TIGHT CLOSURE, PLUS CLOSURE
AND FROBENIUS CLOSURE IN CUBICAL CONES

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Abstract. We consider tight closure, plus closure and Frobenius closure in the rings $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where $K$ is a field of characteristic $p$ and $p \neq 3$. We use a $\mathbb{Z}_3$-grading of these rings to reduce questions about ideals in the quotient rings to questions about ideals in the regular ring $K[[x,y]]$. We show that Frobenius closure is the same as tight closure in certain classes of ideals when $p \equiv 2 \mod 3$. Since $I^F \subseteq IR^+ \cap R \subseteq I^*$, we conclude that $IR^+ \cap R = I^*$ for these ideals. Using injective modules over the ring $R^\infty$, the union of all $p^m$th roots of elements of $R$, we reduce the question of whether $I^F = I^*$ for $\mathbb{Z}_3$-graded ideals to the case of $\mathbb{Z}_3$-graded irreducible modules. We classify the irreducible $m$-primary $\mathbb{Z}_3$-graded ideals. We then show that $I^F = I^*$ for most irreducible $m$-primary $\mathbb{Z}_3$-graded ideals in $K[[x,y,z]]/(x^3+y^3+z^3)$, where $K$ is a field of characteristic $p$ and $p \equiv 2 \mod 3$. Hence $I^* = IR^+ \cap R$ for these ideals.

In this paper we discuss the conjecture that $I^* = IR^+ \cap R$, where $R^+$ denotes the integral closure of a domain $R$ of characteristic $p$ in an algebraic closure of its fraction field and $I^*$ denotes the tight closure of $I$. The ring $R^+$ is characterized by the property that it is a domain integral over $R$ and every monic polynomial with coefficients in $R^+$ factors into monic linear factors. This characterization can be used to prove the following property of $R^+$: If $W$ is a multiplicatively closed set of $R$, then $(W^{-1}R)^+ \cong W^{-1}R^+$. Aside from providing a much more concrete description of tight closure, proving that $I^* = IR^+ \cap R$ would solve the localization problem for tight closure. It is known that $I^* = IR^+ \cap R$ for parameter ideals [Sm1] and for rings in which every ideal of the normalization is tightly closed. Also, for those ideals $I$ of an excellent local domain $R$ such that $R/I$ has finite phantom projective dimension, it is known that $I^* = IR^+ \cap R$ [Ab]. However, the conjecture is open even for two-dimensional normal Gorenstein domains. In particular, the conjecture is open for the cubical cone $K[[x,y,z]]/(x^3+y^3+z^3)$, where $K$ is a field of characteristic $p$ and $p \neq 3$, and more generally for rings of the form $K[[x,y,z]]/(F(x,y,z))$ where $F$ is a homogeneous cubic polynomial.

We consider tight closure, plus closure and Frobenius closure in the rings $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where $K$ is a field of characteristic $p$ and $p \neq 3$. In Section 1 we use a $\mathbb{Z}_3$-grading of these rings to reduce questions about ideals in the quotient rings to questions about ideals in the regular rings $K[[x,y]]$. In Section 2 we show that the Frobenius closure of an ideal $I$, denoted $I^F$, is the same as the tight closure in certain classes of ideals when $p \equiv 2 \mod 3$. Since $I^F \subseteq IR^+ \cap R \subseteq I^*$,

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we conclude that $IR^+ \cap R = I^*$ for these ideals. In Section 3 we use injective modules over the ring $R^\infty$, the union of all $p^i$th roots of elements of $R$, to reduce the question of whether $I^F = I^*$ for $\mathbb{Z}_3$-graded ideals to the case of $\mathbb{Z}_3$-graded irreducible modules. In Section 4 we classify the irreducible $m$-primary $\mathbb{Z}_3$-graded ideals and then show that $I^F = I^*$ for most irreducible $m$-primary $\mathbb{Z}_3$-graded ideals in $K[[x, y, z]]/(x^3 + y^3 + z^3)$, where $K$ is a field of characteristic $p$ and $p \equiv 2 \mod 3$. Hence $I^* = IR^+ \cap R$ for these ideals.

1. Cubical Cones

We denote by $\mathbb{Z}_n$ the ring $\mathbb{Z}/n\mathbb{Z}$. We first describe a $\mathbb{Z}_3$-grading on the cubical cones $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$. We will also discuss tight closure and Frobenius closure in these rings before proving the main results, Theorem 2.1 and Theorem 4.5.

**$\mathbb{Z}_3$-grading.** First we describe a $\mathbb{Z}_n$-grading of rings of the form $R = A[[z]]/(z^n - a)$ where $a \in A$. The ring $R$ has the following decomposition as an $A$-module: $R = A \oplus A z \oplus \cdots \oplus A z^{n-1}$. Every element of $R$ can be uniquely expressed as an element of $A \oplus A z \oplus \cdots \oplus A z^{n-1}$ by replacing every occurrence of $z^n$ by $a$. $R$ is $\mathbb{Z}_n$-graded, where the $k$th piece of $R$ denotes by $R_k$, is $A z^i$, $0 \leq i < n$, since $A z^i A z^j \subseteq A z^{i+j}$ if $i + j < n$ and $A z^i A z^j \subseteq A z^{i+j-n}$ if $i + j \geq n$.

We use this idea to obtain a $\mathbb{Z}_3$-grading on $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ by letting $A = K[[x, y]]$. Let $H$, $I$, and $J$ be ideals of $K[[x, y]]$. Suppose $H \subseteq I \subseteq J \subseteq H: (x^3 + y^3)$. Then $H + I z + J z^2$ is an ideal of $R$. On the other hand, in order for a $\mathbb{Z}_3$-graded ideal to be closed under multiplication by $z$, it must have this form. Thus, it is easy to see that the ideals of $R$ homogeneous with respect to the $\mathbb{Z}_3$-grading are precisely the ideals of this form. We can study the ideal $H + I z + J z^2$ by considering $(H, I, J)$, a triple of ideals in $K[[x, y]]$. Indeed, we will use the notation $(H, I, J)$ to denote the ideal $H + I z + J z^2$, and it is understood that $H$, $I$, and $J$ are ideals of $K[[x, y]]$. For example, the ideal $(x^2, y^2 z, x z^2)$ is represented by the triple $(H, I, J)$ where $H = (x^2, y^2, x y^2)$, $I = (x^2, y^2)$ and $J = (x, y^2)$.

If $R$ is a reduced ring of characteristic $p$, we write $R^{1/q}$ for the ring obtained by adjoining $q$th roots of all elements of $R$. Next we observe that the $\mathbb{Z}_3$-grading on $R$ extends to $R^\infty = \bigcup_q R^{1/q}$. It is enough to show that the grading on $R$ extends to $R^{1/q}$. If $u \in R_1$, then the image of $u$ is in $R_j^{1/q}$ where $qi \equiv j \mod 3$.

We now show that if $I$ is a graded ideal, then so is $I^*$.

**1.1 Lemma.** Let $R$ be a finitely generated $k$-algebra that is $\mathbb{Z}_n$-graded and of characteristic $p$, where $p$ is not a prime factor of $n$ ($p = 0$ is allowed). Then the tight closure of a homogeneous ideal of $R$ is homogeneous.

**Proof.** Without loss of generality, we can assume $R$ is reduced, since the tight closure of $I$ is the preimage of the tight closure of the image of $I$ modulo the nilradical. Because the singular locus of $R$ is defined by a homogeneous ideal not contained in any minimal prime, $R$ has a homogeneous test element, say $c$. Let $I$ be a homogeneous ideal, and suppose that $z = z_0 + z_1 + \cdots + z_{n-1}$ is in $I^*$, where $z_i$ is the homogeneous component of $z$ of degree $i \mod n$. Now we have $c z_i = c z^i_0 + c z^i_1 + \cdots + c z^i_{n-1}$ is in the homogeneous ideal $I^{[q]}$, and hence each of its homogeneous components is in $I^{[q]}$. But each of the elements $c z^i_0$ is homogeneous of degree $qi + \deg c \mod n$, and since $q$ is invertible in $\mathbb{Z}_n$, these all have distinct
degrees. Thus each \( cz_q^i \in I^{[q]} \) for all \( q \gg 0 \) and each \( z_i \in I^* \). This shows that \( I^* \) is homogeneous.

### Tight Closure and Frobenius Closure

We review the definition of tight closure for ideals of rings of characteristic \( p > 0 \). Tight closure is defined more generally for modules and also for rings containing fields of arbitrary characteristic. See [HH1] or [Hu] for more details.

### (1.2) Definition

Let \( R \) be a ring of characteristic \( p \) and \( I \) be an ideal in a Noetherian ring \( R \) of characteristic \( p > 0 \). An element \( u \in R \) is in the tight closure of \( I \), denoted \( I^* \), if there exists an element \( c \in R \), not in any minimal prime of \( R \), such that for all large \( q = p^r \), \( cz^q \in I^{[q]} \) where \( I^{[q]} \) is the ideal generated by the \( q \)th powers of all elements of \( I \).

We denote by \( I^F \) the Frobenius closure of an ideal \( I \). Recall that \( I^F = \{ u \in R : u^q \in I^{[q]} \} \) for some \( q \}. We can also think of \( I^F \) as \( I^{\infty} \cap R \), so \( I^F \subseteq (IR^+ \cap R, \) since \( R^{\infty} \subseteq R^+ \). In addition, we know that \( IR^+ \cap R \subseteq I^* \) [HH2]. Hence \( I^F \subseteq IR^+ \cap R \subseteq I^* \). So, if \( I^F = I^* \), then that implies that \( I^* = IR^+ \cap R \).

