FINITE GROUPS OF MATRICES OVER GROUP RINGS

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Abstract. We investigate certain finite subgroups $\Gamma$ of $GL_n(\mathbb{Z}\Pi)$, where $\Pi$ is a finite nilpotent group. Such a group $\Gamma$ gives rise to a $\mathbb{Z}[\Gamma \times \Pi]$-module; we study the characters of these modules to limit the structure of $\Gamma$. We also exhibit some exotic subgroups $\Gamma$.

1. Introduction

Let $\Pi$ be a finite group. We set

$$SGL_n(\mathbb{Z}\Pi) = \ker \text{aug} : GL_n(\mathbb{Z}\Pi) \to GL_n(\mathbb{Z}),$$

where $\text{aug}$ is the usual augmentation map applied to each entry of $GL_n(\mathbb{Z}\Pi)$.

Suppose that $\alpha$ is a homomorphism from a finite group $\Gamma$ to $SGL_n(\mathbb{Z}\Pi)$. We shall investigate the following problem.

Problem 0. Do there exist group homomorphisms $\sigma_i : \Gamma \to \Pi$, $i = 1, 2, \ldots, n$, and an element $x \in GL_n(\mathbb{Q}\Pi)$ such that $x^{-1} \alpha(\gamma)x = \text{diag}(\sigma_i(\gamma))$, $\gamma \in \Gamma$?

This is analogous to a conjecture of Zassenhaus, who was interested in units of $\mathbb{Z}\Pi$ of augmentation 1, i.e. $SGL_1(\mathbb{Z}\Pi)$. Problem 0 is related to results on units of group rings, as shown in a special case in [MRSW]. There is a positive answer to Problem 0 if $\Pi$ is a $p$-group [WA1] or if $n = 1$ and $\Pi$ is nilpotent [WC].

Given finite groups $\Gamma$ and $\Pi$, set

$$G = \Gamma \times \Pi, \quad N = 1 \times \Pi.$$

A homomorphism $\alpha : \Gamma \to SGL_n(\mathbb{Z}\Pi)$ gives rise to a double action $\mathbb{Z}G$-module $M(\alpha)$, defined as follows: as abelian group, $M(\alpha)$ is equal to the column vectors $\mathbb{Z}\Pi^n$, and the $G$-action is given by

$$m \cdot (\gamma, \pi) = \alpha(\gamma^{-1})m\pi, \quad (\gamma, \pi) \in G, \quad m \in M.$$

There is a bijection between $GL_n(\mathbb{Z}\Pi)$-conjugacy classes of homomorphisms $\alpha$ and isomorphism classes of $\mathbb{Z}G$-lattices $M$ which satisfy

(a) $G/N$ acts trivially on the $N$-fixed points $M^N$, and

(b) $\text{res}_N^G M$ is a free $\mathbb{Z}N$-module.

(See [S, §38.6] for details.)
Problem 1. Is \( \mathcal{P} \) finitely generated? We will show that if \( \Pi \) is a \( p \)-group, then \( \mathcal{D} = \mathcal{P}^+ \). Indeed, it turns out (Proposition 2.3 below) that there is a positive answer to Problem 0 for finite groups \( \Gamma, \Pi \) if and only if \( \mathcal{D} = \mathcal{P}^+ \). Note that \( \mathcal{D} \) is closed under addition, and is a sub-semigroup of \( R^+(G) \).

Problem 0 can be reformulated as

Problem 2. Is the semigroup \( \mathcal{D} \) finitely generated?

We will show that this is indeed true if \( \Gamma \) and \( \Pi \) are nilpotent; this is in \( \S 7 \). If we know that \( \mathcal{D} \) is finitely generated, we are also interested in explicitly finding a generating set.

Analysis of \( \mathcal{D}(\Gamma, \Pi) \) is complicated by questions about locally free class groups, so we are led to replace (b) in the definition of \( \mathcal{D} \) by

\[(b') \text{ res}_N^M \text{ is a locally free } \mathbb{Z}N\text{-module.} \]

Definition. Let \( \mathcal{D}'(\Gamma, \Pi) \) be the set of characters of \( \mathbb{Z}G \)-modules \( M \) which satisfy (a) and (b').

The semigroup \( \mathcal{D}' \) is a good approximation to \( \mathcal{D} \), in the sense that \( r \mathcal{D}' \subseteq \mathcal{D} \subseteq \mathcal{D}' \) for some positive integer \( r \) (Lemma 2.5 below); in other words, if \( M \) is a \( \mathbb{Z}G \)-module satisfying (a) and (b'), then the direct sum \( M^r \) of \( r \) copies of \( M \) satisfies (a) and (b). We get first approximations to results about \( \mathcal{D} \) by considering \( \mathcal{D}' \) instead. All known cases of the equality \( \mathcal{D} = \mathcal{P}^+ \) actually have \( \mathcal{D}' = \mathcal{P}^+ \).

One of our main results, Theorem 3.3 below, is a complete character-theoretic description of \( \mathcal{D}' \) in the case that \( \Pi \) is nilpotent. To explain this we introduce some notation. For a finite nilpotent group \( N \), let \( N_p \) be its Sylow \( p \)-subgroup and \( N_{p'} \) its Sylow \( p' \)-complement. Then we have \( R(G) = R(G_{p'}) \otimes R(G_p) \), and we use this to define

\[ \mathcal{Q}_p(\Gamma, \Pi) = R(G_{p'}) \otimes \mathcal{P}(\Gamma_p, \Pi_p), \]

\[ \mathcal{Q}^+_p(\Gamma, \Pi) = \left\{ \sum \eta \otimes \lambda : \eta \in R^+(G_{p'}), \lambda \in \mathcal{P}^+(\Gamma_p, \Pi_p) \right\}. \]
We show in Theorem 3.3 below that if \( \Pi \) is nilpotent, then
\[
D^*(\Gamma, \Pi) = \bigcap_p Q^*_p(\Gamma, \Pi).
\]

Theorem 3.3 is crucial to most of the results in the rest of the paper.

We are also interested in knowing if Problem 1 has a “virtual” answer, that is, is the \( \mathbb{Z} \)-span of \( D \) equal to \( \mathcal{P}^* \)? This is discussed in §4.

Although the above definitions and problems are sensible for arbitrary finite groups \( \Gamma \) and \( \Pi \), we shall need to assume that these groups are nilpotent for most of our results. This is because we rely on the local results of [WAn], which at this point have no analogues in the general case. From §3 through to the end of the paper, we will assume that \( \Gamma \) and \( \Pi \) are nilpotent.

2. Preliminaries

As in the introduction, let \( G = \Gamma \times \Pi \) and \( N = 1 \times \Pi \). In this section, \( \Pi \) and \( \Gamma \) can be arbitrary finite groups.

**Lemma 2.1.** For \( \sigma \in \text{hom}(\Gamma, \Pi) \), the double action module \( M(\sigma) \) is isomorphic to \( \text{ind}^G_{\sigma}[\mathbb{Z}] \) as \( \mathbb{Z}G \)-modules.

**Proof.** By definition, \( M(\sigma) \) has \( \mathbb{Z} \)-basis \( \Pi \). Consider elements of \( \text{ind}^G_{\sigma}[\mathbb{Z}] \) as linear combinations of the cosets of \( [\sigma] \) in \( G \); use \( 1 \times \Pi \) as coset representatives. Define
\[
f : M(\sigma) \to \text{ind}^G_{\sigma}[\mathbb{Z}], \quad f(\pi) = [\sigma](1, \pi), \quad \pi \in \Pi.
\]
To check that \( f \) is a \( \mathbb{Z}G \)-homomorphism, for \( (\gamma, \pi') \in G, \)
\[
\begin{align*}
f(\pi \cdot (\gamma, \pi')) &= f(\sigma(\gamma^{-1})\pi\pi') = [\sigma](1, \sigma(\gamma^{-1})\pi\pi') \\
&= [\sigma](\gamma, \pi')(1, \sigma(\gamma^{-1})\pi\pi') = [\sigma](\gamma, \pi\pi') = f(\pi) \cdot (\gamma, \pi').
\end{align*}
\]
This proves the lemma. \( \square \)

**Corollary 2.2.** \( \mathcal{P}^* \subseteq D \).

**Proposition 2.3.** For a homomorphism \( \alpha : \Gamma \to SGL_n(\mathbb{Z}\Pi) \), the character \( \chi \) of the double action module \( \mathcal{M}(\alpha) \) is in \( \mathcal{P}^*(\Gamma, \Pi) \) if and only if there exist group homomorphisms \( \sigma_i \in \text{hom}(\Gamma, \Pi) \), \( 1 \leq i \leq n \), and an element \( u \in GL_n(\mathbb{Q}\Pi) \) such that \( u\alpha(\gamma)u^{-1} = \text{diag}(\sigma_1(\gamma), \ldots, \sigma_n(\gamma)) \) for all \( \gamma \in \Gamma \).

**Proof.** Suppose that \( \chi \) is in \( \mathcal{P}^*(\Gamma, \Pi) \). By Lemma 2.1 there are homomorphisms \( \sigma_i : \Gamma \to \Pi \), \( 1 \leq i \leq k \), such that there is a \( \mathbb{Q}G \)-isomorphism
\[
f : \mathbb{Q} \otimes \mathcal{M}(\alpha) \to \mathbb{Q} \otimes \left( \bigoplus_{i=1}^k \mathcal{M}(\sigma_i) \right)
\]
Comparing dimensions over \( \mathbb{Q} \) gives \( k = n \). Let \( \{e_j : 1 \leq j \leq n\} \) be the standard basis of the \( \mathbb{Q}\Pi \)-column vectors \( \mathbb{Q} \otimes \mathcal{M}(\alpha) \), and write elements of \( \mathbb{Q} \otimes \mathcal{M}(\sigma_i) \) as \( \langle x \rangle, \ x \in \mathbb{Q}\Pi \). Then \( \mathbb{Q} \otimes \left( \bigoplus_{i=1}^k \mathcal{M}(\sigma_i) \right) \) has \( \mathbb{Q}N \)-basis \( \{\langle 1 \rangle : 1 \leq i \leq n\} \) and, since \( f \) is a \( \mathbb{Q}N \)-homomorphism, we have
\[
f(e_j) = \sum_i \langle u_{ij} \rangle \text{ where } u \in GL_n(\mathbb{Q}\Pi).
\]
Act by $(\gamma^{-1}, 1)$, giving
\[ f(e_j)(\gamma^{-1}, 1) = \sum_i (u_{ij})_i(\gamma^{-1}, 1) = \sum_i \langle \sigma_i(\gamma)u_{ij} \rangle_i. \]
Since \( f(e_j)(\gamma^{-1}, 1) = f(e_j(\gamma^{-1}, 1)) \), this equals
\[ f(\alpha(\gamma)e_j) = f\left(\sum_k e_k\alpha(\gamma)_{kj}\right) = \sum_k f(e_k)(1, \alpha(\gamma)_{kj}) = \sum_k \langle u_{ik}\alpha(\gamma)_{kj} \rangle_i. \]
Therefore for all \( i, j \) we have
\[ \sigma_i(\gamma)u_{ij} = \sum_k u_{ik}\alpha(\gamma)_{kj} \quad \text{so, as matrix equation,} \quad \text{diag}(\sigma_i(\gamma))u = u\alpha(\gamma). \]
This implies that \( u\alpha(\gamma)u^{-1} = \text{diag}(\sigma_i(\gamma)). \)

