3 \rightarrow x_1x_3$  to the equation  $E_2$ :

$$x_2x_1x_3x_4 = x_3x_1x_3x_2x_1$$

with  $\partial(x_3) \leq \partial(x_2)$  and  $\partial(x_1) > 0$ . The equation  $E_2$  can be reduced by the transformations  $x_2 \rightarrow x_3x_2$ ,  $x_4 \rightarrow x_4x_1$  to the equation  $E_3$ :

$$x_2x_1x_3x_4 = x_1x_3^2x_2$$

with  $\partial(x_1) > 0$ . The equation  $E_3$  can be reduced by the transformation

$$\begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1 \end{cases}$$

to the equation **21** with  $\partial(x_2) > 0$ .

On the other hand, the transformation  $t$  is a parametric solution of **25**.

**Proposition 26.** *The general solution of the equation*

$$\mathbf{26} \quad x_1x_2x_3x_4 = x_2x_3^2x_1$$

is described by the transformation

$$t \quad \begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1, x_2x_3^2)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2x_3^2, x_1)^\mu x_2, \end{cases}$$

where  $\mu$  is a variable for sequences of natural parameters, followed by one of the four transformations

$$\begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow 1, \\ x_4 \rightarrow x_3, \end{cases} \quad \langle \mathbf{23} \rangle, \quad \langle \mathbf{24} \rangle,$$

and the transformation

$$tt \quad \begin{cases} x_1 \rightarrow \mathbf{Fi}(x_3x_2x_1, x_3x_2(x_1x_3)^2)^\mu x_3x_2x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_3x_2(x_1x_3)^2, x_3x_2x_1)^\mu x_3x_2, \\ x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow x_4x_1, \end{cases}$$

where  $\mu$  is a variable for sequences of natural parameters, followed by the transformation

$$\begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1, \end{cases} \quad \langle \mathbf{21} \rangle.$$

*Proof.* Proposition 26 is deduced by means of Propositions 21–25.

**Proposition 27.** *The general solution of the equation*

$$\mathbf{27} \quad x_1x_2x_3x_4 = x_2x_3^2x_1$$

is described by the transformation

$$T \begin{cases} x_1 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_1^\nu x_1, \\ x_2 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_2^\nu x_2, \\ x_3 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_3^\nu x_3, \\ x_4 \rightarrow x_4 \mathbf{Re}(x_1, x_2, x_3)^\nu, \end{cases}$$

where  $\nu$  is a variable whose values are even sequences of variables from the alphabet (4), followed by the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1, x_2 x_3^2)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2 x_3^2, x_1)^\mu x_2, \end{cases}$$

where  $\mu$  is a variable whose values are finite sequences of natural parameters, followed by one of the four transformations

$$\begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow 1, \\ x_4 \rightarrow x_3, \end{cases} \quad \langle \mathbf{23} \rangle, \quad \langle \mathbf{24} \rangle,$$

and the transformation

$$TT \begin{cases} x_1 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_1^\kappa x_1, \\ x_2 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_2^\kappa x_2, \\ x_3 \rightarrow \mathbf{Ro}(x_1, x_2, x_3)_3^\kappa x_3, \\ x_4 \rightarrow x_4 \mathbf{Re}(x_1, x_2, x_3)^\kappa, \end{cases}$$

where  $\kappa$  is a variable whose values are odd sequences of variables from the alphabet (4), followed by the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{Fi}(x_1 x_3^2, x_2)^\mu x_1, \\ x_2 \rightarrow \mathbf{Fi}(x_2, x_1 x_3^2)^\mu x_2, \end{cases}$$

where  $\mu$  is a variable whose values are finite sequences of natural parameters, followed by one of the four transformations

$$\begin{cases} x_1 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_2 \rightarrow 1, \\ x_4 \rightarrow x_3, \end{cases} \quad \begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1, \end{cases} \quad \begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1, \end{cases} \quad \langle \mathbf{23} \rangle, \quad \langle \mathbf{24} \rangle.$$

*Proof.* It is easy to apply the transformations T and TT to the equation **27** and to verify, by using the definition of the functions  $\mathbf{Ro}(x_1, x_2, x_3)_i^{\mu_1, \dots, \mu_t}$  ( $i = 1, 2, 3$ ), that they are parametric solution **27**.

Let

$$S \begin{cases} x_1 \rightarrow X_1, \\ x_2 \rightarrow X_2, \\ x_3 \rightarrow X_3, \\ x_4 \rightarrow X_4, \end{cases}$$

where the  $X_i$  are words in the alphabet (1), be an arbitrary solution of **27**. We prove by induction on  $|X_1 X_2 X_3 X_4|$  that S is contained either in the transformation T or in TT.

The equation **27** can be divided into the following collection of equations:

(j) **27** with  $\partial(x_2) \leq \partial(x_1) \leq \partial(x_2x_3^2)$ ,

(jj) **27** with  $\partial(x_1) \leq \partial(x_2)$ ,

(jjj) **27** with  $\partial(x_2x_3^2) \leq \partial(x_1)$ .

Let  $S$  be a solution of (j). By Proposition 22,  $S$  is contained in **(23)**, **(24)**, **(25)**.

If  $S$  is contained in **(23)** or **(24)**, then  $S$  is contained in  $T$  with  $\nu = \emptyset$  and  $\mu = \emptyset$ .

Let  $S$  be contained in **(25)**. If  $|X_1X_2X_3| = 0$ , then  $S$  is contained in  $T$ . Let  $|X_1X_2X_3| > 0$ . Then  $X_1 = X_2Y_1$  for some  $Y_1$ , and  $Y_1X_2X_3X_4 = X_3^2X_2Y_1$ . Then  $X_3 = Y_1Y_3$  and  $X_2Y_1Y_3X_4 = Y_3Y_1Y_3X_2Y_1$ . But then  $X_2 = Y_3Y_2$  and  $Y_2Y_1Y_3X_4 = Y_1Y_3Y_3Y_2Y_1$ . By Proposition 25 the equation **25** is reduced by the parametric transformation

$$\begin{array}{l} \mathbf{t} \\ \mathbf{c} \end{array} \begin{cases} x_1 \rightarrow x_3x_2x_1, \\ x_2 \rightarrow x_3x_2, \\ x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow x_4x_1, \\ \\ x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1 \end{cases}$$

to the equation **21**, that is, to **27**. The image of the solution  $S$  via the transformation  $\mathbf{t}$  is the coefficient transformation

$$S_1 \begin{cases} x_1 \rightarrow Y_1, \\ x_2 \rightarrow Y_2, \\ x_3 \rightarrow Y_3, \\ x_4 \rightarrow X_4. \end{cases}$$

Since  $|Y_1Y_2Y_3X_4| < |X_1X_2X_3X_4|$ , one can use the inductive proposition to see that  $S_1$  is contained in the parametric solutions  $\mathbf{cT}$ ,  $\mathbf{cTT}$ . According to Lemma 1 the parametric solutions  $\mathbf{tcT}$ ,  $\mathbf{tcTT}$  of **27** contain the solution  $\mathbf{tcS}_1$  of **27**, that is,  $S$ .

According to Theorem **Ro1** the transformations  $\mathbf{tcT}$  and  $\mathbf{tcTT}$  can be obtained from  $T$  and  $TT$  by replacing the parameter  $\nu$  by  $\emptyset, \nu$ .

Let  $S$  be a solution of  $\mathbf{t}$  equation (jj). If  $|X_1| = 0$ , then  $S$  is contained in  $T$  and  $TT$ . Let  $|X_1| > 0$ . Then  $X_2 = X_1Y_2$  for some  $Y_2$ , and  $X_1Y_2X_3X_4 = Y_2X_3^2X_1$ . The equation **27** is reduced by the transformation  $\mathbf{t}: x_2 \rightarrow x_1x_2$  to **27**. The image of the solution  $S$  via the transformation  $\mathbf{t}$  is the coefficient transformation

$$S_1 \begin{cases} x_1 \rightarrow X_1, \\ x_2 \rightarrow Y_2, \\ x_3 \rightarrow X_3, \\ x_4 \rightarrow X_4. \end{cases}$$

Since  $|Y_2| < |X_2|$ , one can use the inductive proposition to see that  $S_1$  is contained in  $T$  and  $TT$ . According to Lemma 1 the parametric solution  $\mathbf{tT}$ ,  $\mathbf{tTT}$  of **27** contains the solution  $\mathbf{tS}_1$ , that is,  $S$ .

According to Theorems **Fi1** and **Ro1** the transformations  $\mathbf{tT}$  and  $\mathbf{tTT}$  can be obtained from  $T$  and  $TT$  by replacing the parameters  $\nu, \mu$  by  $\nu', \mu'$ , where if  $\nu = \emptyset$ , then  $\nu' = \emptyset$  and  $\mu' = 0, 1, \mu$ ; if  $\nu = \mu_1, \dots, \mu_s$  ( $s \geq 1$ ), then  $\nu' = \mu'_1, \dots, \mu'_s$ , where  $\mu'_1 = 0, 1, \mu_1$  and  $\mu' = \mu$ .