An interesting bifurcation of this question in \( R = K[[x, y, z]]/(x^3 + y^3 + z^3) \) depends on the characteristic of \( K \). If \( K \) has characteristic \( p \) and \( p \equiv 1 \mod 3 \), then \( R \) is F-pure [HR, Proposition 5.21(c)] and \( I^F = I \). We know that \( I^* \neq I \) for some ideals of \( R \), so \( I^F \) cannot equal \( I^* \), although it is still possible that \( I^* = IR^+ \cap R \).

If \( p \equiv 2 \mod 3 \), then \( R \) is not F-pure and it is conjectured that \( I^F = I^* \) and hence that \( I^* = IR^+ \cap R \).

The goal of this paper is to show that \( I^* = I^F \), and hence \( I^* = IR^+ \cap R \), for many graded ideals of \( R \) when the characteristic of \( K \) is congruent to \( 2 \mod 3 \).

### Test Elements in Cubical Cones

In many applications one would like to be able to choose the element \( c \) in the definition of tight closure independent of \( x \) or \( I \).

It is very useful when a single choice of \( c \), a test element, can be used for all tight closure tests in a given ring.

### (1.3) Definition

The ideal of all \( c \in R \) such that, for any ideal \( I \subseteq R \), we have \( cu^q \in I^{[q]} \) for all \( q \) whenever \( u \in I^* \) is called the test ideal for \( R \). An element of the test ideal that is not in any minimal prime is called a test element. The ideal of all \( c \in R \) such that for all parameter ideals (ideals generated by \( i \) elements with height at least \( i \)) \( I \subseteq R \), we have \( cu^q \in I^{[q]} \) for all \( q \) whenever \( u \in I^* \) is called the parameter test ideal for \( R \).

We now determine the test ideal for \( K[[x, y, z]]/(x^3 + y^3 + z^3) \). The following proposition is proved for char \( K \neq 2, 3 \) using a somewhat different method in [Sm2].

### (1.4) Proposition

Let \( R = K[[x, y, z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \neq 3 \). Then the maximal ideal, \( m \), is the test ideal.

**Proof.** First note that we can reduce to the case where \( K \) is algebraically closed. Enlarging \( K \) to an algebraic closure is an integral extension and will not affect tight closure.

Let \( \tau \) be the parameter test ideal for \( R \). By Proposition 4.4(iii) of [Sm2], we know that \( \tau = \{ c \in R \) such that \( c(x^t, y^t)^* \subseteq (x^t, y^t)^* \) all \( t \in \mathbb{N} \}. Since \( R \) is Gorenstein, the test ideal is the same as the parameter test ideal [Sm2, Proposition 4.4].

We will show that \( (x^t, y^t)^* = (x^t, y^t, x^{t-1}y^{t-1}z^2) \). Then it is clear that \( \tau = (x, y, z) \) since \( (x^t, y^t): (x^t, y^t, x^{t-1}y^{t-1}z^2) = (x, y, z) \). Let \( I = (x^t, y^t) \) and \( J = \)
Let \( (x^t, y^t, x^{t-1}y^{t-1}z^2) \). The socle mod \( J \) is generated by \( u_1 = x^{t-2}y^{t-2}z^2 \), \( u_2 = x^{t-1}y^{t-2}z^2 \) and \( u_3 = x^{t-1}y^{t-1}z \). To see that \( I^* = J \), it suffices to show that \( \sum K u_i \cap I^* = 0 \), since if \( J \not\subset I^* \), then \( I^* \) has nonzero intersection with \( J : m \).

We would like to see that \( \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 \notin (x^i, y^i) \) where \( \lambda_i \in K \). Using the \( \mathbb{Z}_3 \)-grading, it is enough to show that \( \lambda_3 u_3 \notin (x^i, y^i) \) and \( \lambda_1 u_1 + \lambda_2 u_2 \notin (x^i, y^i) \).

Using the \( \mathbb{Z}_3 \)-grading again, but now letting \( x \) play the role of \( z \) (\( R = A[[x]]/(x^3 - a) \)), \( A = K[[y, z]] \), we can reduce the problem to showing \( \lambda_1 u_1 \notin (x^i, y^i) \), \( \lambda_2 u_2 \notin (x^i, y^i) \) and \( \lambda_3 u_3 \notin (x^i, y^i) \).

Suppose \( u_3 \in (x^i, y^i) \). Then \( z \in (x^i, y^i) \). We claim that \( (x^i, y^i) : x^{t-1}y^{t-1}z \). Let \( u \in (x^i, y^i) \), so \( u x^{t-1}y^{t-1}z \in (x^i, y^i) \). Then there exists \( c \) such that \( cuq \in (x^q, y^q) \) : \( (x^q, y^q) : x^{t(q-1)}y^{t(q-1)}q \). This implies that \( cuq \in (x^q, y^q) \) : \( x^{t(q-1)}y^{t(q)}q \). But \( (x^q, y^q) \) : \( x^{t(q-1)}y^{t(q-1)}q \subseteq (x^q, y^q) \) by a colon capturing argument [HH1, Theorem 7.15a]. So \( cuq \in (x^q, y^q) \), and we can find a test element \( d \) such that \( cdq \in (x^q, y^q) \) for all \( q \). In other words, \( u \in (x^q, y^q) \). Thus \( x^{t-1}y^{t-1}z \in (x^i, y^i) \). We now know that \( z \notin (x, y) \) by a degree argument [Sm3, Theorem 2.2].

Now suppose \( u_1 \in (x^i, y^i) \). This implies that \( z^2 \in (x^i, y^i) \). Using the same argument as before, we can show that \( (x^i, y^i) : x^{t-2}y^{t-1} \). By symmetry, we must also have \( z^2 \in (x^i, y^i) \). So \( z^2 \in (x^i, y^i) \) which is contained in \( (x^2, xy, y^2) \) by Theorem 7.12 of [HH1]. Again, \( z^2 \notin (x^i, y^i) \) by degree arguments [Sm3, Theorem 2.2].

The fact that \( m \) is the test ideal provides quite a lot of information. For example, using the fact that \( m \) is the test ideal, we may conclude that if \( u \in I^* \), then \( u \) is in the socle mod \( I \).

(1.5) Proposition. Let \( (R, m) \) be a local ring. Suppose \( m \) is the test ideal. If \( u \in I^* \), then \( u \) is in the socle mod \( I \).

Proof. Let \( u \in I^* \). Then \( muq \subseteq I^{|q|} \) for all \( q \). In particular, \( mu \subseteq I \). This says exactly that \( u \) is in the socle mod \( I \).

(1.6) Remark. Although determining whether an element is in the tight closure or Frobenius closure of an ideal involves checking certain conditions for infinitely many values of \( p = p^{t} \), there are some instances where one \( q \) is enough. If \( c \) is a test element and \( cuq \notin I^{|q|} \) for some \( q \), then \( u \notin I^* \). Similarly, if \( u^q \in I^{|q|} \) for some \( q \), then \( u^q \in I^{|q|} \) for all \( q \geq q \) and hence \( u \in I^F \).

In either situation, since we only need one \( q \) that works, we can pick whichever value of \( q \) is most helpful. For example, when \( p \equiv 2 \) mod 3, \( p^{2e} \equiv 1 \) mod 3 and \( p^{2e+1} \equiv 2 \) mod 3. It is often easier to work with powers of \( p \) with a particular residue mod 3 and so we may choose \( q \) accordingly.

Applications of the \( \mathbb{Z}_3 \)-grading to Tight Closure. When trying to determine \( I^* \) and \( I^F \) for a given ideal \( I \), we are interested in calculating \( I^{|q|} \) and \( I : m \). We will first calculate \( I : m \).

(1.7) Lemma. Let \( R = K[[x, y, z]]/(x^3 + y^3 + z^3) \), and let \( H + Iz + Jz^2 \) be a \( \mathbb{Z}_3 \)-graded ideal in \( R \). Then \( (H + Iz + Jz^2) : (x, y, z) = ((H : (x, y)) \cap I) + ((I : (x, y)) \cap J) + (J : (x, y)) \cap (H : (x^3 + y^3)) \zeta^2 \).

Proof. Let \( R_0 \) denote the \( i \) mod 3 graded piece of \( R \). Suppose \( r \in R_0 \) and \( r \in (H+Iz+Jz^2) : (x, y, z) \). So we must have \( r(x, y) \subseteq H \) and \( rz \in I \). In other words,
\( r \in (H : (x, y)) \cap I \). Similarly, if \( rz \in R_1 \) and \( rz \in (H + Iz + Jz^2) : (x, y, z) \), we must have \( r \in (I : (x, y)) \cap J \). Let \( rz^2 \in R_2 \) and suppose \( r \in (H + Iz + Jz^2) : (x, y, z) \). Again, we see that \( r \in J : (x, y) \). We also know that \( (rz^2)z = r(x^3 + y^3) \in (H + Iz + Jz^2) \). Since \( r(x^3 + y^3) \in R_0 \), we must have \( r(x^3 + y^3) \in H \). In other words, \( r \in H : (x^3 + y^3) \). So \( r \in ((J : (x, y)) \cap (H : (x^3 + y^3))) \).

Next we will determine \( I^{[q]} \) when \( q \equiv 2 \mod 3 \).

