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TIGHT CLOSURE, PLUS CLOSURE AND FROBENIUS CLOSURE IN CUBICAL CONES

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ABSTRACT. We consider tight closure, plus closure and Frobenius closure in the rings $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$. We use a $\mathbb{Z}_3$-grading of these rings to reduce questions about ideals in the quotient rings to questions about ideals in the regular ring $K[[x, y]]$. We show that Frobenius closure is the same as tight closure in certain classes of ideals when $p \equiv 2 \bmod 3$. Since $I^F \subseteq IR^+ \cap R \subseteq I^*$, we conclude that $IR^+ \cap R = I^*$ for these ideals. Using injective modules over the ring R^∞, the union of all p^eth roots of elements of R, we reduce the question of whether $I^F = I^*$ for $\mathbb{Z}_3$-graded ideals to the case of $\mathbb{Z}_3$-graded irreducible modules. We classify the irreducible m-primary $\mathbb{Z}_3$-graded ideals. We then show that $I^F = I^*$ for most irreducible m-primary $\mathbb{Z}_3$-graded ideals in $K[[x, y, z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \bmod 3$. Hence $I^* = IR^+ \cap R$ for these ideals.

In this paper we discuss the conjecture that $I^* = IR^+ \cap R$, where R^+ denotes the integral closure of a domain R of characteristic p in an algebraic closure of its fraction field and I^* denotes the tight closure of I. The ring R^+ is characterized by the property that it is a domain integral over R and every monic polynomial with coefficients in R^+ factors into monic linear factors. This characterization can be used to prove the following property of R^+: If W is a multiplicatively closed set of R, then $(W^{-1}R)^+ \cong W^{-1}R^+$. Aside from providing a much more concrete description of tight closure, proving that $I^* = IR^+ \cap R$ would solve the localization problem for tight closure. It is known that $I^* = IR^+ \cap R$ for parameter ideals [Sm1] and for rings in which every ideal of the normalization is tightly closed. Also, for those ideals I of an excellent local domain R such that R/I has finite phantom projective dimension, it is known that $I^* = IR^+ \cap R$ [Ab]. However, the conjecture is open even for two-dimensional normal Gorenstein domains. In particular, the conjecture is open for the cubical cone $K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$, and more generally for rings of the form $K[[x, y, z]]/(F(x, y, z))$ where F is a homogeneous cubic polynomial.

We consider tight closure, plus closure and Frobenius closure in the rings $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$. In Section 1 we use a $\mathbb{Z}_3$-grading of these rings to reduce questions about ideals in the quotient rings to questions about ideals in the regular rings $K[[x, y]]$. In Section 2 we show that the Frobenius closure of an ideal I, denoted I^F, is the same as the tight closure in certain classes of ideals when $p \equiv 2 \bmod 3$. Since $I^F \subseteq IR^+ \cap R \subseteq I^*$,

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we conclude that $IR^+ \cap R = I^*$ for these ideals. In Section 3 we use injective modules over the ring R^∞, the union of all p^eth roots of elements of R, to reduce the question of whether $I^F = I^*$ for $\mathbb{Z}_3$-graded ideals to the case of $\mathbb{Z}_3$-graded irreducible modules. In Section 4 we classify the irreducible m-primary $\mathbb{Z}_3$-graded ideals and then show that $I^F = I^*$ for most irreducible m-primary $\mathbb{Z}_3$-graded ideals in $K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \bmod 3$. Hence $I^* = IR^+ \cap R$ for these ideals.

1. Cubical Cones

We denote by $\mathbb{Z}_n$ the ring $\mathbb{Z}/n\mathbb{Z}$. We first describe a $\mathbb{Z}_3$-grading on the cubical cones $R = K[[x,y,z]]/(x^3+y^3+z^3)$. We will also discuss tight closure and Frobenius closure in these rings before proving the main results, Theorem 2.1 and Theorem 4.5.

$\mathbb{Z}_3$-grading. First we describe a $\mathbb{Z}_n$-grading of rings of the form $R = A[[z]]/(z^n - a)$ where $a \in A$. The ring R has the following decomposition as an A-module: $R = A \oplus Az \oplus \cdots \oplus Az^{n-1}$. Every element of R can be uniquely expressed as an element of $A \oplus Az \oplus \cdots \oplus Az^{n-1}$ by replacing every occurrence of z^n by a. R is $\mathbb{Z}_n$-graded, where the ith piece of R, denoted by R_i, is Az^i, $0 \le i < n$, since $Az^iAz^j \subseteq Az^{i+j}$ if $i+j<n$ and $Az^iAz^j \subseteq Az^{i+j-n}$ if $i+j \ge n$.

We use this idea to obtain a $\mathbb{Z}_3$-grading on $R = K[[x,y,z]]/(x^3+y^3+z^3)$ by letting $A = K[[x,y]]$. Let H, I, and J be ideals of $K[[x,y]]$. Suppose $H \subseteq I \subseteq J \subseteq H\colon (x^3+y^3)$. Then $H + Iz + Jz^2$ is an ideal of R. On the other hand, in order for a $\mathbb{Z}_3$-graded ideal to be closed under multiplication by z, it must have this form. Thus, it is easy to see that the ideals of R homogeneous with respect to the $\mathbb{Z}_3$-grading are precisely the ideals of this form. We can study the ideal $H+Iz+Jz^2$ by considering (H,I,J), a triple of ideals in $K[[x,y]]$. Indeed, we will use the notation (H,I,J) to denote the ideal $H+Iz+Jz^2$, and it is understood that H, I, and J are ideals of $K[[x,y]]$. For example, the ideal (x^2, y^2z, xz^2) is represented by the triple (H,I,J) where $H=(x^2,y^5,xy^3)$, $I=(x^2,y^2)$ and $J=(x,y^2)$.

If R is a reduced ring of characteristic p, we write $R^{1/q}$ for the ring obtained by adjoining qth roots of all elements of R. Next we observe that the $\mathbb{Z}_3$-grading on R extends to $R^\infty = \bigcup_q R^{1/q}$. It is enough to show that the grading on R extends to $R^{1/q}$. If $u \in R_i$, then the image of u is in $R_j^{1/q}$ where $qi \equiv j \bmod 3$.

We now show that if I is a graded ideal, then so is I^*.

(1.1) Lemma. *Let R be a finitely generated k-algebra that is $\mathbb{Z}_n$-graded and of characteristic p, where p is not a prime factor of n ($p=0$ is allowed). Then the tight closure of a homogeneous ideal of R is homogeneous.*

Proof. Without loss of generality, we can assume R is reduced, since the tight closure of I is the preimage of the tight closure of the image of I modulo the nilradical. Because the singular locus of R is defined by a homogeneous ideal not contained in any minimal prime, R has a homogeneous test element, say c. Let I be a homogeneous ideal, and suppose that $z = z_0 + z_1 + \cdots + z_{n-1}$ is in I^*, where z_i is the homogeneous component of z of degree i mod n. Now we have $cz^q = cz_0^q + cz_1^q + \cdots + cz_{n-1}^q$ is in the homogeneous ideal $I^{[q]}$, and hence each of its homogeneous components is in $I^{[q]}$. But each of the elements cz_i^q is homogeneous of degree $qi + \deg c \bmod n$, and since q is invertible in $\mathbb{Z}_n$, these all have distinct degrees. Thus each $cz_i^q \in I^{[q]}$ for all $q \gg 0$ and each $z_i \in I^*$. This shows that I^* is homogeneous. □

Tight Closure and Frobenius Closure. We review the definition of tight closure for ideals of rings of characteristic $p > 0$. Tight closure is defined more generally for modules and also for rings containing fields of arbitrary characteristic. See [HH1] or [Hu] for more details.

(1.2) Definition. Let R be a ring of characteristic p and I be an ideal in a Noetherian ring R of characteristic $p > 0$. An element $u \in R$ is in the *tight closure* of I, denoted I^*, if there exists an element $c \in R$, not in any minimal prime of R, such that for all large $q = p^e$, $cx^q \in I^{[q]}$ where $I^{[q]}$ is the ideal generated by the qth powers of all elements of I.

We denote by I^F the Frobenius closure of an ideal I. Recall that $I^F = \{u \in R : u^q \in I^{[q]}$ for some $q\}$. We can also think of I^F as $IR^\infty \cap R$, so $I^F \subseteq IR^+ \cap R$, since $R^\infty \subseteq R^+$. In addition, we know that $IR^+ \cap R \subseteq I^*$ [HH2]. Hence $I^F \subseteq IR^+ \cap R \subseteq I^*$. So, if $I^F = I^*$, then that implies that $I^* = IR^+ \cap R$.

An interesting bifurcation of this question in $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ depends on the characteristic of K. If K has characteristic p and $p \equiv 1 \mod 3$, then R is F-pure [HR, Proposition 5.21(c)] and $I^F = I$. We know that $I^* \neq I$ for some ideals of R, so I^F cannot equal I^*, although it is still possible that $I^* = IR^+ \cap R$. If $p \equiv 2 \mod 3$, then R is not F-pure and it is conjectured that $I^F = I^*$ and hence that $I^* = IR^+ \cap R$.

The goal of this paper is to show that $I^* = I^F$, and hence $I^* = IR^+ \cap R$, for many graded ideals of R when the characteristic of K is congruent to 2 mod 3.

Test Elements in Cubical Cones. In many applications one would like to be able to choose the element c in the definition of tight closure independent of x or I. It is very useful when a single choice of c, a test element, can be used for all tight closure tests in a given ring.

(1.3) Definition. The ideal of all $c \in R$ such that, for any ideal $I \subseteq R$, we have $cu^q \in I^{[q]}$ for all q whenever $u \in I^*$ is called the *test ideal* for R. An element of the test ideal that is not in any minimal prime is called a *test element*. The ideal of all $c \in R$ such that for all parameter ideals (ideals generated by i elements with height at least i) $I \subseteq R$, we have $cu^q \in I^{[q]}$ for all q whenever $u \in I^*$ is called the *parameter test ideal* for R.

We now determine the test ideal for $K[[x, y, z]]/(x^3 + y^3 + z^3)$. The following proposition is proved for char $K \neq 2, 3$ using a somewhat different method in [Sm2].

(1.4) Proposition. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$. Then the maximal ideal, m, is the test ideal.*

Proof. First note that we can reduce to the case where K is algebraically closed. Enlarging K to an algebraic closure is an integral extension and will not affect tight closure.