For the converse, given \( u \in GL_n(\mathbb{Q}\Pi) \) such that \( u\alpha(\gamma)u^{-1} = \text{diag}(\sigma_i(\gamma)) \), define
\[ f : \mathbb{Q} \otimes M(\alpha) \to \mathbb{Q} \otimes \langle \oplus_i M(\sigma_i) \rangle \]
by \( f(e_j) = \sum_i (u_{ij})_i \). It follows as above that \( f \) is a \( \mathbb{Q}G \)-isomorphism. \hfill \( \Box \)

**Lemma 2.4.** Suppose that \( \Pi' \) is a subgroup of \( \Pi \), and set \( G' = \Gamma \times \Pi' \), \( N' = 1 \times \Pi' \). Then \( \text{ind}_{G'}^G \mathcal{D}(\Gamma, \Pi') \subseteq \mathcal{D}(\Gamma, \Pi) \).

**Proof.** We have \( G = NG' \) and \( N \cap G' = N' \). Let \( \chi' \in \mathcal{D}(\Gamma, \Pi') \) be the character of a \( \mathbb{Z}G' \)-lattice \( M' \) which satisfies (a) and (b) for \( G', N \in \mathcal{D} \) in the definition of \( \mathcal{D} \) in §1.

We must check that \( M = \text{ind}_{G'}^G M' \) satisfies (a) and (b) for \( G, N \).

For (a) identify \( G/N \) and \( G'/N' \) with \( \Gamma \); there are \( \Gamma \)-isomorphisms
\[ M^G \cong M \otimes_{\mathbb{Z}G} \mathbb{Z} \Gamma \cong M' \otimes_{\mathbb{Z}G'} \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \Gamma \cong M'^{N'}, \]
so \( \Gamma \) acts trivially.

For (b), use Mackey decomposition to get
\[ \text{res}_{G'}^G M \cong \text{res}_{N'}^N \text{ind}_{G'}^G M' \cong \text{ind}_{N'}^N \text{res}_{N'}^N M', \]
because \( G = NG' \) and \( N \cap G' = N' \). \hfill \( \Box \)

**Lemma 2.5.** \( rD' \subseteq D \subseteq D' \) for some positive integer \( r \).

**Proof.** By definition, we have \( D \subseteq D' \). We will find a positive integer \( r \) so that for any locally free \( \mathbb{Z}N \)-lattice \( X, X^r \) is a free \( \mathbb{Z}N \)-lattice. Let \( e \) be the exponent of the locally free class group of \( \mathbb{Z}N \), as defined in [CR, 49.10]. Then \( Y = X^e \) is stably free. From the Bass Cancellation Theorem [CR, 41.20] \( Y \oplus Y \) is free, so \( X^{2e} \) is free, and the lemma is proved, with \( r = 2e \). \hfill \( \Box \)

To proceed further, we will apply the \( p \)-group results of [WAn]. In order to do this for all primes dividing \( |\Pi| \), we will assume that \( \Pi \) is nilpotent. We next show that under this assumption, \( \alpha(\Gamma) \) is also nilpotent.

**Lemma 2.6.** Suppose that \( \Pi \) is nilpotent; let \( \phi_p \) denote the natural map
\[ \phi_p : GL_n(\mathbb{Z}\Pi) \to GL_n(\mathbb{Z}[\Pi/\Pi_p]). \]
Let \( H \) be a finite subgroup of \( SGL_n(\mathbb{Z}\Pi) \). Then the following hold:
1. \( \ker \phi_p \cap H \) is a \( p \)-group.
2. If $x \in SGL_n(\mathbb{Z}I)$ has prime order $r$, then $r$ divides $|\Pi|$.
3. $\ker \phi_p \cap H$ is a normal Sylow $p$-subgroup of $H$.

Proof. 1. Suppose that $x \in \ker \phi_p \cap H$. Then $x = 1 + \delta$, where all the entries of $\delta$ are in $\mathbb{Z}I\Delta(\Pi_p)$, where $\Delta(\Pi_p)$ is the augmentation ideal of $\mathbb{Z}I$. Since $\mathbb{Z}I\Delta(\Pi_p)$ is a nilpotent ideal mod $p$, it follows that for a suitable positive integer $m$ we have $x^{p^m} = 1 + py_i$, for some element $y_i$ of the matrix ring $M_n(\mathbb{Z}I)$. Then, raising to $p$-powers, we get

$$x^{p^{m+i}} = 1 + p^iy_i, \quad y_i \in M_n(\mathbb{Z}I).$$

Now $x$ has finite order, since it is in the finite group $H$; so there are only finitely many possible values of $x^{p^{m+i}}$ as $i$ varies. Then there is a subsequence of integers $i$ for which $x^{p^{m+i}}$ is constant, say $z$. We see from the last equation that $z - 1$ has coefficients divisible by arbitrarily high powers of $p$. This forces $z = 1$, and $x$ has $p$-power order.

2. Use induction on $\Pi$. If $|\Pi| = 1$, then $|SGL_n(\mathbb{Z}I)| = 1$. Suppose that $|\Pi| > 1$. Let $p$ be a prime dividing $|\Pi|$. If $x \in \ker \phi_p$, then $x$ has order $p$ by 1, and $r = p$. If $x \notin \ker \phi_p$, then $\phi_p(x) \in SGL_n(\mathbb{Z}[\Pi/\Pi_p])$ has order $r$ dividing $|\Pi/\Pi_p|$, by induction.

3. Let $\Phi_p = \ker \phi_p \cap H$. From 1, $\Phi_p$ is a $p$-group. Suppose that $y$ is an element of $p$-power order in $H$. From 2, $SGL_n(\mathbb{Z}[\Pi/\Pi_p])$ has no element of order $p$; therefore $y \in \ker \phi_p$. So $\Phi_p$ contains all Sylow $p$-subgroups of $H$. This completes the proof.

Corollary 2.7. If $\Pi$ is nilpotent then $D(\Gamma, \Pi) = D(\Gamma_{nil}, \Pi)$, where $\Gamma_{nil}$ is the largest nilpotent quotient of $\Gamma$. The same holds for $D'$.

Proof. If $\chi \in D(\Gamma, \Pi)$ is the character of $M(\alpha)$ then $\alpha(\Gamma)$ is nilpotent by Lemma 2.6, since each Sylow subgroup is normal. Thus $\chi \in D(\Gamma_{nil}, \Pi)$. The same assertion for $D'$ follows from Lemma 2.5.

3. Character-theoretic Description of $D'$

For the rest of the paper, we assume that $\Pi$ is nilpotent; then by Corollary 2.7, it is no loss of generality to assume that $\Gamma$ is nilpotent. Thus we shall always assume that $\Gamma$ is nilpotent.

In this section we give a character-theoretic description of $D'(\Gamma, \Pi)$.

Lemma 3.1. Let $\chi \in R(G) = R(G_{p'}) \otimes R(G_p)$ be written uniquely in the form

$$\chi = \sum_{\eta \in \text{irr}(G_{p'})} \eta \otimes \lambda_\eta, \quad \lambda_\eta \in R(G_p).$$

Then $\chi \in Q_p^+(\Gamma, \Pi)$ if and only if $\lambda_\eta \in P^+(\Gamma_p, \Pi_p)$ for all $\lambda$. 

Proof. Since $R(G_{p'})$ has $\mathbb{Z}$-basis the irreducible complex characters $\text{irr}(G_{p'})$, and $R^+(G_{p'})$ is the non-negative linear combinations of $\text{irr}(G_{p'})$, the result follows.

Lemma 3.2. $\bigcap_p Q_p \subseteq Q \otimes P$.

Proof. Take $\chi \in \bigcap_p Q_p$. Let $B_p$ be a maximal linearly independent subset of $\{\text{ind}_{G_{p}}^{G}\sigma_{p} : \sigma_{p} \in \text{hom}(\Gamma_p, \Pi_p)\}$; extend this to a basis $\hat{B}_p$ of $Q \otimes R(G_p)$. Then $\otimes_p \hat{B}_p$ is a $Q$-basis of $Q \otimes R(G_p) = Q \otimes R(G)$, so we can write $\chi$ uniquely as a $Q$-linear combination in this basis. For a fixed $p$, $\chi$ is a linear combination of elements in
We first show that \( \sigma(X) \) is not \( \mathbb{Q} \)-linear.

Proof. Fix a prime \( p \) dividing \( |\Pi| \). We want to apply Theorem 2 of [WAn] to the \( \mathbb{Z}G \)-module \( M_p = \text{res}_{\mathbb{Q}G}(\mathbb{Z} \otimes \mathbb{Z} M) \), relative to the normal subgroup \( N_p = 1 \times \Pi_p \) of \( G_p \). We need to verify that

(a) the \( N_p \)-fixed points of \( M_p \) have trivial \( G_p/N_p \)-action, and
(b) \( \text{res}_{N_p} M_p \) is a free \( \mathbb{Z}_p N_p \)-module.

Now (b) holds because \( \text{res}_N M \) is locally free. To prove (a), it is no loss to replace \( M \) by a direct sum of copies of \( M \), and by Lemma 2.5, we may assume that \( \text{res}_N M \) is free. Then \( M \cong M(\alpha) \) for some \( \alpha : \Gamma \to GL_n(\mathbb{Z}) \); hence \( M_{N_p} \cong M(\phi_p \alpha) \), where \( \phi_p \) is as in Lemma 2.6. Now \( \Gamma_p \) is in the kernel of \( \phi_p \alpha \) by Lemma 2.6, so \( G_p/N_p \) acts trivially on \( M_{N_p} \).