(1.8) Lemma. Let \( R = K[[x, y, z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \). Let \( q = p^{2e+1} = 3h + 2 \) and let \( f = x^3 + y^3 \). Let \( H + Iz + Jz^2 \) be a \( \mathbb{Z}_3 \)-graded ideal in \( R \). Then

\[
(H + Iz + Jz^2)^{[q]} = (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+2}) \\
+ (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})z + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})z^2.
\]

Let \( u = u_0 + u_1z + u_2z^2 \). Then \( u^{\bar{q}} \in (H + Iz + Jz^2)^{[q]} \) in \( R \) if and only if

\[
u_{0,1,2} \in \langle H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+2} \rangle,
\]

\[
u_{1} \in \langle H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1} \rangle,
\]

\[
u_{2} \in \langle H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1} \rangle \quad \text{in} \ K[[x, y]].
\]

Proof. We start by noting that \( (H + Iz + Jz^2)^{[q]} \) is generated by \( H^{[q]} + I^{[q]}z^q + J^{[q]}z^{2q} \). Rewriting this using \( q = 3h + 2 \), and the basic relation in \( R \), \( z^3 = -(x^3 + y^3) \), yields \( H^{[q]} + I^{[q]}f^{h}z^2 + J^{[q]}f^{2h+1}z^2 \). We will first consider \( (H + Iz + Jz^2)^{[q]} \cap R_0 \). If we multiply \( I^{[q]}f^{h}z^2 \) by \( z \), we get \( I^{[q]}f^{h}z^3 = I^{[q]}f^{h+1} \) which is in \( R_0 \). Similarly, multiplying \( J^{[q]}f^{2h+1}z \) by \( z^2 \) gives \( J^{[q]}f^{2h+1}z^3 = J^{[q]}f^{2h+2} \). Thus,

\[
(H + Iz + Jz^2)^{[q]} \cap R_0 = (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+2}z) \]

Similar arguments show that

\[
(H + Iz + Jz^2)^{[q]} \cap R_1 = (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1}z) \quad \text{and}
\]

\[
(H + Iz + Jz^2)^{[q]} \cap R_2 = (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1}z) \]

Since \( u^{\bar{q}} = u_0 + u_1f^{2h+1} + u_2f^{2h}z^2 \), the last statement in the lemma is now clear. □

Next we determine \( I^{[q]} \) when \( q \equiv 1 \mod 3 \).

(1.9) Lemma. Let \( R = K[[x, y, z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \). Let \( q = p^{2e} = 3h + 1 \) and let \( f = x^3 + y^3 \). Let \( H + Iz + Jz^2 \) be a \( \mathbb{Z}_3 \)-graded ideal in \( R \). Then

\[
(H + Iz + Jz^2)^{[q]} = (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h})z^2 \\
+ (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})z + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h})z^2.
\]

Let \( u = u_0 + u_1z + u_2z^2 \). Then \( u^{\bar{q}} \in (H + Iz + Jz^2)^{[q]} \) in \( R \) if and only if

\[
u_{0,1,2} \in \langle H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1} \rangle,
\]

\[
u_{1} \in \langle H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1} \rangle,
\]

\[
u_{2} \in \langle H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h} \rangle \quad \text{in} \ K[[x, y]].
\]
Proof. The proof is identical to the proof of Lemma 1.8 except we use \( q = 3h + 1 \).

Note that \( f^h = (x^3 + y^3)^h \) appears often in the calculations. The question of whether a given element is in the tight closure of an ideal often comes down to whether or not a certain power of \( f \) is contained in \((x^q, y^q)\). To this end, we establish the following lemmas which will be useful in showing that \( I^* = I^F \).

(1.10) Lemma. Let \( A = K[[x, y]] \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \). Let \( p = 3h + 2 \) and let \( f = x^3 + y^3 \). Then \( f^{2h} \notin (x^p, y^p) \). Let \( q = p^{2e} = 3k + 1 \); then \( f^{2k} \in (x^q, y^q) \).

Proof. Expand \( f^{2h} = (x^3 + y^3)^{2h} \) using the binomial theorem. Since \( \binom{2h}{h}x^{3h}y^{3h} \) is a term in the expansion and \( x^{3h}y^{3h} \notin (x^{3h+2}, y^{3h+2}) = (x^p, y^p) \), it suffices to see that \( \binom{2h}{h} \equiv 0 \mod p \). But \( 2h < p \), so \( p \) does not divide \( \binom{2h}{h} \).

As in the above case, \( f^{2k} \in (x^q, y^q) \) if and only if \( \binom{2k}{k} \equiv 0 \mod p \). Suppose we know that \( z^{2q} \in (x^q, y^q) \), we can show that \( z^{2q} \notin (x^q, y^q) \) if and only if \( f^{2k} \notin (x^q, y^q) \). Using the \( \mathbb{Z}_q \)-grading we see that this is equivalent to having \( f^{2k} \notin (x^q, y^q) \). Expand \( f^{2k} \) using the binomial theorem to see that this is equivalent to having \( \binom{2k}{k} \equiv 0 \mod p \). In other words, \( \binom{2k}{k} \equiv 0 \mod p \) if and only if \( z^{2q} \notin (x^q, y^q) \) if and only if \( q = 3k + 1 \). We know that \( z^{2q} \in (x^p, y^p) \) when \( p \equiv 2 \mod 3 \) by the proof of Proposition 4.3. This implies that \( z^{2q} \in (x^q, y^q) \) for all \( q = p^e \), in particular for \( q = 3k + 1 \). Hence \( \binom{2k}{k} \equiv 0 \mod p \), and \( f^{2k} \in (x^q, y^q) \).

We will use the following result about calculating binomial coefficients \( mod \) \( p \) in Lemma 1.12.

(1.11) Lucas’s Theorem. Let \( p \) be a prime and let \( n = \sum a_i p^i \), \( 0 \leq a_j < p \), \( m = \sum b_k p^k \), \( 0 \leq b_k < p \). Then \( \binom{n}{m} \equiv \prod \binom{a_i}{b_k} \mod p \).

Proof. See [Fi, Theorem 1] or [L, p. 230].

(1.12) Lemma. Let \( A = K[[x, y]] \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \). Let \( q = p^{2e} = 3h + 1 \) and \( f = x^3 + y^3 \). Then \( \binom{3h-2}{h-1} \notin (x^q, y^q) \) except when \( q = 25 \).

Proof. Since \( p^{2e} = 3h + 1 \), we can write \( 3h - 2 = p^{2e} - 3 \). So \( \binom{3h-2}{h-1} = (p^{2e} - 3)(p^{2e} - 4) \cdots (p^{2e} - (h + 1))/1 \cdot 2 \cdots (h - 1). \) It is easy to show that \( \binom{3h-2}{h-1} \) is divisible by \( p \) if and only if \( (p^{2e} - (h - 1)) \) is divisible by \( p \). Routine divisibility arguments show that this cannot happen.

To see that \( f^{2h-2} \notin (x^q, y^q) \), we expand \( f^{2h-2} \) using the binomial theorem. It is sufficient to show that \( \binom{2h-2}{h-1} \) and \( \binom{2h-2}{h-2} \) are congruent to zero \( \mod p \). If \( p \neq 2 \), then \( p \) divides \( \binom{2h-2}{h-1} \) if and only if \( p \) divides \( \binom{2h-2}{h-2} \). Next note that if \( p \neq 2, 5 \), then \( p \) divides \( \binom{2h}{h} \) if and only if \( p \) divides \( \binom{2h-2}{h-1} \). We know from Proposition 1.10 that \( p \) divides \( \binom{2h}{h} \) for the values of \( h \) we are considering, so if \( p \neq 2, 5 \), we know that \( p \) also divides \( \binom{2h-2}{h-2} \) and \( \binom{2h-2}{h-1} \). If \( p = 2 \), using (1.1), we can show that \( \binom{2h-2}{h-1} \equiv 0 \mod 2 \) and \( \binom{2h-2}{h-2} = \binom{2h-2}{h-1} \equiv 0 \mod 2 \).

It remains to see that \( \binom{2h-2}{h-1} \equiv 0 \mod 5 \) and \( \binom{2h-2}{h-2} \equiv 0 \mod 5 \). We know from above that if \( p \neq 2 \), then it is enough to show that \( \binom{2h-2}{h-2} \equiv 0 \mod 5 \). Write \( 5^{2e} = 3h + 1 \). Using (1.1), we see that \( \binom{2h-2}{h-2} \equiv 0 \mod 5 \) as long as \( 5^{2e} > 25 \).
At times, we will be able to make use of the fact that we are working over a
regular ring or that $K[[x,y,z]]/(x^3+y^3+z^3)$ is flat as a $K[[x,y]]$-module. The
following lemma and corollary provide useful information in these situations.

(1.13) Lemma. Let $R$, $S$ be arbitrary Noetherian rings such that $S$ is a flat $R$-
alg, and let $I$, $J$ be ideals of $R$. Then $I : SJS = (I : RJ)S$, where $I : RJ =
\{ r \in R : rJ \subseteq I \}$.

Proof. See [N, Theorem 18.1, part 2].

(1.14) Corollary. In a regular ring $R$ of characteristic $p$, for any two ideals $I$, $J$
we have $I^{[q]} : RJ^{[q]} = (I : RJ)^{[q]}$ for all $q$. In particular, $I^{[q]} : x^q = (I : x)^{[q]}$ for all $q$.

Proof. See Corollary 4.3 of [HH1]. The statement follows from Lemma 1.13, since
the iterated Frobenius endomorphism $F^e : R \to R$ is flat when $R$ is regular [K,
Theorem 2.1] and $I^{[q]} = F^e(I)R$.

2. Tight Closure and Frobenius Closure in Cubical Cones

We can now show that $I^* = I^F$ for some not necessarily irreducible ideals. We
will discuss irreducible ideals in Section 4.

(2.1) Proposition. Let $I$ be a $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where
$K$ is a field of characteristic $p$ and $p \equiv 2 \mod 3$. Let $f = x^3+y^3$. If $I$ has any of
the following forms, then $I^* = I^F$.

1. $(H, H, H)$,
2. $(H, H, H : (x,y))$,
3. $(H, H : (x,y), H : f)$,
4. $(H, H : (x,y), H : (x,y))$,
5. $(H, H, H : (x^2, y))$.