Let τ be the parameter test ideal for R. By Proposition 4.4(iii) of [Sm2], we know that $\tau = \{\, c \in R$ such that $c(x^t, y^t)^* \subset (x^t, y^t)$ all $t \in \mathbb{N}\,\}$. Since R is Gorenstein, the test ideal is the same as the parameter test ideal [Sm2, Proposition 4.4].

We will show that $(x^t, y^t)^* = (x^t, y^t, x^{t-1}y^{t-1}z^2)$. Then it is clear that $\tau = (x, y, z)$ since $(x^t, y^t) \colon (x^t, y^t, x^{t-1}y^{t-1}z^2) = (x, y, z)$. Let $I = (x^t, y^t)$ and $J =$

$(x^t, y^t, x^{t-1}y^{t-1}z^2)$. The socle mod J is generated by $u_1 = x^{t-2}y^{t-1}z^2$, $u_2 = x^{t-1}y^{t-2}z^2$ and $u_3 = x^{t-1}y^{t-1}z$. To see that $I^* = J$, it suffices to show that $\sum Ku_i \cap I^* = 0$, since if $J \subsetneq I^*$, then I^* has nonzero intersection with $J : m$.

We would like to see that $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 \notin (x^t, y^t)^*$ where $\lambda_i \in K$. Using the $\mathbb{Z}_3$-grading, it is enough to show that $\lambda_3 u_3 \notin (x^t, y^t)^*$ and $\lambda_1 u_1 + \lambda_2 u_2 \notin (x^t, y^t)^*$. Using the $\mathbb{Z}_3$-grading again, but now letting x play the role of z ($R = A[[x]]/(x^3 - a)$, $A = K[[y, z]]$), we can reduce the problem to showing $\lambda_1 u_1 \notin (x^t, y^t)^*$, $\lambda_2 u_2 \notin (x^t, y^t)^*$ and $\lambda_3 u_3 \notin (x^t, y^t)^*$.

Suppose $u_3 \in (x^t, y^t)^*$. Then $z \in (x^t, y^t)^* : x^{t-1}y^{t-1}$. We claim that $(x^t, y^t)^* : x^{t-1}y^{t-1} \subseteq (x, y)^*$. Let $u \in (x^t, y^t)^* : x^{t-1}y^{t-1}$, so $ux^{t-1}y^{t-1} \in (x^t, y^t)^*$. Then there exists c such that $cu^q x^{(t-1)q} y^{(t-1)q} \in (x^{tq}, y^{tq})$. This implies that $cu^q \in (x^{tq}, y^{tq}) : x^{(t-1)q} y^{(t-1)q}$. But $(x^{tq}, y^{tq}) : x^{(t-1)q} y^{(t-1)q} \subseteq (x^q, y^q)^*$ by a colon capturing argument [HH1, Theorem 7.15a]. So $cu^q \in (x^q, y^q)^*$, and we can find a test element d such that $dcu^q \in (x^q, y^q)$ for all q. In other words, $u \in (x, y)^*$. Thus $x^{t-1}y^{t-1}z \in (x^t, y^t)^*$ implies $z \in (x, y)^*$, but we know that $z \notin (x, y)^*$ by a degree argument [Sm3, Theorem 2.2].

Now suppose $u_1 \in (x^t, y^t)^*$. This implies that $z^2 \in (x^t, y^t)^* : x^{t-2}y^{t-1}$. Using the same argument as before, we can show that $(x^t, y^t)^* : x^{t-2}y^{t-1} \subseteq (x^2, y)^*$. By symmetry, we must also have $z^2 \in (x, y^2)^*$. So $z^2 \in (x^2, y)^* \cap (x, y^2)^*$ which is contained in $(x^2, xy, y^2)^*$ by Theorem 7.12 of [HH1]. Again, $z^2 \notin (x^2, xy, y^2)^*$ by degree arguments [Sm3, Theorem 2.2]. ☐

The fact that m is the test ideal provides quite a lot of information. For example, using the fact that m is the test ideal, we may conclude that if $u \in I^* \backslash I$, then u is in the socle mod I.

(1.5) Proposition. *Let (R, m) be a local ring. Suppose m is the test ideal. If $u \in I^* \backslash I$, then u is in the socle mod I.*

Proof. Let $u \in I^* \backslash I$. Then $mu^q \subseteq I^{[q]}$ for all q. In particular, $mu \subseteq I$. This says exactly that u is in the socle mod I. ☐

(1.6) Remark. Although determining whether an element is in the tight closure or Frobenius closure of an ideal involves checking certain conditions for infinitely many values of $q = p^e$, there are some instances where one q is enough. If c is a test element and $cu^q \notin I^{[q]}$ for some q, then $u \notin I^*$. Similarly, if $u^q \in I^{[q]}$ for some q, then $u^{q'} \in I^{[q']}$ for all $q' \geq q$ and hence $u \in I^F$.

In either situation, since we only need one q that works, we can pick whichever value of q is most helpful. For example, when $p \equiv 2 \bmod 3$, $p^{2e} \equiv 1 \bmod 3$ and $p^{2e+1} \equiv 2 \bmod 3$. It is often easier to work with powers of p with a particular residue mod 3 and so we may choose q accordingly.

Applications of the $\mathbb{Z}_3$-grading to Tight Closure. When trying to determine I^* and I^F for a given ideal I, we are interested in calculating $I^{[q]}$ and $I : m$. We will first calculate $I : m$.

(1.7) Lemma. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, and let $H + Iz + Jz^2$ be a $\mathbb{Z}_3$-graded ideal in R. Then $(H + Iz + Jz^2) : (x, y, z) =$*

$$\big((H : (x, y)) \cap I\big) + \big((I : (x, y)) \cap J\big)\, z + \big((J : (x, y)) \cap (H : (x^3 + y^3))\big)\, z^2.$$

Proof. Let R_i denote the i mod 3 graded piece of R. Suppose $r \in R_0$ and $r \in (H + Iz + Jz^2) : (x, y, z)$. So we must have $r(x, y) \subseteq H$ and $rz \in Iz$. In other words,

$r \in \big(H\colon (x,y)\big) \cap I$. Similarly, if $rz \in R_1$ and $rz \in (H+Iz+Jz^2)\colon (x,y,z)$, we must have $r \in \big(I\colon (x,y)\big) \cap J$. Let $rz^2 \in R_2$ and suppose $r \in (H+Iz+Jz^2)\colon (x,y,z)$. Again, we see that $r \in J\colon (x,y)$. We also know that $(rz^2)z = r(x^3+y^3) \in (H+Iz+Jz^2)$. Since $r(x^3+y^3) \in R_0$, we must have $r(x^3+y^3) \in H$. In other words, $r \in H\colon (x^3+y^3)$. So $r \in \big((J\colon (x,y)) \cap (H\colon (x^3+y^3))\big)$. □

Next we will determine $I^{[q]}$ when $q \equiv 2 \bmod 3$.

(1.8) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2$ mod 3. Let $q = p^{2e+1} = 3h+2$ and let $f = x^3+y^3$. Let $H+Iz+Jz^2$ be a $\mathbb{Z}_3$-graded ideal in R. Then*

$$\begin{aligned}(H+Iz+Jz^2)^{[q]} &= (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+2}) \\ &\quad + (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1})z + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})z^2.\end{aligned}$$

Let $u = u_0 + u_1 z + u_2 z^2$. Then $u^q \in (H+Iz+Jz^2)^{[q]}$ in R if and only if

$$\begin{aligned}{u_0}^q &\in (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+2}),\\ {u_1}^q f^h &\in (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1}),\\ {u_2}^q f^{2h+1} &\in (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1}) \qquad \textit{in } K[[x,y]].\end{aligned}$$

Proof. We start by noting that $(H+Iz+Jz^2)^{[q]}$ is generated by $H^{[q]} + I^{[q]}z^q + J^{[q]}z^{2q}$. Rewriting this using $q = 3h+2$ and the basic relation in R, $z^3 = -(x^3+y^3)$, yields $H^{[q]} + I^{[q]}f^h z^2 + J^{[q]}f^{2h+1}z$. We will first consider $(H+Iz+Jz^2)^{[q]} \cap R_0$. If we multiply $I^{[q]}f^h z^2$ by z, we get $I^{[q]}f^h z^3 = I^{[q]}f^{h+1}$ which is in R_0. Similarly, multiplying $J^{[q]}f^{2h+1}z$ by z^2 gives $J^{[q]}f^{2h+1}z^3 = J^{[q]}f^{2h+2}$. Thus,

$$(H+Iz+Jz^2)^{[q]} \cap R_0 = (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+2}).$$

Similar arguments show that

$$\begin{aligned}(H+Iz+Jz^2)^{[q]} \cap R_1 &= (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1}) \text{ and}\\ (H+Iz+Jz^2)^{[q]} \cap R_2 &= (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1}).\end{aligned}$$

Since $u^q = u_0^q + u_2 f^{2h+1}z + u_1 f^{2h}z^2$, the last statement in the lemma is now clear. □

Next we determine $I^{[q]}$ when $q \equiv 1 \bmod 3$.

(1.9) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2$ mod 3. Let $q = p^{2e} = 3h+1$ and let $f = x^3+y^3$. Let $H+Iz+Jz^2$ be a $\mathbb{Z}_3$-graded ideal in R. Then*

$$\begin{aligned}(H+Iz+Jz^2)^{[q]} &= (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1}) \\ &\quad + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})z + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h})z^2.\end{aligned}$$

Let $u = u_0 + u_1 z + u_2 z^2$. Then $u^q \in (H+Iz+Jz^2)^{[q]}$ in R if and only if

$$\begin{aligned}{u_0}^q &\in (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1}),\\ {u_1}^q f^h &\in (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1}),\\ {u_2}^q f^{2h} &\in (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h}) \quad \textit{in } K[[x,y]].\end{aligned}$$

Proof. The proof is identical to the proof of Lemma 1.8 except we use $q = 3h+1$. ☐

Note that $f^h = (x^3+y^3)^h$ appears often in the calculations. The question of whether a given element is in the tight closure of an ideal often comes down to whether or not a certain power of f is contained in (x^q, y^q). To this end, we establish the following lemmas which will be useful in showing that $I^* = I^F$.

(1.10) Lemma. *Let* $A = K[[x,y]]$, *where* K *is a field of characteristic* p *and* $p \equiv 2$ *mod* 3. *Let* $p = 3h+2$ *and let* $f = x^3+y^3$. *Then* $f^{2h} \notin (x^p, y^p)$. *Let* $q = p^{2e} = 3k+1$*; then* $f^{2k} \in (x^q, y^q)$.

