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FINITE GROUPS OF MATRICES OVER GROUP RINGS

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ABSTRACT. We investigate certain finite subgroups Γ of $GL_n(\mathbf{Z}\Pi)$, where Π is a finite nilpotent group. Such a group Γ gives rise to a $\mathbf{Z}[\Gamma \times \Pi]$-module; we study the characters of these modules to limit the structure of Γ. We also exhibit some exotic subgroups Γ.

1. INTRODUCTION

Let Π be a finite group. We set

$$SGL_n(\mathbf{Z}\Pi) = \ker \operatorname{aug} : GL_n(\mathbf{Z}\Pi) \to GL_n(\mathbf{Z}),$$

where aug is the usual augmentation map applied to each entry of $GL_n(\mathbf{Z}\Pi)$.

Suppose that α is a homomorphism from a finite group Γ to $SGL_n(\mathbf{Z}\Pi)$. We shall investigate the following problem.

Problem 0. *Do there exist group homomorphisms $\sigma_i : \Gamma \to \Pi$, $i = 1, 2, \ldots, n$, and an element $x \in GL_n(\mathbf{Q}\Pi)$ such that $x^{-1}\alpha(\gamma)x = \operatorname{diag}(\sigma_i(\gamma))$, $\gamma \in \Gamma$?*

This is analogous to a conjecture of Zassenhaus, who was interested in units of $\mathbf{Z}\Pi$ of augmentation 1, i.e. $SGL_1(\mathbf{Z}\Pi)$. Problem 0 is related to results on units of group rings, as shown in a special case in [MRSW]. There is a positive answer to Problem 0 if Π is a p-group [WAn] or if $n = 1$ and Π is nilpotent [WCr].

Given finite groups Γ and Π, set

$$G = \Gamma \times \Pi, \qquad N = 1 \times \Pi.$$

A homomorphism $\alpha : \Gamma \to SGL_n(\mathbf{Z}\Pi)$ gives rise to a double action $\mathbf{Z}G$-module $M(\alpha)$, defined as follows: as abelian group, $M(\alpha)$ is equal to the column vectors $\mathbf{Z}\Pi^n$, and the G-action is given by

$$m \cdot (\gamma, \pi) = \alpha(\gamma^{-1})m\pi, \quad (\gamma, \pi) \in G, \quad m \in M.$$

There is a bijection between $GL_n(\mathbf{Z}\Pi)$-conjugacy classes of homomorphisms α and isomorphism classes of $\mathbf{Z}G$-lattices M which satisfy

(a) G/N acts trivially on the N-fixed points M^N, and
(b) $\operatorname{res}_N M$ is a free $\mathbf{Z}N$-module.

(See [S, §38.6] for details.)

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Definition. Let $\mathcal{D}(\Gamma,\Pi)$ ($\mathcal{D}$ for "double-action") be the set of characters of $\mathbf{Z}G$-modules M which satisfy (a) and (b).

If $\sigma : \Gamma \to \Pi$ is a homomorphism, it is easy to show (Lemma 2.1 below) that the double action module $M(\sigma)$ is isomorphic to the permutation module $\mathrm{ind}_{[\sigma]}^{G} \mathbf{Z}$, where $[\sigma]$ denotes the subgroup $\{(\gamma,\sigma(\gamma)) : \gamma \in \Gamma\}$ of G. Define

$$\begin{aligned} R(G) &= \text{the virtual complex characters of } G,\\ R^{+}(G) &= \text{the proper characters of } G,\\ \mathcal{P}(\Gamma,\Pi) &= \mathbf{Z}\text{-span of } \{\mathrm{ind}_{[\sigma]}^{G} 1 : \sigma \in \hom(\Gamma,\Pi)\} \subset R(G),\\ \mathcal{P}^{+}(\Gamma,\Pi) &= \mathbf{Z}_{\geq 0}\text{-span of } \{\mathrm{ind}_{[\sigma]}^{G} 1 : \sigma \in \hom(\Gamma,\Pi)\} \subset R^{+}(G). \end{aligned}$$

Here $\mathcal{P}$ is for "permutation". Then $\mathcal{P}^{+}$ is contained in $\mathcal{D}$; it follows from [WAn] that if Π is a p-group, then $\mathcal{D} = \mathcal{P}^{+}$. Indeed, it turns out (Proposition 2.3 below) that there is a positive answer to Problem 0 for finite groups Γ, Π if and only if $\mathcal{D} = \mathcal{P}^{+}$. Note that $\mathcal{D}$ is closed under addition, and is a sub-semigroup of $R^{+}(G)$. Problem 0 can be reformulated as

Problem 1. *Is $\mathcal{D} = \mathcal{P}^{+}$?*

In §6 we give some conditions under which Problem 1 has a positive solution. In §5 we give examples of groups Γ and Π for which $\mathcal{D} \neq \mathcal{P}^{+}$. A counterexample to Problem 1 for $n = 1$ has been constructed by Roggenkamp and Scott [RS] (see [K]); there are also such examples in [B], but with no general principles of construction for them. We will show that for $n > 1$ counterexamples are so plentiful that our focus shifts to describing all of them. The feasibility of doing so is addressed by

Problem 2. *Is the semigroup $\mathcal{D}$ finitely generated?*

We will show that this is indeed true if Γ and Π are nilpotent; this is in §7. If we know that $\mathcal{D}$ is finitely generated, we are also interested in explicitly finding a generating set.

Analysis of $\mathcal{D}(\Gamma,\Pi)$ is complicated by questions about locally free class groups, so we are led to replace (b) in the definition of $\mathcal{D}$ by

(b′) $\mathrm{res}_N M$ is a locally free $\mathbf{Z}N$-module.

Definition. Let $\mathcal{D}'(\Gamma,\Pi)$ be the set of characters of $\mathbf{Z}G$-modules M which satisfy (a) and (b′).

The semigroup $\mathcal{D}'$ is a good approximation to $\mathcal{D}$, in the sense that $r\mathcal{D}' \subseteq \mathcal{D} \subseteq \mathcal{D}'$ for some positive integer r (Lemma 2.5 below); in other words, if M is a $\mathbf{Z}G$-module satisfying (a) and (b′), then the direct sum M^r of r copies of M satisfies (a) and (b). We get first approximations to results about $\mathcal{D}$ by considering $\mathcal{D}'$ instead. All known cases of the equality $\mathcal{D} = \mathcal{P}^{+}$ actually have $\mathcal{D}' = \mathcal{P}^{+}$.

One of our main results, Theorem 3.3 below, is a complete character-theoretic description of $\mathcal{D}'$ in the case that Π is nilpotent. To explain this we introduce some notation. For a finite nilpotent group N, let N_p be its Sylow p-subgroup and $N_{p'}$ its Sylow p-complement. Then we have $R(G) = R(G_{p'}) \otimes R(G_p)$, and we use this to define

$$\begin{aligned} \mathcal{Q}_p(\Gamma,\Pi) &= R(G_{p'}) \otimes \mathcal{P}(\Gamma_p,\Pi_p),\\ \mathcal{Q}_p^{+}(\Gamma,\Pi) &= \left\{ \sum \eta \otimes \lambda : \eta \in R^{+}(G_{p'}), \lambda \in \mathcal{P}^{+}(\Gamma_p,\Pi_p) \right\}. \end{aligned}$$

We show in Theorem 3.3 below that if Π is nilpotent, then

$$\mathcal{D}'(\Gamma,\Pi) = \bigcap_p \mathcal{Q}_p^+(\Gamma,\Pi).$$

Theorem 3.3 is crucial to most of the results in the rest of the paper.

We are also interested in knowing if Problem 1 has a "virtual" answer, that is, is the $\mathbf{Z}$-span of $\mathcal{D}$ equal to $\mathcal{P}$? This is discussed in §4.

Although the above definitions and problems are sensible for arbitrary finite groups Γ and Π, we shall need to assume that these groups are nilpotent for most of our results. This is because we rely on the local results of [WAn], which at this point have no analogues in the general case. *From §3 through to the end of the paper, we will assume that* Γ *and* Π *are nilpotent.*

2. Preliminaries

As in the introduction, let $G = \Gamma \times \Pi$ and $N = 1 \times \Pi$. In this section, Π and Γ can be arbitrary finite groups.

Lemma 2.1. *For* $\sigma \in \hom(\Gamma,\Pi)$*, the double action module* $M(\sigma)$ *is isomorphic to* $\operatorname{ind}_{[\sigma]}^G \mathbf{Z}$ *as* $\mathbf{Z}G$*-modules.*

Proof. By definition, $M(\sigma)$ has $\mathbf{Z}$-basis Π. Consider elements of $\operatorname{ind}_{[\sigma]}^G \mathbf{Z}$ as linear combinations of the cosets of $[\sigma]$ in G; use $1 \times \Pi$ as coset representatives. Define

$$f : M(\sigma) \to \operatorname{ind}_{[\sigma]}^G \mathbf{Z}, \qquad f(\pi) = [\sigma](1,\pi), \quad \pi \in \Pi.$$

To check that f is a $\mathbf{Z}G$-homomorphism, for $(\gamma,\pi') \in G$,

$$\begin{aligned} f\big(\pi\cdot(\gamma,\pi')\big) &= f(\sigma(\gamma^{-1})\pi\pi') = [\sigma](1,\sigma(\gamma^{-1})\pi\pi') \\ &= [\sigma]\big(\gamma,\sigma(\gamma)\big)\big(1,\sigma(\gamma^{-1})\pi\pi'\big) = [\sigma](\gamma,\pi\pi') = f(\pi)\cdot(\gamma,\pi'). \end{aligned}$$

This proves the lemma. ☐

Corollary 2.2. $\mathcal{P}^+ \subseteq \mathcal{D}$.