From Theorem 2 of [WAn], \( M_p \) is a permutation \( \mathbb{Z}_p G_{p'} \)-lattice; moreover, the permuted basis is a disjoint union of orbits whose point stabilizers are of the form \( \{ \sigma \}_p = \{ \gamma, \sigma(\gamma) : \gamma \in \Gamma_p \} \), where \( \sigma \in \hom(\Gamma_p, \Pi_p) \). Since \( \mathbb{Z}_p \otimes \mathbb{Z} \) is a summand of \( \text{ind}_{\Gamma_p}^{G_p} M_p \), then \( \mathbb{Z}_p \otimes \mathbb{Z} \) is a summand of a permutation \( \mathbb{Z}_p G_{p'} \)-lattice.

Let \( L \) be an indecomposable summand of \( \mathbb{Z}_p \otimes \mathbb{Z} \). Denote it by \( L \). Then \( L \) is isomorphic to a summand of \( \text{ind}_{\Gamma_p}^{G_p} \mathbb{Z}_p \cong \mathbb{Z}_p G_{p'} \otimes \text{ind}_{\Gamma_p}^{G_p} \mathbb{Z}_p \). Write \( \mathbb{Z}_p G_{p'} = \bigoplus X_i \), where \( X_i \) are direct sums of (projective) indecomposables. Then \( L \) is a summand of \( X_i \otimes \text{ind}_{\Gamma_p}^{G_p} \mathbb{Z}_p \) for some \( i \). We claim that \( M = X_i \otimes \text{ind}_{\Gamma_p}^{G_p} \mathbb{Z}_p \) is indecomposable. The \( G_{p'} \)-fixed points of \( M \) are \( X_i \otimes \mathbb{Z}_p \cong X_i \), which is irreducible mod \( p \) as an \( \mathbb{F}_p G_{p'} \)-module, since \( p \) does not divide \( |G_{p'}| \). If \( M \) were decomposable, it would be decomposable mod \( p \), say as \( Z_1 \oplus Z_2 \), where \( Z_1, Z_2 \) are nonzero \( \mathbb{F}_p G_{p'} \)-modules. The \( G_{p'} \)-fixed points of each of \( Z_1, Z_2 \) are non-zero, since \( G_p \) is a \( p \)-group. This contradicts the irreducibility of \( (Y/p)G_{p'} \) as \( \mathbb{F}_p G_{p'} \)-module. So \( M \) is indeed indecomposable, and therefore \( L \), which is a summand of \( M \), is \( Y \) itself. Thus \( \mathbb{Z}_p \otimes \mathbb{Z} \) is isomorphic to a sum of modules of the form \( X \otimes \text{ind}_{\Gamma_p}^{G_p} \mathbb{Z}_p \), where \( X \) is an \( \mathbb{Z}_p G_{p'} \)-module. Since each \( D \) of \( G_{p'} \), \( G_{p} \), it follows that

\[
\mathbb{Z}_p \otimes \mathbb{Z} \cong \bigoplus_{\sigma_p} X_{\sigma_p} \otimes \text{ind}_{\Gamma_p}^{G_p} \mathbb{Z}_p.
\]

where each \( X_{\sigma_p} \) is an \( \mathbb{Z}_p G_{p'} \)-module. Let \( \xi_{\sigma_p} \) be the character of \( X_{\sigma_p} \). Then \( \chi \) has the form

\[
\chi(g) = \sum_{\sigma_p} \xi_{\sigma_p}(g) \cdot \text{ind}_{\Gamma_p}^{G_p} 1(g_p).
\]

In other words, writing \( R(G) = R(G_{p'}) \otimes R(G_p) \), we have

\[
\chi = \sum_{\sigma_p \in \Sigma_p} \xi_{\sigma_p} \otimes \text{ind}_{\Gamma_p}^{G_p} 1, \quad \xi_{\sigma_p} \in R^+(G_{p'}).
\]
Changing the order of summation, we have

$$\chi = \sum_{\eta} \eta \otimes \lambda_{\eta},$$

where

$$\lambda_{\eta} = \sum_{p} b(\rho_p, \eta) \text{ind}_{[\rho_p]}^{G_p} 1 \in P^+(\Gamma_p, \Pi_p).$$

(3.4)

It follows from Lemma 3.1 that $\chi \in \bigcap_p Q^+_p$.

Now suppose that $\chi \in \bigcap_p Q^+_p$; we will show that $\chi \in D$. From Lemma 3.2, $\chi \in Q \otimes P$; since characters in $P$ are characters of permutation modules, we see that $\chi$ is rational valued.

Let $p$ be a prime dividing $|G|$. Since $\chi$ is in $Q^+_p$, we write $\chi$ as

$$\chi = \sum_{\eta \in \text{irr}(G_p)} \eta \otimes \lambda_{\eta}, \quad \lambda_{\eta} \in P^+(\Gamma_p, \Pi_p).$$

(3.5)

Let $\zeta$ be a primitive $|G|$-th root of unity; let $G$ denote the Galois group of $Q(\zeta)$ over $Q$, and let $G_p$ denote the Galois group of $Q_p(\zeta)$ over $Q_p$, where $Q_p$ is the $p$-adic rationals. Since $\chi$ and $\lambda_{\eta}$ are rational valued, we have

$$\chi = \chi^\omega = \sum_{\eta} \eta^\omega \otimes \lambda_{\eta}, \quad \omega \in G_p.$$

By uniqueness of the representation $\chi = \sum_{\eta} \eta \otimes \lambda_{\eta}$, it follows that $\lambda_{\eta} = \lambda_{\eta^\omega}$, $\omega \in G_p$. Partition $\text{irr}(G_p)$ into $G_p$-orbits. For an orbit $O$, $\lambda_{\eta}$ is the same for all $\eta$ in $O$; call this common value $\lambda_O$. Let $\tau_O$ denote $\sum_{\eta \in O} \eta$, which takes values in $Q_p$. Then

$$\chi = \sum_O \sum_{\eta \in O} \eta \otimes \lambda_{\eta} = \sum_O \sum_{\eta \in O} \eta \otimes \lambda_O = \sum_O \tau_O \otimes \lambda_O.$$

(3.6)

We have each $\lambda_O \in P^+(\Gamma_p, \Pi_p)$, so we may choose non-negative integers $b(\rho_p, O)$ such that $\lambda_O = \sum_{\eta \in O} b(\rho_p, O) \text{ind}_{[\rho_p]}^{G_p} 1$. Let

$$\xi_{\rho_p} = \sum_O b(\rho_p, O) \tau_O.$$

Then, changing the order of summation, we have

$$\chi = \sum_O \tau_O \otimes \lambda_O = \sum_{\rho_p} \sum_{\tau_O} \tau_O \otimes b(\rho_p, O) \text{ind}_{[\rho_p]}^{G_p} 1 = \sum_{\rho_p} \xi_{\rho_p} \otimes \text{ind}_{[\rho_p]}^{G_p} 1.$$

Now $\xi_{\rho_p}$ is a $Z_{\geq 0}$-linear combination of $\{\tau_O\}$, and each $\tau_O = \sum_{\eta \in \text{irr}(G_p)} \eta$. Fix $\eta \in \text{irr}(G_p)$ in an orbit $O$. Since the order of $G_p$ is not divisible by $p$, the Schur index of $\eta$ over the field $Q_p(\eta)$ is 1, by [F, IV.9.5]. Let $K = Q_p(\eta)$ and let $X$ be a $KG_p$-module affording $\eta$. By restriction of scalars, $X$ is a $Q_pG_p$-module whose character is the sum of all the algebraic conjugates of $\eta$ over $Q_p$, namely $\tau_O$. Thus $\tau_O$ is the character of a $Q_pG_p$-module; hence so is $\xi_{\rho_p}$. Choose a $Z_pG_p$-lattice $L_{\rho_p}$ in this module, so $\xi_{\rho_p}$ is afforded by $L_{\rho_p}$.

Next, define the $Z_pG$-lattice $M(p)$ by

$$M(p) = \bigoplus_{\rho_p} L_{\rho_p} \otimes Z_p \text{ind}_{[\rho_p]}^{G_p} Z_p.$$

(3.7)

We claim that

(a) the $N$-fixed points $M(p)^N$ have trivial $G/N$-action, and

(b) $\text{res}_N M(p)$ is $Z_pN$-free.
To prove (a), note that $M(p)$ has character $\chi$ which is in $\cap_p Q_p^+$; by Lemma 3.2, $\chi \in Q \otimes P$. Since the $N$-fixed points of $\text{ind}_{[\sigma_p]}^G Z$ have trivial $G/N$-action, (a) holds.

We now prove (b). Since $G_p = N_p[\sigma_p]$ and $N_p \cap [\sigma_p] = 1$, it follows that $\text{res}_{N_p} \text{ind}_{[\sigma_p]}^G Z_p \cong Z_pN_p$. By Mackey decomposition. Since it suffices to show that $\text{res}_{N_p} \text{ind}_{[\sigma_p]}^G Z_p \cong Z_pN_p$. Since $p$ does not divide $|N_p|$, we need only show that the character $\chi_p'$ of $\bigoplus_{\sigma_p} L_{\sigma_p}$ has the property that $\text{res}_{N_p} \chi_p'$ is a multiple of the character $\rho(N_p')$ of the regular representation of $N_p'$.

Since $\chi = \sum_{\sigma_p} \xi_{\sigma_p} \otimes \text{ind}_{[\sigma_p]}^G Z$ is a multiple of the character $\rho(N_p')$ of the regular representation of $N_p'$. Therefore $\chi_p'$, the character of a permutation module, then $\chi_p'$ is a $Q$-multiple of $\text{res}_{N_p} \chi_p'$. Let $y$ be a non-identity element of $N_p'$. Then $y$ has the form $(1, \pi)$, so no conjugate of $y$ lies in $[\sigma_p]$. Since $\chi \in Q \otimes P$ by Lemma 3.2, $\chi$ and therefore $\chi_p'$ both vanish on $y$. It follows that $\text{res}_{N_p} \chi_p'$ is a multiple of $\rho(N_p')$ as desired. This proves (b).

We next show that $\chi$ is afforded by a $QG$-module. As in equation (3.5), we have

$$\chi = \sum_{\sigma_p} \tau_{\sigma'} \otimes \lambda_{\sigma'}$$

where $\tau_{\sigma'}$ is the orbit sum over an orbit of $G$ acting on $\text{irr}(G_p')$, and $\lambda_{\sigma'}$ is the common value of $\lambda_k$ for all $k \in \sigma'$. Let $R_q(G)$ be the characters afforded by $QG$-modules.