In fact, in (2)–(5), $I$ is tightly closed, i.e. $I = I^*$.

Proof. We know that if $u \in I^* \setminus I$, then $u$ is in the socle mod $I$ (Proposition 1.5),
so it is sufficient to check whether elements of the socle are in $I^*$ and $I^F$.

Proof of (1). Let $q = 3h + 2$. Using the $\mathbb{Z}_3$-grading (Lemma 1.7) we know that
$I : (x,y,z) = H + Hz + (H : (x,y))z^2$. So the socle mod $I$ is in $R_2$, the second
graded piece of $R$.

Let $u \in (H : (x,y)) \setminus H$. Then $uz^2 \in R_2$ represents an element
of the socle mod $I$. The test ideal is $(x,y,z)$ by Proposition 1.4. If $uz^2 \in I^*$,
then, using $z$ as a test element, and the grading (Lemma 1.8), we see that this is
equivalent to having $u^q f^{2h+1} \in H^{[q]} + H^{[q]} f^h + H^{[q]} f^{2h+1}$ in $K[[x,y]]$, which implies
that $u^q f^{2h+1} \in H^{[q]}$. This, however, is exactly what is needed to have $(uz^2)^q \in I^{[q]}$
(Lemma 1.8) and hence $uz^2 \in I^F$.

We can also show that $I^* \neq I$ in this case; in other words, $uz^2$ is always in $I^*$.
In fact we can show that $uz^2 \in I^F$. If $uz^2 \in I^F$, then we must have $u^q z^{2q} \in I^{[q]}$.
This is equivalent to having $z^{2q} \in R u^q$. Since $R$ is a flat $K[[x,y]]$-algebra,
$I^{[q]} : u^q = I^{[q]} : k[[x,y]] u^q R$ (Lemma 1.13). Since $K[[x,y]]$ is a regular local ring,
$I^{[q]} : k[[x,y]] u^q R = (I : k[[x,y]] u^q)^{[q]} R$ (Corollary 1.14). Since $I$ is just the expansion
of $H$ to $R$,

$\left( I : k[[x,y]] u^q \right)^{[q]} R = (H : k[[x,y]] u^q)^{[q]} R = (x,y)^{[q]} R = (x^q, y^q) R$. 

Thus, $uz^2 \in I^F$ if and only if $z^{2q} \in (x^q, y^q)$, which it is by the proof of Proposition 4.3.

Proof of (2). Let $q = p^{2e} = 3h + 1$. Using the $\mathbb{Z}_3$-grading we know that $I: (x, y, z) = H + (H: (x, y))z + (H: (x^2, xy, y^2))z^2$ (Lemma 1.7). So the socle has components in $R_1$ and $R_2$. Let $u \in (H: (x, y)) \setminus H$. Then $uz$ represents an element of the socle mod $I$ and $uz$ is in $R_1$. If $uz \in I^*$, then, using $x$ as a test element and the grading (Lemma 1.9), we know that

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + (H: (x, y))^{[q]} f^{2h+1} \in H^{[q]} + H^{[q]} f^h + (H: (x^q, y^q)) f^{2h+1}$$

in $K[[x, y]]$. Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 1.10), we know that

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + H^{[q]} = H^{[q]}.$$ 

This implies that $xf^h \in H^{[q]} : u^q$. Since we are working over a regular ring, $xf^h \in (H: u)^{[q]}$ (Corollary 1.14). Now $(x, y) \subseteq H: u$, and since $u \notin H: u = (x, y)$. So now we have that $xf^h \in (x, y)^{[q]} = (x^q, y^q)$. But $xf^h \notin (x^q, y^q)$. To see this expand $f^h = (x^q + y^q)^h$ using the binomial theorem. Thus $uz \notin I^*$.

Let $u \in (H: (x^2, xy, y^2)) \setminus (H: (x, y))$. Then $uz^2 \in R_2$ represents an element of the socle mod $I$. We will use the grading and other arguments just as above. If $uz^2 \in I^*$, then, using $x$ as a test element, and the fact that $f^{2h} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^{2h} \in H^{[q]}$. This implies that $xf^{2h} \in (H: u)^{[q]}$. Now $(x^2, xy, y^2) \subseteq H: u$, and since $u \notin H: u = (x, y)$. Since $K[x, y]/(x^2, xy, y^2) \cong K + Kx + Ky$, we know that $H: u = (x^2, xy, y^2)$ or $(x, y^2)$ or $(x^2, y)$ or $(x^2, xy, y^2, x + \lambda y)$ where $\lambda \in K$. If we expand $f^{2h} = (x^q + y^q)^{2h}$, it is clear that $xf^{2h} \notin (x^2, xy, y^2)^{[q]}$. Similarly, $xf^{2h} \notin (x, y^2)^{[q]}$ and $xf^{2h} \notin (x^2, y)^{[q]}$. Now suppose $xf^{2h} \in (x^2q, x^qy^q, y^q, x^2 + \lambda y y^q)$. Make a change of variables and replace $x$ by $x - \lambda y$. Now it is sufficient to show that

$$(x - \lambda y)((x - \lambda y)^3 + y^{3})^{2h} \in ((x - \lambda y)^{2q}, (x - \lambda y)^{q} y^q, y^{2q}, x^q).$$

Expanding both sides shows that this cannot happen. Thus $uz^2 \notin I^*$.

Proof of (3). Assume $q = 3h + 1$. Let $u \in (H: (x, y)) \setminus H$. Then $u \in R_2$ represents an element of the socle mod $I$ (Lemma 1.7). We use the same method as in the proof of (2). If $u \in I^*$, then we use $x$ as a test element and multiply by $f^h$ to see that

$$xu^q f^h \in H^{[q]} f^h + (H: (x, y))^{[q]} f^{2h+1} + (H: f)^{[q]} f^q.$$ 

Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^h \in H^{[q]}$. This implies that $xf^h \in (H: u)^{[q]}$. As before $H: u = (x, y)$, and $xf^h \in (x^q, y^q)$, but $xf^h \notin (x^q, y^q)$. Thus $u \notin I^*$.

Let $u \in (H: (x^2, xy, y^2)) \setminus (H: (x, y))$. Then $uz \in R_1$ represents an element of the socle mod $I$. If $uz \in I^*$, then we use $x$ as a test element and multiply by $f^h$. Since $f^{2h} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^{2h} \in H^{[q]}$. This implies that $xf^{2h} \in (H: u)^{[q]}$. But this cannot happen by the second part of case (2). Thus $uz \notin I^*$.

Proof of (4). Let $q = 3h + 1$. Let $u \in (H: (x, y)) \setminus H$. Then $u \in R_0$ represents an element of the socle mod $I$. If $u \in I^*$, then we use $x$ as a test element and multiply by $f^h$. Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 1.10), we can show that $xu^q f^{2h} \in H^{[q]}$. This
imply that \( xf^h \in (H : u)^{[q]} \). As before, \( H : u = (x, y) \), and \( xf^h \in (x^4, y^q) \), but \( xf^h \not\in (x^q, y^q) \). Thus \( u \not\in I^* \).

Let \( u \in (H : (x^2, xy, y^2)) \) \( \setminus \left( H : (x, y) \right) \). Then \( uz^2 \in R_2 \) represents an element of the socle mod \( I \). If \( uz^2 \in I^* \), then we use \( x \) as a test element and then multiply by \( f^{h-2} \) to see that

\[
uxq f^{3h-2} \in H[\phi] f^{h-2} + (H : (x, y))^{[q]} f^{2h-2} + (H : (x, y))^{[q]} f^{3h-2}.
\]

Since \( f^{2h-2} \in (x^q, y^q) \) (Lemma 1.12), we know that \( uxq f^{3h-2} \in H[q] \). This implies that \( xf^{3h-2} \notin H[q] : u^q \). As before we can show that \( (x^2, xy, y^2) \subseteq H : u \subseteq (x, y) \).

As in the proof of (2), we know that \( H : u = (x^2, xy, y^2) \) or \((x^2, y)\) or \((x^2, y^2, x + \lambda y)\) where \( \lambda \in K \). Expand \( f^{3h-2} \) using the binomial theorem. We know that \( \binom{3h-2}{h-1} \neq 0 \) mod \( p \) by Proposition 1.12, so \( xf^{3h-2} \notin (x^2, xy, y^2)^{[q]} \). Similarly, \( xf^{3h-2} \notin (x^2, y)^{[q]} \) and \( xf^{3h-2} \notin (x^2, y)^{[q]} \). Now suppose \( xf^{3h-2} \in (x^2, xy^2, y^2, x + \lambda y) \). Make a change of variables and replace \( x \) by \( x - \lambda y \). An argument similar to the second part of the proof of (2) shows that this is impossible. Thus \( uz^2 \notin I^* \).

Proof of (5). Let \( p = 3h + 2 \). Let \( u \in (H : (x, y)) \setminus H \). Then \( uz \in R_1 \) represents an element of the socle mod \( I \). If \( uz \in I^* \), then, using \( x \) as a test element, we must have \( xu^2 f^h \in H[p] + H[p] f^h + (H : (x^2, y))^{[p]} f^{2h+1} \) in \( K[[x, y]] \) (Lemma 1.8). Let \( A = K[[x, y]] \). Taking \( p \)th roots of both sides yields

(*) \( x^{1/p} uf^{h/p} \in HA^{1/p} + H f^{h/p} A^{1/p} + (H : (x^2, y)) f^{(2h+1)/p} A^{1/p} \).