Proposition 2.3. *For a homomorphism* $\alpha : \Gamma \to SGL_n(\mathbf{Z}\Pi)$*, the character* χ *of the double action module* $M(\alpha)$ *is in* $\mathcal{P}^+(\Gamma,\Pi)$ *if and only if there exist group homomorphisms* $\sigma_i \in \hom(\Gamma,\Pi)$*,* $1 \le i \le n$*, and an element* $u \in GL_n(\mathbf{Q}\Pi)$ *such that* $u\alpha(\gamma)u^{-1} = \operatorname{diag}\big(\sigma_1(\gamma),\dots,\sigma_n(\gamma)\big)$ *for all* $\gamma \in \Gamma$*.*

Proof. Suppose that χ is in $\mathcal{P}^+(\Gamma,\Pi)$. By Lemma 2.1 there are homomorphisms $\sigma_i : \Gamma \to \Pi$, $1 \le i \le k$, such that there is a $\mathbf{Q}G$-isomorphism

$$f : \mathbf{Q}\otimes M(\alpha) \to \mathbf{Q}\otimes\Big(\bigoplus_{i=1}^k M(\sigma_i)\Big).$$

Comparing dimensions over $\mathbf{Q}$ gives $k = n$. Let $\{e_j : 1 \le j \le n\}$ be the standard basis of the $\mathbf{Q}\Pi$-column vectors $\mathbf{Q}\otimes M(\alpha)$, and write elements of $\mathbf{Q}\otimes M(\sigma_i)$ as $\langle x\rangle_i$, $x \in \mathbf{Q}\Pi$. Then $\mathbf{Q}\otimes\big(\bigoplus_{i=1}^k M(\sigma_i)\big)$ has $\mathbf{Q}N$-basis $\{\langle 1\rangle_i : 1 \le i \le n\}$ and, since f is a $\mathbf{Q}N$-homomorphism, we have

$$f(e_j) = \sum_i \langle u_{ij}\rangle_i \quad \text{where } u \in GL_n(\mathbf{Q}\Pi).$$

Act by $(\gamma^{-1},1)$, giving

$$f(e_j)(\gamma^{-1},1)=\sum_i\langle u_{ij}\rangle_i(\gamma^{-1},1)=\sum_i\langle\sigma_i(\gamma)u_{ij}\rangle_i.$$

Since $f(e_j)(\gamma^{-1},1)=f\big(e_j(\gamma^{-1},1)\big)$, this equals

$$\begin{aligned}f(\alpha(\gamma)e_j)=f\Big(\sum_k e_k\alpha(\gamma)_{kj}\Big)&=\sum_k f(e_k)\big(1,\alpha(\gamma)_{kj}\big)\\&=\sum_{k,i}\langle u_{ik}\rangle_i\big(1,\alpha(\gamma)_{kj}\big)=\sum_i\Big\langle\sum_k u_{ik}\alpha(\gamma)_{kj}\Big\rangle_i.\end{aligned}$$

Therefore for all i,j we have

$$\sigma_i(\gamma)u_{ij}=\sum_k u_{ik}\alpha(\gamma)_{kj}\quad\text{so, as matrix equation,}\quad \operatorname{diag}\big(\sigma_i(\gamma)\big)u=u\alpha(\gamma).$$

This implies that $u\alpha(\gamma)u^{-1}=\operatorname{diag}\big(\sigma_i(\gamma)\big)$.

For the converse, given $u\in GL_n(\mathbf{Q}\Pi)$ such that $u\alpha(\gamma)u^{-1}=\operatorname{diag}\big(\sigma_i(\gamma)\big)$, define

$$f:\mathbf{Q}\otimes M(\alpha)\to\mathbf{Q}\otimes\big(\oplus_i M(\sigma_i)\big)$$

by $f(e_j)=\sum_i\langle u_{ij}\rangle_i$. It follows as above that f is a $\mathbf{Q}G$-isomorphism. □

Lemma 2.4. *Suppose that* Π' *is a subgroup of* Π*, and set* $G'=\Gamma\times\Pi'$, $N'=1\times\Pi'$. *Then* $\operatorname{ind}_{G'}^{G}\mathcal{D}(\Gamma,\Pi')\subseteq\mathcal{D}(\Gamma,\Pi)$.

Proof. We have $G=NG'$ and $N\cap G'=N'$. Let $\chi'\in\mathcal{D}(\Gamma,\Pi')$ be the character of a $\mathbf{Z}G'$-lattice M' which satisfies (a) and (b) for G',N in the definition of $\mathcal{D}$ in §1. We must check that $M=\operatorname{ind}_{G'}^{G}M'$ satisfies (a) and (b) for G,N.

For (a) identify G/N and G'/N' with Γ; there are Γ-isomorphisms

$$M^N\cong M\otimes_{\mathbf{Z}G}\mathbf{Z}\Gamma\cong M'\otimes_{\mathbf{Z}G'}\mathbf{Z}G\otimes_{\mathbf{Z}G}\mathbf{Z}\Gamma\cong {M'}^{N'},$$

so Γ acts trivially.

For (b), use Mackey decomposition to get

$$\operatorname{res}_N^G M\cong\operatorname{res}_N^G\operatorname{ind}_{G'}^G M'\cong\operatorname{ind}_{N'}^N\operatorname{res}_{N'}^{G'}M',$$

because $G=NG'$ and $N\cap G'=N'$. □

Lemma 2.5. $r\mathcal{D}'\subseteq\mathcal{D}\subseteq\mathcal{D}'$ *for some positive integer* r.

Proof. By definition, we have $\mathcal{D}\subseteq\mathcal{D}'$. We will find a positive integer r so that for any locally free $\mathbf{Z}N$-lattice X, X^r is a free $\mathbf{Z}N$-lattice. Let e be the exponent of the locally free class group of $\mathbf{Z}N$, as defined in [CR, 49.10]. Then $Y=X^e$ is stably free. From the Bass Cancellation Theorem [CR, 41.20] $Y\oplus Y$ is free, so X^{2e} is free, and the lemma is proved, with $r=2e$. □

To proceed further, we will apply the p-group results of [WAn]. In order to do this for all primes dividing $|\Pi|$, we will assume that Π is nilpotent. We next show that under this assumption, $\alpha(\Gamma)$ is also nilpotent.

Lemma 2.6. *Suppose that* Π *is nilpotent; let* ϕ_p *denote the natural map*

$$\phi_p:GL_n(\mathbf{Z}\Pi)\to GL_n\big(\mathbf{Z}[\Pi/\Pi_p]\big).$$

Let H *be a finite subgroup of* $SGL_n(\mathbf{Z}\Pi)$*. Then the following hold:*

1. $\ker\phi_p\cap H$ *is a* p*-group.*

2. *If $x \in SGL_n(\mathbf{Z}\Pi)$ has prime order r, then r divides $|\Pi|$.*
3. $\ker \phi_p \cap H$ *is a normal Sylow p-subgroup of H.*

Proof. 1. Suppose that $x \in \ker \phi_p \cap H$. Then $x = 1 + \delta$, where all the entries of δ are in $\mathbf{Z}\Pi\Delta(\Pi_p)$, where $\Delta(\Pi_p)$ is the augmentation ideal of $\mathbf{Z}\Pi_p$. Since $\mathbf{Z}\Pi\Delta(\Pi_p)$ is a nilpotent ideal mod p, it follows that for a suitable positive integer m we have $x^{p^m} = 1 + py$, for some element y of the matrix ring $M_n(\mathbf{Z}\Pi)$. Then, raising to p-powers, we get

$$x^{p^{m+i}} = 1 + p^i y_i, \qquad y_i \in M_n(\mathbf{Z}\Pi).$$

Now x has finite order, since it is in the finite group H; so there are only finitely many possible values of $x^{p^{m+i}}$ as i varies. Then there is a subsequence of integers i for which $x^{p^{m+i}}$ is constant, say z. We see from the last equation that $z - 1$ has coefficients divisible by arbitrarily high powers of p. This forces $z = 1$, and x has p-power order.

2. Use induction on Π. If $|\Pi| = 1$, then $|SGL_n(\mathbf{Z}\Pi)| = 1$. Suppose that $|\Pi| > 1$. Let p be a prime dividing $|\Pi|$. If $x \in \ker \phi_p$, then x has order p by 1, and $r = p$. If $x \notin \ker \phi_p$, then $\phi_p(x) \in SGL_n(\mathbf{Z}[\Pi/\Pi_p])$ has order r dividing $|\Pi/\Pi_p|$, by induction.

3. Let $\Phi_p = \ker \phi_p \cap H$. From 1, Φ_p is a p-group. Suppose that y is an element of p-power order in H. From 2, $SGL_n(\mathbf{Z}[\Pi/\Pi_p])$ has no element of order p; therefore $y \in \ker \phi_p$. So Φ_p contains all Sylow p-subgroups of H. This completes the proof. □

Corollary 2.7. *If Π is nilpotent then $\mathcal{D}(\Gamma, \Pi) = \mathcal{D}(\Gamma_{\text{nil}}, \Pi)$, where Γ_{nil} is the largest nilpotent quotient of Γ. The same holds for $\mathcal{D}'$.*

Proof. If $\chi \in \mathcal{D}(\Gamma, \Pi)$ is the character of $M(\alpha)$ then $\alpha(\Gamma)$ is nilpotent by Lemma 2.6, since each Sylow subgroup is normal. Thus $\chi \in \mathcal{D}(\Gamma_{\text{nil}}, \Pi)$. The same assertion for $\mathcal{D}'$ follows from Lemma 2.5. □

3. Character-theoretic Description of $\mathcal{D}'$

For the rest of the paper, we assume that Π is nilpotent; then by Corollary 2.7, it is no loss of generality to assume that Γ is nilpotent. Thus we shall always assume that Γ is nilpotent.

In this section we give a character-theoretic description of $\mathcal{D}'(\Gamma, \Pi)$.

Lemma 3.1. *Let $\chi \in R(G) = R(G_{p'}) \otimes R(G_p)$ be written uniquely in the form*

$$\chi = \sum_{\eta \in \operatorname{irr}(G_{p'})} \eta \otimes \lambda_\eta, \qquad \lambda_\eta \in R(G_p).$$

Then $\chi \in \mathcal{Q}_p^+(\Gamma, \Pi)$ if and only if $\lambda_\eta \in \mathcal{P}^+(\Gamma_p, \Pi_p)$ for all λ.

Proof. Since $R(G_{p'})$ has $\mathbf{Z}$-basis the irreducible complex characters $\operatorname{irr}(G_{p'})$, and $R^+(G_{p'})$ is the non-negative linear combinations of $\operatorname{irr}(G_{p'})$, the result follows. □

Lemma 3.2. $\bigcap_p \mathcal{Q}_p \subseteq \mathbf{Q} \otimes \mathcal{P}$.

Proof. Take $\chi \in \bigcap_p \mathcal{Q}_p$. Let B_p be a maximal linearly independent subset of $\{\operatorname{ind}_{[\sigma_p]}^{G_p} 1 : \sigma_p \in \hom(\Gamma_p, \Pi_p)\}$; extend this to a basis $\widehat{B}_p$ of $\mathbf{Q} \otimes R(G_p)$. Then $\bigotimes_p \widehat{B}_p$ is a $\mathbf{Q}$-basis of $\bigotimes_p (\mathbf{Q} \otimes R(G_p)) = \mathbf{Q} \otimes R(G)$, so we can write χ uniquely as a $\mathbf{Q}$-linear combination in this basis. For a fixed p, χ is a linear combination of elements in

$\left(\bigotimes_{l\neq p}\widehat{B}_l\right)\otimes B_p$. Varying over p shows that χ is a $\mathbf{Q}$-linear combination of elements of $\bigotimes B_p$. A collection $\{\sigma_p:\Gamma_p\to\Pi_p\}$, one for each prime, corresponds to a single homomorphism $\sigma:\Gamma\to\Pi$, whose restriction to Γ_p is σ_p. So $\chi\in\mathbf{Q}\otimes\mathcal{P}(\Gamma,\Pi)$. ☐

Theorem 3.3. *Suppose that* Π *is nilpotent. Then* $\mathcal{D}'=\bigcap_p\mathcal{Q}_p^+$.