Proof. Expand $f^{2h} = (x^3+y^3)^{2h}$ using the binomial theorem. Since $\binom{2h}{h}x^{3h}y^{3h}$ is a term in the expansion and $x^{3h}y^{3h} \notin (x^{3h+2}, y^{3h+2}) = (x^p, y^p)$, it suffices to see that $\binom{2h}{h} \not\equiv 0 \bmod p$. But $2h < p$, so p does not divide $\binom{2h}{h}$.

As in the above case, $f^{2k} \in (x^q, y^q)$ if and only if $\binom{2k}{k} \equiv 0 \bmod p$. Suppose we know that $z^{2q} \in (x^q, y^q)R$ where $q = 3k+1$. Using the basic relation in R we see that $z^{2q} \in (x^q, y^q)R$ if and only if $f^{2k}z^2 \in (x^q, y^q)R$. Using the $\mathbb{Z}_3$-grading we see that this is equivalent to having $f^{2k} \in (x^q, y^q)A$. Expand f^{2k} using the binomial theorem to see that this is equivalent to having $\binom{2k}{k} \equiv 0 \bmod p$. In other words, $\binom{2k}{k} \equiv 0 \bmod p$ if and only if $z^{2q} \in (x^q, y^q)R$ where $q = 3k+1$. We know that $z^{2p} \in (x^p, y^p)R$ when $p \equiv 2 \bmod 3$ by the proof of Proposition 4.3. This implies that $z^{2q} \in (x^q, y^q)$ for all $q = p^e$, in particular for $q = 3k+1$. Hence $\binom{2k}{k} \equiv 0 \bmod p$, and $f^{2k} \in (x^q, y^q)$. ☐

We will use the following result about calculating binomial coefficients mod p in Lemma 1.12.

(1.11) Lucas's Theorem. *Let* p *be a prime and let* $n = \sum_0^s a_i p^i$, $0 \le a_j < p$, $m = \sum_0^s b_i p_i$, $0 \le b_k < p$. *Then* $\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_s}{b_s} \mod p$.

Proof. See [Fi, Theorem 1] or [L, p. 230]. ☐

(1.12) Lemma. *Let* $A = K[[x,y]]$, *where* K *be a field of characteristic* p *and* $p \equiv 2$ *mod* 3. *Let* $q = p^{2e} = 3h+1$ *and* $f = x^3+y^3$. *Then* $\binom{3h-2}{h-1} \not\equiv 0$ *mod* p *and* $f^{2h-2} \in (x^q, y^q)$ *except when* $q = 25$.

Proof. Since $p^{2e} = 3h+1$, we can write $3h-2 = p^{2e}-3$. So $\binom{3h-2}{h-1} = (p^{2e}-3)(p^{2e}-4)\cdots(p^{2e}-(h+1))/1\cdot 2\cdots(h-1)$. It is easy to show that $\binom{3h-2}{h-1}$ is divisible by p if and only if $((p^{2e}-h)(p^{2e}-(h+1)))/2$ is divisible by p. Routine divisibility arguments show that this cannot happen.

To see that $f^{2h-2} \in (x^q, y^q)$, we expand f^{2h-2} using the binomial theorem. It is sufficient to show that $\binom{2h-2}{h-1}$ and $\binom{2h-2}{h}$ are congruent to zero mod p. If $p \neq 2$, then p divides $\binom{2h-2}{h}$ if and only if p divides $\binom{2h-2}{h-1}$. Next note that if $p \neq 2, 5$, then p divides $\binom{2h}{h}$ if and only if p divides $\binom{2h-2}{h-1}$.

We know from Proposition 1.10 that p divides $\binom{2h}{h}$ for the values of h we are considering, so if $p \neq 2, 5$, we know that p also divides $\binom{2h-2}{h-1}$ and $\binom{2h-2}{h}$. If $p = 2$, using (1.11), we can show that $\binom{2h-2}{h} = \binom{\text{even}}{\text{odd}} \equiv 0 \mod 2$ and $\binom{2h-2}{h-1} = \binom{2(h-1)}{h-1} \equiv 0 \mod 2$.

It remains to see that $\binom{2h-2}{h} \equiv 0 \bmod 5$ and $\binom{2h-2}{h-1} \equiv 0 \bmod 5$. We know from above that if $p \neq 2$, then it is enough to show that $\binom{2h-2}{h} \equiv 0 \bmod 5$. Write $5^{2e} = 3h+1$. Using (1.11), we see that $\binom{2h-2}{h} \equiv 0 \mod 5$ as long as $5^{2e} > 25$. ☐

At times, we will be able to make use of the fact that we are working over a regular ring or that $K[[x,y,z]]/(x^3+y^3+z^3)$ is flat as a $K[[x,y]]$-module. The following lemma and corollary provide useful information in these situations.

(1.13) Lemma. *Let R, S be arbitrary Noetherian rings such that S is a flat R-algebra, and let I, J be ideals of R. Then $IS\colon {}_S JS = (I\colon {}_R J)S$, where $I\colon {}_R J = \{\, r \in R : rJ \subseteq I \,\}$.*

Proof. See [N, Theorem 18.1, part 2]. ☐

(1.14) Corollary. *In a regular ring R of characteristic p, for any two ideals I, J we have $I^{[q]} : {}_R J^{[q]} = (I : {}_R J)^{[q]}$ for all q. In particular, $I^{[q]} \colon x^q = (I\colon x)^{[q]}$ for all q.*

Proof. See Corollary 4.3 of [HH1]. The statement follows from Lemma 1.13, since the iterated Frobenius endomorphism $F^e : R \to R$ is flat when R is regular [K, Theorem 2.1] and $I^{[q]} = F^e(I)R$. ☐

2. Tight Closure and Frobenius Closure in Cubical Cones

We can now show that $I^* = I^F$ for some not necessarily irreducible ideals. We will discuss irreducible ideals in Section 4.

(2.1) Proposition. *Let I be a $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2$ mod 3. Let $f = x^3+y^3$. If I has any of the following forms, then $I^* = I^F$.*

(1) (H,H,H),
(2) $\big(H,H,H\colon (x,y)\big)$,
(3) $\big(H,H\colon (x,y),H\colon f\big)$,
(4) $\big(H,H\colon (x,y),H\colon (x,y)\big)$,
(5) $\big(H,H,H\colon (x^2,y)\big)$.

In fact, in (2)–(5), I is tightly closed, i.e. $I = I^*$.

Proof. We know that if $u \in I^* \backslash I$, then u is in the socle mod I (Proposition 1.5), so it is sufficient to check whether elements of the socle are in I^* and I^F.

Proof of (1). Let $q = 3h+2$. Using the $\mathbb{Z}_3$-grading (Lemma 1.7) we know that $I\colon (x,y,z) = H + Hz + \big(H\colon (x,y)\big)z^2$. So the socle mod I is in R_2, the second graded piece of R. Let $u \in \big(H\colon (x,y)\big)\backslash H$. Then $uz^2 \in R_2$ represents an element of the socle mod I. The test ideal is (x,y,z) by Proposition 1.4. If $uz^2 \in I^*$, then, using z as a test element, and the grading (Lemma 1.8), we see that this is equivalent to having $u^q f^{2h+1} \in H^{[q]} + H^{[q]} f^h + H^{[q]} f^{2h+1}$ in $K[[x,y]]$, which implies that $u^q f^{2h+1} \in H^{[q]}$. This, however, is exactly what is needed to have $(uz^2)^q \in I^{[q]}$ (Lemma 1.8) and hence $uz^2 \in I^F$.

We can also show that $I^* \neq I$ in this case; in other words, uz^2 is always in I^*. In fact we can show that $uz^2 \in I^F$. If $uz^2 \in I^F$, then we must have $u^q z^{2q} \in I^{[q]}$. This is equivalent to having $z^{2q} \in I^{[q]}\colon {}_R u^q$. Since R is a flat $K[[x,y]]$-algebra, $I^{[q]}\colon {}_R u^q = (I^{[q]}\colon {}_{K[[x,y]]} u^q)R$ (Lemma 1.13). Since $K[[x,y]]$ is a regular local ring, $(I^{[q]}\colon {}_{K[[x,y]]} u^q)R = (I\colon {}_{K[[x,y]]} u)^{[q]}R$ (Corollary 1.14). Since I is just the expansion of H to R,

$$(I\colon {}_{K[[x,y]]} u)^{[q]}R = (H\colon {}_{K[[x,y]]} u)^{[q]}R = (x,y)^{[q]}R = (x^q,y^q)R.$$

Thus, $uz^2 \in I^F$ if and only if $z^{2q} \in (x^q, y^q)$, which it is by the proof of Proposition 4.3.

Proof of (2). Let $q = p^{2e} = 3h + 1$. Using the $\mathbb{Z}_3$-grading we know that $I\colon (x, y, z) = H + \big(H\colon (x, y)\big)z + \big(H\colon (x^2, xy, y^2)\big)z^2$ (Lemma 1.7). So the socle has components in R_1 and R_2. Let $u \in \big(H\colon (x, y)\big)\backslash H$. Then uz represents an element of the socle mod I and uz is in R_1. If $uz \in I^*$, then, using x as a test element and the grading (Lemma 1.9), we know that

$$\begin{aligned} xu^q f^h &\in H^{[q]} + H^{[q]} f^h + \big(H\colon (x, y)\big)^{[q]} f^{2h+1} \\ &\in H^{[q]} + H^{[q]} f^h + \big(H^{[q]}\colon (x^q, y^q)\big) f^{2h+1} \end{aligned}$$

in $K[[x, y]]$. Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 1.10), we know that

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + H^{[q]} = H^{[q]}.$$

This implies that $xf^h \in H^{[q]}\colon u^q$. Since we are working over a regular ring, $xf^h \in (H\colon u)^{[q]}$ (Corollary 1.14). Now $(x, y) \subseteq H\colon u$, and since $u \notin H$, $H\colon u = (x, y)$. So now we have that $xf^h \in (x, y)^{[q]} = (x^q, y^q)$. But $xf^h \notin (x^q, y^q)$. To see this expand $f^h = (x^3 + y^3)^h$ using the binomial theorem. Thus $uz \notin I^*$.