Since $\tau_{\sigma'}$ is a rational valued character of $G_p'$, then $|G_p'| \tau_{\sigma'} \in R^+_Q(G_p')$; since $\lambda_{\sigma'}$ is the character of a permutation module, then $|G_p'| \chi = R_q^+(G)$ is free over $p$ since the greatest common divisor of $|G_p'|$ is 1, it follows that $\chi \in R_q^+(G)$. Let $V$ be a $QG$-module affording $\chi$. For each prime $p$ we have an isomorphism

$$\phi_p : Q_p \otimes Q V \rightarrow Q_p \otimes Z_p M(p).$$

For each $p$ let

$$V(p) = \{ v \in V : \phi_p (1 \otimes v) \in 1 \otimes M(p) \}.$$ 

Then let $M = \bigcap_p V(p)$. From [R, 5.3] we see that $M$ is a $ZG$-lattice such that $Z_p \otimes M \cong M(p)$. Then $M$ affords $\chi$, the fixed points $M^N$ have trivial $G/N$-action, and $\text{res}_N(Z_p \otimes M) \cong \text{res}_N M(p)$, so $\text{res}_N M$ is locally free. Therefore $\chi$ is in $D'$, as desired. This completes the proof.

For later use, we record the following result, which is proved in the second paragraph of the proof of Theorem 3.3.

**Proposition 3.4.** If $M$ satisfies (a) and (b'), then for each prime $p$, $Z_p \otimes M$ is a summand of a permutation lattice for $Z_p G$.

### 4. The Lattice Spanned by $D'$

In this section we show that the $Z$-span $ZD'$ of $D'$ is equal to $\bigcap_p Q_p$. We also show that $ZD'$ is equal to $P$, if $\Gamma$ is cyclic. We do not know if this result holds for arbitrary $\Gamma$. 
Proposition 4.1. The $\mathbb{Z}$-span of $\mathcal{D}'$ is equal to $\bigcap_p Q_p$.

Proof. It follows from Theorem 3.3 that

$$\mathbb{Z}\mathcal{D}' \subseteq \bigcap_p Q_p.$$  

For the reverse inclusion, suppose that $\chi \in \bigcap_p Q_p$. For a fixed $p$ dividing $|G|$, write

$$\chi = \sum_{\eta \in \text{irr}(G_{p'})} \eta \otimes \lambda_\eta \text{ for unique } \lambda_\eta \in \mathcal{P}(\Gamma_p, \Pi_p).$$

By Lemma 3.2, $\chi \in Q \otimes \mathcal{P}$. For a given $\eta \in \text{irr}(G_{p'})$, suppose that

$$\langle \eta, \text{ind}^{G_{p'}}_{\sigma_{p'}} 1 \rangle_{G_{p'}} = 0 \text{ for all } \sigma_{p'} \in \text{hom}(\Gamma_{p'}, \Pi_{p'}).$$

Then $\eta$ is orthogonal to $\mathcal{P}(\Gamma_{p'}, \Pi_{p'})$, and hence

$$0 = \langle \chi, \eta \otimes \lambda_\eta \rangle_G = \langle \eta \otimes \lambda_\eta, \eta \otimes \lambda_\eta \rangle_G = \langle \eta, \eta \rangle_{G_{p'}} \langle \lambda_\eta, \lambda_\eta \rangle_{G_{p'}}.$$

It follows that $\lambda_\eta = 0$. Thus in equation (4.1), we need only sum over $\eta \in \text{irr}(G_{p'})$ for which (4.2) does not hold. For such an $\eta$, if $\langle \eta, \text{ind}^{G_{p'}}_{\sigma_{p'}} 1 \rangle_{G_{p'}} \neq 0$, decompose $\text{ind}^{G_{p'}}_{\sigma_{p'}} 1$, giving

$$-\eta = \sum \tilde{\eta} - \text{ind}^{G_{p'}}_{\sigma_{p'}} 1$$

for some $\tilde{\eta} \in \text{irr}(G_{p'})$. Write $\lambda_\eta = \lambda'_\eta - \lambda''_\eta$ with $\lambda'_\eta, \lambda''_\eta \in \mathcal{P}^+(\Gamma_p, \Pi_p)$; we get

$$\chi = \sum \eta \otimes \lambda'_\eta + \sum (-\eta) \otimes \lambda''_\eta.$$  

In this equation, replace $-\eta$ using equation (4.3). We get $\chi = \chi'_p - \xi_p$ with $\chi'_p \in Q^+_p$, $\xi_p \in \mathcal{P}^+$. Let $\zeta = \sum_{p | |G|} \xi_p \in \mathcal{P}^+$; then $\chi + \zeta = \chi'_p + \sum_{p \neq p} \xi_t$, which is in $Q^+_p$ for all $p$. Thus

$$\chi = (\chi + \zeta) - \xi$$

with $\chi + \zeta \in \bigcap_p Q^+_p$ and $\xi \in \mathcal{P}^+ \subseteq \bigcap_p Q^+_p$. Since $\bigcap_p Q^+_p = \mathcal{D}'$ by Theorem 3.3, the result is proved.

We next show that $\mathbb{Z}\mathcal{D}' = \mathcal{P}$ if $\Gamma$ is cyclic. Let $\Sigma$ be a complete set of homomorphisms from $\Gamma$ to $\Pi$ up to conjugacy in $\Pi$. Our proof that $\mathbb{Z}\mathcal{D}' = \mathcal{P}$ for cyclic $\Gamma$ uses the next result, that $\{\text{ind}^G_{\sigma} 1 : \sigma \in \Sigma\}$ is linearly independent over $Q$ if $\Gamma$ is cyclic. This lemma can fail if $\Gamma$ is not cyclic, but it is possible that $\mathbb{Z}\mathcal{D}' = \mathcal{P}$ can be proved by some other method in the non-cyclic case.

Lemma 4.2. The set $\{\text{ind}^G_{\sigma} 1 : \sigma \in \Sigma\}$ is a basis of $Q \otimes \mathcal{P}$ if $\Gamma$ is cyclic.

Proof. For $g = (\gamma, \pi) \in G$ and $\sigma \in \text{hom}(\Gamma, \Pi)$, we have

$$\text{ind}^G_{\sigma} 1(g) = \begin{cases} |C_{\Pi}(\pi)|, & \sigma(\gamma) \sim \pi, \\ 0, & \text{else,} \end{cases}$$

where $\sim$ denotes conjugacy in $\Pi$. If $\tau \in \text{hom}(\Gamma, \Pi)$ then $\tau$ is conjugate to some $\sigma \in \Sigma$. From the formula for $\text{ind}^G_{\sigma} 1(g)$ above, then $\text{ind}^G_{\sigma} 1 = \text{ind}^G_{\tau} 1$, so $Q \otimes \mathcal{P}$ is spanned by $\{\text{ind}^G_{\sigma} 1 : \sigma \in \Sigma\}$.
Suppose that $\sum_{\sigma \in \Sigma} a_\sigma \text{ind}_{[\sigma]} G^1 1 = 0$ with $a_\sigma \in Q$. Given $\tau \in \Sigma$, let $\gamma$ be a generator of $\Gamma$, and evaluate at $(\gamma, \tau(\gamma))$. We get

$$\text{ind}_{[\sigma]} G^1(\gamma, \tau(\gamma)) = \begin{cases} |C_{\Pi}(\tau(\gamma))|, & \sigma(\gamma) \sim \tau(\gamma), \\ 0, & \text{else}. \end{cases}$$

If $\tau$ and $\sigma$ are distinct elements of $\Sigma$, then $\tau(\gamma)$ and $\sigma(\gamma)$ are not conjugate in $\Pi$, so $\text{ind}_{[\sigma]} G^1(\gamma, \tau(\gamma)) = 0$. It follows that $a_\tau = 0$, for all $\tau$ in $\Sigma$, and the result is proved.\hfill\Box

**Proposition 4.3.** The $Z$-span of $D'$ is equal to $P$ if $\Gamma$ is cyclic.

**Proof.** Since $P^+ \subseteq D'$ it suffices to show that $D' \subseteq P$. Suppose that $\chi \in D'$. Then $\chi \in Q \otimes P$ from Theorem 3.3 and Lemma 3.2. Then $\chi = \sum_{\sigma \in \Sigma} a_\sigma \text{ind}_{[\sigma]} G^1$, where $a_\sigma \in Q$, and by Lemma 4.2, the $a_\sigma \in Q$ are unique. We must show that each $a_\sigma \in Z$.

Since $G$ is nilpotent, we pick $\Sigma$ by picking complete sets $\Sigma_p \subseteq \text{hom}(\Gamma_p, \Pi_p)$ up to conjugacy in $\Pi_p$, and then letting $\Sigma$ be those homomorphisms whose restrictions to $\Gamma_p$ are in $\Sigma_p$.

Fix a prime $p$ dividing $|G|$. From Theorem 3.3, $\chi \in Q_p(\Gamma, \Pi) = R(G_{p'}) \otimes P(\Gamma_p, \Pi_p)$. For $\sigma \in \Sigma$, we have $\text{ind}_{[\sigma]} G^1 = \text{ind}_{[\sigma_{p'}]} G_{p'}^1 \otimes \text{ind}_{[\sigma_p]} G_p^1$. Then

$$\chi = \sum_{\tau \in \Sigma_p} \left( \sum_{\sigma \in \Sigma} a_\sigma \text{ind}_{[\sigma_{p'}]} G_{p'}^1 \right) \otimes \text{ind}_{[\tau]} G_p^1.$$

But $\{\text{ind}_{[\tau]} G_p^1\}$ is a $Z$-basis of $P(\Gamma_p, \Pi_p)$, from Lemma 4.2, so it follows that

$$\phi_\tau = \sum_{\sigma \in \Sigma} a_\sigma \text{ind}_{[\sigma_{p'}]} G_{p'}^1 1 \in R(G_{p'})$$

for all $\tau \in \Sigma_p$. Let $\gamma_{p'}$ be a generator of $\Gamma_{p'}$. Evaluate $\phi_\tau$ at $(\gamma_{p'}, \sigma_{p'}(\gamma_{p'}))$. For the unique $\sigma \in \Sigma$ whose restriction to $\Gamma_p$ is $\tau$ and whose restriction to $\Gamma_{p'}$ is $\sigma_{p'}$, we get, as in the proof of Lemma 4.1,

$$|C_{\Pi_{p'}}(\sigma_{p'}(\gamma_{p'}))| a_\sigma \in Z$$

since the values of $\phi_\tau$ are algebraic integers in $Q$. Thus

$$|\Pi_{p'}| a_\sigma \in Z, \quad \text{for all } \sigma \in \Sigma.$$

This holds for all $p$. Since the greatest common divisor of $|\Pi_{p'}|$ is 1, it follows that $a_\sigma \in Z$ for all $\sigma \in \Sigma$, and the proof is complete.\hfill\Box

Here is an example showing that Lemma 4.2 can fail if $\Gamma$ is not cyclic. Let $\Gamma$ and $\Pi$ each be the direct product $C_p \times C_p$ of cyclic groups of prime order. Consider the $p^4 \times p^4$ matrix whose rows are indexed by $\Sigma = \text{hom}(\Gamma, \Pi)$ and columns by $G = \Gamma \times \Pi$, with $(\sigma, g)$-entry given by $\text{ind}_{[\sigma]} G^1(1, x)$. Since $\text{ind}_{[\sigma]} G^1(1, x) = 0$ if $x$ is a nontrivial element of $\Pi$, then there is a column of zeros, and so the rows are linearly dependent.
5. Examples of \( D \neq P^+ \)

In this section we produce examples of groups \( \Gamma, \Pi \) and modules \( M \) having character which is in \( D(\Gamma, \Pi) \) but not in \( P^+(\Gamma, \Pi) \).