Proof. We first show that $\mathcal{D}'\subseteq\bigcap_p\mathcal{Q}_p^+$. Suppose that $\chi\in\mathcal{D}'$, and let M be a $\mathbf{Z}G$-lattice affording χ. Fix a prime p dividing $|\Pi|$. We want to apply Theorem 2 of [WAn] to the $\mathbf{Z}_pG_p$-module $M_p=\operatorname{res}_{G_p}(\mathbf{Z}_p\otimes_{\mathbf{Z}}M)$, relative to the normal subgroup $N_p=1\times\Pi_p$ of G_p. We need to verify that

(a) the N_p-fixed points $M_p^{N_p}$ have trivial G_p/N_p-action, and

(b) $\operatorname{res}_{N_p}M_p$ is a free $\mathbf{Z}_pN_p$-module.

Now (b) holds because $\operatorname{res}_N M$ is locally free. To prove (a), it is no loss to replace M by a direct sum of copies of M, and by Lemma 2.5, we may assume that $\operatorname{res}_N M$ is free. Then $M\cong M(\alpha)$ for some $\alpha:\Gamma\to GL_n(\mathbf{Z}\Pi)$; hence $M^{N_p}\cong M(\phi_p\alpha)$, where ϕ_p is as in Lemma 2.6. Now Γ_p is in the kernel of $\phi_p\alpha$ by Lemma 2.6, so G_p/N_p acts trivially on M^{N_p}.

From Theorem 2 of [WAn], M_p is a permutation $\mathbf{Z}_pG_p$-lattice; moreover, the permuted basis is a disjoint union of orbits whose point stabilizers are of the form $[\sigma_p]=\{(\gamma,\sigma_p(\gamma)):\gamma\in\Gamma_p\}$, where $\sigma_p\in\hom(\Gamma_p,\Pi_p)$. Since $\mathbf{Z}_p\otimes M$ is a summand of $\operatorname{ind}_{G_p}^G M_p$, then $\mathbf{Z}_p\otimes M$ is a summand of a permutation $\mathbf{Z}_pG$-lattice.

Let L be an indecomposable summand of $\mathbf{Z}_p\otimes M$. Denote its vertex by $1\times D\subseteq G_{p'}\times G_p$; then L is isomorphic to a summand of $\operatorname{ind}_{1\times D}^{G_{p'}\times G_p}\mathbf{Z}_p\cong\mathbf{Z}_pG_{p'}\otimes\operatorname{ind}_D^{G_p}\mathbf{Z}_p$. Write $\mathbf{Z}_pG_{p'}=\bigoplus X_i$ as a direct sum of (projective) indecomposables. Then L is a summand of $X_i\otimes\operatorname{ind}_D^{G_p}\mathbf{Z}_p$ for some i. We claim that $Y=X_i\otimes\operatorname{ind}_D^{G_p}\mathbf{Z}_p$ is indecomposable. The G_p-fixed points of Y are $X_i\otimes\mathbf{Z}_p\cong X_i$, which is irreducible mod p as an $\mathbf{F}_pG_{p'}$-module, since p does not divide $|G_{p'}|$. If Y were decomposable, it would be decomposable mod p, say as $Z_1\oplus Z_2$, where Z_1,Z_2 are nonzero $\mathbf{F}_pG$-modules. The G_p-fixed points of each of Z_1,Z_2 are non-zero, since G_p is a p-group. This contradicts the irreduciblity of $(Y/pY)^{G_p}$ as $\mathbf{F}_pG_{p'}$-module. So Y is indeed indecomposable, and therefore L, which is a summand of Y, is Y itself. Thus $\mathbf{Z}_p\otimes M$ is isomorphic to a sum of modules of the form $X\otimes\operatorname{ind}_D^{G_p}\mathbf{Z}_p$, where X is a $\mathbf{Z}_pG_{p'}$-module. Since each D is of the form $[\sigma_p]$, it follows that

$$\mathbf{Z}_p\otimes M\cong\bigoplus_{\sigma_p\in\hom(\Gamma_p,\Pi_p)}X_{\sigma_p}\otimes\operatorname{ind}_{[\sigma_p]}^{G_p}\mathbf{Z}_p,\tag{3.1}$$

where each X_{σ_p} is a $\mathbf{Z}_pG_{p'}$-module. Let ξ_{σ_p} be the character of X_{σ_p}. Then χ has the form

$$\chi(g)=\sum_{\sigma_p}\xi_{\sigma_p}(g_{p'})\operatorname{ind}_{[\sigma_p]}^{G_p}1(g_p).$$

In other words, writing $R(G)=R(G_{p'})\otimes R(G_p)$, we have

$$\chi=\sum_{\sigma_p\in\Sigma_p}\xi_{\sigma_p}\otimes\operatorname{ind}_{[\sigma_p]}^{G_p}1,\quad \xi_{\sigma_p}\in R^+(G_{p'}).\tag{3.2}$$

Write ξ_{σ_p} as a sum of irreducible characters of $G_{p'}$:

$$\xi_{\sigma_p}=\sum_{\eta\in\operatorname{irr}(G_{p'})}b(\sigma_p,\eta)\eta,\qquad\text{with unique integers } b(\sigma_p,\eta)\geq 0.\tag{3.3}$$

Changing the order of summation, we have $\chi = \sum_\eta \eta \otimes \lambda_\eta$, where

$$\lambda_\eta = \sum_{\sigma_p} b(\sigma_p, \eta) \operatorname{ind}_{[\sigma_p]}^{G_p} 1 \in \mathcal{P}^+(\Gamma_p, \Pi_p). \tag{3.4}$$

It follows from Lemma 3.1 that $\chi \in \bigcap_p \mathcal{Q}_p^+$.

Now suppose that $\chi \in \bigcap_p \mathcal{Q}_p^+$; we will show that $\chi \in \mathcal{D}'$. From Lemma 3.2, $\chi \in \mathbf{Q} \otimes \mathcal{P}$; since characters in $\mathcal{P}$ are characters of permutation modules, we see that χ is rational valued.

Let p be a prime dividing $|G|$. Since χ is in $\mathcal{Q}_p^+$, write χ as

$$\chi = \sum_{\eta \in \operatorname{irr}(G_{p'})} \eta \otimes \lambda_\eta, \qquad \lambda_\eta \in \mathcal{P}^+(\Gamma_p, \Pi_p).$$

Let ζ be a primitive $|G|$-th root of unity; let $\mathcal{G}$ denote the Galois group of $\mathbf{Q}(\zeta)$ over $\mathbf{Q}$, and let $\mathcal{G}_p$ denote the Galois group of $\mathbf{Q}_p(\zeta)$ over $\mathbf{Q}_p$, where $\mathbf{Q}_p$ is the p-adic rationals. Since χ and λ_η are rational valued, we have

$$\chi = \chi^\omega = \sum_\eta \eta^\omega \otimes \lambda_\eta, \qquad \omega \in \mathcal{G}_p.$$

By uniqueness of the representation $\chi = \sum_\eta \eta \otimes \lambda_\eta$, it follows that $\lambda_\eta = \lambda_{\eta^\omega}$, $\omega \in \mathcal{G}_p$. Partition $\operatorname{irr}(G_{p'})$ into $\mathcal{G}_p$-orbits. For an orbit $\mathcal{O}$, λ_η is the same for all η in $\mathcal{O}$; call this common value $\lambda_{\mathcal{O}}$. Let $\tau_{\mathcal{O}}$ denote $\sum_{\eta \in \mathcal{O}} \eta$, which takes values in $\mathbf{Q}_p$. Then

$$\chi = \sum_{\mathcal{O}} \sum_{\eta \in \mathcal{O}} \eta \otimes \lambda_\eta = \sum_{\mathcal{O}} \sum_{\eta \in \mathcal{O}} \eta \otimes \lambda_{\mathcal{O}} = \sum_{\mathcal{O}} \tau_{\mathcal{O}} \otimes \lambda_{\mathcal{O}}. \tag{3.5}$$

We have each $\lambda_{\mathcal{O}} \in \mathcal{P}^+(\Gamma_p, \Pi_p)$, so we may choose non-negative integers $b(\sigma_p, \mathcal{O})$ such that $\lambda_{\mathcal{O}} = \sum_{\sigma_p} b(\sigma_p, \mathcal{O}) \operatorname{ind}_{[\sigma_p]}^{G_p} 1$. Let

$$\xi_{\sigma_p} = \sum_{\mathcal{O}} b(\sigma_p, \mathcal{O}) \tau_{\mathcal{O}}. \tag{3.6}$$

Then, changing the order of summation, we have

$$\chi = \sum_{\mathcal{O}} \tau_{\mathcal{O}} \otimes \lambda_{\mathcal{O}} = \sum_{\mathcal{O}} \sum_{\sigma_p} \tau_{\mathcal{O}} \otimes b(\sigma_p, \mathcal{O}) \operatorname{ind}_{[\sigma_p]}^{G_p} 1 = \sum_{\sigma_p} \xi_{\sigma_p} \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1.$$

Now ξ_{σ_p} is a $\mathbf{Z}_{\geq 0}$-linear combination of $\{\tau_{\mathcal{O}}\}$, and each $\tau_{\mathcal{O}} = \sum_{\eta \in \mathcal{O}} \eta$. Fix $\eta \in \operatorname{irr}(G_{p'})$ in an orbit $\mathcal{O}$. Since the order of $G_{p'}$ is not divisible by p, the Schur index of η over the field $\mathbf{Q}_p(\eta)$ is 1, by [F, IV.9.5]. Let $K = \mathbf{Q}_p(\eta)$ and let X be a $KG_{p'}$-module affording η. By restriction of scalars, X is a $\mathbf{Q}_p G_{p'}$-module whose character is the sum of all the algebraic conjugates of η over $\mathbf{Q}_p$, namely $\tau_{\mathcal{O}}$. Thus $\tau_{\mathcal{O}}$ is the character of a $\mathbf{Q}_p G_{p'}$-module; hence so is ξ_{σ_p}. Choose a $\mathbf{Z}_p G_{p'}$-lattice L_{σ_p} in this module, so ξ_{σ_p} is afforded by L_{σ_p}.

Next, define the $\mathbf{Z}_p G$-lattice $M(p)$ by

$$M(p) = \bigoplus_{\sigma_p} L_{\sigma_p} \otimes_{\mathbf{Z}_p} \operatorname{ind}_{[\sigma_p]}^{G_p} \mathbf{Z}_p. \tag{3.7}$$

We claim that

(a) the N-fixed points $M(p)^N$ have trivial G/N-action, and

(b) $\operatorname{res}_N M(p)$ is $\mathbf{Z}_p N$-free.

To prove (a), note that $M(p)$ has character χ which is in $\bigcap_p \mathcal{Q}_p^+$; by Lemma 3.2, $\chi \in \mathbf{Q} \otimes \mathcal{P}$. Since the N-fixed points of $\operatorname{ind}_{[\sigma]}^G \mathbf{Z}$ have trivial G/N-action, (a) holds.