Let  $S$  be a solution of the equation (jjj). If  $|X_2X_3^2| = 0$ , then  $S$  is contained in  $T$ ,  $TT$ . Let  $|X_2X_3^2| > 0$ . Then  $X_1 = X_2X_3^2Y_1$  for some  $Y_1$ , and  $Y_1X_2X_3X_4 =$

$X_2X_3^2Y_1$ . The equation **27** is reduced by the transformation  $t: x_1 \rightarrow x_2x_3^2x_1$  to the equation **27**. The image of the solution  $S$  via the transformation  $t$  is the coefficient transformation

$$S_1 \begin{cases} x_1 \rightarrow Y_1, \\ x_2 \rightarrow X_2, \\ x_3 \rightarrow X_3, \\ x_4 \rightarrow X_4. \end{cases}$$

Since  $|Y_1| < |X_1|$ , one can use the inductive proposition to see that  $S_1$  is contained in  $T$  and  $TT$ . According to Lemma 1 a parametric solution  $tT, tTT$  of **27** contains the solution  $tS_1$ , that is,  $S$ .

According to Theorems **Fi1** and **Ro1** the transformations  $tT$  and  $tTT$  can be obtained from  $T$  and  $TT$  by replacing the parameters  $\nu, \mu$  by parameters  $\nu', \mu'$ , where if  $\nu = \emptyset$ , then  $\nu' = \emptyset$  and  $\mu' = 1, 0, \mu$ ; if  $\nu = \mu_1, \dots, \mu_s$  ( $s \geq 1$ ), then  $\nu' = \mu'_1, \dots, \mu'_s$ , where  $\mu'_1 = 1, 0, \mu_1$  and  $\mu' = \mu$ .

8. EQUATIONS AND SOLUTIONS

**Proposition 28.** *The general solution of the equation*

**28**  $x_1x_2x_3x_4 = x_3(x_2x_3)^{\alpha+1}x_1$  with  $\partial(x_1) < \partial(x_3(x_2x_3)^{\alpha+1})$ ,

where  $\lambda$  is a natural parameter, is described by the transformations

$$t \begin{cases} x_1 \rightarrow (x_3x_2)^\tau x_1, \\ x_3 \rightarrow x_1x_3, \\ x_4 \rightarrow (x_3x_2x_1)^{\alpha-\tau} x_3x_1(x_3x_2x_1)^\tau, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$tt \begin{cases} x_1 \rightarrow (x_3x_2)^\tau x_3x_1, \\ x_2 \rightarrow x_1x_2, \\ x_4 \rightarrow x_2x_3(x_1x_2x_3)^{\alpha-\tau-1} x_3x_1(x_2x_3x_1)^\tau, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$ttt \begin{cases} x_1 \rightarrow x_3(x_2x_3)^\alpha x_1, \\ x_2 \rightarrow x_1x_2, \\ x_4 \rightarrow x_4(x_2x_3x_1)^\alpha, \end{cases} \quad \langle \mathbf{27} \rangle,$$

$$tttt \begin{cases} x_1 \rightarrow (x_3x_2)^{\alpha+1} x_1, \\ x_3 \rightarrow x_1x_3, \\ x_2 \rightarrow x_3x_2, \\ x_4 \rightarrow x_4(x_1x_3^2x_2)^\alpha x_1, \end{cases} \quad \langle \mathbf{27} \rangle,$$

$$ttttt \begin{cases} x_1 \rightarrow (x_3x_2)^{\alpha+1} x_1, \\ x_3 \rightarrow x_1x_2x_3, \\ x_4 \rightarrow x_3x_2x_1(x_2x_3x_2x_1)^\alpha, \end{cases} \quad \langle \mathbf{2} \rangle.$$

*Proof.* The equation **28** can be divided into the collection of equations:

- (j) **28** with  $\partial((x_3x_2)^\tau) \leq \partial(x_1) < \partial(x_3(x_2x_3)^\tau)$ ,  $\tau \leq \alpha$ ,
- (jj) **28** with  $\partial(x_3(x_2x_3)^\tau) \leq \partial(x_1) < \partial(x_3x_2)^{\tau+1}$ ,  $\tau \leq \alpha$ ,
- (jjj) **28** with  $\partial((x_3x_2)^{\alpha+1}) \leq \partial(x_1) < \partial(x_3(x_2x_3)^{\alpha+1})$ .

The equation (j) can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow (x_3x_2)^\tau x_1, \\ x_3 &\rightarrow x_1x_3, \\ x_4 &\rightarrow (x_3x_2x_1)^{\alpha-\tau} x_3x_1(x_3x_2x_1)^\tau \end{aligned}$$

to the equation  $x_2x_1x_3 = x_3x_2x_1$  with  $\partial(x_3) > 0$ , that is, to **2** with  $\partial(x_3) > 0$ . On the other hand, the transformation t is a parametric solution of **28**.

The equation (jj) can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_3(x_2x_3)^\tau x_1, \\ x_2 &\rightarrow x_1x_2 \end{aligned}$$

to the equation  $E$ :

$$x_1x_2x_3x_4 = x_2x_3(x_1x_2x_3)^{\alpha-\tau} x_3x_1(x_2x_3x_1)^\tau$$

with  $\partial(x_2) > 0$ . If  $\tau < \alpha$ , then the equation  $E$  can be reduced by the transformation

$$x_4 \rightarrow x_2x_3(x_1x_2x_3)^{\alpha-\tau-1} x_3x_1(x_2x_3x_1)^\tau$$

to the equation  $x_1x_2x_3 = x_2x_3x_1$  with  $\partial(x_2) > 0$ , that is, to **2** with  $\partial(x_2) > 0$ . On the other hand, the transformation tt is a parametric solution of **28**. If  $\tau = \alpha$ , then the equation  $E$  can be reduced by the transformation

$$x_4 \rightarrow x_4(x_2x_3x_1)^\tau$$

to the equation  $x_1x_2x_3x_4 = x_2x_3^2x_1$  with  $\partial(x_2) > 0$ , that is, to **27** with  $\partial(x_2) > 0$ . On the other hand, the transformation ttt is a parametric solution of **28**.

The equation (jjj) can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow (x_3x_2)^{\alpha+1} x_1, \\ x_3 &\rightarrow x_1x_3 \end{aligned}$$

to the equation  $E_1$ :

$$x_2x_1x_3x_4 = x_3x_1(x_3x_2x_1)^{\alpha+1} \quad \text{with } \partial(x_3) > 0.$$

The equation  $E_1$  can be divided by means of the conditions  $\partial(x_2) \geq \partial(x_3)$  and  $\partial(x_3) \geq \partial(x_2)$ .

The equation  $E_1$  with  $\partial(x_2) \geq \partial(x_3)$  can be reduced by the transformation

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_4 &\rightarrow x_4(x_1x_3^2x_2)^\alpha x_1 \end{aligned}$$

to the equation  $x_2x_1x_3x_4 = x_1x_3^2x_2$  with  $\partial(x_3) > 0$ , that is, to **27** with  $x_1 \leftrightarrow x_2$  and  $\partial(x_3) > 0$ . On the other hand, the transformation tttt is a parametric solution of **28**.

The equation  $E_1$  with  $\partial(x_3) \geq \partial(x_2)$  can be reduced by the transformation

$$\begin{aligned} x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_3x_2x_1(x_2x_3x_2x_1)^\alpha \end{aligned}$$

to the equation  $x_1x_2x_3 = x_3x_1x_2$ , that is, to **2**.

**Proposition 29.** *The general solution of the equation*

$$\mathbf{29} \quad x_1 x_2 x_3 x_4 = x_3 (x_2 x_3)^{\alpha+1} x_1,$$

where  $\lambda$  is a natural parameter, is described by the transformations

$$t \quad \begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad x_1 \rightarrow (x_3 (x_2 x_3)^{\alpha+1})^\beta x_1, \quad \langle \mathbf{28} \rangle,$$

where  $\beta$  is a natural parameter.

*Proof.* The equation **29** can be divided into a collection of equations by means of the conditions  $\partial(x_2 x_3) = 0$  and  $\partial(x_2 x_3) > 0$ .

The equation **29** with  $\partial(x_2 x_3) = 0$  can be reduced by the transformation  $t$  to the equation **1**.

The equation **29** with  $\partial(x_2 x_3) > 0$  by Proposition 8 can be reduced by the transformation  $x_1 \rightarrow (x_3 (x_2 x_3)^{\alpha+1})^\beta x_1$ , where  $\beta$  is a natural parameter, to **28**.