**Lemma 5.1.** For distinct primes \( p_1 \) and \( p_2 \), let \( \Gamma = C_{p_1p_2} \), the cyclic group of order \( p_1p_2 \), and let \( \Pi = C_{p_1p_2} \times C_{p_1p_2} \). Then \( D(\Gamma, \Pi) \neq P^+(\Gamma, \Pi) \).

**Proof.** As before, let \( G = \Gamma \times \Pi, N = 1 \times \Pi \). Choose sets \( \Sigma_1, \Sigma_2 \) of homomorphisms \( \sigma : \Gamma \to \Pi \) whose images \( \sigma(\Gamma) \) are precisely the subgroups of \( \Pi \) of order \( p_1 \), respectively, \( p_2 \). Let \( \Sigma = \Sigma_1 \cup \Sigma_2 \). Let \( \tau : \Gamma \to \Pi \) be the trivial map. The module \( M \) we will construct has character

\[
\chi = -\text{ind}_{[\tau]}^G 1 + \sum_{\sigma \in \Sigma} \text{ind}_{[\sigma]}^G 1.
\]

It will follow from our construction that \( \chi \in D(\Gamma, \Pi) \). Let \( \gamma \) be a generator of \( \Gamma \), and let \( g = (\gamma, 1) \in G \). Then \( (\gamma, 1) \in [\tau] \), so \( \text{ind}_{[\tau]}^G 1(g) = |\Pi| \), but \( (\gamma, 1) \notin [\sigma] \) for all \( \sigma \in \Sigma \), so \( \text{ind}_{[\sigma]}^G 1(g) = 0 \). Hence \( \chi(g) < 0 \) and \( \chi \) is not the character of a permutation module, so \( \chi \notin P^+(\Gamma, \Pi) \).

For \( \sigma \in \Sigma \), \( M(\sigma) \) is the corresponding double-action module. Define

\[
M(\Sigma) = \bigoplus_{\sigma \in \Sigma} M(\sigma).
\]

For each \( \sigma \in \Sigma \), let \( s(\sigma) = \sum_{x \in \sigma(\Gamma)} x \in \mathbb{Z}^{\Pi} \), and define the map

\[
f_\sigma : M(\sigma) \to M(\tau), \quad f_\sigma(m) = s(\sigma)m, \quad m \in M(\sigma).
\]

Define \( f : M(\Sigma) \to M(\tau) \) by \( f = \sum_\sigma f_\sigma \). The key to the proof is the claim that \( f \) is an epimorphism of \( \mathbb{Z}G \)-modules.

We now prove that the claim holds. Since \( s(\sigma)\sigma(\gamma^{-1}) = s(\sigma) = \sigma(\gamma^{-1})s(\sigma) \), then each \( f_\sigma \) is a \( \mathbb{Z}G \)-homomorphism, and so is \( f \). To prove that \( f \) is surjective, it suffices to find \( v \in M(\Sigma) \) such that \( f(v) = 1 \in M(\tau) \). Pick two distinct elements \( \phi_1, \psi_1 \) from \( \Sigma_i, i = 1, 2 \). Find integers \( n_1, n_2 \) such that \( n_1p_1 + n_2p_2 = 1 \). Define \( v = \sum_{\sigma \in \Sigma} v_\sigma \) with \( v_\sigma \in M(\sigma) \) given by

\[
v_\sigma = \begin{cases} n_{i_1}, & \sigma \in \Sigma_i, \sigma \neq \phi_1, \\ n_{i_1}(1 - s(\psi_i)), & \sigma = \phi_i. \end{cases}
\]

To compute \( f(v) \),

\[
f(v) = n_1 \sum_{\sigma \in \Sigma_1, \sigma \neq \phi_1} s(\sigma) + n_1s(\phi_1)(1 - s(\psi_1)) + n_2 \sum_{\sigma \in \Sigma_2, \sigma \neq \phi_2} s(\sigma) + n_2s(\phi_2)(1 - s(\psi_2)).
\]

Since \( s(\sigma)s(\tilde{\sigma}) = \hat{\Pi}, \sigma, \tilde{\sigma} \in \Sigma, \sigma \neq \tilde{\sigma} \), we get

\[
f(v) = n_1 \left( \sum_{\sigma \in \Sigma_1} s(\sigma) - \hat{\Pi}p_1 \right) + n_2 \left( \sum_{\sigma \in \Sigma_2} s(\sigma) - \hat{\Pi}p_2 \right).
\]
In the sum $\sum_{\sigma \in \Sigma} s(\sigma) \in \mathbb{Z} \Pi_{p_i}$, non-identity elements $y \in \Pi_{p_i}$ occur exactly once, whereas 1 occurs $p_i + 1$ times and

$$\sum_{\sigma \in \Sigma} s(\sigma) = p_i \cdot 1 + \hat{\Pi}_{p_i}.$$ 

We obtain

$$f(v) = n_1 p_1 \cdot 1 + n_2 p_2 \cdot 1 = 1.$$ 

Therefore $f$ is indeed a $\mathbb{Z}G$-epimorphism.

Now define $M$ to be the kernel of $f : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\tau)$, so we have the exact sequence

$$0 \rightarrow M \rightarrow \mathcal{M}(\Sigma) \xrightarrow{f} \mathcal{M}(\tau) \rightarrow 0.$$ 

Since $\text{res}_N M(\sigma)$ is free for all $\sigma$, this sequence splits when restricted to $N$; hence $\text{res}_N M$ is stably free. Since $\mathbb{Z}N$ satisfies the Eichler condition, it follows that $\text{res}_N M$ is $\mathbb{Z}N$-free by Jacobinski’s Cancellation Theorem [CR, 51.24]. (In fact, it can be shown directly that $\text{res}_N M$ is $\mathbb{Z}N$-free by exhibiting a basis; this is done in a special case below.) Since $G/N$ acts trivially on $\mathcal{M}(\Sigma)^N$, it does so on $\mathcal{M}^N$.

The character $\chi$ of $M$ is therefore in $\mathcal{D}$, and it follows from the exact sequence that $\chi = -\text{ind}_{[1]}^G 1 + \sum_{\sigma \in \Sigma} \text{ind}^G_{[\sigma]} 1$. This completes the proof.

From Proposition 2.4, there exists $U \in SGL_{p_1p_2}(\mathbb{Z} \Pi)$ with $U^{p_1p_2} = 1$ and $U$ not conjugate in $GL(\mathbb{Q} \Pi)$ to a diagonal matrix of group elements. We can exhibit such a matrix $U$ by computing the action of a generator of $\Gamma$ on an explicit $\mathbb{Z}N$-basis of the free module $\text{res}_N M$. To simplify the exposition, we assume that

$$p_1 = 2, \quad p_2 = 3.$$ 

Then pick

$$n_1 = -1, \quad n_2 = 1.$$ 

Write $\Gamma = \langle c \rangle$ of order 6, $\Pi = \langle a \rangle \times \langle b \rangle$, where $a$ and $b$ each have order 6. Let

$$\Sigma_1 = \{\sigma_1, \sigma_2, \sigma_3\}, \quad \sigma_1(c) = a^3, \sigma_2(c) = b^3, \sigma_3(c) = a^3b^3;$$

$$\Sigma_2 = \{\sigma_4, \sigma_5, \sigma_6, \sigma_7\}, \quad \sigma_4(c) = a^2, \sigma_5(c) = b^2, \sigma_6(c) = a^2b^2, \sigma_7(c) = a^2b^4.$$ 

Pick

$$\psi_1 = \sigma_1, \quad \phi_1 = \sigma_3, \quad \psi_2 = \sigma_4, \quad \phi_2 = \sigma_7.$$ 

Define

$$s_i = s(\sigma_i), \quad c_i = \sigma_i(c).$$ 

Identify $\mathcal{M}(\Sigma)$ with $\mathbb{Z} \Pi^7$ and $M(\tau)$ with $\mathbb{Z} \Pi$, Then our map $f$ takes $\mathbb{Z} \Pi^7$ to $\mathbb{Z} \Pi$, given by

$$f(z_1, \ldots, z_7)^T = \sum_{i=1}^{7} s_i z_i.$$ 

The $G$-action on $\mathbb{Z} \Pi^7$ becomes

$$(z_1, \ldots, z_7)^T (x, y) = (\sigma_1(x^{-1}) z_1 y \cdots, \sigma_7(x^{-1}) z_7 y)^T.$$
The action of $\chi$ strategy of [MRSW]. Actually every $\{Q, M, \ldots, \}$. In this notation the element $v$ in the proof of Lemma 5.1 is

$$v = (-1, -1, s_1 - 1, 1, 1, 1 - s_4)^T.$$ 

Let $\{e_1, e_2, \ldots, e_7\}$ be the standard basis of $\mathbb{Z}^7$. Extend the “unimodular column” $v$ to a $\mathbb{Z}$-basis $\{v, e_2, \ldots, e_7\}$ of $\mathbb{Z}^7$. Then do “elementary operations” by setting

$$m_i = e_{i+1} - v f(e_{i+1}) = e_{i+1} - v s_{i+1}, \quad 1 \leq i \leq 6,$$

and then $\{v, m_1, m_2, \ldots, m_6\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^7$. The $m_i$ are in $M$ by construction, so $\{m_1, \ldots, m_6\}$ is a $\mathbb{Z}$-basis of $M$.