We now prove (b). Since $G_p = N_p[\sigma_p]$ and $N_p \cap [\sigma_p] = 1$, it follows that $\operatorname{res}_{N_p} \operatorname{ind}_{[\sigma_p]}^{G_p} \mathbf{Z}_p \cong \mathbf{Z}_p N_p$ by Mackey decomposition. Since

$$\operatorname{res}_N M(p) \cong \bigoplus_{\sigma_p} (\operatorname{res}_{N_{p'}} L_{\sigma_p}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p N_p \cong \operatorname{res}_{N_{p'}} \Big(\bigoplus_{\sigma_p} L_{\sigma_p} \Big) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p N_p,$$

it suffices to show that $\operatorname{res}_{N_{p'}} \left(\bigoplus_{\sigma_p} L_{\sigma_p} \right)$ is free over $\mathbf{Z}_p N_{p'}$. Since p does not divide $|N_{p'}|$, we need only show that the character χ_p' of $\bigoplus_{\sigma_p} L_{\sigma_p}$ has the property that $\operatorname{res}_{N_{p'}} \chi_p'$ is a multiple of the character $\rho(N_{p'})$ of the regular representation of $N_{p'}$. Since $\chi = \sum_{\sigma_p} \xi_{\sigma_p} \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1$ relative to $R(G_{p'}) \otimes R(G_p)$ and since the degree of each $\operatorname{ind}_{[\sigma_p]}^{G_p} 1$ is $|G_p : [\sigma_p]| = |\Pi_p|$, then $\operatorname{res}_{G_{p'}} \chi = |\Pi_p| \chi_p'$; hence $\operatorname{res}_{N_{p'}} \chi_p'$ is a $\mathbf{Q}$-multiple of $\operatorname{res}_{N_{p'}} \chi$. Let y be a non-identity element of $N_{p'}$. Then y has the form $(1, \pi)$, so no conjugate of y lies in $[\sigma_p]$. Since $\chi \in \mathbf{Q} \otimes \mathcal{P}$ by Lemma 3.2, χ and therefore χ_p' both vanish on y. It follows that $\operatorname{res}_{N_{p'}} \chi_p'$ is a multiple of $\rho(N_{p'})$, as desired. This proves (b).

We next show that χ is afforded by a $\mathbf{Q}G$-module. As in equation (3.5), we have

$$\chi = \sum_{\mathcal{O}'} \tau_{\mathcal{O}'} \otimes \lambda_{\mathcal{O}'},$$

where $\tau_{\mathcal{O}'}$ is the orbit sum over an orbit of $\mathcal{G}$ acting on $\operatorname{irr}(G_{p'})$, and $\lambda_{\mathcal{O}'}$ is the common value of λ_η for all $\eta \in \mathcal{O}'$. Let $R_{\mathbf{Q}}^+(G)$ be the characters afforded by $\mathbf{Q}G$-modules.

Since $\tau_{\mathcal{O}'}$ is a rational valued character of $G_{p'}$, then $|G_{p'}| \tau_{\mathcal{O}'} \in R_{\mathbf{Q}}^+(G_{p'})$; since $\lambda_{\mathcal{O}'}$ is the character of a permutation module, then $|G_{p'}| \chi \in R_{\mathbf{Q}}^+(G)$. Varying over p, since the greatest common divisor of the $|G_{p'}|$ is 1, it follows that $\chi \in R_{\mathbf{Q}}^+(G)$.

Let V be a $\mathbf{Q}G$-module affording χ. For each prime p we have an isomorphism

$$\phi_p : \mathbf{Q}_p \otimes_{\mathbf{Q}} V \to \mathbf{Q}_p \otimes_{\mathbf{Z}_p} M(p).$$

For each p let

$$V(p) = \{ v \in V : \phi_p(1 \otimes v) \in 1 \otimes M(p) \}.$$

Then let $M = \bigcap_p V(p)$. From [R, 5.3] we see that M is a $\mathbf{Z}G$-lattice such that $\mathbf{Z}_p \otimes M \cong M(p)$. Then M affords χ, the fixed points M^N have trivial G/N-action, and $\operatorname{res}_N (\mathbf{Z}_p \otimes M) \cong \operatorname{res}_N M(p)$, so $\operatorname{res}_N M$ is locally free. Therefore χ is in $\mathcal{D}'$, as desired. This completes the proof. ☐

For later use, we record the following result, which is proved in the second paragraph of the proof of Theorem 3.3.

Proposition 3.4. *If M satisfies* (a) *and* (b′)*, then for each prime p, $\mathbf{Z}_p \otimes M$ is a summand of a permutation lattice for $\mathbf{Z}_p G$.*

4. The Lattice Spanned by $\mathcal{D}'$

In this section we show that the $\mathbf{Z}$-span $\mathbf{Z}\mathcal{D}'$ of $\mathcal{D}'$ is equal to $\bigcap_p \mathcal{Q}_p$. We also show that $\mathbf{Z}\mathcal{D}'$ is equal to $\mathcal{P}$, if Γ is cyclic. We do not know if this result holds for arbitrary Γ.

Proposition 4.1. *The* $\mathbf{Z}$*-span of* $\mathcal{D}'$ *is equal to* $\bigcap_p \mathcal{Q}_p$.

Proof. It follows from Theorem 3.3 that

$$\mathbf{Z}\mathcal{D}' \subseteq \bigcap_p \mathcal{Q}_p.$$

For the reverse inclusion, suppose that $\chi \in \bigcap_p \mathcal{Q}_p$. For a fixed p dividing $|G|$, write

$$\chi = \sum_{\eta \in \mathrm{irr}(G_{p'})} \eta \otimes \lambda_\eta \quad \text{for unique } \lambda_\eta \in \mathcal{P}(\Gamma_p, \Pi_p). \tag{4.1}$$

By Lemma 3.2, $\chi \in \mathbf{Q} \otimes \mathcal{P}$. For a given $\eta \in \mathrm{irr}(G_{p'})$, suppose that

$$\langle \eta, \mathrm{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \rangle_{G_{p'}} = 0 \text{ for all } \sigma_{p'} \in \mathrm{hom}(\Gamma_{p'}, \Pi_{p'}). \tag{4.2}$$

Then η is orthogonal to $\mathcal{P}(\Gamma_{p'}, \Pi_{p'})$, and hence

$$0 = \langle \chi, \eta \otimes \lambda_\eta \rangle_G = \langle \eta \otimes \lambda_\eta, \eta \otimes \lambda_\eta \rangle_G = \langle \eta, \eta \rangle_{G_{p'}} \langle \lambda_\eta, \lambda_\eta \rangle_{G_p}.$$

It follows that $\lambda_\eta = 0$. Thus in equation (4.1), we need only sum over $\eta \in \mathrm{irr}(G_{p'})$ for which (4.2) does not hold. For such an η, if $\langle \eta, \mathrm{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \rangle_{G_{p'}} \neq 0$, decompose $\mathrm{ind}_{[\sigma_{p'}]}^{G_{p'}} 1$, giving

$$-\eta = \sum \tilde{\eta} - \mathrm{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \tag{4.3}$$

for some $\tilde{\eta} \in \mathrm{irr}(G_{p'})$. Write $\lambda_\eta = \lambda'_\eta - \lambda''_\eta$ with $\lambda'_\eta, \lambda''_\eta \in \mathcal{P}^+(\Gamma_p, \Pi_p)$; we get

$$\chi = \sum \eta \otimes \lambda'_\eta + \sum (-\eta) \otimes \lambda''_\eta.$$

In this equation, replace $-\eta$ using equation (4.3). We get $\chi = \chi'_p - \xi_p$ with $\chi'_p \in \mathcal{Q}_p^+$, $\xi_p \in \mathcal{P}^+$. Let $\xi = \sum_{p||G|} \xi_p \in \mathcal{P}^+$; then $\chi + \xi = \chi'_p + \sum_{l \neq p} \xi_l$, which is in $\mathcal{Q}_p^+$ for all p. Thus

$$\chi = (\chi + \xi) - \xi$$

with $\chi + \xi \in \bigcap_p \mathcal{Q}_p^+$ and $\xi \in \mathcal{P}^+ \subseteq \bigcap_p \mathcal{Q}_p^+$. Since $\bigcap_p \mathcal{Q}_p^+ = \mathcal{D}'$ by Theorem 3.3, the result is proved. □

We next show that $\mathbf{Z}\mathcal{D}' = \mathcal{P}$ if Γ is cyclic. Let Σ be a complete set of homomorphisms from Γ to Π up to conjugacy in Π. Our proof that $\mathbf{Z}\mathcal{D}' = \mathcal{P}$ for cyclic Γ uses the next result, that $\{\mathrm{ind}_{[\sigma]}^G 1 : \sigma \in \Sigma\}$ is linearly independent over $\mathbf{Q}$ if Γ is cyclic. This lemma can fail if Γ is not cyclic, but it is possible that $\mathbf{Z}\mathcal{D}' = \mathcal{P}$ can be proved by some other method in the non-cyclic case.

Lemma 4.2. *The set* $\{\mathrm{ind}_{[\sigma]}^G 1 : \sigma \in \Sigma\}$ *is a basis of* $\mathbf{Q} \otimes \mathcal{P}$ *if* Γ *is cyclic.*

Proof. For $g = (\gamma, \pi) \in G$ and $\sigma \in \mathrm{hom}(\Gamma, \Pi)$, we have

$$\mathrm{ind}_{[\sigma]}^G 1(g) = \begin{cases} |C_\Pi(\pi)|, & \sigma(\gamma) \sim \pi, \\ 0, & \text{else}, \end{cases}$$

where $\sim$ denotes conjugacy in Π. If $\tau \in \mathrm{hom}(\Gamma, \Pi)$ then τ is conjugate to some $\sigma \in \Sigma$. From the formula for $\mathrm{ind}_{[\sigma]}^G 1(g)$ above, then $\mathrm{ind}_{[\tau]}^G 1 = \mathrm{ind}_{[\sigma]}^G 1$, so $\mathbf{Q} \otimes \mathcal{P}$ is spanned by $\{\mathrm{ind}_{[\sigma]}^G 1 : \sigma \in \Sigma\}$.

Suppose that $\sum_{\sigma\in\Sigma} a_\sigma \operatorname{ind}_{[\sigma]}^G 1 = 0$ with $a_\sigma \in \mathbf{Q}$. Given $\tau \in \Sigma$, let γ be a generator of Γ, and evaluate at $\bigl(\gamma, \tau(\gamma)\bigr)$. We get

$$\operatorname{ind}_{[\sigma]}^G 1\bigl(\gamma, \tau(\gamma)\bigr) = \begin{cases} |C_\Pi\bigl(\tau(\gamma)\bigr)|, & \sigma(\gamma) \sim \tau(\gamma), \\ 0, & \text{else.} \end{cases}$$

If τ and σ are distinct elements of Σ, then $\tau(\gamma)$ and $\sigma(\gamma)$ are not conjugate in Π, so $\operatorname{ind}_{[\sigma]}^G 1(\gamma, \tau(\gamma)) = 0$. It follows that $a_\tau = 0$, for all τ in Σ, and the result is proved. □

Proposition 4.3. *The* $\mathbf{Z}$*-span of* $\mathcal{D}'$ *is equal to* $\mathcal{P}$ *if* Γ *is cyclic.*

Proof. Since $\mathcal{P}^+ \subseteq \mathcal{D}'$ it suffices to show that $\mathcal{D}' \subseteq \mathcal{P}$. Suppose that $\chi \in \mathcal{D}'$. Then $\chi \in \mathbf{Q} \otimes \mathcal{P}$ from Theorem 3.3 and Lemma 3.2. Then $\chi = \sum_{\sigma\in\Sigma} a_\sigma \operatorname{ind}_{[\sigma]}^G 1$, where $a_\sigma \in \mathbf{Q}$, and by Lemma 4.2, the $a_\sigma \in \mathbf{Q}$ are unique. We must show that each $a_\sigma \in \mathbf{Z}$.