We claim that \( x^{1/p} f^{h/p} \) is part of a free basis for \( A^{1/p} \) over \( A \); equivalently \( xf^h \) is part of a free basis for \( A \) over \( A^P = K[[x^p, y^p]] \). It is sufficient to see that \( xf^h \) is not in the expansion of the maximal ideal of \( A \) to \( A^P \). If we expand \( f^h = (x^3 + y^3)^h \), it is clear that \( xf^h \not\in (x^p, y^p) \). Since \( x^{1/p} f^{h/p} \) is part of a free basis for \( A^{1/p} \) over \( A \), we have an \( A \)-linear map \( \theta : A^{1/p} \to A \), sending \( x^{1/p} f^{h/p} \) to \( 1 \). It is clear that \( \theta(f^{h/p} A^{1/p}) \subseteq A \). If we expand \( f^{(2h+1)/p} \) and write it in terms of the basis, we see that \( \theta(f^{(2h+1)/p} A^{1/p}) \subseteq (x^2, xy, y^2)A \). Thus applying \( \theta \) to (*) gives \( u \in H + H + (H : (x^2, y))(x^2, xy, y^2) \). Since \( (x^2, xy, y^2) \subseteq (x^2, y) \), this implies that \( u \in H \) which is a contradiction. Hence \( uz \notin I^* \).

Now let \( u \in (H : (x^3, y^3, y^2)) \setminus (H : (x, y)) \). So \( uz^2 \in R_2 \) represents an element of the socle mod \( I \). Suppose \( uz^2 \in I^* \). The argument is the same as above except we use \( y \) as a test element and show that there exists an \( A \)-linear map \( \theta : A^{1/p} \to A \), sending \( y^{1/p} f^{2h/p} \) to \( 1 \). This shows that \( uz^2 \notin I^* \).

In addition, in the following cases we can prove that if \( u \in I^* \), then \( u \in I^P \) for some but not all elements of the socle.

(2.2) Proposition. Let \( I \) be a \( \mathbb{Z}_3 \)-graded ideal of \( K[[x, y, z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \) mod \( 3 \).

(1) If \( I = (H, H, H : (x^2, xy, y^2)) \), then \( uz \notin I^* \) where \( u \in H : (x, y) \).

(2) If \( I = (H, J, J) \), then \( u \in I^* \) implies \( u \in I^P \) where \( u \in (H : (x, y)) \setminus H \).

(3) If \( I = (H, H, J) \), then \( uz^2 \in I^* \) implies \( uz^2 \in I^P \) where \( u \in H : (x, y) \).

Proof. Proof of (1). Let \( p = 3h + 2 \). Let \( u \in (H : (x, y)) \setminus H \). If \( uz \in R_1 \) represents an element of the socle mod \( I \). Suppose \( uz \in I^* \). We use the same argument as in 2.1 (5) with \( x \) as a test element to show that \( uz \notin I^* \). A similar
technique does not work when trying to determine whether a socle element in $R_2$ is in $I^*$.

Proof of (2). Let $q = 3h + 2$. Let $u \in ((H : (x, y)) \cap J) \setminus H$, so $u \in R_0$ represents an element of the socle mod $I$. If $u \in I^*$, then, using $z$ as a test element, and the grading (Lemma 1.8), we determine that this is equivalent to showing that
\[ u^q \in H^{[a]} + J^{[a]} f^{h+1} + J^{[a]} f^{2h+1} = H^{[a]} + J^{[a]} f^{h+1}. \]
In order to have $u \in I^F$, we need $u^q \in I^{[a]}$ for $q \gg 0$, or equivalently,
\[ u^q \in H^{[a]} + J^{[a]} f^{h+1} + J^{[a]} f^{2h+2} = H^{[a]} + J^{[a]} f^{h+1} \]
(Lemma 1.8). As before, this technique provides no information about the contribution to the socle from $R_2$.

Proof of (3). Let $q = 3h + 2$. Let $u \in ((J : (x, y)) \cap (H : (x^3 + y^3))) \setminus H$, so $uz^2 \in R_0$ represents the socle mod $I$. If $uz^2 \in I^*$, then, using $z$ as a test element and the grading we see that this is equivalent to showing that
\[ u^q f^{2h+1} \in H^{[a]} + H^{[a]} f^h + J^{[a]} f^{2h+1} = H^{[a]} + J^{[a]} f^{2h+1} \]
in $K[[x, y]]$ (Lemma 1.8). In order to have $uz^2 \in I^F$, we need
\[ u^q f^{2h+1} \in H^{[a]} + H^{[a]} f^h + J^{[a]} f^{2h+1} = H^{[a]} + J^{[a]} f^{2h+1} \]
in $K[[x, y]]$ (Lemma 1.8). So, if $uz^2 \in I^*$, then $uz^2 \in I^F$. As before, this technique provides no information about the contribution to the socle from $R_1$. 

3. Injective Modules over $R^\infty$

We can study the question of whether $I^* = I^F$ in a ring $R$ by looking at injective modules over $R^\infty$. For example, if it were true that one could write the injective hull of $K$ over $R^\infty$ as a direct limit of cyclic modules, $R^\infty/I_\nu$, then we could reduce the problem for modules to studying the ideals $I_\nu$. At this point we can find a $Z_3$-graded injective $R^\infty$-module that contains a copy of $K$. This is enough to give certain reductions in the problem of whether tight closure is the same as plus closure. We will use the following general lemma.

(3.1) Lemma. If $R$ is an $A$-algebra and $E$ is injective over $A$, then $\text{Hom}_A(R, E)$ is an injective $R$-module.

Proof. See [E, Lemma A3.8].

(3.2) Comment. With $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, $A = K$ and $E = K$, we see that $E_{R^\infty} = \text{Hom}_K(R^\infty, K)$ is an injective $R^\infty$-module. In order to use this injective to reduce the problem of whether $I^* = I^F$ to the graded irreducible case, we will show that it contains a copy of $K$ and that it is $Z_3$-graded.

(3.3) Lemma. Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ and $E_{R^\infty} = \text{Hom}_K(R^\infty, K)$. Then $K \hookrightarrow E_{R^\infty}$.

Proof. Let $\phi \in \text{Hom}_K(R^\infty, K)$ be the map $\phi : R^\infty \rightarrow R^\infty/m_{R^\infty} \hookrightarrow K$. Then $R^\infty \phi \cong K$, since $m_{R^\infty} \phi(x) = \phi(m_{R^\infty}x) = 0$. 

Next we would like to see that $E_{R^\infty}$ is $Z_3$-graded.

(3.4) Lemma. Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ and $E_{R^\infty} = \text{Hom}_K(R^\infty, K)$. Then $E_{R^\infty}$ is $Z_3$-graded.
Proof. Recall that the grading on $R$ extends to $R^\infty$ (see Section 1). Next, to see that $E_{R^\infty}$ is graded, write $R^\infty = R_0 + R_1 + R_2$. Then
\[
\text{Hom}_K(R^\infty, K) = \text{Hom}_K(R_0, K) \oplus \text{Hom}_K(R_1, K) \oplus \text{Hom}_K(R_2, K).
\]
Let $E_{R^\infty} = W_0 + W_1 + W_2$ where $W_i = \text{Hom}_K(R_{2i}, K)$. Any subscripts that indicate a graded piece of a module or ring, e.g. $2i$, will be reduced mod 3. If $\phi_i \in W_i$ and $r_i \in R_i$, then $\phi_i(r_i) \in K$ and $\phi_i(r_j) = 0$ when $i \neq j$.

Let $f_i \in R_i$ and $\phi_j \in W_j$. We want to see that $f_i \phi_j \in W_{i+j}$. Recall that $W_{i+j} = \text{Hom}_K(R_{2(i+j)}, K)$, so we need to show that $f_i \phi_j \in \text{Hom}_K(R_{2(i+j)}, K)$. Since $f_i \phi_j(r_{2(i+j)}) = \phi_j(f_i r_{2(i+j)})$ and $f_i r_{2(i+j)} \in R_{i+2(i+j)} = R_{3i+j} = R_j$, we know that $f_i \phi_j(r_{2(i+j)}) \in W_{i+j}$ as required. Similarly, if $k \neq 2(i + j)$, then $f_i \phi_j(r_k) = 0$ and hence $f_i \phi_j \in W_{i+j}$.

\[\square\]

(3.5) Theorem (Reduction to $\mathbb{Z}_3$-graded module case). Let
\[R = K[[x, y, z]]/(x^3 + y^3 + z^3),\]
where $K$ is a field of characteristic $p$. Let $I \subseteq R$ be an $m$-primary ideal such that $I^* \neq I^F$. Then there exist a $\mathbb{Z}_3$-graded $R$-module $M$ and an irreducible $\mathbb{Z}_3$-graded submodule $N$ such that $N^* \neq N^F$.

Proof. Suppose $I \subseteq R$ is an $m$-primary ideal such that $I^* \neq I^F$. Then there exists $u \in I^* R^\infty \setminus IR^\infty$. Expand $IR^\infty$ to an ideal of $R^\infty$ maximal with respect to not containing $u$. Then $u$ is the socle mod $IR^\infty$ and $IR^\infty$ is irreducible. To see that $um_{R^\infty} = 0$, note that $m_{R^\infty} = \bigcup m_{R^{1/q}}$. Also, $u \in (I \cap R^{1/q})^*$ for some $q$. This implies that $m_{R^{1/q}} u \subseteq I \cap R^{1/q}$. Thus $m_{R^\infty} u \subseteq IR^\infty$.