Let $u \in \big(H\colon (x^2, xy, y^2)\big)\backslash\big(H\colon (x, y)\big)$. Then $uz^2 \in R_2$ represents an element of the socle mod I. We will use the grading and other arguments just as above. If $uz^2 \in I^*$, then, using x as a test element, and the fact that $f^{2h} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^{2h} \in H^{[q]}$. This implies that $xf^{2h} \in (H\colon u)^{[q]}$. Now $(x^2, xy, y^2) \subseteq H\colon u$, and since $u \notin H\colon (x, y)$, $H\colon u \neq (x, y)$. Since $K[x, y]/(x^2, xy, y^2) \cong K + Kx + Ky$, we know that $H\colon u = (x^2, xy, y^2)$ or (x, y^2) or (x^2, y) or $(x^2, xy, y^2, x + \lambda y)$ where $\lambda \in K$. If we expand $f^{2h} = (x^3 + y^3)^{2h}$, it is clear that $xf^{2h} \notin (x^2, xy, y^2)^{[q]}$. Similarly, $xf^{2h} \notin (x, y^2)^{[q]}$ and $xf^{2h} \notin (x^2, y)^{[q]}$. Now suppose $xf^{2h} \in (x^{2q}, x^q y^q, y^{2q}, x^q + \lambda^q y^q)$. Make a change of variables and replace x by $x - \lambda y$. Now it is sufficient to show that

$$(x - \lambda y)[(x - \lambda y)^3 + y^3]^{2h} \in ((x - \lambda y)^{2q}, (x - \lambda y)^q y^q, y^{2q}, x^q).$$

Expanding both sides shows that this cannot happen. Thus $uz^2 \notin I^*$.

Proof of (3). Assume $q = 3h+1$. Let $u \in \big(H\colon (x, y)\big)\backslash H$. Then $u \in R_0$ represents an element of the socle mod I (Lemma 1.7). We use the same method as in the proof of (2). If $u \in I^*$, then we use x as a test element and multiply by f^h to see that

$$xu^q f^h \in H^{[q]} f^h + \big(H\colon (x, y)\big)^{[q]} f^{2h+1} + (H\colon f)^{[q]} f^q.$$

Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^h \in H^{[q]}$. This implies that $xf^h \in (H\colon u)^{[q]}$. As before $H\colon u = (x, y)$, and $xf^h \in (x^q, y^q)$, but $xf^h \notin (x^q, y^q)$. Thus $u \notin I^*$.

Let $u \in \big(H\colon (x^2, xy, y^2)\big)\backslash(H\colon (x, y))$. Then $uz \in R_1$ represents an element of the socle mod I. If $uz \in I^*$, then we use x as a test element and multiply by f^h. Since $f^{2h} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^{2h} \in H^{[q]}$. This implies that $xf^{2h} \in (H\colon u)^{[q]}$. But this cannot happen by the second part of case (2). Thus $uz \notin I^*$.

Proof of (4). Let $q = 3h+1$. Let $u \in \big(H\colon (x, y)\big)\backslash H$. Then $u \in R_0$ represents an element of the socle mod I. If $u \in I^*$, then we use x as a test element and multiply by f^h. Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^h \in H^{[q]}$. This implies that $xf^h \in (H\colon u)^{[q]}$. As before, $H\colon u = (x, y)$, and $xf^h \in (x^q, y^q)$, but $xf^h \notin (x^q, y^q)$. Thus $u \notin I^*$.

Let $u \in \big(H\colon (x^2, xy, y^2)\big)\backslash\big(H\colon (x, y)\big)$. Then $uz^2 \in R_2$ represents an element of the socle mod I. If $uz^2 \in I^*$, then we use x as a test element and then multiply by f^{h-2} to see that

$$xu^q f^{3h-2} \in H^{[q]} f^{h-2} + \big(H\colon (x, y)\big)^{[q]} f^{2h-2} + \big(H\colon (x, y)\big)^{[q]} f^{3h-2}.$$

Since $f^{2h-2} \in (x^q, y^q)$ (Lemma 1.12), we know that $xu^q f^{3h-2} \in H^{[q]}$. This implies that $xf^{3h-2} \in H^{[q]}\colon u^q$. As before we can show that $(x^2, xy, y^2) \subseteq H\colon u \subsetneq (x, y)$. As in the proof of (2), we know that $H\colon u = (x^2, xy, y^2)$ or (x, y^2) or (x^2, y) or $(x^2, xy, y^2, x + \lambda y)$ where $\lambda \in K$. Expand f^{3h-2} using the binomial theorem. We know that $\binom{3h-2}{h-1} \not\equiv 0 \bmod p$ by Proposition 1.12, so $xf^{3h-2} \notin (x^2, xy, y^2)^{[q]}$. Similarly, $xf^{3h-2} \notin (x, y^2)^{[q]}$ and $xf^{3h-2} \notin (x^2, y)^{[q]}$. Now suppose $xf^{3h-2} \in (x^{2q}, x^q y^q, y^{2q}, x^q + \lambda^q y^q)$. Make a change of variables and replace x by $x - \lambda y$. An argument similar to the second part of the proof of (2) shows that this is impossible. Thus $uz^2 \notin I^*$.

Proof of (5). Let $p = 3h + 2$. Let $u \in \big(H\colon (x, y)\big)\backslash H$. Then $uz \in R_1$ represents an element of the socle mod I. If $uz \in I^*$, then, using x as a test element, we must have $xu^p f^h \in H^{[p]} + H^{[p]} f^h + \big(H\colon (x^2, y)\big)^{[p]} f^{2h+1}$ in $K[[x, y]]$ (Lemma 1.8). Let $A = K[[x, y]]$. Taking pth roots of both sides yields

$$(*) \qquad x^{1/p} u f^{h/p} \in HA^{1/p} + Hf^{h/p} A^{1/p} + \big(H\colon (x^2, y)\big) f^{(2h+1)/p} A^{1/p}.$$

We claim that $x^{1/p} f^{h/p}$ is part of a free basis for $A^{1/p}$ over A; equivalently xf^h is part of a free basis for A over $A^p = K[[x^p, y^p]]$. It is sufficient to see that xf^h is not in the expansion of the maximal ideal of A to A^p. If we expand $f^h = (x^3 + y^3)^h$, it is clear that $xf^h \notin (x^p, y^p)$. Since $x^{1/p} f^{h/p}$ is part of a free basis for $A^{1/p}$ over A, we have an A-linear map $\theta\colon A^{1/p} \to A$, sending $x^{1/p} f^{h/p}$ to 1. It is clear that $\theta(f^{h/p} A^{1/p}) \subseteq A$. If we expand $f^{(2h+1)/p}$ and write it in terms of the basis, we see that $\theta(f^{(2h+1)/p} A^{1/p}) \subseteq (x^2, xy, y^2)A$. Thus applying θ to $(*)$ gives $u \in H + H + \big(H\colon (x^2, y)\big)(x^2, xy, y^2)$. Since $(x^2, xy, y^2) \subseteq (x^2, y)$, this implies that $u \in H$ which is a contradiction. Hence $uz \notin I^*$.

Now let $u \in \big(H\colon (x^3, xy, y^2)\big)\backslash\big(H\colon (x^2, y)\big)$. So $uz^2 \in R_2$ represents an element of the socle mod I. Suppose $uz^2 \in I^*$. The argument is the same as above except we use y as a test element and show that there exists an A-linear map $\theta\colon A^{1/p} \to A$, sending $y^{1/p} f^{2h/p}$ to 1. This shows that $uz^2 \notin I^*$. □

In addition, in the following cases we can prove that if $u \in I^*$, then $u \in I^F$ for some but not all elements of the socle.

(2.2) Proposition. *Let I be a $\mathbb{Z}_3$-graded ideal of $K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \equiv 2$ mod 3.*

1. *If $I = \big(H, H, H\colon (x^2, xy, y^2)\big)$, then $uz \notin I^*$ where $u \in H\colon (x, y)$.*
2. *If $I = (H, J, J)$, then $u \in I^*$ implies $u \in I^F$ where $u \in (H\colon (x, y))\backslash H$.*
3. *If $I = (H, H, J)$, then $uz^2 \in I^*$ implies $uz^2 \in I^F$ where $u \in H\colon (x, y)$.*

Proof. Proof of (1). Let $p = 3h + 2$. Let $u \in \big(H\colon (x, y)\big)\backslash H$. Then $uz \in R_1$ represents an element of the socle mod I. Suppose $uz \in I^*$. We use the same argument as in 2.1 (5) with x as a test element to show that $uz \notin I^*$. A similar technique does not work when trying to determine whether a socle element in R_2 is in I^*.

Proof of (2). Let $q = 3h+2$. Let $u \in \big((H\colon (x,y)) \cap J\big)\backslash H$, so $u \in R_0$ represents an element of the socle mod I. If $u \in I^*$, then, using z as a test element, and the grading (Lemma 1.8), we determine that this is equivalent to having

$$u^q \in H^{[q]} + J^{[q]}f^{h+1} + J^{[q]}f^{2h+1} = H^{[q]} + J^{[q]}f^{h+1}.$$

In order to have $u \in I^F$, we need $u^q \in I^{[q]}$ for $q \gg 0$, or equivalently,

$$u^q \in H^{[q]} + J^{[q]}f^{h+1} + J^{[q]}f^{2h+2} = H^{[q]} + J^{[q]}f^{h+1}$$

(Lemma 1.8). As before, this technique provides no information about the contribution to the socle from R_2.

Proof of (3). Let $q = 3h+2$. Let $u \in \big((J\colon (x,y)) \cap (H\colon (x^3+y^3))\big)\backslash H$, so $uz^2 \in R_2$ represents the socle mod I. If $uz^2 \in I^*$, then, using z as a test element and the grading we see that this is equivalent to showing that

$$u^q f^{2h+1} \in H^{[q]} + H^{[q]}f^{h} + J^{[q]}f^{2h+1} = H^{[q]} + J^{[q]}f^{2h+1}$$

in $K[[x,y]]$ (Lemma 1.8). In order to have $uz^2 \in I^F$, we need

$$u^q f^{2h+1} \in H^{[q]} + H^{[q]}f^{h+1} + J^{[q]}f^{2h+1} = H^{[q]} + J^{[q]}f^{2h+1}$$

in $K[[x,y]]$ (Lemma 1.8). So, if $uz^2 \in I^*$, then $uz^2 \in I^F$. As before, this technique provides no information about the contribution to the socle from R_1. □

3. Injective Modules over R^∞

We can study the question of whether $I^* = I^F$ in a ring R by looking at injective modules over R^∞. For example, if it were true that one could write the injective hull of K over R^∞ as a direct limit of cyclic modules, R^∞/I_ν, then we could reduce the problem for modules to studying the ideals I_ν. At this point we can find a $\mathbb{Z}_3$-graded injective R^∞-module that contains a copy of K. This is enough to give certain reductions in the problem of whether tight closure is the same as plus closure. We will use the following general lemma.