Since G is nilpotent, we pick Σ by picking complete sets $\Sigma_p \subseteq \hom(\Gamma_p, \Pi_p)$ up to conjugacy in Π_p, and then letting Σ be those homomorphisms whose restrictions to Γ_p are in Σ_p.

Fix a prime p dividing $|G|$. From Theorem 3.3, $\chi \in \mathcal{Q}_p(\Gamma, \Pi) = R(G_{p'}) \otimes \mathcal{P}(\Gamma_p, \Pi_p)$. For $\sigma \in \Sigma$, we have $\operatorname{ind}_{[\sigma]}^G 1 = \operatorname{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1$. Then

$$\chi = \sum_{\tau\in\Sigma_p} \Bigl(\sum_{\substack{\sigma\in\Sigma \\ \sigma_p = \tau}} a_\sigma \operatorname{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \Bigr) \otimes \operatorname{ind}_{[\tau]}^{G_p} 1.$$

But $\{\operatorname{ind}_{[\tau]}^{G_p} 1\}$ is a $\mathbf{Z}$-basis of $\mathcal{P}(\Gamma_p, \Pi_p)$, from Lemma 4.2, so it follows that

$$\phi_\tau = \sum_{\substack{\sigma\in\Sigma \\ \sigma_p = \tau}} a_\sigma \operatorname{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \in R(G_{p'})$$

for all $\tau \in \Sigma_p$. Let $\gamma_{p'}$ be a generator of $\Gamma_{p'}$. Evaluate ϕ_τ at $\bigl(\gamma_{p'}, \sigma_{p'}(\gamma'_p)\bigr)$. For the unique $\sigma \in \Sigma$ whose restriction to Γ_p is τ and whose restriction to $\Gamma_{p'}$ is $\sigma_{p'}$, we get, as in the proof of Lemma 4.1,

$$|C_{\Pi_{p'}}\bigl(\sigma_{p'}(\gamma_{p'})\bigr)| a_\sigma \in \mathbf{Z}$$

since the values of ϕ_τ are algebraic integers in $\mathbf{Q}$. Thus

$$|\Pi_{p'}| a_\sigma \in \mathbf{Z}, \quad \text{for all } \sigma \in \Sigma.$$

This holds for all p. Since the greatest common divisor of $|\Pi_{p'}|$ is 1, it follows that $a_\sigma \in \mathbf{Z}$ for all $\sigma \in \Sigma$, and the proof is complete. □

Here is an example showing that Lemma 4.2 can fail if Γ is not cyclic. Let Γ and Π each be the direct product $C_p \times C_p$ of cyclic groups of prime order. Consider the $p^4 \times p^4$ matrix whose rows are indexed by $\Sigma = \hom(\Gamma, \Pi)$ and columns by $G = \Gamma \times \Pi$, with (σ, g)-entry given by $\operatorname{ind}_{[\sigma]}^G 1(g)$. Since $\operatorname{ind}_{[\sigma]}^G 1(1, x) = 0$ if x is a nontrivial element of Π, then there is a column of zeros, and so the rows are linearly dependent.

5. Examples of $\mathcal{D} \neq \mathcal{P}^+$

In this section we produce examples of groups Γ, Π and modules M having character which is in $\mathcal{D}(\Gamma, \Pi)$ but not in $\mathcal{P}^+(\Gamma, \Pi)$.

Lemma 5.1. *For distinct primes p_1 and p_2, let $\Gamma = C_{p_1 p_2}$, the cyclic group of order $p_1 p_2$, and let $\Pi = C_{p_1 p_2} \times C_{p_1 p_2}$. Then $\mathcal{D}(\Gamma, \Pi) \neq \mathcal{P}^+(\Gamma, \Pi)$.*

Proof. As before, let $G = \Gamma \times \Pi$, $N = 1 \times \Pi$. Choose sets Σ_1, Σ_2 of homomorphisms $\sigma : \Gamma \to \Pi$ whose images $\sigma(\Gamma)$ are precisely the subgroups of Π of order p_1, respectively, p_2. Let $\Sigma = \Sigma_1 \cup \Sigma_2$. Let $\tau : \Gamma \to \Pi$ be the trivial map. The module M we will construct has character

$$\chi = -\operatorname{ind}_{[\tau]}^G 1 + \sum_{\sigma \in \Sigma} \operatorname{ind}_{[\sigma]}^G 1.$$

It will follow from our construction that $\chi \in \mathcal{D}(\Gamma, \Pi)$. Let γ be a generator of Γ, and let $g = (\gamma, 1) \in G$. Then $(\gamma, 1) \in [\tau]$, so $\operatorname{ind}_{[\tau]}^G 1(g) = |\Pi|$, but $(\gamma, 1) \notin [\sigma]$ for all $\sigma \in \Sigma$, so $\operatorname{ind}_{[\sigma]}^G 1(g) = 0$. Hence $\chi(g) < 0$ and χ is not the character of a permutation module, so $\chi \notin \mathcal{P}^+(\Gamma, \Pi)$.

For $\sigma \in \Sigma$, $M(\sigma)$ is the corresponding double-action module. Define

$$M(\Sigma) = \bigoplus_{\sigma \in \Sigma} M(\sigma).$$

For each $\sigma \in \Sigma$, let $s(\sigma) = \sum_{x \in \sigma(\Gamma)} x \in \mathbf{Z}\Pi$, and define the map

$$f_\sigma : M(\sigma) \to M(\tau), \quad f_\sigma(m) = s(\sigma)m, \quad m \in M(\sigma).$$

Define $f : M(\Sigma) \to M(\tau)$ by $f = \sum_\sigma f_\sigma$. The key to the proof is the claim that f is an epimorphism of $\mathbf{Z}G$-modules.

We now prove that the claim holds. Since $s(\sigma)\sigma(\gamma^{-1}) = s(\sigma) = \sigma(\gamma^{-1})s(\sigma)$, then each f_σ is a $\mathbf{Z}G$-homorphism, and so is f. To prove that f is surjective, it suffices to find $v \in M(\Sigma)$ such that $f(v) = 1 \in M(\tau)$. Pick two distinct elements ϕ_i, ψ_i from Σ_i, $i = 1, 2$. Find integers n_1, n_2 such that $n_1 p_1 + n_2 p_2 = 1$. Define $v = \sum_{\sigma \in \Sigma} v_\sigma$ with $v_\sigma \in M(\sigma)$ given by

$$v_\sigma = \begin{cases} n_i \cdot 1, & \sigma \in \Sigma_i, \quad \sigma \neq \phi_i, \\ n_i\big(1 - s(\psi_i)\big), & \sigma = \phi_i. \end{cases}$$

To compute $f(v)$,

$$f(v) = n_1 \sum_{\substack{\sigma \in \Sigma_1 \\ \sigma \neq \phi_1}} s(\sigma) + n_1 s(\phi_1)\big(1 - s(\psi_1)\big) + n_2 \sum_{\substack{\sigma \in \Sigma_2 \\ \sigma \neq \phi_2}} s(\sigma) + n_2 s(\phi_2)\big(1 - s(\psi_2)\big).$$

Since

$$s(\sigma)s(\tilde\sigma) = \widehat{\Pi}_{p_i}, \quad \sigma, \tilde\sigma \in \Sigma_i, \sigma \neq \tilde\sigma,$$

we get

$$f(v) = n_1\Big(\sum_{\sigma \in \Sigma_1} s(\sigma) - \widehat{\Pi}_{p_1}\Big) + n_2\Big(\sum_{\sigma \in \Sigma_2} s(\sigma) - \widehat{\Pi}_{p_2}\Big).$$

In the sum $\sum_{\sigma\in\Sigma_i} s(\sigma) \in \mathbf{Z}\Pi_{p_i}$, non-identity elements $y \in \Pi_{p_i}$ occur exactly once, whereas 1 occurs $p_i + 1$ times and

$$\sum_{\sigma\in\Sigma_i} s(\sigma) = p_i \cdot 1 + \widehat{\Pi}_{p_i}.$$

We obtain

$$f(v) = n_1 p_1 \cdot 1 + n_2 p_2 \cdot 1 = 1.$$

Therefore f is indeed a $\mathbf{Z}G$-epimorphism.

Now define M to be the kernel of $f : M(\Sigma) \to M(\tau)$, so we have the exact sequence

$$0 \to M \to M(\Sigma) \xrightarrow{f} M(\tau) \to 0.$$

Since $\operatorname{res}_N M(\sigma)$ is free for all σ, this sequence splits when restricted to N; hence $\operatorname{res}_N M$ is stably free. Since $\mathbf{Z}N$ satisfies the Eichler condition, it follows that $\operatorname{res}_N M$ is $\mathbf{Z}N$-free by Jacobinski's Cancellation Theorem [CR, 51.24]. (In fact, it can be shown directly that $\operatorname{res}_N M$ is $\mathbf{Z}N$-free by exhibiting a basis; this is done in a special case below.) Since G/N acts trivially on $M(\Sigma)^N$, it does so on M^N. The character χ of M is therefore in $\mathcal{D}$, and it follows from the exact sequence that $\chi = -\operatorname{ind}_{[\tau]}^G 1 + \sum_{\sigma\in\Sigma} \operatorname{ind}_{[\sigma]}^G 1$. This completes the proof. □

From Proposition 2.4, there exists $U \in SGL_{p_1p_2}(\mathbf{Z}\Pi)$ with $U^{p_1p_2} = 1$ and U not conjugate in $GL(\mathbf{Q}\Pi)$ to a diagonal matrix of group elements. We can exhibit such a matrix U by computing the action of a generator of Γ on an explicit $\mathbf{Z}N$-basis of the free module $\operatorname{res}_N M$. To simplify the exposition, we assume that

$$p_1 = 2, \qquad p_2 = 3.$$

Then pick

$$n_1 = -1, \qquad n_2 = 1.$$

Write $\Gamma = \langle c\rangle$ of order 6, $\Pi = \langle a\rangle \times \langle b\rangle$, where a and b each have order 6. Let

$$\Sigma_1 = \{\sigma_1, \sigma_2, \sigma_3\}, \quad \sigma_1(c) = a^3, \sigma_2(c) = b^3, \sigma_3(c) = a^3b^3,$$

$$\Sigma_2 = \{\sigma_4, \sigma_5, \sigma_6, \sigma_7\}, \quad \sigma_4(c) = a^2, \sigma_5(c) = b^2, \sigma_6(c) = a^2b^2, \sigma_7(c) = a^2b^4.$$