Let $E_{R^\infty}$ be a $\mathbb{Z}_3$-graded injective $R^\infty$-module that contains a copy of $K$. We know one exists by Lemmas 3.3 and 3.4. We have an injective map $R^\infty / IR^\infty \to E_{R^\infty}$ sending 1 to $\alpha$. We can find a finitely generated ideal $I_0 \subseteq R^{1/q}$ such that $u \in I_0^{1/q}$, the finitistic tight closure. Here $I_0^{1/q} = \bigcup J(I_0 \cap J)^*$ where $J$ ranges over all finitely generated ideals of $R^{1/q}/IR^\infty$. Let $\tilde{u}$ be the image of $u$ in $R^{1/q}/I_0$. Let $M$ be the submodule of $E_{R^\infty}$ generated by $\alpha$. Then we have a map $R^{1/q}/I_0 \to M$. $M$ is a finitely generated $R^{1/q}$-module that contains the image of $R^{1/q}/I_0$ and is graded. It is still true that $\tilde{u} \in I_0^*$ in $M$ since $u \in 0^*$ in $E_{R^\infty}$. If $u \in 0^F$ in $M$, then we would have $u \in 0^F$ in $E_{R^\infty}$ and an element is in $0^F$ in an $R^\infty$-module if and only if it is zero. Thus $u \notin 0^F$ in $M$. \[\square\]

4. Irreducible Ideals

As we saw in Section 3, we can reduce the question of whether $I^* = I^F$ in $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ to the graded irreducible module case. Given this reduction, it seems likely that understanding the graded irreducible ideal case will be helpful. In this section we will show that $I^* = I^F$ for most $\mathbb{Z}_3$-graded irreducible ideals in $R$ when $K$ has characteristic $p$ and $p \equiv 2 \mod 3$. In the course of proving the main result, Theorem 4.5, we develop a number of techniques for determining when an element of the socle is in the tight closure or the Frobenius closure of a given ideal.

Preliminary Techniques. The following proposition provides a useful tool for determining whether or not a given irreducible $m$-primary ideal, $I$, is tightly closed. If we can find an irreducible ideal contained in $I$ which is tightly closed, then we
know that \( I \) is also tightly closed. Similarly, if we can find an irreducible ideal containing \( I \) which is not tightly closed, then we know that \( I \) is not tightly closed.

(4.1) Proposition. Let \( R \) be a local Gorenstein ring. Let \( m \) be the maximal ideal of \( R \) and let \( J \) and \( I \) be irreducible \( m \)-primary ideals of \( R \) with \( J \subseteq I \). Then \( R/I \hookrightarrow R/J \), and if \( I^* \neq I \), then \( J^* \\neq J \). Also, if \( I^F \neq I \), then \( J^F \neq J \).

Proof. Since \( I \) and \( J \) are \( m \)-primary, \( R/I \) and \( R/J \) are zero-dimensional. As \( I \) and \( J \) are irreducible and \( m \)-primary, \( \dim_K \text{ Soc } R/J = 1 \) and \( R/J \) is Gorenstein, and similarly for \( R/I \). So \( R/J \) is a zero-dimensional Gorenstein local ring, which implies that \( R/J \) is injective as a module over itself and \( R/J \cong E_{R/J}(K) \). Similarly, \( R/I \cong E_{R/I}(K) \). So \( \text{Ann}_{R/J} I \cong \text{Ann}_{E_{R/J}(K)} I \cong E_{R/J}(K) \cong E_{R/I}(K) \cong R/I \), and thus \( \text{Ann}_{R/J} I \cong R/I \). Composing this isomorphism with the natural inclusion \( \text{Ann}_{R/J} I \hookrightarrow R/J \) gives the inclusion \( \phi: R/I \hookrightarrow R/J \). We also know that \( \phi(0)_{R/I} \subseteq (0)_{R/J} \). If \( I^* \neq I \), then \( (0)_{R/I} = I^F/I \neq 0 \), and so \( (0)_{R/J} = 0 \). Then \( J^*/J \neq 0 \) and \( J^* \neq J \) as required. The same argument applies for \( I^F \) and \( J^F \) since \( \phi(0)^F_{R/J} \subseteq (0)^F_{R/J} \).

In fact, even if one or both of \( I \) and \( J \) is not irreducible, if we can show that we have an injection \( R/I \hookrightarrow R/J \), then \( J^* = J \) implies that \( I^* = I \). The following lemma gives a criterion for when such an injection exists.

(4.2) Lemma. Let \( R \) be a Noetherian ring. Let \( I \) and \( J \) be ideals of \( R \) with \( J \subseteq I \), \( I \) irreducible and let \( u \) be the socle mod \( I \). Then \( R/I \hookrightarrow (R/J)^h \) if and only if there exists \( v \in R \) such that \( vI \subseteq J \) and \( vu \notin J \). If, in addition, \( J = J^* \), then \( I = I^* \).

Proof. Let \( u_1, \ldots, u_h \) generate \( J : I \). Let \( \bar{u}_1, \ldots, \bar{u}_h \) be the images of the generators in \( R/J \). Then \( \bar{u}_1, \ldots, \bar{u}_h \) generate \( (J : I)/J \cong \text{Ann}_{R/J} I \). We have a map \( R \rightarrow (R/J)^h \) taking \( \bar{r} \) to \( (ru_1, \ldots, ru_h) \). Now \( \bar{r} \) gets mapped to 0 if and only if \( r(J : I) \subseteq J \). This is equivalent to having \( r \in J : (J : I) \). So the map is injective if and only if \( I = (J : I) \). This is equivalent to having \( u \notin J : (J : I) \) or \( u(J : I) \subseteq J \). Finally, this is true if and only if there exists \( v \in J : I \) such that \( uv \notin J \).

Suppose \( u \in 0^*_{R/J} \). Then the image of \( u \) is contained in \( 0^*_{R/J} \). Thus if \( J \) is tightly closed, so is \( I \).

(4.3) Proposition. Let \( R = K[[x,y,z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \). Let \( I \) be an irreducible \( m \)-primary ideal of \( R \) and let \( u \) represent the socle mod \( I \). If \( I \subseteq (x,y) \), then \( u \in I^F \). Let \( (f,g) \) be generated by a system of parameters. If \( I \subseteq (f,g) \), then \( u \in I^F \).

Proof. Since \( I \) and \( (x,y) \) are both irreducible \( m \)-primary ideals, we have an injection \( R/(x,y) \hookrightarrow R/I \) sending \( z^2 \), the socle in \( R/(x,y) \), to \( u \) (Proposition 4.1). It is enough to see that \( z^2 \in (x,y)^F \), for then \( u \in I^F \). For this it is sufficient to show that \( z^{2p} \) is contained in \( (x^p,y^p) \). Let \( p = 3h + 2 \). Using the basic relation in \( R \) and the \( \mathbb{Z}_3 \)-grading it is sufficient to show that \( (x^3 + y^3)^{2h+1} \in (x^p,y^p) \) (Lemma 1.8). This is routine if we expand using the binomial theorem. Thus \( z^2 \in (x,y)^F \).

Let \( v \) represent the socle in \( R/(f,g) \). Since \( I \) and \( (f,g) \) are both irreducible \( m \)-primary ideals, we have an injection \( R/(f,g) \hookrightarrow R/I \) sending \( v \) to \( u \) (Proposition 4.1). It is enough to see that \( v \in (x,y)^F \), for then \( u \in I^F \). We know that \( (f^q,g^q) \subseteq (x,y) \) for some \( q \). The socle mod \( (f^q,g^q) \) is \( f^{q-1}g^{q-1}v \). Since \( (f^q,g^q) \) is an \( m \)-primary irreducible ideal contained in \( (x,y) \), we know that \( f^{q-1}g^{q-1}v \in (f^q,g^q)^F \).
This implies that $f^{(q-1)Q}g^{(q-1)Q}u^Q \in (f^Q,g^Q)$ for some $Q = p^e$. Dividing by powers of $f$ and $g$ yields $v^Q \in (f^Q,g^Q)$, and hence $v \in (f,g)^F$.

Classification of Irreducibles. The $\mathbb{Z}_3$-grading on $R$ allows us to characterize the irreducible ideals.

(4.4) Proposition. Let $I$ be an irreducible $m$-primary $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where $K$ is a field of characteristic $p$. Then $I$ corresponds to one of the following triples of ideals in $K[[x,y]]$ where $H$ is an irreducible $m$-primary ideal of $K[[x,y]]$ and $f = x^3 + y^3$: $(H, H, H)$, $(H, H : f, H : f)$, $(H, H, f : f)$.

Proof. We know that $(H_0 + H_1z + H_2z^2) : (x, y, z)$ can be decomposed into graded pieces as follows:

$$(H_0 : (x, y)) \cap H_2 + ((H_1 : (x, y)) \cap H_2)z + ((H_2 : (x, y)) \cap (H_0 : (x^3 + y^3)))z^2$$

(Lemma 1.7). Suppose $u$, the socle mod $I$, is contained in $R_0$, the zero graded piece of $R$. Then in order for $I$ to have a one-dimensional socle, there must be no contribution from $R_1$ or $R_2$. This requires that $(H_1 : (x, y)) \cap H_2 = H_1$ and $(H_2 : (x, y)) \cap (H_0 : f) = H_2$. These conditions imply that $H_1 = H_2$ and $H_2 = H_0$; $f$, respectively. To see this, just note that if $H_1$ were strictly contained in $H_2$, since $H_1 : (x, y)$ is strictly larger than $H_1$, their intersection would strictly contain $H_1$. In other words, $I$ corresponds to the triple $(H_0, H_0 : f, H_0 : f)$. The annihilator of $(x, y, z)$ is now $(H_0 : (x, y)) \cap (H_0 : f)$. Since $(f) \subseteq (x, y)$, we know that $(H_0 : (x, y)) \subseteq (H_0 : f)$, and so the intersection is just $H_0 : (x, y)$. The socle is then $(H_0 : (x, y)) \setminus H_0$ or just the socle mod $H_0$ in $K[[x,y]]$. Thus, if $H_0$ is an irreducible ideal of $K[[x,y]]$, then $I$ has a one-dimensional socle and is irreducible. Similar arguments are used if the socle mod $I$ is contained in $R_1$ or $R_2$. □

Tight Closure and Frobenius Closure of Irreducible Ideals. Now we can prove the main result of this section.