We want the action of a generator of $\Gamma$ on $M$ in this basis. The matrix of $(c^{-1}, 1)$ in the standard basis of $\mathbb{Z}^7$ is $D = \text{diag}(c_1, \ldots, c_7)$. So we need only see the effect of two changes of basis. Set

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ s_1 - 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 - s_4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -s_2 & -s_3 & -s_4 & -s_5 & -s_6 & -s_7 \\ & 1 & 1 & & & & \\ & * & & & & & \\ & * & & & & & \\ & * & & & & & \\ & * & & & & & \end{pmatrix}. $$

The action of $(c^{-1}, 1)$ in the basis $\{v, m_1, m_2, \ldots, m_6\}$ is given by the $\mathbb{Z}$-matrix $X = B^{-1} A^{-1} D A B$. Since $M$ is a $G$-submodule of $\mathbb{Z}^7$, this matrix has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ & & & & & U \end{pmatrix}$$

for some $U \in \text{GL}_6(\mathbb{Z})$, and it is this $U$ that we want. Using $c_i s_i = s_i$, and denoting $c_{ij} = c_i - c_j$, $s_1^* = s_1 - 1$, $s_4^* = s_4 - 1$, we get $U =$

$$\begin{pmatrix} c_2 + c_1 s_2 & c_21 s_3 & c_21 s_4 & c_21 s_5 & c_21 s_6 & c_21 s_7 \\ c_{13}s_1^* s_2 & c_3 + c_{13} s_1^* s_3 & c_{13}s_1^* s_4 & c_{13}s_1^* s_5 & c_{13}s_1^* s_6 & c_{13}s_1^* s_7 \\ c_{14}s_2 & c_4 + c_{14} s_4 & c_{14} s_4 & c_{14} s_6 & c_{14} s_6 & c_{14} s_7 \\ c_{15}s_2 & c_{15} s_3 & c_{15} s_4 & c_{15} s_5 & c_{15} s_6 & c_{15} s_7 \\ c_{16}s_2 & c_{16} s_3 & c_{16} s_4 & c_{16} s_5 & c_6 + c_{16} s_6 & c_{16} s_7 \\ c_{71}s_2 & c_{71}s_4 & c_{71}s_4 & c_{71}s_4 & c_{71}s_4 & c_{71}s_4 \end{pmatrix}. $$

Note that this matrix has trace $-1 + \sum_{i=1}^7 c_i$, so it is a counterexample to the strategy of [MRSW]. Actually every $\chi \in D' - P^+$ gives such a counterexample, by Lemma 1 of [WCr] generalized to matrices.

**Lemma 5.2.** Let $p$ be an odd prime, and let $\Gamma = C_4 \times C_p$ and $\Pi = Q_8 \times C_p \times C_p$, where $Q_8$ is the quaternion group of order 8. Then $D(\Gamma, \Pi) \neq P^+(\Gamma, \Pi)$.
Proof. Pick \( \tau \in \hom(\Gamma, \Pi) \) whose image is one of the cyclic subgroups of order 4 of \( Q_8 \). Choose \( \Sigma_2 \) to consist of three homomorphisms \( \Gamma \to \Pi \) whose images are the two other subgroups of order 4 in \( Q_8 \) as well as the subgroup of order 2. Choose \( \Sigma_p \) to consist of \( p+1 \) elements of \( \hom(\Gamma, \Pi) \) so that \( \im \sigma_p \) are the nontrivial cyclic \( p \)-subgroups and so that \( \sigma_2 = \tau_2 \). Set \( \Sigma = \Sigma_2 \cup \Sigma_p \), and

\[
\chi' = -\ind_{[\tau]}^{G_1} 1 + \sum_{\sigma \in \Sigma} \ind_{[\sigma]}^{G} 1.
\]

Relative to \( R(G) = R(G_p) \otimes R(G_2) \),

\[
\chi' = \left(-\ind_{[\tau]}^{G_1} 1 + \sum_{\sigma \in \Sigma_p} \ind_{[\sigma]}^{G_p} 1\right) \otimes \ind_{[\tau_2]}^{G_2} 1 + \sum_{\sigma \in \Sigma_2} \ind_{[\sigma]}^{G_2} 1 \otimes \ind_{[\tau_2]}^{G_2} 1,
\]

where the expression in parentheses is in \( R^+(G_p) \). So \( \chi \in Q_8^+ \). Relative to \( R(G) = R(G_2) \otimes R(G_p) \) we have

\[
\chi' = \left(-\ind_{[\tau_2]}^{G_2} 1 + \sum_{\sigma \in \Sigma_2} \ind_{[\sigma_2]}^{G_2} 1\right) \otimes \ind_{[\tau]}^{G_1} 1 + \sum_{\sigma \in \Sigma_p} \ind_{[\sigma]}^{G_p} 1 \otimes \ind_{[\tau_2]}^{G_2} 1,
\]

where the expression in parentheses is in \( R^+(G_2) \). Therefore \( \chi \in Q_8^+ \).

By Theorem 3.3, \( \chi' \in D' \); hence \( \chi = r\chi' \in D \) by Lemma 2.5. We shall show that \( \chi \notin \mathcal{P}^+ \). By Lemma 4.2, it is enough to check that \( \tau \) is not \( \Pi \)-conjugate to any \( \sigma \in \Sigma \). But the image of \( \tau \), which is normal in \( \Pi \), is different from the images of all \( \sigma \in \Sigma \). This completes the proof.

\[\square\]

6. On Problem 1

In this section we prove that \( D' = \mathcal{P}^+ \) if \( \Pi \) is nilpotent and \( \Gamma \) has prime-prower order. Since \( \mathcal{P}^+ \subseteq D \subseteq D' \), then Problem 1 has a positive answer in this case. We also completely deal with Problem 1 if \( \Gamma \) is cyclic.

Theorem 6.1. Suppose that \( \Gamma \) is an \( l \)-group for some prime \( l \) and that \( \Pi \) is nilpotent. Then \( D' = \mathcal{P}^+ = \mathcal{P}^+(\Gamma, \Pi) \).

Proof. Suppose that \( \chi \in D' \). For each prime \( p \neq l \), use Theorem 3.3 to write \( \chi \) relative to \( R(G_p) \otimes R(G_p) \) as \( \chi = \sum_{\sigma_p} \xi_{\sigma_p} \otimes \ind_{[\sigma_p]}^{G_p} 1 \). Since \( \Gamma_p = 1 \), the only \( \sigma_p : \Gamma_p \to \Pi_p \) in this sum is the trivial map; hence \( \ind_{[\sigma_p]}^{G_p} 1 \) is the character \( \rho(G_p) \) of the regular representation and \( \chi = \xi_{\sigma_p} \otimes \rho(G_p) \). In particular, \( \chi \) vanishes off \( G_{p'} \). Varying \( p \neq l \), it follows that \( \chi \) vanishes off \( G_l \).

Define the class function \( \lambda \) on \( G_l \) by \( \lambda(g) = \chi(g)/|G_l| \). Then relative to \( R(G) = R(G_{l'}) \otimes R(G_l) \) (actually with scalars extended to \( Q \)) we have \( \chi = \rho(G_{l'}) \otimes \lambda \).

At the prime \( l \) write \( \chi = \sum_{\eta \in \irr(G_{l'})} \eta \otimes \chi_\eta \) with \( \chi_\eta \in \mathcal{P}^+(\Gamma_l, \Pi_l) \). Since \( \rho(G_{l'}) = \sum_{\eta \in \irr(G_{l'})} \eta(1) \eta \), we get

\[
\chi = \rho(G_{l'}) \otimes \lambda = \sum_{\eta} \eta \otimes \eta(1) \lambda = \sum_{\eta} \eta \otimes \chi_\eta,
\]

so we deduce that \( \chi_\eta = \eta(1) \lambda \) for all \( \eta \in \irr(G_{l'}) \). Take \( \eta \) to be the trivial character; then \( \lambda = \chi_1 \in \mathcal{P}^+(\Gamma_l, \Pi_l) \), and we can write \( \lambda = \sum_{\sigma_1} a_{\sigma_1} \ind_{[\sigma_1]}^{G_l} 1 \), where each \( a_{\sigma_1} \) is a non-negative integer.
Since \( \rho(G_{\nu}) \otimes \text{ind}^{G}_{[\sigma]} 1 = \text{ind}^{G}_{[\sigma]} 1 \), we have
\[
\chi = \rho(G_{\nu}) \otimes \lambda = \sum_{\sigma} a_{\sigma} \rho(G_{\nu}) \otimes \text{ind}^{G}_{[\sigma]} 1 = \sum_{\sigma} a_{\sigma} \text{ind}^{G}_{[\sigma]} 1.
\]
Thus \( \chi \in \mathcal{P}^{+}(\Gamma, \Pi) \), and the proof is complete. \( \square \)

**Corollary 6.2.** \( \Gamma \) is a subgroup of \( SGL_{n}(\mathbb{Z}I) \) if and only if \( \Gamma \) is isomorphic to a subgroup of \( \Pi^{n} \) (the direct product of \( n \) copies of \( \Pi \)).

**Proof.** Suppose that \( \Gamma \subseteq SGL_{n}(\mathbb{Z}I) \). From Corollary 2.7, \( \Gamma \) is nilpotent, so in order to prove that \( \Gamma \) is a subgroup of \( \Pi^{n} \) it suffices to prove that \( \Gamma_{l} \subseteq \Pi^{n} \) for each prime \( l \) dividing \( \Gamma \). Hence we may assume that \( \Gamma \) is an \( l \)-group. Then Theorem 6.1 implies that \( D(\Gamma, \Pi) = \mathcal{P}^{+}(\Gamma, \Pi) \), and then from Proposition 2.3, \( \mathcal{P}^{+}(\Gamma, \Pi) \).

The converse is clear. \( \square \)

**Theorem 6.3.** Suppose that \( \Pi \) is nilpotent. Then \( \mathcal{D} = \mathcal{P}^{+} \) for all cyclic \( \Gamma \) if and only if \( \Pi \) has at most one non-cyclic Sylow \( p \)-subgroup.