(3.1) Lemma. *If R is an A-algebra and E is injective over A, then* $\operatorname{Hom}_A(R,E)$ *is an injective R-module.*

Proof. See [E, Lemma A3.8]. □

(3.2) Comment. With $R = K[[x,y,z]]/(x^3+y^3+z^3)$, $A = K$ and $E = K$, we see that $E_{R^\infty} = \operatorname{Hom}_K(R^\infty, K)$ is an injective R^∞-module. In order to use this injective to reduce the problem of whether $I^* = I^F$ to the graded irreducible case, we will show that it contains a copy of K and that it is $\mathbb{Z}_3$-graded.

(3.3) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$ and $E_{R^\infty} = \operatorname{Hom}_K(R^\infty, K)$. Then $K \hookrightarrow E_{R^\infty}$.*

Proof. Let $\phi \in \operatorname{Hom}_K(R^\infty, K)$ be the map $\phi\colon R^\infty \to R^\infty/m_{R^\infty} \hookrightarrow K$. Then $R^\infty\phi \cong K$, since $m_{R^\infty}\phi(x) = \phi(m_{R^\infty}x) = 0$. □

Next we would like to see that E_{R^∞} is $\mathbb{Z}_3$-graded.

(3.4) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$ and $E_{R^\infty} = \operatorname{Hom}_K(R^\infty, K)$. Then E_{R^∞} is $\mathbb{Z}_3$-graded.*

Proof. Recall that the grading on R extends to R^∞ (see Section 1). Next, to see that E_{R^∞} is graded, write $R^\infty = R_0 + R_1 + R_2$. Then

$$\operatorname{Hom}_K(R^\infty, K) = \operatorname{Hom}_K(R_0, K) \oplus \operatorname{Hom}_K(R_1, K) \oplus \operatorname{Hom}_K(R_2, K).$$

Let $E_{R^\infty} = W_0 + W_1 + W_2$ where $W_i = \operatorname{Hom}_K(R_{2i}, K)$. Any subscripts that indicate a graded piece of a module or ring, e.g. $2i$, will be reduced mod 3. If $\phi_i \in W_i$ and $r_i \in R_i$, then $\phi_i(r_i) \in K$ and $\phi_i(r_j) = 0$ when $i \neq j$.

Let $f_i \in R_i$ and $\phi_j \in W_j$. We want to see that $f_i\phi_j \in W_{i+j}$. Recall that $W_{i+j} = \operatorname{Hom}_K(R_{2(i+j)}, K)$, so we need to show that $f_i\phi_j \in \operatorname{Hom}_K(R_{2(i+j)}, K)$. Since $f_i\phi_j(r_{2(i+j)}) = \phi_j(f_i r_{2(i+j)})$ and $f_i r_{2(i+j)} \in R_{i+2(i+j)} = R_{3i+j} = R_j$, we know that $f_i\phi_j(r_{2(i+j)}) \in W_{i+j}$ as required. Similarly, if $k \neq 2(i+j)$, then $f_i\phi_j(r_k) = 0$ and hence $f_i\phi_j \in W_{i+j}$. □

(3.5) Theorem (Reduction to $\mathbb{Z}_3$-graded module case)**.** *Let*

$$R = K[[x, y, z]]/(x^3 + y^3 + z^3),$$

where K is a field of characteristic p. Let $I \subseteq R$ be an m-primary ideal such that $I^* \neq I^F$. Then there exist a $\mathbb{Z}_3$-graded R-module M and an irreducible $\mathbb{Z}_3$-graded submodule N such that $N^* \neq N^F$.

Proof. Suppose $I \subseteq R$ is an m-primary ideal such that $I^* \neq I^F$. Then there exists $u \in I^*R^\infty \backslash IR^\infty$. Expand IR^∞ to an ideal of R^∞ maximal with respect to not containing u. Then u is the socle mod IR^∞ and IR^∞ is irreducible. To see that $um_{R^\infty} = 0$, note that $m_{R^\infty} = \bigcup m_{R^{1/q}}$. Also, $u \in (I \cap R^{1/q})^*$ for some q. This implies that $m_{R^{1/q}}u \subseteq I \cap R^{1/q}$. Thus $m_{R^\infty}u \subseteq IR^\infty$.

Let E_{R^∞} be a $\mathbb{Z}_3$-graded injective R^∞-module that contains a copy of K. We know one exists by Lemmas 3.3 and 3.4. We have an injective map $R^\infty/IR^\infty \rightarrow E_{R^\infty}$ sending 1 to α. We can find a finitely generated ideal $I_0 \subseteq R^{1/q}$ such that $u \in I_0^{*fg}$, the finitistic tight closure. Here $I_0^{*fg} = \bigcup_J (I_0 \cap J)^*$ where J ranges over all finitely generated ideals of R^∞/IR^∞. Let $\tilde{u}$ be the image of u in $R^{1/q}/I_0$. Let M be the submodule of E_{R^∞} generated by α. Then we have a map $R^{1/q}/I_0 \longrightarrow M$. M is a finitely generated $R^{1/q}$-module that contains the image of $R^{1/q}/I_0$ and is graded. It is still true that $\tilde{u} \in I_0^*$ in M since $u \in 0^*$ in E_{R^∞}. If $u \in 0^F$ in M, then we would have $u \in 0^F$ in E_{R^∞} and an element is in 0^F in an R^∞-module if and only if it is zero. Thus $u \notin 0^F$ in M. □

4. IRREDUCIBLE IDEALS

As we saw in Section 3, we can reduce the question of whether $I^* = I^F$ in $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ to the graded irreducible module case. Given this reduction, it seems likely that understanding the graded irreducible ideal case will be helpful. In this section we will show that $I^* = I^F$ for most $\mathbb{Z}_3$-graded irreducible ideals in R when K has characteristic p and $p \equiv 2 \bmod 3$. In the course of proving the main result, Theorem 4.5, we develop a number of techniques for determining when an element of the socle is in the tight closure or the Frobenius closure of a given ideal.

Preliminary Techniques. The following proposition provides a useful tool for determining whether or not a given irreducible m-primary ideal, I, is tightly closed. If we can find an irreducible ideal contained in I which is tightly closed, then we know that I is also tightly closed. Similarly, if we can find an irreducible ideal containing I which is not tightly closed, then we know that I is not tightly closed.

(4.1) Proposition. *Let R be a local Gorenstein ring. Let m be the maximal ideal of R and let J and I be irreducible m-primary ideals of R with $J \subseteq I$. Then $R/I \hookrightarrow R/J$, and if $I^* \neq I$, then $J^* \neq J$. Also, if $I^F \neq I$, then $J^F \neq J$.*

Proof. Since I and J are m-primary, R/I and R/J are zero-dimensional. As I and J are irreducible and m-primary, $\dim_K \operatorname{Soc} R/J = 1$ and R/J is Gorenstein, and similarly for R/I. So R/J is a zero-dimensional Gorenstein local ring, which implies that R/J is injective as a module over itself and $R/J \cong E_{R/J}(K)$. Similarly, $R/I \cong E_{R/I}(K)$. So $\operatorname{Ann}_{R/J} I \cong \operatorname{Ann}_{E_{R/J}(K)} I \cong E_{(R/J)/I}(K) \cong E_{R/I}(K) \cong R/I$, and thus $\operatorname{Ann}_{(R/J)} I \cong R/I$. Composing this isomorphism with the natural inclusion $\operatorname{Ann}_{(R/J)} I \hookrightarrow R/J$ gives the inclusion $\phi\colon R/I \to R/J$. We also know that $\phi((0)^*_{R/I}) \subseteq (0)^*_{R/J}$. If $I^* \neq I$, then $(0)^*_{R/I} = I^*/I \neq 0$, and so $(0)^*_{R/J} \neq 0$. Then $J^*/J \neq 0$ and $J^* \neq J$ as required. The same argument applies for I^F and J^F since $\phi((0)^F_{R/I}) \subseteq (0)^F_{R/J}$. □

In fact, even if one or both of I and J is not irreducible, if we can show that we have an injection $R/I \hookrightarrow R/J$, then $J^* = J$ implies that $I^* = I$. The following lemma gives a criterion for when such an injection exists.

(4.2) Lemma. *Let R be a Noetherian ring. Let I and J be ideals of R with $J \subseteq I$, I irreducible and let u be the socle mod I. Then $R/I \hookrightarrow (R/J)^h$ if and only if there exists $v \in R$ such that $vI \subseteq J$ and $vu \notin J$. If, in addition, $J = J^*$, then $I = I^*$.*

Proof. Let $u_1, \dots, u_h$ generate $J\colon I$. Let $\bar{u}_1, \dots, \bar{u}_h$ be the images of the generators in R/J. Then $\bar{u}_1, \dots, \bar{u}_h$ generate $(J\colon I)/J \cong \operatorname{Ann}_{R/J} I$. We have a map $R \to (R/J)^h$ taking $\bar{r}$ to $(r\bar{u}_1, \dots, r\bar{u}_h)$. Now $\bar{r}$ gets mapped to 0 if and only if $r(J\colon I) \subseteq J$. This is equivalent to having $r \in J\colon (J\colon I)$. So the map is injective if and only if $I = J\colon (J\colon I)$. This is equivalent to having $u \notin J\colon (J\colon I)$ or $u(J\colon I) \not\subseteq J$. Finally, this is true if and only if there exists $v \in J\colon I$ such that $uv \notin J$.

Suppose $u \in 0^*_{R/I}$. Then the image of u is contained in $0^*_{R/J}$. Thus if J is tightly closed, so is I. □

(4.3) Proposition. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \bmod 3$. Let I be an irreducible m-primary ideal of R and let u represent the socle mod I. If $I \subseteq (x,y)$, then $u \in I^F$. Let (f,g) be generated by a system of parameters. If $I \subseteq (f,g)$, then $u \in I^F$.*

Proof. Since I and (x,y) are both irreducible m-primary ideals, we have an injection $R/(x,y) \hookrightarrow R/I$ sending z^2, the socle in $R/(x,y)$, to u (Proposition 4.1). It is enough to see that $z^2 \in (x,y)^F$, for then $u \in I^F$. For this it is sufficient to show that z^{2p} is contained in (x^p, y^p). Let $p = 3h+2$. Using the basic relation in R and the $\mathbb{Z}_3$-grading it is sufficient to show that $(x^3+y^3)^{2h+1} \in (x^p, y^p)$ (Lemma 1.8). This is routine if we expand using the binomial theorem. Thus $z^2 \in (x,y)^F$.