**Proposition 30.** *The general solution of the equation*

$$\mathbf{30} \quad x_1 x_2 x_3 x_4 = x_2 x_3^{\alpha+3} x_1 \quad \text{with } \partial(x_2) < \partial(x_1) < \partial(x_2 x_3^{\alpha+3}),$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\begin{aligned} x_1 &\rightarrow x_2 x_3^{\alpha+2} x_1, & x_1 &\rightarrow x_2 x_3^\beta x_1, \\ x_3 &\rightarrow x_1 x_3, & x_3 &\rightarrow x_1 x_3, \\ x_4 &\rightarrow (x_3 x_1)^{\alpha+2}, & x_4 &\rightarrow x_4 (x_1 x_3)^\beta x_1, \\ &\langle \mathbf{2} \rangle, & &\langle \mathbf{29} \rangle, \end{aligned}$$

where  $\beta$  is a natural parameter with  $\beta \leq \alpha + 2$ .

*Proof.* The equation **30** can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_2 x_3^\beta x_1 & (\beta \leq \alpha + 2), \\ x_3 &\rightarrow x_1 x_3 \end{aligned}$$

to the equation  $E$ :

$$x_2 x_1 x_3 x_4 = x_3 (x_1 x_3)^{\alpha+2-\beta} x_2 (x_1 x_3)^\beta x_1.$$

The equation  $E$  can be divided into a collection of equations by means of the conditions  $\beta = \alpha + 2$  and  $\beta < \alpha + 2$ .

The equation  $E$  with  $\beta = \alpha + 2$  can be reduced by the transformation  $x_4 \rightarrow (x_3 x_1)^{\alpha+2}$  to the equation  $x_2 x_1 x_3 = x_3 x_2 x_1$ , that is, to **2**.

The equation  $E$  with  $\beta < \alpha + 2$  can be reduced by the transformation  $x_4 \rightarrow x_4 (x_1 x_3)^\beta x_1$  to the equation

$$x_2 x_1 x_3 x_4 = x_3 (x_1 x_3)^{\alpha+2-\beta} x_2,$$

that is, to **29**.

**Proposition 31.** *The general solution of the equation*

$$\mathbf{31} \quad x_1 x_2 x_3 x_4 = x_2 x_3^{\alpha+3} x_1,$$

where  $\alpha$  is a natural parameter, is described by the transformation

$$\begin{cases} x_1 \rightarrow \mathbf{F}^i(x_1, x_2 x_3^{\alpha+3})^\mu x_1, \\ x_2 \rightarrow \mathbf{F}^i(x_2 x_3^{\alpha+3}, x_1)^\mu x_2, \end{cases}$$

where  $\mu$  is a variable for sequences of natural parameters, followed by one of the three transformations

$$\begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1, \end{cases} \quad \begin{cases} x_1 \rightarrow 1, \\ x_4 \rightarrow x_3^{\alpha+2}, \end{cases} \quad \langle \mathbf{30} \rangle.$$

*Proof.* This follows directly from Propositions 9 and 30.

**Proposition 32.** *The general solution of the equation*

$$\mathbf{32} \quad x_1 x_2 x_3 x_4 = x_2 x_3^\alpha x_1 x_5,$$

where  $\lambda$  is a natural parameter, is described by the transformations

$$\begin{array}{ll} \mathbf{t} & \begin{cases} \alpha \rightarrow 0, \\ x_5 \rightarrow x_3 x_4, \\ \langle \mathbf{1} \rangle, \end{cases} & \mathbf{tt} & \begin{cases} \alpha \rightarrow 1, \\ x_5 \rightarrow x_4, \\ \langle \mathbf{2} \rangle, \end{cases} \\ \mathbf{ttt} & \begin{cases} \alpha \rightarrow 2, \\ x_4 \rightarrow x_4 x_5, \\ \langle \mathbf{27} \rangle, \end{cases} & \mathbf{tttt} & \begin{cases} \alpha \rightarrow \gamma + 3, \\ x_4 \rightarrow x_4 x_5, \\ \langle \mathbf{31} \rangle, \end{cases} \end{array}$$

where  $\gamma$  is a natural parameter.

*Proof.* The equation **32** is divided into a collection of equations by means of the conditions  $\alpha = 0$ ,  $\alpha = 1$ ,  $\alpha = 2$ ,  $\alpha = \gamma + 3$ , where  $\gamma$  is a natural parameter.

The equation **32** with  $\alpha = 0$  can be reduced by the transformation  $\mathbf{t}$  to the equation  $x_1 x_2 = x_2 x_1$ , that is, to **1**.

The equation **32** with  $\alpha = 1$  can be reduced by the transformation  $\mathbf{tt}$  to the equation  $x_1 x_2 x_3 = x_2 x_3 x_1$ , that is, to **2**.

The equation **32** with  $\alpha = 2$  can be reduced by the transformation  $\mathbf{ttt}$  to the equation  $x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1$ , that is, to **27**.

The equation **32** with  $\alpha = \gamma + 3$  can be reduced by the transformation  $\mathbf{tttt}$  to the equation  $x_1 x_2 x_3 x_4 = x_2 x_3^{\gamma+3} x_1$ , that is, to **31**.

**Proposition 33.** *The general solution the equation*

$$\mathbf{33} \quad x_1 x_2 = x_3^\alpha x_4,$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \rightarrow x_3^\alpha x_4, \\ x_4 \rightarrow x_4 x_2, \end{cases} \quad \begin{cases} x_1 \rightarrow (x_1 x_3)^\beta x_1, \\ x_2 \rightarrow x_3 (x_1 x_3)^{\gamma-\beta} x_4, \\ x_3 \rightarrow x_1 x_3, \\ \alpha \rightarrow \gamma + 1, \end{cases} \quad \begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow x_4, \\ \alpha \rightarrow 0, \end{cases}$$

where  $\beta, \gamma$  are natural parameters.

*Proof.* The equation **33** can be divided by means of the conditions  $\partial(x_2) \leq \partial(x_4)$  and  $\partial(x_2) \geq \partial(x_4)$ .

The equation **33** with  $\partial(x_2) \leq \partial(x_4)$  can be reduced by the transformation  $x_4 \rightarrow x_4 x_2$  to the equation  $x_1 = x_3^\alpha x_4$ .

The equation **33** with  $\partial(x_2) \geq \partial(x_4)$  and  $\alpha > 0$  can be reduced by the transformation

$$\begin{cases} x_3 \rightarrow x_1x_3, \\ x_1 \rightarrow (x_1x_3)^\beta x_1, \\ x_2 \rightarrow x_3(x_1x_3)^{\alpha-\beta-1}x_4 \end{cases}$$

to the equation **1**.

The equation **33** with  $\partial(x_2) \geq \partial(x_4)$  and  $\alpha = 0$  can be reduced by the transformation

$$\begin{cases} x_1 \rightarrow 1, \\ x_2 \rightarrow x_4, \\ \alpha \rightarrow 0 \end{cases}$$

to the equation **1**.

**Proposition 34.** *The general solution of the equation*

$$\mathbf{34} \quad x_1x_2x_3x_4 = x_2^{\alpha+1}x_5,$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \rightarrow x_2^{\alpha+1}x_1, \\ x_5 \rightarrow x_1x_2x_3x_4, \end{cases} \quad \begin{cases} x_1 \rightarrow x_1^{(\eta+\theta)\beta+\eta}, \\ x_2 \rightarrow x_1^{\eta+\theta}, \\ \alpha \rightarrow \gamma + 1, \end{cases} \quad \begin{cases} x_1 \rightarrow (x_1x_2)^\alpha x_1, \\ x_2 \rightarrow x_1x_2, \\ x_5 \rightarrow x_5x_3x_4, \end{cases}$$

**(33)**, **(4)**,

where  $\beta, \gamma, \eta, \theta$  are natural parameters with  $\beta \leq \gamma$ .

*Proof.* The equation **34** can be divided by means of the conditions  $\partial(x_1) \geq \partial(x_2^{\alpha+1})$  and  $\partial(x_1) < \partial(x_2^{\alpha+1})$ .

The equation **33** with  $\partial(x_1) \geq \partial(x_2^{\alpha+1})$  can be reduced by the transformation  $x_1 \rightarrow x_2^{\alpha+1}x_1$  to the equation  $x_1x_2x_3x_4 = x_5$ .

The equation **34** with  $\partial(x_1) < \partial(x_2^{\alpha+1})$  can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_2^\beta x_1 & (\beta < \alpha + 1), \\ x_2 &\rightarrow x_1x_2 \end{aligned}$$

to the equation *E*:

$$x_1x_2x_3x_4 = x_2(x_1x_2)^{\alpha-\beta}x_5 \quad \text{with } \beta \leq \alpha, \partial(x_2) > 0.$$

If  $\beta < \alpha$ , then by Proposition 1 the equation *E* is reduced by the parametric transformation

$$\begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta, \end{cases}$$

where  $\eta, \theta$  are natural parameters, to the equation

$$x_3x_4 = x_1^{\theta+(\eta+\theta)(\alpha-\beta-1)}x_5$$

with  $\partial(x_1^\theta) > 0$ , that is, to **33**.