Pick

$$\psi_1 = \sigma_1, \quad \phi_1 = \sigma_3, \quad \psi_2 = \sigma_4, \quad \phi_2 = \sigma_7.$$

Define

$$s_i = s(\sigma_i), \qquad c_i = \sigma_i(c).$$

Identify $M(\Sigma)$ with $\mathbf{Z}\Pi^7$ and $M(\tau)$ with $\mathbf{Z}\Pi$, Then our map f takes $\mathbf{Z}\Pi^7$ to $\mathbf{Z}\Pi$, given by

$$f(z_1, \dots, z_7)^T = \sum_{i=1}^{7} s_i z_i.$$

The G-action on $\mathbf{Z}\Pi^7$ becomes

$$(z_1, \dots, z_7)^T(x, y) = (\sigma_1(x^{-1})z_1 y \cdots, \sigma_7(x^{-1})z_7 y)^T.$$

(Transposes are used because $M(\Sigma)$ is a right $\mathbf{Z}\Pi$-module and endomorphisms are matrices over $\mathbf{Z}\Pi$ acting on the left.) In this notation the element v in the proof of Lemma 5.1 is

$$v = (-1, -1, s_1 - 1, 1, 1, 1, 1 - s_4)^T.$$

Let $\{e_1, e_2, \dots, e_7\}$ be the standard basis of $\mathbf{Z}\Pi^7$. Extend the "unimodular column" v to a $\mathbf{Z}\Pi$-basis $\{v, e_2, \dots, e_7\}$ of $\mathbf{Z}\Pi^7$. Then do "elementary operations" by setting

$$m_i = e_{i+1} - vf(e_{i+1}) = e_{i+1} - vs_{i+1}, \quad 1 \le i \le 6,$$

and then $\{v, m_1, m_2, \dots, m_6\}$ is a $\mathbf{Z}\Pi$-basis of $\mathbf{Z}\Pi^7$. The m_i are in M by construction, so $\{m_1, \dots, m_6\}$ is a $\mathbf{Z}\Pi$-basis of M.

We want the action of a generator of Γ on M in this basis. The matrix of $(c^{-1}, 1)$ in the standard basis of $\mathbf{Z}\Pi^7$ is $D = \operatorname{diag}(c_1, \dots, c_7)$. So we need only see the effect of two changes of basis. Set

$$A = \begin{pmatrix} -1 & & & & & & \\ -1 & 1 & & & & & \\ s_1 - 1 & & 1 & & & & \\ 1 & & & 1 & & & \\ 1 & & & & 1 & & \\ 1 & & & & & 1 & \\ 1 - s_4 & & & & & & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -s_2 & -s_3 & -s_4 & -s_5 & -s_6 & -s_7 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}.$$

The action of $(c^{-1}, 1)$ in the basis $\{v, m_1, m_2, \dots, m_6\}$ is given by the $\mathbf{Z}\Pi$-matrix $X = B^{-1}A^{-1}DAB$. Since M is a G-submodule of $\mathbf{Z}\Pi^7$, this matrix has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & & & & & & \\ * & & & & & & \\ * & & & U & & & \\ * & & & & & & \\ * & & & & & & \\ * & & & & & & \end{pmatrix}$$

for some $U \in GL_6(\mathbf{Z}\Pi)$, and it is this U that we want. Using $c_i s_i = s_i$, and denoting $c_{ij} = c_i - c_j$, $s_1^* = s_1 - 1$, $s_4^* = s_4 - 1$, we get $U =$

$$\begin{pmatrix} c_2 + c_{21}s_2 & c_{21}s_3 & c_{21}s_4 & c_{21}s_5 & c_{21}s_6 & c_{21}s_7 \\ c_{13}s_1^*s_2 & c_3 + c_{13}s_1^*s_3 & c_{13}s_1^*s_4 & c_{13}s_1^*s_5 & c_{13}s_1^*s_6 & c_{13}s_1^*s_7 \\ c_{14}s_2 & c_{14}s_3 & c_4 + c_{14}s_4 & c_{14}s_5 & c_{14}s_6 & c_{14}s_7 \\ c_{15}s_2 & c_{15}s_3 & c_{15}s_4 & c_5 + c_{15}s_5 & c_{15}s_6 & c_{15}s_7 \\ c_{16}s_2 & c_{16}s_3 & c_{16}s_4 & c_{16}s_5 & c_6 + c_{16}s_6 & c_{16}s_7 \\ c_{71}s_4^*s_2 & c_{71}s_4^*s_3 & c_{71}s_4^*s_4 & c_{71}s_4^*s_5 & c_{71}s_4^*s_6 & c_7 + c_{71}s_4^*s_7 \end{pmatrix}.$$

Note that this matrix has trace $-1 + \sum_{i=1}^{7} c_i$, so it is a counterexample to the strategy of [MRSW]. Actually every $\chi \in \mathcal{D}' - \mathcal{P}^+$ gives such a counterexample, by Lemma 1 of [WCr] generalized to matrices.

Lemma 5.2. *Let p be an odd prime, and let $\Gamma = C_4 \times C_p$ and $\Pi = Q_8 \times C_p \times C_p$, where Q_8 is the quaternion group of order 8. Then $\mathcal{D}(\Gamma, \Pi) \neq \mathcal{P}^+(\Gamma, \Pi)$.*

Proof. Pick $\tau \in \hom(\Gamma, \Pi)$ whose image is one of the cyclic subgroups of order 4 of Q_8. Choose Σ_2 to consist of three homomorphisms $\Gamma \to \Pi$ whose images are the two other subgroups of order 4 in Q_8 as well as the subgroup of order 2. Choose Σ_p to consist of $p+1$ elements of $\hom(\Gamma, \Pi)$ so that $\operatorname{im}\sigma_p$ are the nontrivial cyclic p-subgroups and so that $\sigma_2 = \tau_2$. Set $\Sigma = \Sigma_2 \cup \Sigma_p$, and

$$\chi' = -\operatorname{ind}_{[\tau]}^{G} 1 + \sum_{\sigma \in \Sigma} \operatorname{ind}_{[\sigma]}^{G} 1.$$

Relative to $R(G) = R(G_p) \otimes R(G_2)$,

$$\chi' = \left(-\operatorname{ind}_{[1]}^{G_p} 1 + \sum_{\sigma \in \Sigma_p} \operatorname{ind}_{[\sigma_p]}^{G_p} 1\right) \otimes \operatorname{ind}_{[\tau_2]}^{G_2} 1 + \sum_{\sigma \in \Sigma_2} \operatorname{ind}_{[1]}^{G_p} 1 \otimes \operatorname{ind}_{[\sigma_2]}^{G_2} 1,$$

where the expression in parentheses is in $R^+(G_p)$. So $\chi \in \mathcal{Q}_2^+$. Relative to $R(G) = R(G_2) \otimes R(G_p)$ we have

$$\chi' = \left(-\operatorname{ind}_{[\tau_2]}^{G_2} 1 + \sum_{\sigma \in \Sigma_2} \operatorname{ind}_{[\sigma_2]}^{G_2}\right) \otimes \operatorname{ind}_{[1]}^{G_p} 1 + \sum_{\sigma \in \Sigma_p} \operatorname{ind}_{[\sigma_p]}^{G_p} 1 \otimes \operatorname{ind}_{[\tau_2]}^{G_2} 1,$$

where the expression in parentheses is in $R^+(G_2)$. Therefore $\chi \in \mathcal{Q}_p^+$.

By Theorem 3.3, $\chi' \in \mathcal{D}'$; hence $\chi = r\chi' \in \mathcal{D}$ by Lemma 2.5. We shall show that $\chi \notin \mathcal{P}^+$. By Lemma 4.2, it is enough to check that τ is not Π-conjugate to any $\sigma \in \Sigma$. But the image of τ, which is normal in Π, is different from the images of all $\sigma \in \Sigma$. This completes the proof.

□

6. On Problem 1

In this section we prove that $\mathcal{D}' = \mathcal{P}^+$ if Π is nilpotent and Γ has prime-prower order. Since $\mathcal{P}^+ \subseteq \mathcal{D} \subseteq \mathcal{D}'$, then Problem 1 has a positive answer in this case. We also completely deal with Problem 1 if Γ is cyclic.

Theorem 6.1. *Suppose that* Γ *is an* l*-group for some prime* l *and that* Π *is nilpotent. Then* $\mathcal{D}'(\Gamma, \Pi) = \mathcal{P}^+(\Gamma, \Pi)$.

Proof. Suppose that $\chi \in \mathcal{D}'$. For each prime $p \neq l$, use Theorem 3.3 to write χ relative to $R(G_{p'}) \otimes R(G_p)$ as $\chi = \sum_{\sigma_p} \xi_{\sigma_p} \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1$. Since $\Gamma_p = 1$, the only $\sigma_p : \Gamma_p \to \Pi_p$ in this sum is the trivial map; hence $\operatorname{ind}_{[\sigma_p]}^{G_p} 1$ is the character $\rho(G_p)$ of the regular representation and $\chi = \xi_{\sigma_p} \otimes \rho(G_p)$. In particular, χ vanishes off $G_{p'}$. Varying $p \neq l$, it follows that χ vanishes off G_l.

Define the class function λ on G_l by $\lambda(g) = \chi(g)/|G_{l'}|$. Then relative to $R(G) = R(G_{l'}) \otimes R(G_l)$ (actually with scalars extended to $\mathbf{Q}$) we have $\chi = \rho(G_{l'}) \otimes \lambda$.

At the prime l write $\chi = \sum_{\eta \in \operatorname{irr}(G_{l'})} \eta \otimes \chi_\eta$ with $\chi_\eta \in \mathcal{P}^+(\Gamma_l, \Pi_l)$. Since $\rho(G_{l'}) = \sum_{\eta \in \operatorname{irr}(G_{l'})} \eta(1)\eta$, we get

$$\chi = \rho(G_{l'}) \otimes \lambda = \sum_{\eta} \eta \otimes \eta(1)\lambda = \sum_{\eta} \eta \otimes \chi_\eta,$$

so we deduce that $\chi_\eta = \eta(1)\lambda$ for all $\eta \in \operatorname{irr}(G_{l'})$. Take η to be the trivial character; then $\lambda = \chi_1 \in \mathcal{P}^+(\Gamma_l, \Pi_l)$, and we can write $\lambda = \sum_{\sigma_l} a_{\sigma_l} \operatorname{ind}_{[\sigma_l]}^{G_l} 1$, where each a_{σ_l} is a non-negative integer.