Let v represent the socle in $R/(f,g)$. Since I and (f,g) are both irreducible m-primary ideals, we have an injection $R/(f,g) \hookrightarrow R/I$ sending v to u (Proposition 4.1). It is enough to see that $v \in (x,y)^F$, for then $u \in I^F$. We know that $(f^q, g^q) \subseteq (x,y)$ for some q. The socle mod (f^q, g^q) is $f^{q-1}g^{q-1}v$. Since (f^q, g^q) is an m-primary irreducible ideal contained in (x,y), we know that $f^{q-1}g^{q-1}v \in (f^q, g^q)^F$.

This implies that $f^{(q-1)Q}g^{(q-1)Q}v^Q \in (f^{qQ}, g^{qQ})$ for some $Q = p^e$. Dividing by powers of f and g yields $v^Q \in (f^Q, g^Q)$, and hence $v \in (f,g)^F$. □

Classification of Irreducibles. The $\mathbb{Z}_3$-grading on R allows us to characterize the irreducible ideals.

(4.4) Proposition. *Let I be an irreducible m-primary $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p. Then I corresponds to one of the following triples of ideals in $K[[x,y]]$ where H is an irreducible m-primary ideal of $K[[x,y]]$ and $f = x^3+y^3$: (H,H,H), $(H, H\colon f, H\colon f)$, $(H,H,H\colon f)$.*

Proof. We know that $(H_0+H_1z+H_2z^2)\colon (x,y,z)$ can be decomposed into graded pieces as follows:

$$\big((H_0\colon (x,y))\cap H_2\big) + \big((H_1\colon (x,y))\cap H_2\big)z + \big((H_2\colon (x,y))\cap \big(H_0\colon (x^3+y^3)\big)\big)z^2$$

(Lemma 1.7). Suppose u, the socle mod I, is contained in R_0, the zero graded piece of R. Then in order for I to have a one-dimensional socle, there must be no contribution from R_1 or R_2. This requires that $\big(H_1\colon (x,y)\big)\cap H_2 = H_1$ and $\big(H_2\colon (x,y)\big)\cap (H_0\colon f) = H_2$. These conditions imply that $H_1 = H_2$ and $H_2 = H_0\colon f$, respectively. To see this, just note that if H_1 were strictly contained in H_2, since $H_1\colon (x,y)$ is strictly larger than H_1, their intersection would strictly contain H_1. In other words, I corresponds to the triple $(H_0, H_0\colon f, H_0\colon f)$. The annihilator of (x,y,z) is now $\big(H_0\colon (x,y)\big)\cap (H_0\colon f)$. Since $(f)\subseteq (x,y)$, we know that $\big(H_0\colon (x,y)\big)\subseteq (H_0\colon f)$, and so the intersection is just $H_0\colon (x,y)$. The socle is then $\big(H_0\colon (x,y)\big)\backslash H_0$ or just the socle mod H_0 in $K[[x,y]]$. Thus, if H_0 is an irreducible ideal of $K[[x,y]]$, then I has a one-dimensional socle and is irreducible. Similar arguments are used if the socle mod I is contained in R_1 or R_2. □

Tight Closure and Frobenius Closure of Irreducible Ideals. Now we can prove the main result of this section.

(4.5) Theorem. *Let I be an irreducible m-primary $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p\equiv 2$ mod 3. Let $f = (x^3+y^3)$. If I has any of the following forms, then $I^* = I^F$.*

1. (H,H,H),
2. $(H, H\colon f, H\colon f)$,
3. $(H,H,H\colon f)$ *and* $f\notin H$,
4. $(H,H,H\colon f)$ *and* $f\in H$ *and* H *contains an element with a linear form.*

Proof of (1)–(3). First observe that $(H,H,H)\subseteq (x,y)$. The ideals $(H,H\colon f,H\colon f)$ and $(H,H,H\colon f)$ are also contained in (x,y) so long as $f\notin H$. If $f\in H$, then $H\colon f = K[[x,y]] = A$. In that case, $(H,H\colon f,H\colon f) = (H,A,A) = H+Az$ and $(H,H,H\colon f) = (H,H,A) = H+Az^2$. When the ideals are contained in (x,y) we know that $I^* = I^F$ by Proposition 4.3. In fact, we know that $I^*\neq I$ in those cases.

We will now consider the case $I = (H,H\colon f,H\colon f)$ where $f\in H$. As noted before, $I = H+Az$ in this case. Let $q = 3h+1$. Suppose $u\in I^*$. Then, using z as a test element, and the grading (Lemma 1.9), we see that this is equivalent to having $u^q\in H^{[q]}+(f^{h+1})+(f^{2h+1})$ in $K[[x,y]]$ which implies that $u^q\in H^{[q]}+(f^{h+1})$. This, however, is exactly what is needed to have $u^q\in I^{[q]}$ (Lemma 1.9). Thus $u\in I^F$. □

The proof of (4) requires several different techniques. We begin with an analysis of the possible forms for H.

(4.6) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let I be a $\mathbb{Z}_3$-graded irreducible ideal of the form $(H, H, H\colon f)$ with $f = (x^3+y^3) \in H$. If H contains an element with a linear form, then H has one of the following forms:*

(1) (x, y),
(2) $(x^2, y - cx)$, $c \in K\backslash\{0\}$,
(3) $(x^k, y + x)$, $k \geq 3$,
(4) $(x^k, y + x - dx^{k-1})$, $k \geq 3$, $d \in K\backslash\{0\}$.

Proof. Let $q = 3h + 1$. We can assume that $H \nsubseteq (x, y^3)$ in $K[[x,y]]$; otherwise I would be contained in a parameter ideal of R and we would done by Proposition 4.3. Suppose an element of H has a term $\alpha y + \cdots$ with $\alpha \neq 0$. Using Weierstrass preparation, we can find a unique monic associate $u = y - g(x)$. Now $K[[x,y]]/u \cong K[[x]]$, a principal ideal domain. $H/(u)$ is an ideal of $K[[x]]$, and since $K[[x]]$ is a PID, $H/(u) = x^k$ for some k. Lifting back to $K[[x,y]]$ we see that $H = (x^k, y - g(x))$. We can also assume that $x^k \notin (y - g(x), z)$; otherwise I would be contained in the ideal $(y - g(x), z)$ which is a parameter ideal. Suppose $x^k \notin (y - g(x), z)$ in R. Using the $\mathbb{Z}_3$-grading (Lemma 1.9) we see that this is equivalent to having $x^k \notin (y - g(x), x^3 + y^3)$ in $K[[x,y]]$, which is equivalent to having $x^k \notin (x^3 + g(x)^3)$ in $K[[x,y]]$ modulo $u = y - g(x)$. In order to have $x^k \notin (x^3, g(x)^3)$, we need the order of $x^3 + g(x)^3$ to be greater than k. Assume $ord_x g(x) \geq 2$ or $c \neq -1$ where $g(x) = cx + \cdots$. If $k = 1$, then $H = (x, y - g(x)) = (x, y)$. If $k = 2$, then $H = (x^2, y - g(x)) = (x^2, y - cx)$.

Now suppose that $k > 2$. We still need the order of $x^3 + g(x)^3$ to be greater than k. We can assume that $ord_x g(x) = 1$ and $g(x) = -x + dx^h + \cdots$. Then $x^3 + g(x)^3 = 3dx^{2+h} +$ lower degree terms. So we need $h + 2 > k$. If $k < h + 1$, then $(x^k, y - g(x)) = (x^k, y + x)$. If $k = h + 1$, then $(x^k, y - g(x)) = (x^k, y + x - dx^{k-1})$. In each case $k \geq 3$. □

We can now deal with these cases separately.

(4.7) Remark. Let R be a Noetherian ring and m a maximal ideal. If I is an m-primary ideal of R, then $R/I \cong \hat{R}/I\hat{R}$. If we are interested in whether $u \in I\hat{R}$, it is sufficient to check whether $u \in I$. We will make use of this idea in several of the following propositions by reducing questions about ideal membership in $K[[x,y]]$ to the polynomial ring $K[x,y]$.

(4.8) Proposition. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$.*

(1) *Let $I = (x, y, z^2)$. Then $I^* = I^F = I$.*
(2) *Let $I = (x^2, y - cx, z^2)$, $c \in K\backslash\{0\}$. Then $I^* = I^F = I$.*
(3) *Let $I = (x^k, y + x, z^2)$ with $k \geq 3$. Then $I^* = I^F = I$.*

Proof. Let $p = 3h + 2$ and $f = x^3 + y^3$.

(1) The socle mod I is z. Using z as a test element, it suffices to see that $zz^p \notin (x^p, y^p, z^{2p})$. Suppose $zz^p \in (x^p, y^p, z^{2p})$. Using the basic relation in R, and the $\mathbb{Z}_3$-grading (Lemma 1.8), we see that this is equivalent to having $f^{h+1} \in (x^p, y^p, f^{2h+2})$ in $K[[x,y]]$. A degree argument shows that this cannot hold.

(2) The socle mod I is xz. Using z as a test element, it suffices to see that $z(xz)^p \notin (x^{2p}, y^p - c^p x^p, z^{2p})$. Suppose $z(xz)^p \in (x^{2p}, y^p - c^p x^p, z^{2p})$. Using the basic relation in R and the $\mathbb{Z}_3$-grading (Lemma 1.8) we see that this is equivalent to having $x^p f^{h+1} \in (x^{2p}, y^p - c^p x^p, f^{2h+2})$ in $K[[x, y]]$. The degree of $x^p f^{h+1}$ is $2p + 1$, while the degree of f^{2h+2} is $2p + 2$. Since we are in the homogeneous case, we may conclude that $x^p f^{h+1} = (a_1 x + a_2 y)x^{2p} + B(y^p - c^p x^p)$ where a_1, $a_2 \in K$ and $B \in K[x, y]$ (4.7). Since $x^p f^{h+1}$ has no term with the degree of x less than p, $B = (b_1 x^{p+1} + b_2 x^p y)$, b_1, $b_2 \in K$. Expanding $x^p f^{h+1}$ shows that the equality cannot hold.