If  $\beta = \alpha$ , then the equation *E* is reduced by the parametric transformation  $x_5 \rightarrow x_5x_3x_4$  to the equation  $x_1x_2 = x_2x_5$ , that is, to **4**.

**Proposition 35.** *The general solution of the equation*

$$\mathbf{35} \quad x_1 x_2 x_3 x_4 = x_3 (x_2 x_3)^{\alpha+1} x_5,$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\mathbf{t} \quad \begin{cases} x_1 \rightarrow (x_1 x_3 x_2)^\tau x_1, \\ x_3 \rightarrow x_1 x_3, \\ x_4 \rightarrow x_3 (x_2 x_1 x_3)^{\alpha-\tau} x_5, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$\mathbf{tt} \quad \begin{cases} x_1 \rightarrow (x_1 x_3 x_2)^{\alpha+1} x_1, \\ x_3 \rightarrow x_1 x_3, \\ x_5 \rightarrow x_5 x_4, \end{cases} \quad \langle \mathbf{6} \rangle,$$

$$\mathbf{ttt} \quad \begin{cases} x_1 \rightarrow x_3 (x_1 x_2 x_3)^\tau x_1, \\ x_2 \rightarrow x_1 x_2, \\ x_4 \rightarrow x_2 x_3 (x_1 x_2 x_3)^{\alpha-\tau-1} x_5, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$\mathbf{tttt} \quad \begin{cases} x_1 \rightarrow x_3 (x_1 x_2 x_3)^\alpha x_1, \\ x_2 \rightarrow x_1 x_2, \\ x_5 \rightarrow x_5 x_4, \end{cases} \quad \langle \mathbf{7} \rangle,$$

$$\mathbf{ttttt} \quad \begin{cases} x_1 \rightarrow x_3 (x_2 x_3)^{\alpha+1} x_1, \\ x_5 \rightarrow x_1 x_2 x_3 x_4, \end{cases}$$

where  $\tau$  is a natural parameter.

*Proof.* The equation **35** can be divided into the collection of equations

(j) **35** with  $\partial((x_2 x_3)^\tau) \leq \partial(x_1) \leq \partial(x_3 (x_2 x_3)^\tau)$ ,  $\tau \leq \alpha + 1$ ,

(jj) **35** with  $\partial(x_3 (x_2 x_3)^\tau) \leq \partial(x_1) \leq \partial((x_2 x_3)^{\tau+1})$ ,  $\tau \leq \alpha$ ,

(jjj) **35** with  $\partial(x_3 (x_2 x_3)^{\alpha+1}) \leq \partial(x_1)$ .

The equation (j) can be reduced by the transformation

$$x_1 \rightarrow (x_3 x_2)^\tau x_1,$$

$$x_3 \rightarrow x_1 x_3$$

to the equation  $E$ :

$$x_2 x_1 x_3 x_4 = x_3 (x_2 x_1 x_3)^{\alpha+1-\tau} x_5.$$

If  $\tau < \alpha + 1$ , then the equation  $E$  can be reduced by the transformation  $x_4 \rightarrow x_3 (x_2 x_1 x_3)^{\alpha-\tau} x_5$  to the equation  $x_2 x_1 x_3 = x_3 x_2 x_1$ , that is, to **2**.

If  $\tau = \alpha + 1$ , then the equation  $E$  can be reduced by the transformation  $x_5 \rightarrow x_5 x_4$  to the equation  $x_2 x_1 x_3 = x_3 x_5$ , that is, to **6**.

The equation (jj) can be reduced by the transformation

$$x_1 \rightarrow x_3 (x_2 x_3)^\tau x_1,$$

$$x_2 \rightarrow x_1 x_2$$

to the equation  $E$ :

$$x_1 x_2 x_3 x_4 = x_2 x_3 (x_1 x_2 x_3)^{\alpha-\tau} x_5.$$

If  $\tau < \alpha$ , then the equation  $E$  can be reduced by the transformation  $x_4 \rightarrow x_2x_3(x_1x_2x_3)^{\alpha-\tau-1}x_5$  to the equation  $x_1x_2x_3 = x_2x_3x_1$ , that is, to **2**.

If  $\tau = \alpha$ , then the equation  $E$  can be reduced by the transformation  $x_5 \rightarrow x_5x_4$  to the equation  $x_1x_2x_3 = x_2x_3x_5$ , that is, to **7**.

The equation (jjj) can be reduced by the transformation

$$x_1 \rightarrow x_3(x_2x_3)^{\alpha+1}x_1$$

to the equation  $x_1x_2x_3x_4 = x_5$ .

**Proposition 36.** *The general solution of the equation*

$$x_1x_2x_3x_4 = x_3^{\alpha+1}x_5,$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\begin{aligned} \text{t} & \begin{cases} x_1 \rightarrow x_3^{\alpha+1}x_1, \\ x_5 \rightarrow x_1x_2x_3x_4, \end{cases} \\ \text{tt} & \begin{cases} x_1 \rightarrow (x_1x_3)^\beta x_1, \\ x_3 \rightarrow x_1x_3, \end{cases} \\ & \langle \mathbf{35} \rangle, \\ \text{ttt} & \begin{cases} x_1 \rightarrow (x_1x_3)^\alpha x_1, \\ x_3 \rightarrow x_1x_3, \\ x_5 \rightarrow x_5x_4, \end{cases} \\ & \langle \mathbf{6} \rangle, \end{aligned}$$

where  $\beta$  is a natural parameter with  $\beta < \alpha$ .

*Proof.* The equation **36** can be divided by means of the conditions  $\partial(x_1) \geq \partial(x_3^{\alpha+1})$  and  $\partial(x_1) \leq \partial(x_3^{\alpha+1})$ .

The equation **36** with  $\partial(x_1) \geq \partial(x_3^{\alpha+1})$  can be reduced by the transformation  $x_1 \rightarrow x_3^{\alpha+1}x_1$  to the equation  $x_1x_2x_3x_4 = x_5$ .

The equation **36** with  $\partial(x_1) \leq \partial(x_3^{\alpha+1})$  can be reduced by the transformation

$$\begin{aligned} x_1 & \rightarrow x_3^\beta x_1 & (\beta < \alpha + 1), \\ x_3 & \rightarrow x_1x_3 \end{aligned}$$

to the equation  $E$ :

$$x_2x_1x_3x_4 = x_3(x_1x_3)^{\alpha-\beta}x_5 \quad \text{with } \beta \leq \alpha.$$

If  $\beta < \alpha$ , then  $E$  is **35**.

If  $\beta = \alpha$ , then  $E$  can be reduced by the transformation  $x_5 \rightarrow x_5x_4$  to the equation  $x_2x_1x_3 = x_3x_5$ , that is, to **6**.

**Proposition 37.** *The general solution of the equation*

**37** 
$$x_1x_2x_3x_4 = x_2x_3^{\alpha+1}x_5,$$

where  $\alpha$  is a natural parameter, is described by the transformations

$$\text{t} \begin{cases} x_1 \rightarrow 1, \\ x_4 \rightarrow x_3^\alpha x_5, \end{cases} \quad \text{tt} \begin{cases} x_1 \rightarrow x_2x_1, \\ x_2 \rightarrow (x_2x_1)^\lambda x_2, \end{cases} \\ \langle \mathbf{36} \rangle,$$

where  $\lambda$  is a natural parameter.

*Proof.* The equation **37** can be divided by means of the conditions  $\partial(x_1) = 0$  and  $\partial(x_1) > 0$ .

The equation **37** with  $\partial(x_1) = 0$  can be reduced by the transformation  $x_1 \rightarrow 1$  to the equation  $x_4 = x_3^\alpha x_5$ .

The equation **37** with  $\partial(x_1) > 0$  by Proposition 8 can be reduced by the transformation  $x_2 \rightarrow x_1^\lambda x_2$  to the equation **37** with  $\partial(x_1) > \partial(x_2)$ , which in turn can be reduced by the transformation  $x_1 \rightarrow x_2 x_1$  to the equation

$$x_1 x_2 x_3 x_4 = x_3^{\alpha+1} x_5 \quad \text{with } \partial(x_1) > 0,$$

that is, to **36** with  $\partial(x_1) > 0$ . On the other hand, the transformation tt, **36** is a parametric solution of **37**.