Since $\rho(G_{l'}) \otimes \operatorname{ind}_{[\sigma_l]}^{G_l} 1 = \operatorname{ind}_{[\sigma_l]}^{G} 1$, we have

$$\chi = \rho(G_{l'}) \otimes \lambda = \sum_{\sigma_l} a_{\sigma_l} \rho(G_{l'}) \otimes \operatorname{ind}_{[\sigma_l]}^{G_l} 1 = \sum_{\sigma_l} a_{\sigma_l} \operatorname{ind}_{[\sigma_l]}^{G} 1.$$

Thus $\chi \in \mathcal{P}^+(\Gamma, \Pi)$, and the proof is complete. □

Corollary 6.2. *Γ is a subgroup of $SGL_n(\mathbf{Z}\Pi)$ if and only if Γ is isomorphic to a subgroup of Π^n (the direct product of n copies of Π.)*

Proof. Suppose that $\Gamma \subseteq SGL_n(\mathbf{Z}\Pi)$. From Corollary 2.7, Γ is nilpotent, so in order to prove that Γ is a subgroup of Π^n it suffices to prove that $\Gamma_l \subseteq \Pi^n$ for each prime l dividing Γ. Hence we may assume that Γ is an l-group. Then Theorem 6.1 implies that $\mathcal{D}(\Gamma, \Pi) = \mathcal{P}^+(\Gamma, \Pi)$, and then from Proposition 2.3, $u\Gamma u^{-1} \subseteq \Pi^n$.

The converse is clear. □

Theorem 6.3. *Suppose that Π is nilpotent. Then $\mathcal{D} = \mathcal{P}^+$ for all cyclic Γ if and only if Π has at most one non-cyclic Sylow p-subgroup.*

Proof. Suppose that Π has at most one non-cyclic Sylow p-subgroup. We will show that $\mathcal{D}' = \mathcal{P}^+$, and therefore that $\mathcal{D} = \mathcal{P}^+$. Fix a prime p, which exists by hypothesis, such that $\Pi_{p'}$ is cyclic, and therefore has a faithful character λ of degree 1. Let χ be in $\mathcal{D}'$. Choose Σ as in the proof of Proposition 4.3, namely

$$\Sigma = \{\sigma \in \hom(\Gamma, \Pi) : \sigma_p \in \Sigma_p\},$$

where $\Sigma_p \subset \hom(\Gamma_p, \Pi_p)$ is a complete set of homomorphisms up to conjugacy in Π_p. By Proposition 4.3, $\chi \in \mathcal{P}$, and since Γ is cyclic, we may write, by Lemma 4.2,

$$\chi = \sum_{\sigma \in \Sigma} a_\sigma \operatorname{ind}_{[\sigma]}^{G} 1 \quad \text{for unique } a_\sigma \in \mathbf{Z}.$$

We must show that $a_\sigma \geq 0$ for all σ.

Relative to $R(G) = R(G_{p'}) \otimes R(G_p)$, we have $\operatorname{ind}_{[\sigma]}^{G} 1 = \operatorname{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1$, giving

$$\chi = \sum_{\sigma} a_\sigma \operatorname{ind}_{[\sigma_{p'}]}^{G_{p'}} 1 \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1.$$

From equation (3.2), we have

$$\chi = \sum_{\sigma_p \in \Sigma_p} \xi_{\sigma_p} \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1, \quad \xi_{\sigma_p} \in R^+(G_{p'}).$$

Comparing these equations, using linear independence of $\{\operatorname{ind}_{[\sigma]}^{G} 1 : \sigma \in \Sigma\}$ from Lemma 4.2, we get

$$\xi_{\sigma_p} = \sum_{\tau} a_\tau \operatorname{ind}_{[\tau_{p'}]}^{G_{p'}} 1 \in R^+(G_{p'})$$

where the sum is over $\tau \in \Sigma$ such that $\tau_p = \sigma_p$. Then $\langle \xi_{\sigma_p}, \eta \rangle_{G_{p'}} \geq 0$ for all irreducible characters η of $G_{p'} = \Gamma_{p'} \times \Pi_{p'}$, in particular for $\eta = \lambda^* \sigma_{p'} \otimes \lambda$, where λ^* is the contragredient of λ. But

$$\begin{aligned}\langle \xi_{\sigma_p}, \lambda^* \sigma_{p'} \otimes \lambda \rangle &= \sum_{\tau} a_\tau \langle 1, \operatorname{res}_{[\tau_{p'}]}^{G_{p'}} (\lambda^* \sigma_{p'} \otimes \lambda) \rangle_{[\tau_{p'}]} \\ &= \sum_{\tau} a_\tau / |\Gamma_{p'}| \sum_{\gamma \in \Gamma_{p'}} \lambda(\sigma_{p'} \gamma^{-1}) \lambda(\tau_{p'} \gamma) = \sum_{\tau} a_\tau \langle \lambda \sigma_{p'}, \lambda \tau_{p'} \rangle_{\Gamma_{p'}},\end{aligned}$$

and this equals a_σ since $\lambda\sigma_{p'}$ and $\lambda\tau_{p'}$ are different irreducible characters of $\Gamma_{p'}$ unless $\sigma_{p'} = \tau_{p'}$, that is, $\tau = \sigma$. Thus $a_\sigma \geq 0$.

Conversely, suppose that Π has at least 2 non-cyclic Sylow p-subgroups. First suppose that Π has a subgroup of the form $\Pi' = C_{p_1} \times C_{p_1} \times C_{p_2} \times C_{p_2} \cong C_{p_1p_2} \times C_{p_1p_2}$, where p_1, p_2 are distinct primes. Using the construction in Lemma 5.1 with $\Gamma = C_{p_1p_2}$, $G' = \Gamma \times \Pi'$, $N' = 1 \times \Pi'$, there exists a $\mathbf{Z}G'$-lattice M' satisfying (a) and (b) whose character is $\chi' = -\operatorname{ind}_{[\tau]}^{G'} 1 + \sum_{\sigma\in\Sigma'} \operatorname{ind}_{[\sigma]}^{G'} 1$. By Lemma 2.4, $\chi = \operatorname{ind}_{G'}^{G} \chi'$ is in $\mathcal{D}(\Gamma, \Pi)$. But $\chi = -\operatorname{ind}_{[\tau]}^{G} 1 + \sum_{\sigma\in\Sigma'} \operatorname{ind}_{[\sigma]}^{G} 1$ is not in $\mathcal{P}^+$ by Lemma 4.2, since $\tau = 1$ is not Π-conjugate to an element of Σ'. Thus $\mathcal{D}(\Gamma, \Pi) \neq \mathcal{P}^+(\Gamma, \Pi)$.

If Π has at least 2 non-cyclic Sylow p-subgroups but does not have a subgroup isomorphic to $C_{p_1} \times C_{p_1} \times C_{p_2} \times C_{p_2}$, where p_1, p_2 are distinct primes, then Π_2 is a quaternion group, and Π has a subgroup of the form $\Pi' = C_p \times C_p \times Q_8$ where p is an odd prime. Apply the construction of Lemma 5.2, where $\Gamma = C_p \times C_4 \cong C_{4p}$, $G' = \Gamma \times \Pi'$, $N' = 1 \times \Pi'$, to get

$$\chi' = r(-\operatorname{ind}_{[\tau]}^{G'} 1 + \sum_{\sigma\in\Sigma'} \operatorname{ind}_{[\sigma]}^{G'} 1) \in \mathcal{D}(\Gamma, \Pi') \text{ for some } r \geq 1.$$

As above, $\chi = \operatorname{ind}_{G'}^{G} \chi'$ is in $\mathcal{D}(\Gamma, \Pi)$ by Lemma 2.4, but to show that $\chi \notin \mathcal{P}^+$ we must be careful in our choice of τ. Let A be a cyclic normal subgroup of index 2 in Π_2 and choose τ whose image is $A \cap \Pi'$. Then the construction of Lemma 5.2 applies, since $\operatorname{im}\tau$, which is normal in Π, is not Π-conjugate to any $\operatorname{im}\sigma$ with $\sigma \in \Sigma'$. Applying Lemma 4.2 as before completes the proof. □

7. Finite generation of $\mathcal{D}$

Theorem 7.1. *If Π is nilpotent, then $\mathcal{D}'$ and $\mathcal{D}$ are finitely generated semigroups.*

Proof. Set

$$X = \bigoplus_{H \leq G} \operatorname{ind}_H^G \mathbf{Z}, \quad \text{where } H \text{ varies over all subgroups of } G.$$

For each prime p dividing $|G|$, enumerate the distinct non-isomorphic indecomposable summands of $\mathbf{Z}_p \otimes X$: suppose they are $X(p,i)$, $1 \leq i \leq n_p$, and suppose that $X(p,i)$ affords the character $\chi(p,i)$ of G. Let $n = \sum n_p$, summed over primes dividing $|G|$. We shall use some ideas in the proof of a result of Jones [CR, 33.2]. Following some of the notation of [CR], let C denote the additive semigroup of n-tuples of non-negative integers; partially order C by writing $(a_i) \leq (b_i)$ in C if $a_i \leq b_i$ for $1 \leq i \leq n$. If the $\mathbf{Z}G$-lattice M satisfies (a) and (b′), then by Proposition 3.4 and the Krull-Schmidt Theorem for $\mathbf{Z}_pG$-lattices, $\mathbf{Z}_p \otimes M$ can be written uniquely as a direct sum of modules $X(p,i)$:

$$\mathbf{Z}_p \otimes M \cong \bigoplus_{1 \leq i \leq n_p} X(p,i)^{m(p,i)}, \quad \text{for unique non-negative integers } m(p,i).$$

Let $\theta(M)$ denote the ordered n-tuple in C whose entries are the integers $m(p,i)$. Let $\widehat{\mathcal{D}}'$ be the set of n-tuples $\theta(M) \in C$ where M ranges over all $\mathbf{Z}G$-lattices which satisfy (a) and (b′); similarly, let $\widehat{\mathcal{D}}$ be the set of $\theta(M) \in C$ where M satisfies (a) and (b). Given $\theta(M) = (a(p,i))$ in $\widehat{\mathcal{D}}$, we associate to $\theta(M)$ the character $\sum_p a(p,i)\chi(p,i)$; this gives us a mapping of $\widehat{\mathcal{D}}$ onto $\mathcal{D}$. Similarly we have a mapping of $\widehat{\mathcal{D}}'$ onto $\mathcal{D}'$. Thus the theorem will follow if we can prove finite generation of $\widehat{\mathcal{D}}'$ and $\widehat{\mathcal{D}}$.