(3) The socle mod I is $x^{k-1}z$. Using z as a test element, it suffices to see that $z(x^{k-1}z)^p \notin (x^{kp}, y^p + x^p, z^{2p})$. Suppose $z(x^{k-1}z)^p \in (x^{kp}, y^p + x^p, z^{2p})$. Using the basic relation in R and the $\mathbb{Z}_3$-grading (Lemma 1.8) shows that this is equivalent to having $x^{(k-1)p} f^{h+1} \in (x^{kp}, y^p + x^p, f^{2h+2})$ in $K[[x, y]]$. Since we are in the homogeneous case,

$$x^{(k-1)p} f^{h+1} = (a_1 x + a_2 y)x^{kp} + B(x^p + y^p) + C f^{2h+2}$$

where $a_1, a_2 \in K$ and B, $C \in K[x, y]$ (4.7). Let $x^3 + y^3 = (x + y)Q$, where Q is the quadratic form $x^2 - xy + y^2$. It is clear that $a_1 = a_2$ since $(x + y)$ must divide the term $(a_1 x + a_2 y)x^{kp}$. So

$$x^{(k-1)p}(x + y)^{h+1} Q^{h+1} = a(x + y)x^{kp} + B(x + y)^p + C(x + y)^{2h+2} Q^{2h+1}.$$

Dividing both sides by $(x + y)$ implies that $(x + y)^h$ divides ax^{kp} which is clearly false. □

(4.9) Proposition. *Let* $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$*, where* K *is a field of characteristic* p *and* $p \equiv 2$ *mod* 3*. Let* $I = (x^k, x + y - dx^{k-1}, z^2)$*,* $k \geq 3$*,* $d \in K \setminus \{0\}$*. Then* $I^* = I^F = I$*.*

Proof. The socle mod I is $x^{k-1}z$. Using z as a test element, it suffices to show that $zx^{(k-1)p} z^p \notin (x^{kp}, x^p + y^p - d^p x^{(k-1)p}, z^{2p})$. We will reduce to the case $d = 1$. Apply the following map to R: $x \to \lambda x$, $y \to \lambda y$, and $z \to \lambda z$, where $\lambda \in K$. Then

$$zx^{k-1}z \in (x^k, x + y - dx^{k-1}, z^2)^*$$

if and only if

$$\lambda^k z x^{k-1} \in (\lambda^k x^k, \lambda x + \lambda y - \lambda^{k-1} d x^{k-1}, \lambda^2 z^2)^*.$$

By factoring out the λs, we are left with $zx^{k-1} \in (x^k, x + y - \lambda^{k-2} d x^{k-1}, z^2)^*$. If $d \neq 0$, let $\lambda = d^{-1/(k-2)}$. So if $x^{k-1}z$ is in the tight closure of the ideal for one value of $d \neq 0$, then it is in for all $d \neq 0$. We have reduced to the case where $I = (x^k, x + y - x^{k-1}, z^2)$. By Lemma 4.2 it is enough to find an ideal $J \subseteq I$ such that J is tightly closed and $R/I \hookrightarrow R/J$. Let

$$J_0 = ((x + y)^2, x^{2k-2}, (x + y)x^k, x^{2k-1}, (x + y)z^2, x^{k-1}z^2).$$

The desired J is $J_0{}^*$. In order to show that $R/I \hookrightarrow R/J_0^*$, it is sufficient to find $v \in J_0^* : I$ such that $vu \notin J_0^*$ where v is the socle mod I (Lemma 4.2).

First we want to see that $J_0{}^* \subseteq I$. Let $J_1 = (y(x + y), x(x + y), x^k, z^2)$. The socle mod J_1 is generated by $(x + y)z$ and $x^{k-1}z$. We would like to show that $J_1 = J_1{}^*$. We know that $(x + y)z \notin J_1{}^*$ by a degree argument [Sm3, Theorem 2.2]. To show that $x^{k-1}z \notin J_1{}^*$ we will consider the ideal $J_2 = (x + y, x^k, z^2)$. We know that $x^{k-1}z \notin J_2{}^*$ and $J_2 = J_2{}^*$ by a previous case (Proposition 4.8 (3)). As $J_1 \subseteq J_2$ and $x^{k-1}z \notin J_2{}^*$, we may conclude that $x^{k-1}z \notin J_1{}^*$. Thus $J_1 = J_1{}^*$. We also know that $J_0 \subseteq J_1$ implies ${J_0}^* \subseteq {J_1}^*$ [HH1, Proposition 4.1]. Now we have ${J_0}^* \subseteq {J_1}^* = J_1 \subseteq I$, which guarantees that ${J_0}^* \subseteq I$.

Next we would like to show that $x + y + x^{k-1} \in {J_0}^* \colon I$. First we note that

$$(x + y + x^{k-1})I \subseteq ((x + y)x^k, x^{2k-1}, (x + y)^2 - x^{2k-2}, (x + y)z^2, x^{k-1}z^2).$$

Certainly $((x + y)x^k, x^{2k-1}, (x + y)^2 - x^{2k-2}, (x + y)z^2, x^{k-1}z^2) \subseteq J_0 \subseteq J_0^*$.

Recall that $x^{k-1}z$ is the socle mod I. We want to show that $(x+y+x^{k-1})x^{k-1}z \notin {J_0}^*$. Since J_0 and hence J_0^* are homogeneous, it is enough to show that $(x + y)x^{k-1}z \notin {J_0}^*$. Using z as a test element, it suffices to see that $z(x+y)^p x^{(k-1)p} z^p \notin {J_0}^{[p]}$. Suppose $z(x + y)^p x^{(k-1)p} z^p \in {J_0}^{[p]}$. Using the basic relation in R and the $\mathbb{Z}_3$-grading (Lemma 1.8) shows that this is equivalent to having

$$(x + y)^p x^{(k-1)p} f^{h+1} \in ((x + y)^p x^{kp}, x^{(2k-2)p}, (x + y)^{2p}, (x + y)^p f^{h+2}, x^{(k-1)p} f^{h+2}).$$

Since we are in the homogeneous case, routine degree arguments show that

$$(x + y)^p x^{(k-1)p} f^{h+1} \in \big((x + y)^p x^{kp}, (x + y)^{2p}, (x + y)^p f^{h+2}\big)$$

as long as $k > 3$. Dividing by $(x + y)^p$ yields $x^{(k-1)p} f^{h+1} \in \big(x^{kp}, (x + y)^p, f^{h+2}\big)$. But this is equivalent to having $x^{k-1}z \in \big(x^k, (x + y), z^2\big)^*$. We know that $x^{k-1}z \notin (x^k, x + y, z^2)^*$ by a previous result (Proposition 4.8 (3)).

Let $k = 3$ and suppose that

$$(x + y)^p x^{2p} f^{h+1} \in \big((x + y)^p x^{3p}, x^{4p}, (x + y)^{2p}, (x + y)^p f^{h+2}, x^{2p} f^{h+2}\big).$$

The degree of $(x + y)^p x^{2p} f^{h+1}$ is $4p + 1$. Since we are in the homogeneous case, this implies that

$$(x + y)^p x^{2p} f^{h+1} = A(x + y)^p + (\beta_1 x + \beta_2 y)x^{4p} + Cx^{2p} f^{h+2}$$

where $\beta_1, \beta_2 \in K$ and A, $C \in K[x, y]$ (4.7). But this implies that $(x+y)^{h+2}$ divides $(\beta_1 x + \beta_2 y)x^{4p}$ which is impossible.

So with $v = x + y + x^{k-1}$, we have $v \in J_0^* \colon I$ and $x^{k-1}zv \notin J_0^*$. This is enough to show $R/I \hookrightarrow R/J_0^*$ by Lemma 4.2. Since J_0^* is tightly closed, we know that I is tightly closed, also by Lemma 4.2. ☐

In addition to the cases where $I \subseteq (x, y)$, we can determine whether or not an irreducible ideal is tightly closed, not just that $I^* = I^F$, in the following cases.

(4.10) Proposition. *Let* $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, *where* K *is a field of characteristic* p. *Let* I *be an irreducible* $\mathbb{Z}_3$*-graded ideal of the form* $(H, H \colon f, H \colon f)$, *where* $f = x^3 + y^3 \in H$, *and* H *is generated by elements whose leading forms are relatively prime quadratic forms. Then* $I = I^*$.

Proof. I is of the form $(Q_1 + C_1, Q_2 + C_2, z)$. Here we mean the ideal generated by $Q_1 + C_1$, $Q_2 + C_2$, and z, not a triple of ideals. Let Q_3 be the third independent quadratic form. By considering the associated graded ring we can see that $K[[x, y]]/(Q_1 + C_1, Q_2 + C_2)$ has dimension four over K, and it follows that $1, x, y, Q_3$ give a basis. Everything of degree three or more will be in H and Q_3 will represent the socle mod I. This also guarantees that $f \in H$. We would like to show that $Q_3 \notin (Q_1 + C_1, Q_2 + C_2, z)^*$. Using the grading and x as a test element, it is sufficient to show that $xQ_3^p \notin (Q_1^p, Q_2^p, f^{h+1})$. This is equivalent to showing that $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is not divisible by f^{h+1} where L_1 and L_2 are linear forms. We will dehomogenize the equation by setting $y = 1$. If $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is divisible by f^{h+1}, then $x\overline{Q_3}^p + \overline{L_1Q_1}^p + \overline{L_2Q_2}^p$ is divisible by $\bar{f}^{h+1}$. This implies the derivative with respect to x is divisible by $\bar{f}^h$. Using the fact that we are in characteristic p, we see that the derivative is $\overline{Q_3}^p + \overline{L_1}'\overline{Q_1}^p + \overline{L_2}'\overline{Q_2}^p$. So we need that $(\overline{Q_3} + (\overline{L_1}')^{1/p}\overline{Q_1} + (\overline{L_2}')^{1/p}\overline{Q_2})^p$ is divisible by $\bar{f}^h$. If we rewrite $\bar{f}^h$ as $(x-1)^h(x-\omega)^h(x-\bar{\omega})^h$, we conclude that all three linear factors of $\bar{f}$ divide $(\overline{Q_3} + (\overline{L_1}')^{1/p}\overline{Q_1} + (\overline{L_2}')^{1/p}\overline{Q_2})$. Since $\overline{Q_1}$ and $\overline{Q_2}$ are still independent over K, this cannot happen. $\square$

(4.11) Comment. Let (H, z) and (H, z^2) be two irreducible m-primary ideals of $K[[x, y, z]]/(x^3 + y^3 + z^3)$. Since $(H, z^2) \subseteq (H, z)$, we know that if (H, z^2) is tightly closed, then so is (H, z) (Proposition 4.1). In particular, if $I = (x, y, z^2)$, $(x^2, y - cx, z^2)$, $(x^k, y + x, z^2)$, or $(x^k, x + y - x^{k-1}, z^2)$, we know that $I = I^*$. So if $I = (x, y, z)$, $(x^2, y - cx, z)$, $(x^k, y + x, z)$, or $(x^k, x + y - x^{k-1}, z)$, we know that $I = I^*$ also. $\square$

Next we classify the cases of m-primary irreducible $\mathbb{Z}_3$-graded ideals not included in Theorem 4.5. To do this we need the following proposition which gives a characterization of the m-primary irreducible ideals in $K[[x, y]]$.