**Proposition 38.** *The general solution of the equation*

$$\mathbf{38} \quad x_2 x_1 x_3 x_4 = x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^{\kappa+1} x_5,$$

where  $\lambda, \kappa$  are natural parameters, is described by the transformations

$$\text{t1} \quad \begin{cases} x_3 \rightarrow x_2 x_3, \\ x_4 \rightarrow x_4 x_3 (x_1 x_2 x_3)^\lambda x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^\kappa x_5, \end{cases} \quad \langle \mathbf{12} \rangle,$$

$$\text{t2} \quad \begin{cases} x_1 \rightarrow x_1^{\eta+\theta}, \\ x_2 \rightarrow x_3 x_1^\theta, \\ x_4 \rightarrow x_4 (x_1^{\eta+\theta} x_3)^\lambda x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^\kappa x_5, \end{cases} \quad \langle \mathbf{5} \rangle$$

$$\text{t3} \quad \begin{cases} x_1 \rightarrow x_2 x_1, \\ x_2 \rightarrow x_3 x_2 x_1 ((x_1 x_2 x_3)^{\lambda+1} x_2 x_1)^\tau (x_2 x_1 x_3)^\sigma x_2, \\ x_4 \rightarrow x_1 (x_2 x_3 x_1)^{\lambda-\sigma} ((x_2 x_1 x_3)^{\lambda+1} x_2 x_1)^{\kappa-\tau} x_5, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$\text{t4} \quad \begin{cases} x_1 \rightarrow x_1^{\eta+\theta}, \\ x_2 \rightarrow x_3 x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^\tau (x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^\theta, \\ x_4 \rightarrow x_4 x_1^{\eta+\theta} (x_3 x_1^{\eta+\theta})^\lambda ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^{\kappa-\tau-1} x_5, \end{cases} \quad \langle \mathbf{5} \rangle,$$

$$\text{t5} \quad \begin{cases} x_1 \rightarrow (x_1 x_2)^{\beta+1} x_1, \\ x_2 \rightarrow x_3 (x_1 x_2)^\beta x_1 ((x_1 x_2)^\beta x_1 x_3)^{\lambda+1} (x_1 x_2)^\beta x_1^\kappa \\ \quad \cdot ((x_1 x_2)^\beta x_1 x_3)^{\lambda+1} x_1 x_2, \\ x_5 \rightarrow x_2 x_1 x_3 x_4, \end{cases}$$

$$\text{t6} \quad \begin{cases} x_2 \rightarrow x_2 x_3 x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^\tau (x_1 x_2 x_3)^\sigma x_1 x_2, \\ x_3 \rightarrow x_2 x_3, \\ x_4 \rightarrow x_3 x_1 (x_2 x_3 x_1)^{\lambda-\sigma-1} ((x_1 x_2 x_3)^{\lambda+1} x_1)^{\kappa-\tau} x_5, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$t7 \quad \begin{cases} x_2 \rightarrow x_2 x_3 x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^\tau (x_1 x_2 x_3)^\lambda x_1 x_2, \\ x_3 \rightarrow x_2 x_3, \\ x_4 \rightarrow x_4 x_3 (x_1 x_2 x_3)^\lambda x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^{\kappa-\tau-1} x_5, \end{cases} \quad \langle \mathbf{12} \rangle,$$

$$t8 \quad \begin{cases} x_2 \rightarrow x_2 x_3 x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^\kappa (x_1 x_2 x_3)^\lambda x_1 x_2, \\ x_3 \rightarrow x_2 x_3, \\ x_5 \rightarrow x_5 x_4, \end{cases} \quad \langle \mathbf{10} \rangle,$$

$$t9 \quad \begin{cases} x_2 \rightarrow x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^{\kappa+1} x_2, \\ x_5 \rightarrow x_2 x_1 x_3 x_4, \end{cases}$$

where  $\eta, \theta, \tau, \sigma, \beta$  are natural parameters.

*Proof.* The equation **38** can be divided into the collection of equations:

- (j) **38** with  $\partial(x_2) < \partial(x_3)$ ,
- (jj) **38** with  $\partial(x_3) \leq \partial(x_2) \leq \partial(x_3 x_1)$ ,
- (jjj) **38** with

$$\begin{aligned} \partial(x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^\tau (x_1 x_3)^\sigma) &\leq \partial(x_2) \\ &\leq \partial(x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^\tau (x_1 x_3)^\sigma x_1), \quad \tau < \kappa + 1, \sigma \leq \lambda + 1, \end{aligned}$$

- (jjjj) **38** with

$$\begin{aligned} \partial(x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^\tau (x_1 x_3)^\sigma x_1) &\leq \partial(x_2) \\ &< \partial(x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^\tau (x_1 x_3)^{\sigma+1}), \quad \tau < \kappa + 1, \sigma < \lambda + 1, \end{aligned}$$

- (jjjjj) **38** with  $\partial(x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^{\kappa+1}) \leq \partial(x_2)$ .

The equation (j) can be reduced by the transformation

$$\begin{aligned} x_3 &\rightarrow x_2 x_3, \\ x_4 &\rightarrow x_4 x_3 (x_1 x_2 x_3)^\lambda x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^\kappa x_5 \end{aligned}$$

to the equation  $x_1 x_2 x_3 x_4 = x_3 x_1^2 x_2$  with  $\partial(x_3) > 0$ , that is, to **12**.

The equation (jj) can be reduced by the transformation

$$\begin{aligned} x_2 &\rightarrow x_3 x_2, \\ x_1 &\rightarrow x_2 x_1, \\ \begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta, \end{cases} \\ x_4 &\rightarrow x_4 (x_1^{\eta+\theta} x_3)^\lambda x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^\kappa x_5 \end{aligned}$$

to the equation  $x_3 x_4 = x_1^\eta x_3$ , that is,  $(x_2 x_3 x_1)^{\beta+1} x_2)^\kappa$  to **5**.

The equation (jjj) can be reduced by the transformation

$$\begin{aligned} x_2 &\rightarrow x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^\tau (x_1 x_3)^\sigma x_2, \\ x_1 &\rightarrow x_2 x_1 \end{aligned}$$

to the equation  $E$ :

$$x_2 x_1 x_3 x_4 = x_1 (x_3 x_2 x_1)^{\lambda+1-\sigma} ((x_2 x_1 x_3)^{\lambda+1} x_2 x_1)^{\kappa-\tau} x_5.$$

The equation  $E$  with  $\lambda + 1 > \sigma$  can be reduced by the transformation

$$x_4 \rightarrow x_1(x_2x_3x_1)^{\lambda-\sigma}((x_2x_1x_3)^{\lambda+1}x_2x_1)^{\kappa-\tau}x_5$$

to the equation  $x_2x_1x_3 = x_1x_3x_2$ , that is, to **2**.

The equation  $E$  with  $\lambda + 1 = \sigma$  and  $\tau < \kappa$  can be reduced by the transformation

$$\begin{cases} x_1 \rightarrow x_1^\eta, \\ x_2 \rightarrow x_1^\theta, \end{cases}$$

$$x_4 \rightarrow x_4x_1^{\eta+\theta}(x_3x_1^{\eta+\theta})^\lambda((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\kappa-\tau-1}x_5$$

to the equation  $x_3x_4 = x_1^\eta x_3$ , that is, **5**.

The equation  $E$  with  $\lambda + 1 = \sigma$  and  $\tau = \kappa$  can be reduced by the transformation

$$\begin{aligned} x_1 &\rightarrow x_2^\beta x_1, \\ x_2 &\rightarrow x_1x_2, \\ x_5 &\rightarrow x_2x_1x_3x_4 \end{aligned}$$

to **1**.

The equation (jjjj) can be reduced by the transformation

$$\begin{aligned} x_2 &\rightarrow x_3x_1((x_1x_3)^{\lambda+1}x_1)^\tau(x_1x_3)^\sigma x_1x_2, \\ x_3 &\rightarrow x_2x_3 \end{aligned}$$

to the equation  $E$ :

$$x_1x_2x_3x_4 = x_3x_1(x_2x_3x_1)^{\lambda-\sigma}((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau}x_5$$

with  $\partial(x_3) > 0$ .

The equation  $E$  with  $\lambda > \sigma$  can be reduced by the transformation

$$x_4 \rightarrow x_3x_1(x_2x_3x_1)^{\lambda-\sigma-1}((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau}x_5$$

to the equation  $x_2x_1x_3 = x_1x_3x_2$ , that is, to **2** with  $\partial(x_3) > 0$ . On the other hand, t6 is a parametric solution of **38**.

The equation  $E$  with  $\lambda = \sigma$  and  $\kappa > \tau$  can be reduced by the transformation

$$x_4 \rightarrow x_4x_3(x_1x_2x_3)^\lambda x_1((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau-1}x_5$$

to the equation  $x_1x_2x_3x_4 = x_3x_1^2x_2$  with  $\partial(x_3) > 0$ , that is, to **12**.

The equation  $E$  with  $\lambda = \sigma$  and  $\kappa = \tau$  can be reduced by the transformation  $x_5 \rightarrow x_5x_4$  to the equation  $x_1x_2x_3 = x_3x_1x_5$  with  $\partial(x_3) > 0$ , that is, to **10** with  $\partial(x_3) > 0$ . On the other hand, t8 is a parametric solution of **38**.

The equation (jjjjj) can be reduced by the transformation

$$\begin{aligned} x_2 &\rightarrow x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\kappa+1}x_2, \\ x_5 &\rightarrow x_2x_1x_3x_4 \end{aligned}$$

to the equation **1**.