(4.5) Theorem. Let $I$ be an irreducible $m$-primary $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where $K$ is a field of characteristic $p$ and $p \equiv 2 \mod 3$. Let $f = (x^3 + y^3)$. If $I$ has any of the following forms, then $I^* = I^F$.

1. $(H, H, H)$,
2. $(H, H : f, H : f)$,
3. $(H, H, f : f)$ and $f \notin H$,
4. $(H, H, f : f)$ and $f \in H$ and $H$ contains an element with a linear form.

Proof of (1)–(3). First observe that $(H, H, H) \subseteq (x, y)$. The ideals $(H, H : f, H : f)$ and $(H, H, H : f)$ are also contained in $(x, y)$ so long as $f \notin H$. If $f \in H$, then $H : f = K[[x,y]] = A$. In that case, $(H, H : f, H : f) = (H, A, A) = H + Az$ and $(H, H, H : f) = (H, H, A) = H + Az^2$. When the ideals are contained in $(x, y)$ we know that $I^* = I^F$ by Proposition 4.3. In fact, we know that $I^* \neq I$ in those cases.

We will now consider the case $I = (H, H : f, H : f)$ where $f \in H$. As noted before, $I = H + Az$ in this case. Let $q = 3h + 1$. Suppose $u \in I^*$. Then, using $z$ as a test element, and the grading (Lemma 1.9), we see that this is equivalent to having $u^q \in H^{[q]} + (f^{h+1}) + (f^{2h+1})$ in $K[[x,y]]$ which implies that $u^q \in H^{[q]} + (f^{h+1})$. This, however, is exactly what is needed to have $u^q \in I^{[q]}$ (Lemma 1.9). Thus $u \in I^F$. □
4.3 Suppose an element of \( H \) would be contained in a parameter ideal of characteristic \( p \). We can also assume that \( x \) preparation, we can find a unique monic associate
\[ \text{order of } x = k: \]
\[ \text{let } g \in K \] the ideal \(( x, y - g(x), z )\) which is a parameter ideal. Suppose \( x^k \notin ( y - g(x), z ) \) in \( R \). Using the \( \mathbb{Z}_3 \)-grading (Lemma 1.9) we see that this is equivalent to having \( x^k \notin ( y - g(x), x^3 + y^3 ) \) in \( K[x,y] \) modulo \( u = y - g(x) \). In order to have \( x^k \notin ( x^3, g(x)^3 ) \), we need the order of \( x^3 + g(x)^3 \) to be greater than \( k \). Assume \( \text{ord}_x g(x) \geq 2 \) or \( c \neq -1 \) where \( g(x) = cx + \cdots \). If \( k = 1 \), then \( H = ( x, y - g(x) ) = ( x, y ) \). If \( k = 2 \), then \( H = ( x^2, y - g(x) ) = ( x^2, y - cx ) \).

Now suppose that \( k > 2 \). We still need the order of \( x^3 + g(x)^3 \) to be greater than \( k \). We can assume that \( \text{ord}_x g(x) = 1 \) and \( g(x) = -x + dx^h + \cdots \). Then \( x^3 + g(x)^3 = 3dx^{2h+1} + \text{lower degree terms} \). So we need \( h + 2 > k \). If \( k < h + 1 \), then \( ( x^k, y - g(x) ) = ( x^k, y + x ) \). If \( k = h + 1 \), then \( ( x^k, y - g(x) ) = ( x^k, y + x - dx^{k-1} ) \). In each case \( k \geq 3 \).

We can now deal with these cases separately.

4.7 Remark. Let \( R \) be a Noetherian ring and \( m \) a maximal ideal. If \( I \) is an \( m \)-primary ideal of \( R \), then \( R/I \cong \hat{R}/I\hat{R} \). If we are interested in whether \( u \in I\hat{R} \), it is sufficient to check whether \( u \in I \). We will make use of this idea in several of the following propositions by reducing questions about ideal membership in \( K[x,y] \) to the polynomial ring \( K[x,y] \).

4.8 Proposition. Let \( R = K[[x,y,z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \).
\[ \begin{align*}
(1) & \quad \text{Let } I = ( x, y, z^2 ). \text{ Then } I^* = I^F = I. \\
(2) & \quad \text{Let } I = ( x^2, y - cx, z^2 ), c \in K \setminus \{0\}. \text{ Then } I^* = I^F = I. \\
(3) & \quad \text{Let } I = ( x^k, y + x, z^2 ) \text{ with } k \geq 3. \text{ Then } I^* = I^F = I.
\end{align*} \]

Proof. Let \( p = 3h + 2 \) and \( f = x^3 + y^3 \).

(1) The socle mod \( I \) is \( z \). Using \( z \) as a test element, it suffices to see that \( zz^p \notin ( x^p, y^p, z^{2p} ) \). Suppose \( zz^p \in ( x^p, y^p, z^{2p} ) \). Using the basic relation in \( R \), and the \( \mathbb{Z}_3 \)-grading (Lemma 1.8), we see that this is equivalent to having \( f^{h+1} \in ( x^p, y^p, f^{2h+2} ) \) in \( K[x,y] \). A degree argument shows that this cannot hold.
The desired relation in \( R \) is
\[
(x, y^2, x^2, x, y, x^2, y, x, y, z, x^2, y, z, x, y, z)
\]
where \( x, y, z \) are the variables. By factoring out the terms involving \( x, y, z \), we may conclude that \( I \) is generated by \( \langle x, y, z \rangle \).

Expanding \( x^p h^{k+1} \) shows that the equality cannot hold.

(3) The socle mod \( I \) is \( x^{k-1}z \). Using \( z \) as a test element, it suffices to see that
\[
z(x^{k-1}z)^p \notin (x^{k+p}, y^p + x^p, z^{2p})
\]
Suppose \( z(x^{k-1}z)^p \in (x^{k+p}, y^p + x^p, z^{2p}) \). Using the basic relation in \( R \) and the \( Z_3 \)-grading (Lemma 1.8) we see that this is equivalent to having \( x^p f^{k+1} \in (x^{2p}, y^p - c^p x^p, z^{2p}) \). The degree of \( f^{k+1} \) is \( 2p + 1 \), while the degree of \( f^{2h+2} \) is \( 2p + 2 \). Since we are in the homogeneous case, we may conclude that \( x^p f^{k+1} = (a_1 x + a_2 y)x^{2p} + B(y^p - c^p x^p) \) where \( a_1, a_2 \in K \) and \( B \in K[x, y] \). Since \( x^p f^{k+1} \) has no term with the degree of \( x \) less than \( p \), \( B = (b_1 x^{p+1} + b_2 x^p y), b_1, b_2 \in K \). Expanding \( x^p f^{k+1} \) shows that the equality

\[
\text{cannot hold.}
\]

(4.9) Proposition. Let \( R = K[[x, y, z]]/(x^3 + y^3 + z^3) \), where \( K \) is a field of characteristic \( p \) and \( p \equiv 2 \mod 3 \). Let \( I = (x^k, x + y - dx^{k-1}, z^2), k \geq 3, d \in K \setminus \{0\} \). Then \( I^* = I^p = I \).

Proof. The socle mod \( I \) is \( x^{k-1}z \). Using \( z \) as a test element, it suffices to show that \( z(x^{k-1}z)^p \notin (x^{k+p}, y^p + x^p, z^{2p}) \). We will reduce to the case \( d = 1 \). Apply the following map to \( R: x \rightarrow \lambda x, y \rightarrow \lambda y, \) and \( z \rightarrow \lambda z \), where \( \lambda \in K \). Then
\[
z(x^{k-1}z)^p \rightarrow \lambda^p z(x^{k-1}z)^p \in (x^k, x + y - dx^{k-1}, z^2)
\]
if and only if
\[
\lambda^k z(x^{k-1}z)^p \in (\lambda^k x^k, \lambda x + \lambda y - \lambda^k dx^{k-1}, \lambda^2 z^2).
\]
By factoring out the \( \lambda^k \), we are left with \( z(\lambda x^{k-1})^p \in (x^k, x + y - \lambda^{k-2} dx^{k-1}, z^2) \). If \( d \neq 0 \), let \( \lambda = d^{-1/\lambda^{k-2}} \). So if \( x^{k-1}z \) is in the tight closure of the ideal for one value of \( d \neq 0 \), then it is in for all \( d \neq 0 \). We have reduced to the case where
\[
I = (x^k, x + y - x^{k-1}, z^2).
\]
By Lemma 4.2 it is enough to find an ideal \( J \subseteq I \) such that \( J \) is tightly closed and \( R/I \rightarrow R/J \). Let
\[
J_0 = ((x + y)^2, x^2, (x + y)x^k, x^2k, (x + y)^2, x^k, x^{k-1}z, (x + y)^2, x^{k-1}z^2).
\]
The desired ideal is \( J_0^* \). In order to show that \( R/I \rightarrow R/J_0^* \), it is sufficient to find \( v \in J_0^* \) such that \( vu \notin J_0^* \) for \( u \) the socle mod \( I \) (Lemma 4.2).