**Proof.** Suppose that \( \Pi \) has at most one non-cyclic Sylow \( p \)-subgroup. We will show that \( \mathcal{D}' = \mathcal{P}^{+} \), and therefore that \( \mathcal{D} = \mathcal{P}^{+} \). Fix a prime \( p \), which exists by hypothesis, such that \( \Pi_{p'} \) is cyclic, and therefore has a faithful character \( \lambda \) of degree 1. Let \( \chi \) be in \( \mathcal{D}' \). Choose \( \Sigma \) as in the proof of Proposition 4.3, namely
\[
\Sigma = \{ \sigma \in \text{hom}(\Gamma, \Pi) : \sigma_{p} \in \Sigma_{p} \},
\]
where \( \Sigma_{p} \subseteq \text{hom}(\Gamma_{p}, \Pi_{p}) \) is a complete set of homomorphisms up to conjugacy in \( \Pi_{p} \). By Proposition 4.3, \( \chi \in \mathcal{P} \), and since \( \Gamma \) is cyclic, we may write, by Lemma 4.2,
\[
\chi = \sum_{\sigma \in \Sigma} a_{\sigma} \text{ind}^{G}_{[\sigma]} 1 \quad \text{for unique} \quad a_{\sigma} \in \mathbb{Z}.
\]
We must show that \( a_{\sigma} \geq 0 \) for all \( \sigma \).

Relative to \( R(G) = R(G_{p'}) \otimes R(G_{p}) \), we have \( \text{ind}^{G}_{[\sigma]} 1 = \text{ind}^{G_{p'}}_{[\sigma_{p}]} 1 \otimes \text{ind}^{G_{p}}_{[\sigma_{p}]} 1 \), giving
\[
\chi = \sum_{\sigma} a_{\sigma} \text{ind}^{G_{p'}}_{[\sigma_{p}]} 1 \otimes \text{ind}^{G_{p}}_{[\sigma_{p}]} 1.
\]
From equation (3.2), we have
\[
\chi = \sum_{\sigma_{p} \in \Sigma_{p}} \xi_{\sigma_{p}} \otimes \text{ind}^{G_{p'}}_{[\sigma_{p}]} 1, \quad \xi_{\sigma_{p}} \in R^{+}(G_{p'}).
\]
Comparing these equations, using linear independence of \( \{ \text{ind}^{G}_{[\sigma]} 1 : \sigma \in \Sigma \} \) from Lemma 4.2, we get
\[
\xi_{\sigma_{p}} = \sum_{\tau} a_{\tau} \text{ind}^{G_{p'}}_{[\tau_{p'}]} 1 \in R^{+}(G_{p'}),
\]
where the sum is over \( \tau \in \Sigma \) such that \( \tau_{p} = \sigma_{p} \). Then \( \langle \xi_{\sigma_{p}}, \eta \rangle_{G_{p'}} \geq 0 \) for all irreducible characters \( \eta \) of \( G_{p'} = \Gamma_{p'} \times \Pi_{p'} \), in particular for \( \eta = \lambda^{*} \sigma_{p'} \otimes \lambda \), where \( \lambda^{*} \) is the contragredient of \( \lambda \). But
\[
\langle \xi_{\sigma_{p}}, \lambda^{*} \sigma_{p'} \otimes \lambda \rangle = \sum_{\tau} a_{\tau} (1, \text{res}^{G_{p'}}_{[\tau_{p'}]}(\lambda^{*} \sigma_{p'} \otimes \lambda)) \langle [\tau_{p'}] \rangle
\]
\[
= \sum_{\tau} a_{\tau} [\Gamma_{p'}][\gamma_{p'}] \lambda(\sigma_{p'} \gamma^{-1}) \lambda(\tau_{p'} \gamma) = \sum_{\tau} a_{\tau} \langle \lambda \sigma_{p'}, \lambda \tau_{p'} \rangle_{\Gamma_{p'}},
\]
where
and this equals \( a_\sigma \) since \( \lambda \sigma \) and \( \lambda \tau \) are different irreducible characters of \( \Gamma \) unless \( \sigma = \tau \), that is, \( \tau = \sigma \). Thus \( a_\sigma \geq 0 \).

Conversely, suppose that \( \Pi \) has at least 2 non-cyclic Sylow \( p \)-subgroups. First suppose that \( \Pi \) has a subgroup of the form \( \Pi' = C_{p_1} \times C_{p_2} \times C_{p_2} \cong C_{p_1} p_2 \times C_{p_1} p_2 \), where \( p_1, p_2 \) are distinct primes. Using the construction in Lemma 5.1 with \( G = C_{p_1} p_2 \), \( G' = \Gamma \times \Pi' \), \( N' = 1 \times \Pi' \), there exists a \( \mathbb{Z}G' \)-lattice \( M' \) satisfying (a) and (b) whose character is \( \chi' = - \text{ind}^{G'}_{[\tau]} 1 + \sum_{\sigma \in \Sigma'} \text{ind}^{G'}_{[\sigma]} 1 \). By Lemma 2.4, \( \chi = \text{ind}^{G'}_{[\tau]} \chi' \) is in \( D(\Gamma, \Pi) \). But \( \chi = - \text{ind}^{G'}_{[\tau]} 1 + \sum_{\sigma \in \Sigma'} \text{ind}^{G'}_{[\sigma]} 1 \) is not in \( \mathcal{P}^+ \) by Lemma 4.2, since \( \tau = 1 \) is not \( \Pi \)-conjugate to an element of \( \Sigma' \). Thus \( D(\Gamma, \Pi) \neq \mathcal{P}^+(\Gamma, \Pi) \).

If \( \Pi \) has at least 2 non-cyclic Sylow \( p \)-subgroups but does not have a subgroup isomorphic to \( C_{p_1} \times C_{p_2} \times C_{p_2} \), then \( \Pi \) is a quaternion group, and \( \Pi \) has a subgroup of the form \( \Pi' = C_p \times C_p \times Q_8 \) where \( p \) is an odd prime. Apply the construction of Lemma 5.2, where \( \Gamma = C_p \times C_4 \cong C_{4p} \), \( G' = \Gamma \times \Pi' \), \( N' = 1 \times \Pi' \), to get

\[
\chi' = r(- \text{ind}^{G'}_{[\tau]} 1 + \sum_{\sigma \in \Sigma'} \text{ind}^{G'}_{[\sigma]} 1) \in D(\Gamma, \Pi') \text{ for some } r \geq 1.
\]

As above, \( \chi = \text{ind}^{G'}_{[\tau]} \chi' \) is in \( D(\Gamma, \Pi) \) by Lemma 2.4, but to show that \( \chi \notin \mathcal{P}^+ \) we must be careful in our choice of \( \tau \). Let \( A \) be a cyclic normal subgroup of index \( 2 \) in \( \Pi_2 \) and choose \( \tau \) whose image is \( A \cap \Pi' \). Then the construction of Lemma 5.2 applies, since \( \text{im} \tau \), which is normal in \( \Pi \), is not \( \Pi \)-conjugate to any \( \text{im} \sigma \) with \( \sigma \in \Sigma' \). Applying Lemma 4.2 as before completes the proof.

\[\square\]

### 7. Finite Generation of \( \mathcal{D} \)

**Theorem 7.1.** If \( \Pi \) is nilpotent, then \( \mathcal{D}' \) and \( \mathcal{D} \) are finitely generated semigroups.

**Proof.** Set

\[
X = \bigoplus_{H \leq G} \text{ind}_{H}^{G} \mathbb{Z}, \text{ where } H \text{ varies over all subgroups of } G.
\]

For each prime \( p \) dividing \( |G| \), enumerate the distinct non-isomorphic indecomposable summands of \( \mathbb{Z} \otimes X \): suppose they are \( X(p, i), 1 \leq i \leq n_p \), and suppose that \( X(p, i) \) affords the character \( \chi(p, i) \) of \( G \). Let \( n = \sum n_p \), summed over primes dividing \( |G| \). We shall use some ideas in the proof of a result of Jones [CR, 33.2].

Following some of the notation of [CR], let \( C \) denote the additive semigroup of \( n \)-tuples of non-negative integers; partially order \( C \) by writing \( (a_i) \leq (b_i) \) in \( C \) if \( a_i \leq b_i \) for \( 1 \leq i \leq n \). If the \( \mathbb{Z}G \)-lattice \( M \) satisfies (a) and (b)', then by Proposition 3.4 and the Krull-Schmidt Theorem for \( \mathbb{Z}_pG \)-lattices, \( \mathbb{Z}_p \otimes M \) can be written uniquely as a direct sum of modules \( X(p, i) \):

\[
\mathbb{Z}_p \otimes M \cong \bigoplus_{1 \leq i \leq n_p} X(p, i)^{m(p, i)}, \text{ for unique non-negative integers } m(p, i).
\]

Let \( \theta(M) \) denote the ordered \( n \)-tuple in \( C \) whose entries are the integers \( m(p, i) \). Let \( \tilde{D}' \) be the set of \( n \)-tuples \( \theta(M) \in C \) where \( M \) ranges over all \( \mathbb{Z}G \)-lattices which satisfy (a) and (b)'; similarly, let \( \tilde{D} \) be the set of \( \theta(M) \in C \) where \( M \) satisfies (a) and (b). Given \( \theta(M) = (a(p, i)) \) in \( \tilde{D} \), we associate to \( \theta(M) \) the character \( \sum p a(p, i) \chi(p, i) \); this gives us a mapping of \( \tilde{D} \) onto \( \mathcal{D} \). Similarly we have a mapping...
of $\hat{D}'$ onto $D'$. Thus the theorem will follow if we can prove finite generation of $\hat{D}'$ and $\hat{D}$.

We first prove that $\hat{D}'$ is finitely generated. From Step 3 in the proof of Jones’ Theorem [CR, p. 689], any subset of $C$ has a finite set of minimal elements in the partial order we have given $C$. Let $S$ be the finite set of minimal elements of $\hat{D}' - \{0\}$. We claim that this set $S$ generates $\hat{D}'$. To prove this, let $\theta(M)$ be an element of $\hat{D}'$; we shall show that $\theta(M)$ is a sum of elements of $S$. This is true if $\theta(M) \in S$, so assume that there is an element $s \in S$ which is strictly smaller than $\theta(M)$. Also assume that if $L$ is a $\mathbb{Z}G$-lattice such that $\theta(L) \in \hat{D}'$ and which has smaller $\mathbb{Z}$-rank than $M$, then $\theta(L)$ is a sum of elements of $S$. Suppose that $s = \theta(M')$. Then locally at each prime, $M'$ is a direct summand of $M$, so [CR, 31.12], there is a lattice $M''$ in the same genus as $M'$ such that $M \cong M'' \oplus M_0$ for some $\mathbb{Z}G$-lattice $M_0$. Then $\theta(M'') = \theta(M')$, and $M_0$ satisfies (a) and (b$'$). Moreover, by our assumption on lattices with ranks smaller than that of $M$, $\theta(M_0)$ is a sum of elements of $S$. Since $\theta(M) = s + \theta(M_0)$, then $\theta(M)$ is a sum of elements of $S$, as claimed.