**Proposition 39.** *The general solution of the equation*

$$\mathbf{39} \quad x_1x_2x_3x_4 = x_3x_1^{\alpha+1}x_5,$$



The equation  $E_2$  with  $\alpha > 0$  is **38** with  $\partial(x_3) > 0$ . On the other hand, **t3** is a parametric solution of **39**.

The equation  $E_2$  with  $\alpha = 0$  and  $\partial(x_2) \geq \partial(x_3)$  can be reduced by the transformation  $x_2 \rightarrow x_3x_2$ ,  $x_5 \rightarrow x_5x_3x_4$  to the equation  $x_2x_1 = x_1x_5$ , that is, to **4**.

The equation  $E_2$  with  $\alpha = 0$  and  $\partial(x_2) \leq \partial(x_3)$  can be reduced by the transformation  $x_3 \rightarrow x_2x_3$  to the equation  $x_1x_2x_3 = x_3x_1x_5$ , that is, to **10**.

The equation  $E_1$  with  $\partial(x_3) = 0$  can be reduced by the transformation  $x_3 \rightarrow 1$  to the equation  $E_3$ :

$$x_2x_4 = x_1^\alpha x_5.$$

The equation  $E_3$  with  $\partial(x_4) \leq \partial(x_5)$  can be reduced by the transformation  $x_5 \rightarrow x_5x_4$  to the equation  $x_2 = x_1^\alpha x_5$ .

The equation  $E_3$  with  $\partial(x_4) \geq \partial(x_5)$  and  $\alpha = 0$  can be reduced by the transformation  $x_4 = x_4x_5$ ,  $\alpha \rightarrow 0$  to the equation  $x_2x_4 = 1$ .

The equation  $E_3$  with  $\partial(x_4) \geq \partial(x_5)$  and  $\alpha = \beta + 1$  can be reduced by the transformation

$$\begin{aligned} \alpha &\rightarrow \beta + 1, \\ x_1 &\rightarrow x_2x_1, \\ x_2 &\rightarrow (x_2x_1)^\gamma x_2, \\ x_4 &\rightarrow (x_1x_2)^{\beta-\gamma} x_1x_5 \end{aligned}$$

to the equation **1**.

## 9. MORE EQUATIONS AND SOLUTIONS

In this section we will denote the words in the alphabet of primitive parametric words  $P_1, \dots, P_k$  by  $\zeta(P_1, \dots, P_k)$ ,  $v(P_1, \dots, P_k)$ ,  $\varphi(P_1, \dots, P_k)$ . The natural numbers will be denoted by  $q, r, s$ , the natural parameters by  $\alpha, \beta, \lambda, \tau$ , and the primitive parametric word by  $P$ .

The proofs of Propositions 40–55 are simple and very similar to proofs of the previous propositions, so we shall omit them.

**Proposition 40.** *The general solution of the equation*

$$\mathbf{40} \quad x_1x_2x_3x_4 = x_2\zeta(x_1x_2, x_3)x_1 \quad \text{with } \partial(x_2) > 0,$$

where  $x_3$  occurs in  $\zeta(x_1x_2, x_3)$ , is described by the transformations

$$\begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \\ x_4 \rightarrow x_4v(x_1^{\alpha+\beta}, x_3)x_1^\alpha, \end{cases} \quad \langle \mathbf{5} \rangle,$$

where  $\zeta(x_1x_2, x_3)$  is  $(x_1x_2)^{q+1}x_3v(x_1x_2, x_3)$ ;

$$x_4 \rightarrow x_4x_2v(x_1x_2, x_3)x_1, \quad \langle \mathbf{32} \rangle,$$

where  $\zeta(x_1x_2, x_3)$  is  $x_3^{q+1}x_1x_2v(x_1x_2, x_3)$ ; and

$$\langle \mathbf{32} \rangle,$$

where  $\zeta(x_1x_2, x_3)$  is  $x_3^{q+1}$ .

**Proposition 41.** *The general solution of the equation*

$$41 \quad x_2x_1x_3x_4 = x_3\zeta(x_1x_3, x_2)x_1 \quad \text{with } \partial(x_3) > 0,$$

where  $x_2$  occurs in  $\zeta(x_1x_3, x_2)$ , is described by the transformations

$$x_4 \rightarrow x_4v(x_1x_3, x_2)x_1, \quad \langle 29 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $(x_1x_3)^{q+1}x_2v(x_1x_3, x_2)$ ;

$$x_4 \rightarrow x_4x_3v(x_1x_3, x_2), \quad \langle 20 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2^{r+1}x_1x_3v(x_1x_3, x_2)$ ;

$$x_4 \rightarrow x_4x_3v(x_1x_3, x_2), \quad \langle 12 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2^2x_1x_3v(x_1x_3, x_2)$ ;

$$x_4 \rightarrow x_4x_3v(x_1x_3, x_2), \quad \langle 2 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2x_1x_3v(x_1x_3, x_2)$ ;

$$\langle 20 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2^{r+3}$ ;

$$\langle 12 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2^2$ ; and

$$\langle 2 \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2$ .

**Proposition 42.** *The general solution of the equation*

$$42 \quad x_1x_2x_3x_4 = \zeta(x_2, x_3)x_1$$

where  $x_2$  and  $x_3$  occur in  $\zeta(x_2, x_3)$ , is described by the transformations

$$\begin{cases} x_2 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1; \\ x_1 \rightarrow (\varphi(x_2, x_3)x_2v(x_2, x_3))^\alpha \varphi(x_2, x_3)x_1, \\ x_2 \rightarrow x_1x_2, \end{cases} \quad \langle 40 \rangle,$$

where  $\zeta(x_2, x_3)$  is  $\varphi(x_2, x_3)x_2v(x_2, x_3)$ ; and

$$\begin{cases} x_1 \rightarrow (\varphi(x_2, x_3)x_3v(x_2, x_3))^\alpha \varphi(x_2, x_3)x_1, \\ x_3 \rightarrow x_1x_3, \end{cases} \quad \langle 41 \rangle,$$

where  $\zeta(x_2, x_3)$  is  $\varphi(x_2, x_3)x_3v(x_2, x_3)$ .

**Proposition 43.** *The general solution of the equation*

$$\mathbf{43} \quad x_1 x_2 x_3 x_4 = x_2 \zeta(x_1 x_2, x_3) x_5 \quad \text{with } \partial(x_2) > 0$$

*is described by the transformations*

$$\langle \mathbf{34} \rangle$$

where  $\zeta(x_1 x_2, x_3)$  is 1;

$$\begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \end{cases} \quad \langle \mathbf{33} \rangle,$$

where  $\zeta(x_1 x_2, x_3)$  is  $(x_1 x_2)^{q+1}$ ;

$$x_4 \rightarrow x_4 x_2 v(x_1 x_2, x_3) x_5, \quad \langle \mathbf{40} \rangle,$$

where  $\zeta(x_1 x_2, x_3)$  is  $(x_1 x_2)^q x_3^{r+1} x_1 x_2 v(x_1 x_2, x_3) x_5$ ;

$$\langle \mathbf{37} \rangle,$$

where  $\zeta(x_1 x_2, x_3)$  is  $x_3^{r+1}$ ; and

$$\begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \\ x_4 \rightarrow x_4 x_3^r x_5, \end{cases} \quad \langle \mathbf{5} \rangle,$$

where  $\zeta(x_1, x_2)$  is  $(x_1 x_2)^{q+1} x_3^{r+1}$ .

**Proposition 44.** *The general solution of the equation*

$$\mathbf{44} \quad x_2 x_1 x_3 x_4 = x_3 \zeta(x_1 x_3, x_2) x_5 \quad \text{with } \partial(x_3) > 0$$

*is described by the transformations*

$$\langle \mathbf{36} \rangle,$$

where  $\zeta(x_1 x_3, x_2)$  is 1;

$$\langle \mathbf{35} \rangle,$$

where  $\zeta(x_1 x_3, x_2)$  is  $(x_1 x_3)^{q+1}$ ;

$$\langle \mathbf{39} \rangle,$$

where  $\zeta(x_1 x_3, x_2)$  is  $x_2^{r+1}$ ;

$$x_4 \rightarrow x_4 v(x_1 x_3, x_2) x_5, \quad \langle \mathbf{29} \rangle,$$

where  $\zeta(x_1 x_3, x_2)$  is  $(x_1 x_3)^{q+1} x_2 v(x_1 x_3, x_2)$ ;

$$x_4 \rightarrow x_4 x_3 v(x_1 x_3, x_2) x_5, \quad \langle \mathbf{2} \rangle,$$

where  $\zeta(x_1 x_3, x_2)$  is  $x_2 x_1 x_3 v(x_1 x_3, x_2) x_5$ ;

$$\langle \mathbf{12} \rangle,$$

where  $\zeta(x_1 x_3, x_2)$  is  $x_2^2 x_1 x_3 v(x_1 x_3, x_2) x_5$ ; and

$$\langle \mathbf{20} \rangle,$$

where  $\zeta(x_1x_3, x_2)$  is  $x_2^{s+3}x_1x_3v(x_1x_3, x_2)x_5$ .