We first prove that $\widehat{\mathcal{D}}'$ is finitely generated. From Step 3 in the proof of Jones' Theorem [CR, p. 689], any subset of C has a finite set of minimal elements in the partial order we have given C. Let S be the finite set of minimal elements of $\widehat{\mathcal{D}}' - \{0\}$. We claim that this set S generates $\widehat{\mathcal{D}}'$. To prove this, let $\theta(M)$ be an element of $\widehat{\mathcal{D}}'$; we shall show that $\theta(M)$ is a sum of elements of S. This is true if $\theta(M) \in S$, so assume that there is an element $s \in S$ which is strictly smaller than $\theta(M)$. Also assume that if L is a $\mathbf{Z}G$-lattice such that $\theta(L) \in \widehat{\mathcal{D}}'$ and which has smaller $\mathbf{Z}$-rank than M, then $\theta(L)$ is a sum of elements of S. Suppose that $s = \theta(M')$. Then locally at each prime, M' is a direct summand of M, so [CR, 31.12], there is a lattice M'' in the same genus as M' such that $M \cong M'' \oplus M_0$ for some $\mathbf{Z}G$-lattice M_0. Then $\theta(M'') = \theta(M')$, and M_0 satisfies (a) and (b$'$). Moreover, by our assumption on lattices with ranks smaller than that of M, $\theta(M_0)$ is a sum of elements of S. Since $\theta(M) = s + \theta(M_0)$, then $\theta(M)$ is a sum of elements of S, as claimed.

We next show that $\widehat{\mathcal{D}}$ is finitely generated. Let S_0 be the set of $s \in S$ so that $s = \theta(M)$ for some M such that $\operatorname{res}_N M$ is stably free but $s = \theta(M')$ for no M' such that $\operatorname{res}_N M'$ is free. Let r be as in Lemma 2.5. Set

$$T = \left(\widehat{\mathcal{D}} \cap \left\{\sum_{s\in S} a_s s : 0 \le a_s \le r, s \in S\right\}\right) \cup \{(r+1)s : s \in S_0\}.$$

Note that if $\operatorname{res}_N M$ is stably free, then since $r+1 \ge 2$, $\operatorname{res}_N M^{r+1}$ is free by [CR, 41.20], so $T \subseteq \widehat{\mathcal{D}}$. We claim that T generates $\widehat{\mathcal{D}}$. Suppose that $d = \theta(D) \in \widehat{\mathcal{D}}$. As above, we assume that if $\theta(L) \in \widehat{\mathcal{D}}$ and L has smaller $\mathbf{Z}$-rank than D, then $\theta(L)$ is a sum of elements of T. Since $\widehat{\mathcal{D}}'$ is generated by S, we can write

$$d = \sum_{s\in I} a_s s \text{ for a subset } I \subseteq S \text{ with } a_s \ge 1, s \in I.$$

Write $a_s = b_s + rc_s$ where $1 \le b_s \le r$ and $c_s \ge 0$, and set

$$e = \sum_{s\in I} b_s s, \quad f = r\sum_{s\in I} c_s s,$$

so $d = e + f$. We have $d \in \widehat{\mathcal{D}}$, and since $r\mathcal{D}' \subseteq \mathcal{D}$, then $f \in \widehat{\mathcal{D}}$, so we can find $\mathbf{Z}G$ lattices D and F satisfying (a) and (b) with $d = \theta(D)$ and $f = \theta(F)$. Also, $e = \theta(E')$ for a lattice E' satisfying (a) and (b$'$). Now $\theta(D) = \theta(E' \oplus F)$, so D and $E' \oplus F$ are in the same genus, and locally for all p, F is a direct summand of D.

We will apply a result of Roiter and Jacobinski [CR, 31.32]; we must check that every irreducible $\mathbf{Q}G$-composition factor of $\mathbf{Q}F$ occurs more often as a composition factor of $\mathbf{Q}D$. This is so because $d = e + f$ and $e = \sum_{s\in I} b_s s$, where $b_s > 0$ for $s \in I$. Thus $D \cong E \oplus F$ for some $\mathbf{Z}G$-lattice E. Restricting to N, we see that $\operatorname{res}_N E$ is stably free. If it is actually free (so, in particular, if we have the Eichler condition for $\mathbf{Q}N$), then $e = \theta(E) \in T$, and we are done.

From [CR, 41.20], if $\sum_{s\in I} b_s > 1$, then $\operatorname{res}_N E$ is free; so we may assume that $\sum_{s\in I} b_s = 1$. Thus I contains a single element s_0, and $e = s_0$. If $s_0 \notin S_0$, then writing $s_0 = \theta(E')$ where E' satisfies (a) and (b), it follows that D is in the same genus as $E' \oplus F$; we replace D by $E' \oplus F$ and we are done as before. So we assume that $s_0 \in S_0$.

Suppose that $f = 0$. Then $d = s_0 \notin \widehat{\mathcal{D}}$. Hence $f \neq 0$. Then $c_s \geq 1$ for some $s \in I$; since $I = \{s_0\}$, then $f = c_{s_0} r s_0$ with $c_{s_0} \geq 1$. Then $e = (r+1)s_0 + (c_{s_0} - 1)(rs_0)$ with $(r+1)s_0 \in T$ and $(c_{s_0} - 1)(rs_0) = \theta(L) \in \widehat{\mathcal{D}}$, where L has smaller rank than D. Thus d is indeed a sum of elements of T, and the proof is complete. □

8. Complements

Assume we have Γ, Π and $G = \Gamma \times \Pi$ as above. As in the proof of Theorem 3.3, $\mathcal{G}$ is the Galois group of $\mathbf{Q}(\zeta)$ over $\mathbf{Q}$, where ζ be a primitive $|G|$-th root of unity, and $\mathcal{G}_p$ denotes the Galois group of $\mathbf{Q}_p(\zeta)$ over $\mathbf{Q}_p$. We identify $\mathcal{G}_p$ as a subgroup of $\mathcal{G}$, namely the decomposition group at any prime of $\mathbf{Q}(\zeta)$ above p. Let Σ_p be a complete set of elements of $\hom(\Gamma_p, \Pi_p)$ up to conjugation by Π_p. As in [WCr], define a *label* for χ to be a collection $\mathbf{b} = \{b_p\}$ of functions

$$b_p : \Sigma_p \times \operatorname{irr}(G_{p'}) \to \mathbf{Z}_{\geq 0},$$

one for each prime p, so that on writing

$$\chi = \sum_{\eta \in \operatorname{irr}(G_{p'})} \eta \otimes \lambda_\eta \text{ relative to } R(G) = R(G_{p'}) \otimes R(G_p)$$

we have

$$\text{(i)}_p \quad \lambda_\eta = \sum_{\sigma_p \in \Sigma_p} b_p(\sigma_p, \eta) \operatorname{ind}_{[\sigma_p]}^{G_p} 1,$$

$$\text{(ii)}_p \quad b_p(\sigma_p, \eta^\omega) = b_p(\sigma_p, \eta) \text{ for all } \omega \in \mathcal{G}_p.$$

Theorem 8.1. *Suppose that* Γ *and* Π *are nilpotent. Then labels for* χ *are in bijection with genera of* $\mathbf{Z}G$*-lattices with character* χ *which satisfy* (a) *and* (b′).

Proof. The proof comes from a closer look at the proof of Theorem 3.3. Suppose that M is a $\mathbf{Z}G$-lattice which satisfies (a) and (b′). From equation (3.1), we have

$$\mathbf{Z}_p \otimes M \cong \bigoplus_{\sigma_p \in \hom(\Gamma_p, \Pi_p)} X_{\sigma_p} \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} \mathbf{Z}_p.$$

Instead of summing over $\sigma_p \in \hom(\Gamma_p, \Pi_p)$, we may sum over Σ_p, because replacing σ_p by a Π_p-conjugate gives a G_p-conjugate of $[\sigma_p]$, hence a module isomorphic to $\operatorname{ind}_{[\sigma_p]}^{G_p} \mathbf{Z}_p$. Then the different groups $[\sigma_p]$, $\sigma_p \in \Sigma_p$, are the vertices of the summands of $\mathbf{Z}_p \otimes M$, so the modules X_{σ_p} are unique up to isomorphism, and their characters ξ_{σ_p} give the well-defined equation

$$\chi = \sum_{\sigma_p \in \Sigma_p} \xi_{\sigma_p} \otimes \operatorname{ind}_{[\sigma_p]}^{G_p} 1, \quad \xi_{\sigma_p} \in R^+(G_{p'}).$$

Then (ii)_p follows from equation (3.3) and (i)_p comes from equation (3.4). Since these depend only on $\mathbf{Z}_p \otimes M$, the same label would be attached to any lattice in the same genus as M.

Conversely, suppose that $\mathbf{b}$ is a label. From (ii)_p, $b(\sigma_p, \eta)$ just depends on the $\mathcal{G}_p$-orbit $\mathcal{O}$ containing η. Then as in equation (3.6), we let $\xi_{\sigma_p} = \sum_{\mathcal{O}} b(\sigma_p, \mathcal{O}) \tau_{\mathcal{O}}$, and $\xi_{\sigma_p} \in R^+_{\mathbf{Q}_p}(G_{p'})$. The lattice

$$M(b_p) = \bigoplus_{\sigma_p} L_{\sigma_p} \otimes_{\mathbf{Z}_p} \operatorname{ind}_{[\sigma_p]}^{G_p} \mathbf{Z}_p$$

in equation (3.7) satisfies the local versions (a_p) and (b_p) of (a) and (b), and has character χ by $(\mathrm{i})_p$. Then the $\mathbf{Z}G$-lattice $M = M(\mathbf{b})$ at the end of the proof has $\mathbf{Z}_p \otimes M \cong M(b_p)$ for all p, hence it satisfies (a) and (b′); the construction of M depends on the identifications ϕ_p, but the genus of M is well-defined. □

Corollary 8.2. *If Γ is cyclic, there is only one genus of $\mathbf{Z}G$-modules having a given character in $\mathcal{D}'$.*

Proof. By Lemma 4.2, for each p the set $\{\mathrm{ind}_{[\sigma_p]}^{G_p} 1 : \sigma_p \in \Sigma_p\}$ is linearly independent; hence there is only one solution to $(\mathrm{i})_p$, and only one label for a character χ. □

Remarks. Given $\chi \in \mathcal{D}'$, we want to decide whether $\chi \in \mathcal{D}$. We begin by determining all labels $\mathbf{b}$ for χ: this is a purely character-theoretic problem. For each $\mathbf{b}$ one then constructs a lattice $M = M(\mathbf{b})$ in the genus of the label. Deciding whether the genus of M contains an M' with $\mathrm{res}_N M'$ stably free can then be approached by genus class group methods, generalizing Theorem 3 of [WCr]. Carrying this out is a long computation which will answer the existence question "Is χ in $\mathcal{D}$?" (at least when we have the Eichler condition for $\mathbf{Z}\Pi$). However, the construction of $\alpha : \Gamma \to SGL_n(\mathbf{Z}\Pi)$ with double-action character χ takes still more calculation.

Nevertheless this is how the first example of §5 was found. It is typical of such computations that once an M satisfying (a) and (b) is found, it is simpler to describe it directly, as we have done in §5.

Finally there is the issue of finding $\chi \in \mathcal{D}'$, no multiple of which is in $\mathcal{P}^+$, in the first place. This amounts to finding generators of $\mathcal{D}'$, and this again is a problem of character theory, by Theorem 3.3. In examples we have considered, $\mathcal{D}'$ has many generators, even when Γ is cyclic. More exploration of this, perhaps by computer, is still needed.

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