We next show that $\hat{D}$ is finitely generated. Let $S_0$ be the set of $s \in S$ so that $s = \theta(M)$ for some $M$ such that $\text{res}_N M$ is stably free but $s = \theta(M')$ for no $M'$ such that $\text{res}_N M'$ is free. Let $r$ be as in Lemma 2.5. Set

$$T = \left(\hat{D} \cap \left\{ \sum_{s \in S} a_s s : 0 \leq a_s \leq r, s \in S \right\} \right) \cup \{(r + 1)s : s \in S_0\}.$$

Note that if $\text{res}_N M$ is stably free, then since $r + 1 \geq 2$, $\text{res}_N M^{r+1}$ is free by [CR, 41.20], so $T \subseteq \hat{D}$. We claim that $T$ generates $\hat{D}$. Suppose that $d = \theta(D) \in \hat{D}$. As above, we assume that if $\theta(L) \in \hat{D}$ and $L$ has smaller $\mathbb{Z}$-rank than $D$, then $\theta(L)$ is a sum of elements of $T$. Since $\hat{D}'$ is generated by $S$, we can write

$$d = \sum_{s \in I} a_s s \text{ for a subset } I \subseteq S \text{ with } a_s \geq 1, s \in I.$$

Write $a_s = b_s + rc_s$ where $1 \leq b_s \leq r$ and $c_s \geq 0$, and set

$$e = \sum_{s \in I} b_s s, \quad f = r \sum_{s \in I} c_s s,$$

so $d = e + f$. We have $d = \hat{E}$, and since $rD' \subseteq D$, then $f = \hat{E}$, so we can find $\mathbb{Z}G$-lattices $D$ and $F$ satisfying (a) and (b) with $d = \theta(D)$ and $f = \theta(F)$. Also, $e = \theta(E')$ for a lattice $E'$ satisfying (a) and (b$'$). Now $\theta(D) = \theta(E' \oplus F)$, so $D$ and $E' \oplus F$ are in the same genus, and locally for all $p$, $F$ is a direct summand of $D$.

We will apply a result of Roiter and Jacobinski [CR, 31.32]; we must check that every irreducible $\mathbb{Q}G$-composition factor of $\mathbb{Q}F$ occurs more often as a composition factor of $\mathbb{Q}D$. This is so because $d = e + f$ and $e = \sum_{s \in I} b_s s$, where $b_s > 0$ for $s \in I$. Thus $D \cong E \oplus F$ for some $\mathbb{Z}G$-lattice $E$. Restricting to $N$, we see that $\text{res}_N E$ is stably free. If it is actually free (so, in particular, if we have the Eichler condition for $\mathbb{Q}N$), then $e = \theta(E) \in T$, and we are done.

From [CR, 41.20], if $\sum_{s \in I} b_s > 1$, then $\text{res}_N E$ is free; so we may assume that $\sum_{s \in I} b_s = 1$. Thus $I$ contains a single element $s_0$, and $e = s_0$. If $s_0 \notin S_0$, then writing $s_0 = \theta(E')$ where $E'$ satisfies (a) and (b$'$), it follows that $D$ is in the same genus as $E' \oplus F$; we replace $D$ by $E' \oplus F$ and we are done as before. So we assume that $s_0 \in S_0$. 

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Suppose that \( f = 0 \). Then \( d = s_0 \notin \mathcal{D} \). Hence \( f \neq 0 \). Then \( c_s \geq 1 \) for some \( s \in I \); since \( I = \{ s_0 \} \), then \( f = c_s r s_0 \) with \( c_s \geq 1 \). Then \( e = (r + 1)s_0 + (c_s - 1)(rs_0) \) with \( (r + 1)s_0 \in T \) and \( (c_s - 1)(rs_0) = \theta(L) \in \mathcal{D} \), where \( L \) has smaller rank than \( D \). Thus \( d \) is indeed a sum of elements of \( T \), and the proof is complete. \( \Box \)

8. Complements

Assume we have \( \Gamma, \Pi \) and \( G = \Gamma \times \Pi \) as above. As in the proof of Theorem 3.3, \( \mathcal{G} \) is the Galois group of \( \mathbb{Q}(\zeta) \) over \( \mathbb{Q} \), where \( \zeta \) be a primitive \( |G| \)-th root of unity, and \( \mathcal{G}_p \) denotes the Galois group of \( \mathbb{Q}_p(\zeta) \) over \( \mathbb{Q}_p \). We identify \( \mathcal{G}_p \) as a subgroup of \( \mathcal{G} \), namely the decomposition group at any prime of \( \mathbb{Q}(\zeta) \) above \( p \). Let \( \Sigma_p \) be a complete set of elements of hom(\( \mathcal{G}_p, \Pi_p \)) up to conjugation by \( \Pi_p \). As in [WCt], define a label for \( \chi \) to be a collection \( b = \{ b_p \} \) of functions

\[
b_p : \Sigma_p \times \text{irr}(\mathcal{G}_p) \to \mathbb{Z}_{\geq 0},
\]

one for each prime \( p \), so that on writing

\[
\chi = \sum_{\eta \in \text{irr}(\mathcal{G}_p)} \eta \otimes \lambda_\eta \text{ relative to } R(G) = R(\mathcal{G}_p) \otimes R(\mathcal{G}_p),
\]

we have

\[
(i)_p \quad \lambda_\eta = \sum_{\sigma_p \in \Sigma_p} b_p(\sigma_p, \eta) \text{ ind}_{[\sigma_p]}^{\mathcal{G}_p} 1,
\]

\[
(ii)_p \quad b_p(\sigma_p, \eta^\omega) = b_p(\sigma_p, \eta) \text{ for all } \omega \in \mathcal{G}_p.
\]

**Theorem 8.1.** Suppose that \( \Gamma \) and \( \Pi \) are nilpotent. Then labels for \( \chi \) are in bijection with genera of \( \mathbb{Z}G \)-lattices with character \( \chi \) which satisfy (a) and (b').

**Proof.** The proof comes from a closer look at the proof of Theorem 3.3. Suppose that \( M \) is a \( \mathbb{Z}G \)-lattice which satisfies (a) and (b'). From equation (3.1), we have

\[
\mathbb{Z}_p \otimes M \cong \bigoplus_{\sigma_p \in \text{hom}(\Gamma_p, \Pi_p)} X_{\sigma_p} \otimes \text{ind}_{[\sigma_p]}^{\mathcal{G}_p} \mathbb{Z}_p.
\]

Instead of summing over \( \sigma_p \in \text{hom}(\Gamma_p, \Pi_p) \), we may sum over \( \Sigma_p \), because replacing \( \sigma_p \) by a \( \Pi_p \)-conjugate gives a \( \mathcal{G}_p \)-conjugate of \( [\sigma_p] \), hence a module isomorphic to \( \text{ind}_{[\sigma_p]}^{\mathcal{G}_p} \mathbb{Z}_p \). Then the different groups \( [\sigma_p] \), \( \sigma_p \in \Sigma_p \), are the vertices of the summands of \( \mathbb{Z}_p \otimes M \), so the modules \( X_{\sigma_p} \) are unique up to isomorphism, and their characters \( \xi_{\sigma_p} \) give the well-defined equation

\[
\chi = \sum_{\sigma_p \in \Sigma_p} \xi_{\sigma_p} \otimes \text{ind}_{[\sigma_p]}^{\mathcal{G}_p} 1, \quad \xi_{\sigma_p} \in R^+(\mathcal{G}_p).
\]

Then \( (ii)_p \) follows from equation (3.3) and \( (i)_p \) comes from equation (3.4). Since these depend only on \( \mathbb{Z}_p \otimes M \), the same label would be attached to any lattice in the same genus as \( M \).

Conversely, suppose that \( b \) is a label. From \( (ii)_p \), \( b(\sigma_p, \eta) \) just depends on the \( \mathcal{G}_p \)-orbit \( \mathcal{O} \) containing \( \eta \). Then as in equation (3.6), we let \( \xi_{\sigma_p} = \sum_{\mathcal{O}} b(\sigma_p, \mathcal{O}) r_{\mathcal{O}} \), and \( \xi_{\sigma_p} \in R^+_{\mathcal{Q}_p} (\mathcal{G}_p) \). The lattice

\[
M(b_p) = \bigoplus_{\sigma_p} L_{\sigma_p} \otimes_{\mathbb{Z}_p} \text{ind}_{[\sigma_p]}^{\mathcal{G}_p} \mathbb{Z}_p
\]
in equation (3.7) satisfies the local versions \((a_p)\) and \((b_p)\) of (a) and (b), and has character \(\chi\) by \((i)_p\). Then the \(\mathbb{Z}G\)-lattice \(M = M(b)\) at the end of the proof has \(\mathbb{Z}_p \otimes M \cong M(b_p)\) for all \(p\), hence it satisfies (a) and (b'); the construction of \(M\) depends on the identifications \(\phi_p\), but the genus of \(M\) is well-defined.

\[\tag{\*} \]

**Corollary 8.2.** If \(\Gamma\) is cyclic, there is only one genus of \(\mathbb{Z}G\)-modules having a given character in \(D'\).

**Proof.** By Lemma 4.2, for each \(p\) the set \(\{\text{ind}^G_{\sigma_p} : \sigma_p \in \Sigma_p\}\) is linearly independent; hence there is only one solution to \((i)_p\), and only one label for a character \(\chi\).

**Remarks.** Given \(\chi \in D'\), we want to decide whether \(\chi \in D\). We begin by determining all labels \(b\) for \(\chi\); this is a purely character-theoretic problem. For each \(b\) one then constructs a lattice \(M = M(b)\) in the genus of the label. Deciding whether the genus of \(M\) contains an \(M'\) with \(\text{res}_\chi M'\) stably free can then be approached by genus class group methods, generalizing Theorem 3 of \([WC]\). Carrying this out is a long computation which will answer the existence question “Is \(\chi\) in \(D'\)” (at least when we have the Eichler condition for \(\mathbb{Z}\Pi\)). However, the construction of \(\alpha : \Gamma \to SGL_n(\mathbb{Z}\Pi)\) with double-action character \(\chi\) takes still more calculation.

Nevertheless this is how the first example of \(\S 5\) was found. It is typical of such computations that once an \(M\) satisfying (a) and (b) is found, it is simpler to describe it directly, as we have done in \(\S 5\).

Finally there is the issue of finding \(\chi \in D'\), no multiple of which is in \(P^+\), in the first place. This amounts to finding generators of \(D'\), and this again is a problem of character theory, by Theorem 3.3. In examples we have considered, \(D'\) has many generators, even when \(\Gamma\) is cyclic. More exploration of this, perhaps by computer, is still needed.

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