**Proposition 45.** *The general solution of the equation*

$$45 \quad x_1x_2x_3x_4 = \zeta(x_2, x_3)x_5$$

where  $x_2, x_3$  occur in  $\zeta(x_2, x_3)$  is described by the transformations

$$\begin{cases} x_1 \rightarrow \zeta(x_2, x_3)x_1, \\ x_5 \rightarrow x_1x_2x_3x_4; \\ x_1 \rightarrow \varphi(x_2, x_3)x_1, \\ x_2 \rightarrow x_1x_2, \end{cases} \quad \langle 43 \rangle,$$

where  $\zeta(x_2, x_3)$  is  $\varphi(x_2, x_3)x_2v(x_2, x_3)$ ; and

$$\begin{cases} x_1 \rightarrow \varphi(x_2, x_3)x_1, \\ x_3 \rightarrow x_1x_3, \end{cases} \quad \langle 44 \rangle,$$

where  $\zeta(x_2, x_3)$  is  $\varphi(x_2, x_3)x_3v(x_2, x_3)$ .

**Proposition 46.** *The general solution of the equation*

$$46 \quad x_1x_2x_3x_4 = \zeta(x_1, x_2)x_3,$$

where  $x_1, x_2$  occur in  $\zeta(x_1, x_2)$ , is described by the transformation

$$\begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \end{cases} \quad \langle 5 \rangle.$$

**Proposition 47.** *The general solution of the equation*

$$47 \quad x_1x_2x_3x_4 = \zeta(x_1, x_2)x_5,$$

where  $x_1, x_2$  occur in  $\zeta(x_1, x_2)$ , is described by the transformation

$$\begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \end{cases} \quad \langle 33 \rangle.$$

**Proposition 48.** *The general solution of the equation*

$$48 \quad x_2x_1x_3x_4 = (x_3x_1)^{r+1}\zeta(x_1x_3, (x_1x_3)^\lambda x_1)x_2$$

with  $\partial(x_2) \leq \partial(x_3x_1)$  is described by the transformations

$$\begin{cases} x_3 \rightarrow x_2x_3, \\ x_4 \rightarrow (x_3x_1x_2)^r, \end{cases} \quad \langle 2 \rangle,$$

where  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{cases} x_3 \rightarrow x_2x_3, \\ x_4 \rightarrow x_4P, \end{cases} \quad \langle 12 \rangle,$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_2$  can be  $x_1x_2P$ ;

$$\begin{aligned} x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{20} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_2$  can be  $x_1^{q+2}x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow (x_1x_3x_2)^r, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow x_1(x_3x_2x_1)^{r-1}\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_3x_2, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $r > 0$ ; and

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow x_4P, \\ \begin{cases} x_1 &\rightarrow x_1^\alpha, \\ x_2 &\rightarrow x_1^\beta, \end{cases} \\ &\langle \mathbf{5} \rangle, \end{aligned}$$

where  $r = 0$ , and  $x_1\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_3x_2$  can be  $(x_2x_1)^{s+1}x_3P$ .

**Proposition 49.** *The general solution of the equation*

$$\mathbf{49} \quad x_2x_1x_3x_4 = x_1x_3\zeta(x_1x_3, (x_1x_3)^\lambda x_1)x_2$$

with  $\partial(x_2) \leq \partial(x_1x_3)$  is described by the transformations

$$\begin{aligned} x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow P, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_2$  can be  $x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow P, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_1x_2$  can be  $x_1x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{12} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_1x_2$  can be  $x_1^2x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{20} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_1x_2$  can be  $x_1^{q+3}x_2P$ ; and

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{29} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_1x_2$  is  $x_2x_3x_1P$ .

**Proposition 50.** *The general solution of the equation*

$$\mathbf{50} \quad x_2x_1x_3x_4 = (x_1x_3)^\lambda x_1\zeta(x_1x_3, x_3x_1, (x_1x_3)^\lambda x_1)x_2$$

with  $\partial(x_2) \leq \partial((x_1x_3)^\lambda x_1)$  is described by the transformations

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow P, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3, x_3x_2x_1, (x_2x_1x_3)^\lambda x_2x_1)(x_2x_1x_3)^\tau x_2$  can be  $x_3x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_3 &\rightarrow 1, \\ x_4 &\rightarrow 1, \\ &\langle \mathbf{1} \rangle, \end{aligned}$$

where  $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3, x_3x_2x_1, (x_2x_1x_3)^\lambda x_2x_1)(x_2x_1x_3)^\tau x_2$  can be  $x_2$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow x_4P, \\ \begin{cases} x_1 &\rightarrow x_1^\alpha, \\ x_2 &\rightarrow x_1^\beta, \end{cases} \\ &\langle \mathbf{5} \rangle, \end{aligned}$$

where  $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3, x_3x_2x_1, (x_2x_1x_3)^\lambda x_2x_1)(x_2x_1x_3)^\tau x_2$  can be  $x_2x_1x_3P$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow P, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $(x_2x_3x_1)^{\lambda-\tau-1}\zeta(x_1x_2x_3, x_2x_3x_1, (x_1x_2x_3)^\lambda x_1)(x_1x_2x_3)^\tau x_1x_2$  is  $x_1x_2P$ ; and

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{29} \rangle, \end{aligned}$$

where  $(x_2x_3x_1)^{\lambda-\tau-1}\zeta(x_1x_2x_3, x_2x_3x_1, (x_1x_2x_3)^\lambda x_1)(x_1x_2x_3)^\tau x_1x_2 = x_2x_3x_1P$ .

**Proposition 51.** *The general solution of the equation*

$$\mathbf{51} \quad x_2x_1x_3x_4 = (x_3x_1)^{r+1}\zeta(x_2x_3, (x_1x_3)^\lambda x_1)x_5$$

with  $\partial(x_2) \leq \partial(x_3x_1)$  is described by the transformations

$$\begin{aligned} x_3 &\rightarrow x_2x_3, \\ x_5 &\rightarrow x_5x_4, \\ &\langle \mathbf{10} \rangle, \end{aligned}$$

where  $r = 0$  and  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{aligned} x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow P, \\ &\langle \mathbf{2} \rangle \end{aligned}$$

where  $(x_2x_3x_1)^r\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_5$  is  $x_2P$ ;

$$\begin{aligned} x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{12} \rangle, \end{aligned}$$

where  $(x_2x_3x_1)^r\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_5$  can be  $x_1x_2P$ ;

$$\begin{aligned} x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{20} \rangle, \end{aligned}$$

where  $(x_2x_3x_1)^r\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_5$  can be  $x_1^{q+2}x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_5 &\rightarrow x_5x_3x_4, \\ &\langle \mathbf{4} \rangle, \end{aligned}$$

where  $r = 0$ ,  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1) = 1$ ;

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow x_1(x_3x_2x_1)^{r-1}\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x - 5, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $r > 0$ ;

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow x_4P, \\ \begin{cases} x_1 &\rightarrow x_1^\alpha, \\ x_2 &\rightarrow x_1^\beta, \end{cases} \\ &\langle \mathbf{5} \rangle, \end{aligned}$$

where  $r = 0$  and  $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_5$  can be  $(x_2x_1)^{q+1}x_3P$ ; and

$$\begin{aligned} x_2 &\rightarrow x_3x_2, \\ x_1 &\rightarrow x_2x_1, \\ \begin{cases} x_1 \rightarrow x_1^\alpha, \\ x_2 \rightarrow x_1^\beta, \end{cases} \\ &\langle \mathbf{33} \rangle, \end{aligned}$$

where  $r = 0$  and  $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)$  can be  $(x_2x_1)^{q+1}$ .

**Proposition 52.** *The general solution of the equation*

$$\mathbf{52} \quad x_2x_1x_3x_4 = x_1x_3\zeta(x_1x_3, (x_1x_3)^\lambda x_1)x_5$$

with  $\partial(x_2) \leq \partial(x_1x_3)$  is described by the transformations

$$\begin{aligned} x_1 &\rightarrow x_2x_1, \\ x_5 &\rightarrow x_5x_4, \\ &\langle \mathbf{7} \rangle, \end{aligned}$$

where  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{aligned} x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow Px_5, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)$  can be  $x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_5 &\rightarrow x_5x_4, \\ &\langle \mathbf{6} \rangle, \end{aligned}$$

where  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ &\langle \mathbf{39} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1^{q+1}$ ;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow P \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{12} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1^2x_2P$ ; and

$$\begin{aligned} x_2 &\rightarrow x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{20} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1^{s+3}x_2P$ .