(4.12) Lemma. *Let $A = K[[x, y]]$. Let I be an irreducible m-primary ideal in A. Then I is generated by parameters.*

Proof. First note that I is a height two ideal and the quotient, A/I, is Cohen-Macaulay and has finite projective dimension. This means that A/I must have a resolution that looks like $0 \to A^{r-1} \to A^r \to A \to A/I \to 0$ where the entries of the matrix of the map from A^r to A can be taken to be minimal generators of I. Then I must be the ideal generated by the $r-1$ size minors of of the second matrix. This implies that the type of A/I is one smaller than the number of generators of I. Since A/I has type one, we must have $r = 2$. $\square$

We are now able to classify the remaining cases.

(4.13) Proposition. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ and $A = K[[x, y]]$, where K is a field of characteristic $p \neq 3$. Let I be an m-primary irreducible $\mathbb{Z}_3$-graded ideal of R corresponding to the triple of ideals $(H, H, H : f)$, where $f = x^3 + y^3 \in H$. Suppose H does not contain an element with a linear leading form. Then I has one of the following forms:*

1. *$I = (Q_1, Q_2, z^2)$ where Q_1, Q_2 are relatively prime quadratic forms in A;*
2. *$I = ({L_1}^2 + C, L_1L_2 + D, z^2)$ where L_1 and L_2 are independent linear forms, L_1 divides f, and C and D have cubic or higher leading forms;*
3. *$I = (L_1L_2 + C, D, z^2)$ where L_1, L_2, C and D are as in (2).*

Proof. We know that $I = H + Az^2$ where H is an m-primary irreducible ideal of A. Also, we know that H is generated by two parameters by Lemma 4.12.

Suppose $H = (Q_1 + C_1, Q_2 + C_2)$ where Q_1 and Q_2 are quadratic forms and C_1 and C_2 are higher order terms. If Q_1 and Q_2 are relatively prime, then by considering the associated graded ring, we can see that everything of degree three or higher is contained in H. Thus $H = (Q_1, Q_2)$ and the third independent quadratic form will be the socle mod H.

If Q_1 and Q_2 are not relatively prime, we can write $H = (LL_1 + C_1, LL_2 + C_2)$. If L and L_1 are independent over K, then they span the space of linear forms and we can write $L_2 = aL + bL_1$. This implies that $LL_2 = aL^2 + bLL_1$. Hence we may rewrite H as $(LL_1 + C_1, L^2 + {C_2}')$. A similar argument applies if L and L_2 are independent. If L, L_1 and L_2 are all dependent, then $H = (L^2 + C_1, L^2 + C_2) = (L^2 + C_1, C_2)$.

If $H = (LL_1 + C_1, L^2 + C_2')$, since we must have $f \in H$, either LL_1 divides f or L^2 divides f. Suppose L does not divide f. Then the associated graded ring must contain everything of order three or higher and $f = (L + D_2)(LL_1 + C_1) - (L_1 + D_1)(L^2 + C_2')$. But everything on the right-hand side has order three or higher; hence L divides f.

If $H = (L^2 + C_1, C_2)$, then L^2 must divide f. To see this, note that if C_2 divides f, then f will be a minimal generator of H. Since $z^2 \in I$ and $z^3 = -f$, if f is a minimal generator of H, then I will be generated by z^2 and the other minimal generator of H, $L^2 + C_1$. In other words, I will be generated by parameters and we know that the socle mod I is contained in I^F by Proposition 4.3. $\square$

(4.14) Comment. The remaining cases have proved to be very challenging. In particular, even the question of whether $xyz \in (x^2, y^2, z^2)^*$ is quite difficult. A. Singh has given an argument using determinants of matrices of binomial coefficients to show that indeed $xyz \in (x^2, y^2, z^2)^*$ for all p and $xyz \in (x^2, y^2, z^2)^F$ for $p \equiv 2 \bmod 3$ [Si].

5. GENERALIZATIONS TO OTHER RINGS

Many of the results in this paper can be generalized to rings of the form $K[[x, y, z]]/(z^3 - F(x, y))$ where $F(x, y)$ is a homogeneous polynomial of degree three, K is a field of characteristic p and $p \neq 3$. We first note that the maximal ideal, m, is the test ideal for these rings. For $p > 3$ this is a consequence of a tight closure interpretation of the Kodaira Vanishing Theorem for Gorenstein rings in dimension two [HuS, (4.5) and (5.4)].

We give a proof for all positive prime characteristics here.

(5.1) Proposition. *Let $R = K[[x, y, z]]/(z^3 - F(x, y))$, where K is a field of characteristic p, and $F(x, y)$ is a homogeneous polynomial of degree three. Then m is the test ideal for R.*

Proof. The beginning of the proof is the same as the beginning of the proof of Proposition 1.4. We can show that it is sufficient to check that $\lambda_3 x^{t-1} y^{t-1} z \notin (x^t, y^t)^*$ and $\lambda_1 x^{t-2} y^{t-1} z^2 + \lambda_2 x^{t-1} y^{t-2} z^2 \notin (x^t, y^t)^*$. The proof that $\lambda_3 x^{t-1} y^{t-1} z \notin (x^t, y^t)^*$ is also the same as the proof in Proposition 1.4.

Suppose $\lambda_1 x^{t-2} y^{t-1} z^2 + \lambda_2 x^{t-1} y^{t-2} z^2 \in (x^t, y^t)^*$. This implies that $\lambda_1 x z^2 + \lambda_2 y z^2 \in (x^t, y^t)^* \colon x^{t-2} y^{t-2}$. We know that $(x^t, y^t)^* \colon x^{t-2} y^{t-2} \subseteq (x^2, y^2)^*$ by the usual colon capturing argument [HH1, Theorem 7.15a]. If $\lambda_1 x z^2 + \lambda_2 y z^2 \in (x^2, y^2)^*$, then we can find $c \neq 0$ such that $c(\lambda_1 x + \lambda_2 y)^q z^{2q} \in (x^{2q}, y^{2q})$ for all q. Write $2q = 3h + 2$. Using the basic relation in R, this implies that $c(\lambda_1 x + \lambda_2 y)^q F^h \in (x^{2q}, y^{2q})$ or $cF^h \in (x^{2q}, y^{2q}) \colon (\lambda_1 x + \lambda_2 y)^q$. This is equivalent to having $cF^h \in (x^{2q}, y^{2q}, x^q y^q, (\lambda_1 x - \lambda_2 y)^q) = (x^{2q}, (\lambda_1 x - \lambda_2 y)^q)$. We can use F as a test element and then $F^{h+1} \in (x^{2q}, (\lambda_1 x - \lambda_2 y)^q)$. Suppose $F^{h+1} = Ax^{2q} + B(\lambda_1 x - \lambda_2 y)^q$, where A, $B \in K[[x, y]]$. By a degree argument we must have $F^{h+1} = (a_1 x + a_2 y) x^{2q} + B(\lambda_1 x - \lambda_2 y)^q$, where a_1, $a_2 \in K$. Let F_x denote the partial derivative of F with respect to x. Taking derivatives twice yields $(h+1)hF^{h-1}F_x^2 + (h+1)F^h F_{xx} = B_{xx}(\lambda_1 x - \lambda_2 y)^q$. This implies that $(h+1)hF_x^2 + (h+1)FF_{xx} = 0$, because after we divide both sides by F^{h-1}, we still have a high power of $(\lambda_1 x - \lambda_2 y)$ that must divide the left-hand side. This implies that $hF_x^2 + FF_{xx} = 0$. We can assume that F has distinct linear factors, and by making a change of variable if necessary, we can assume that $F = xy(ax+by)$ with a, $b \neq 0$. Write L for $(ax+by)$. Then $F = xyL$, $F_x = y(ax+L)$ and $F_{xx} = 2ay$. Substituting yields $hy^2(ax+L)^2 + xyL(2ay) = 0$ or $hy^2(a^2x^2+2axL+L^2)+2axy^2L = 0$. If $p \neq 2$, then this implies that L divides $ha^2x^2y^2$ which is not possible. If $p = 2$, then we must have $hy^2(ax+L)^2 = hy^2(by)^2 = 0$. This implies that $hb^2 = 0$, but h can be chosen larger than 2 and b was assumed to be non-zero. □

As m is the test ideal, we know that if $u \in I^*\backslash I$, then u is in the socle mod I (Proposition 1.5). We will combine this fact with the following analogue of Proposition 4.3 in order to make the generalizations.

(5.2) Proposition. *Let $R = K[[x,y,z]]/(z^3 - F(x,y))$, where K is a field of characteristic p, $p \equiv 2 \bmod 3$, and $F(x,y)$ is a homogeneous polynomial of degree three. Let I be an irreducible m-primary ideal of R with $I \subseteq (x,y)$. Suppose u represents the socle mod I. Then $u \in I^F$.*

Proof. We know that there is an injection $R/(x,y) \hookrightarrow R/I$ (Proposition 4.1). It suffices to see that $z^{2p} \in (x^p, y^p)$. Suppose $p = 3h+2$. Then $z^{2p} = F^{2h+1}z$. Now it is enough to see that $F^{2h+1} \in (x^p, y^p)$ in $K[[x,y]]$. The degree of F^{2h+1} is $2p-1$, so every term of F^{2h+1} has a factor of x^p or y^p. In other words, $F^{2h+1} \in (x^p, y^p)$. Hence $u \in I^F$ (4.1). □

The classification of irreducibles (Proposition 4.4) also follows essentially unchanged. Thus the irreducible m-primary ideals of $R = K[[x,y,z]]/(z^3 - F(x,y))$ are exactly the ideals of the form (H,H,H), $(H,H\colon F, H\colon F)$ and $(H,H,H\colon F)$ where H is an irreducible m-primary ideal of $K[[x,y]]$ and $F = F(x,y)$. As before, $(H,H,H) \subseteq (x,y)$ and $(H,H\colon F,H\colon F)$ and $(H,H,H\colon F)$ are both contained in the ideal (x,y) as long as $F \notin H$. We know then that $I^{\mathrm{F}} = I^*$ and $I \neq I^*$ in these cases. More generally, for any irreducible m-primary ideal of R contained in (x,y) we have that $I^{\mathrm{F}} = I^*$.

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