**Proposition 53.** *The general solution of the equation*

$$\mathbf{53} \quad x_2x_1x_3x_4 = (x_1x_3)^\lambda x_1 \zeta(x_1x_3, (x_1x_3)^\lambda x_1)x_5$$

with  $\partial(x_2) \leq \partial((x_1x_3)^\lambda x_1)$  is described by the transformations

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\lambda x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_5 &\rightarrow x_5x_3x_4, \\ &\langle \mathbf{4} \rangle, \end{aligned}$$

where  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_2, \\ x_1 &\rightarrow x_2x_1, \\ x_4 &\rightarrow x_4P, \\ \begin{cases} x_1 &\rightarrow x_1^\alpha, \\ x_2 &\rightarrow x_1^\beta, \end{cases} \\ &\langle \mathbf{5} \rangle, \end{aligned}$$

where  $(x_2x_1x_3)^{\lambda-\tau} \zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_5$  can be  $(x_2x_1)^{q+1}x_3P$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^\tau x_2, \\ x_1 &\rightarrow x_2x_1, \\ \begin{cases} x_1 &\rightarrow x_1^\alpha, \\ x_2 &\rightarrow x_1^\beta, \end{cases} \\ &\langle \mathbf{33} \rangle, \end{aligned}$$

where  $(x_2x_1x_3)^{\lambda-\tau} \zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)$  can be  $(x_2x_1)^{q+1}$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^{\lambda-1}x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ x_5 &\rightarrow x_5x_4, \\ &\langle \mathbf{10} \rangle, \end{aligned}$$

where  $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$  is 1;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^{\lambda-1}x_1x_2, \\ x_3 &\rightarrow x_2x_3, \\ &\langle \mathbf{39} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1^{s+1}$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^{\lambda-1}, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow P, \\ &\langle \mathbf{2} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1x_2P$ ;

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^{\lambda-1}, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{12} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1^2x_2P$ ; and

$$\begin{aligned} x_2 &\rightarrow (x_1x_3)^{\lambda-1}, \\ x_3 &\rightarrow x_2x_3, \\ x_4 &\rightarrow x_4P, \\ &\langle \mathbf{20} \rangle, \end{aligned}$$

where  $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$  can be  $x_1^{s+3}x_2$ .

**Proposition 54.** *The general solution of the equation*

$$\mathbf{54} \quad x_1x_2x_3x_4 = x_3^{r+1}x_1\zeta(x_1, x_3)x_2,$$

where  $r$  is a natural number, is described by the transformations

$$\begin{cases} x_1 \rightarrow 1, \\ x_3 \rightarrow 1, \\ x_4 \rightarrow 1; \\ \\ x_1 \rightarrow (x_1x_3)^\lambda x_1, \\ x_3 \rightarrow x_1x_3, \\ x_2 \rightarrow ((x_3x_1)^{r+1}\zeta((x_1x_3)^\lambda x_1, x_1x_3))^\alpha v((x_1x_3)^\lambda x_1, x_3x_1, x_1x_3)x_2, \end{cases}$$

where  $x_5^{r+1}\zeta(x_1, x_3)$  is  $v(x_1, x_5, x_3)\varphi(x_1, x_5, x_3)$ , followed by one of the three transformations  $\langle \mathbf{48} \rangle$ ,  $\langle \mathbf{49} \rangle$ ,  $\langle \mathbf{50} \rangle$ .

**Proposition 55.** *The general solution of the equation*

$$\mathbf{55} \quad x_1x_2x_3x_4 = x_3^{r+1}x_1\zeta(x_1, x_3)x_5,$$

where  $r$  is a natural number, is described by the transformations

$$\begin{cases} x_1 \rightarrow (x_1x_3)^\lambda x_1, \\ x_3 \rightarrow x_1x_3, \\ x_2 \rightarrow \varphi((x_1x_3)^\lambda x_1, x_3x_1, x_1x_3)x_2, \end{cases}$$

where  $x_6^{r+1}\zeta(x_1, x_3)$  is  $\varphi(x_1, x_6, x_3)v(x_1, x_6, x_3)$ , followed by one of the four transformations  $\langle \mathbf{51} \rangle$ ,  $\langle \mathbf{52} \rangle$ ,  $\langle \mathbf{53} \rangle$ , or  $x_5 \rightarrow x_2x_1x_3x_4$ , where  $v(x_1, x_6, x_3)$  is 1.

**Proposition 56.** *The general solution of the equation*

$$\mathbf{56} \quad x_1x_2x_3x_4 = \zeta(x_1, x_2, x_3)x_5$$

is described by the transformations  $\langle \mathbf{42} \rangle$ ;  $\langle \mathbf{46} \rangle$ ;  $\langle \mathbf{54} \rangle$ ;  $\langle \mathbf{45} \rangle$ ;  $\langle \mathbf{47} \rangle$ ;  $\langle \mathbf{55} \rangle$ ;  $\langle \mathbf{34} \rangle$ ;  $\langle \mathbf{36} \rangle$ ;  $x_5 \rightarrow x_1x_2x_3x_4$ , where  $\zeta(x_1, x_2, x_3) = 1$ .

*Proof.* If  $x_1, x_2, x_3$  occur in  $\zeta(x_1, x_2, x_3)$ , then the equation **56** has one of the following three forms:

$$E_1: \quad x_1x_2x_3x_4 = v(x_2, x_3)x_1\varphi(x_1, x_2, x_3)x_5,$$

where  $x_2$  and  $x_3$  occur in  $v(x_2, x_3)$ ;

$$E_2: \quad x_1x_2x_3x_4 = v(x_1, x_2)x_3\varphi(x_1, x_2, x_3)x_5,$$

where  $x_1$  and  $x_2$  occur in  $v(x_1, x_2)$ ;

$$E_3: \quad x_1x_2x_3x_4 = v(x_1, x_3)x_2\varphi(x_1, x_2, x_3)x_5,$$

where  $x_1$  and  $x_3$  occur in  $v(x_1, x_3)$ .

The equation  $E_1$  can be reduced by the transformation

$$x_4 \rightarrow x_4\varphi(x_1, x_2, x_3)x_5$$

to the equation **42**.

The equation  $E_2$  can be reduced by the transformation

$$x_4 \rightarrow x_4\varphi(x_1, x_2, x_3)x_5$$

to the equation **46**.

The equation  $E_3$  can be reduced by the transformation

$$x_4 \rightarrow x_4\varphi(x_1, x_2, x_3)x_5$$

to the equation **54**.

If only  $x_2, x_3$  occur in  $\zeta(x_1, x_2, x_3)$ , then **56** is **45**.

If only  $x_1, x_2$  occur in  $\zeta(x_1, x_2, x_3)$ , then **56** is **47**.

If only  $x_1, x_3$  occur in  $\zeta(x_1, x_2, x_3)$ , then **56** is **55**.

If only  $x_2$  occurs in  $\zeta(x_1, x_2, x_3)$ , then **56** is **34**.

If only  $x_3$  occurs in  $\zeta(x_1, x_2, x_3)$ , then **56** is **36**.

If  $\zeta(x_1, x_2, x_3)$  is the empty word, then **56** has the form  $x_1x_2x_3x_4 = x_5$ .

#### REFERENCES

- [1] R. C. Lyndon, *Equations in free groups*, Trans. Amer. Math. Soc. **96** (1960), 445–457. MR **27**:1488
- [2] ———, *Groups with parametric exponents*, Trans. Amer. Math. Soc. **96** (1960), 518–533. MR **27**:1487
- [3] Yu. I. Khmelevskii, *Solution of word equations in three unknowns*, Dokl. Akad. Nauk SSSR **177** (1967), 1023–1025; English transl., Soviet Math. Dokl. **8** (1967), 1554–1556. MR **36**:3899
- [4] ———, *Equation in free semigroups*, Trudy Mat. Inst. Steklov. **107** (1971); English transl., Proc. Steklov Inst. Math. **107** (1971) (1976). MR **51**:5808; MR **52**:14094
- [5] G. S. Makanin and P. Goralcik, *Structure of the solutions of one parametric equation*, Report of research, LIR, INSA of Rouen, April 1996.
- [6] G. S. Makanin and H. Abdulrab, *Functions for the general solution of parametric word equations*, Report of research, LIR, INSA of Rouen, June 1996.
- [7] G. S. Makanin, *Systems of standard equations in words of an  $n$ -layer alphabet with unknowns*, Sibirsk. Mat. Zh. **19** (1978), 637–645; English transl., Siberian Math. J. **19** (1978), 448–454. MR **80a**:20039
- [8] ———, *Recognition of the rank of equations in a free semigroup*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 547–602; English transl., Math. USSR Izv. **14** (1980), 499–545. MR **80h**:20084
- [9] M. Lothaire, *Combinatorics on words*, Addison-Wesley, 1983. MR **84g**:05002

STEKLOV MATHEMATICAL INSTITUTE, VA VILOVA 42, 117 966, MOSCOW GSP-1, RUSSIA