First we want to see that \( J_0^* \subseteq I \). Let \( J_1 = (y(x + y), x(x + y), x^k, z^2) \). The socle mod \( J_1 \) is generated by \( (x + y)z \) and \( x^{k-1}z \). We would like to show that \( J_1 = J_1^* \). We know that \( (x + y)z \notin J_1^* \) by a degree argument [Sm3, Theorem 2.2]. To show that \( x^{k-1}z \notin J_1^* \) we will consider the ideal \( J_2 = (x + y, x^k, z^2) \). We know that \( x^{k-1}z \notin J_2^* \) and \( J_2 = J_2^* \) by a previous case (Proposition 4.8 (3)). As \( J_1 \subseteq J_2 \) and \( x^{k-1}z \notin J_2^* \), we may conclude that \( x^{k-1}z \notin J_1^* \). Thus \( J_1 = J_1^* \). We
also know that $J_0 \subseteq J_1$ implies $J_0^* \subseteq J_1^*$ [HH1, Proposition 4.1]. Now we have $J_0^* \subseteq J_1^* = J_1 \subseteq I$, which guarantees that $J_0^* \subseteq I$.

Next we would like to show that $x + y + x^{k-1} \in J_0^*$. First we note that
\[(x + y + x^{k-1})I \subseteq ((x + y)x^k, x^{2k-1}, (x + y)^2 - x^{2k-2}, (x + y)z^2, x^{k-1}z^2).
\]
Certainly $(x + y)x^k, x^{2k-1}, (x + y)^2 - x^{2k-2}, (x + y)z^2, x^{k-1}z^2 \subseteq J_0 \subseteq J_0^*$.

Recall that $x^{k-1}z$ is the socle mod $I$. We want to show that $(x+y(x^{k-1})x^{k-1}z \notin J_0^*$. Since $J_0$ and hence $J_0^*$ are homogeneous, it is enough to show that $(x + y)x^{k-1}z \notin J_0$. Using $z$ as a test element, it suffices to see that $z(x+y)p_{x^{(k-1)p}zy^p} \notin J_0$. Suppose $z(x+y)p_{x^{(k-1)p}zy^p} \in J_0$. Using the basic relation in $R$ and the $\mathbb{Z}_3$-grading (Lemma 1.8) shows that this is equivalent to having $(x + y)^p x^{(k-1)p} f^{h+1} \in ((x + y)^p x^{kp}, x^{(2k-2)p}, (x + y)^p f^{h+2})$

Since we are in the homogeneous case, routine degree arguments show that $(x + y)^p x^{(k-1)p} f^{h+1} \in ((x + y)^p x^{kp}, (x + y)^p f^{h+2})$

as long as $k > 3$. Dividing by $(x + y)^p$ yields $x^{(k-1)p} f^{h+1} \in (x^{kp}, (x + y)^p f^{h+2})$. But this is equivalent to having $x^{k-1}z \in (x^k, (x + y), z^2)^*$. We know that $x^{k-1}z \notin (x^k, (x + y), z^2)^*$ by a previous result (Proposition 4.8 (3)).

Let $k = 3$ and suppose that
\[(x + y)^p z^p f^{h+1} = ((x + y)^p x^{kp}, (x + y)^p f^{h+2}) \text{ is divisible by } (x^3, (x + y), z^2)^*.
\]
The degree of $(x + y)^p z^p f^{h+1}$ is $4p + 1$. Since we are in the homogeneous case, this implies that $(x + y)^p z^p f^{h+1} = A(x + y)^p + (\beta_1 x + \beta_2 y)x^{4p} + Cx^{2p} f^{h+2}$

where $\beta_1, \beta_2 \in K$ and $A, C \in K[x, y]$ (4.7). This but implies that $(x + y)^{h+2}$ divides $(\beta_1 x + \beta_2 y)x^{4p}$ which is impossible.

So with $v = x + y + x^{k-1}$, we have $v \in J_0^*$. I and $x^{k-1}zv \notin J_0^*$. This is enough to show $R/I \hookrightarrow R/J_0^*$ by Lemma 4.2. Since $J_0^*$ is tightly closed, we know that $I$ is tightly closed, also by Lemma 4.2. □

In addition to the cases where $I \subseteq (x, y)$, we can determine whether or not an irreducible ideal is tightly closed, not just that $I^* = I^F$, in the following cases.

(4.10) Proposition. Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where $K$ is a field of characteristic $p$. Let $I$ be an irreducible $\mathbb{Z}_3$-graded ideal of the form $(H; H; f)$, where $f = x^3 + y^3 \in H$, and $H$ is generated by elements whose leading forms are relatively prime quadratic forms. Then $I = I^*$.

Proof. $I$ is of the form $(Q_1 + C_1, Q_2 + C_2, z)$. Here we mean the ideal generated by $Q_1 + C_1, Q_2 + C_2,$ and $z$, not a triple of ideals. Let $Q_3$ be the third independent quadratic form. By considering the associated graded ring we can see that $K[[x, y, z]]/(Q_1 + C_1, Q_2 + C_2)$ has dimension four over $K$, and it follows that $1, x, y, Q_3$ give a basis. Everything of degree three or more will be in $H$ and $Q_3$ will represent the socle mod $I$. This also guarantees that $f \in H$. We would like to show that $Q_3 \notin (Q_1 + C_1, Q_2 + C_2, z)^*$. Using the grading and $x$ as a test element, it is sufficient to show that $xQ_3^p \notin (Q_1^p, Q_2^p, f^{h+1})$. This is equivalent to showing that $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is not divisible by $f^{h+1}$ where $L_1$ and $L_2$ are linear forms. We will dehomogenize the equation by setting $y = 1$. If $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is divisible by $f^{h+1}$, then $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is divisible by $f^{h+1}$.
the derivative with respect to $x$ is divisible by $\bar{f}^h$. Using the fact that we are in characteristic $p$, we see that the derivative is $Q_3^h + \frac{1}{h}Q_1^h + \frac{1}{h}Q_2^h$. So we need that $(Q_3^h + \frac{1}{h}Q_1^h + \frac{1}{h}Q_2^h)^p$ is divisible by $\bar{f}^h$. If we rewrite $\bar{f}^h$ as $(x - 1)^h(x - \omega)^h(x - \bar{\omega})^h$, we conclude that all three linear factors of $\bar{f}$ divide $(Q_3^h + \frac{1}{h}Q_1^h + \frac{1}{h}Q_2^h)^p$. Since $Q_1$ and $Q_2$ are still independent over $K$, this cannot happen.

$(4.11)$ Comment. Let $(H, z)$ and $(H, z^2)$ be two irreducible $m$-primary ideals of $K[[x, y, z]]/(x^3 + y^3 + z^3)$. Since $(H, z^2) \subseteq (H, z)$, we know that if $(H, z^2)$ is tightly closed, then so is $(H, z)$ (Proposition 4.1). In particular, if $I = (x, y, z^2)$, $(x^2, y - cx, z^2)$, $(x^k, y + x, z^2)$, $(x^k, x + y - x^{k-1}, z^2)$, we know that $I = I'$. So if $I = (x, y, z)$, $(x^2, y - cx, z)$, $(x^k, y + x, z)$, $(x^k, x + y - x^{k-1}, z)$, we know that $I = I'$ also.

Next we classify the cases of $m$-primary irreducible $\mathbb{Z}_3$-graded ideals not included in Theorem 4.5. To do this we need the following proposition which gives a characterization of the $m$-primary irreducible ideals in $K[[x, y]]$.

$(4.12)$ Lemma. Let $A = K[[x, y]]$. Let $I$ be an irreducible $m$-primary ideal in $A$. Then $I$ is generated by parameters.

Proof. First note that $I$ is a height two ideal and the quotient, $A/I$, is Cohen-Macaulay and has finite projective dimension. This means that $A/I$ must have a resolution that looks like $0 \to A'^{-1} \to A' \to A \to A/I \to 0$ where the entries of the matrix of the map from $A'$ to $A$ can be taken to be minimal generators of $I$. Then $I$ must be the ideal generated by the $r - 1$ size minors of the second matrix. This implies that the type of $A/I$ is one smaller than the number of generators of $I$. Since $A/I$ has type one, we must have $r = 2$.

We are now able to classify the remaining cases.

$(4.13)$ Proposition. Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ and $A = K[[x, y]]$, where $K$ is a field of characteristic $p \neq 3$. Let $I$ be an $m$-primary irreducible $\mathbb{Z}_3$-graded ideal of $R$ corresponding to the triple of ideals $(H, H, H; f)$, where $f = x^3 + y^3 \in H$.

Suppose $H$ does not contain an element with a linear leading form. Then $I$ has one of the following forms:

1. $I = (Q_1, Q_2, z^2)$ where $Q_1$, $Q_2$ are relatively prime quadratic forms in $A$;
2. $I = (L_1^2 + C, L_1L_2 + D, z^2)$ where $L_1$ and $L_2$ are independent linear forms, $L_1$ divides $f$, and $C$ and $D$ have cubic or higher leading forms;
3. $I = (L_1L_2 + C, D, z^2)$ where $L_1$, $L_2$, $C$ and $D$ are as in $(2)$.

Proof. We know that $I = H + A^2$ where $H$ is an $m$-primary irreducible ideal of $A$. Also, we know that $H$ is generated by two parameters by Lemma 4.12.

Suppose $H = (Q_1 + C_1, Q_2 + C_2)$ where $Q_1$ and $Q_2$ are quadratic forms and $C_1$ and $C_2$ are higher order terms. If $Q_1$ and $Q_2$ are relatively prime, then by considering the associated graded ring, we can see that everything of degree three or higher is contained in $H$. Thus $H = (Q_1, Q_2)$ and the third independent quadratic form will be the socle mod $H$.

If $Q_1$ and $Q_2$ are not relatively prime, we can write $H = (LL_1 + C_1, LL_2 + C_2)$. If $L$ and $L_1$ are independent over $K$, then they span the space of linear forms and we can write $L_2 = aL + bL_1$. This implies that $LL_2 = aL^2 + bLL_1$. Hence we may rewrite $H$ as $(LL_1 + C_1, L^2 + C_2')$. A similar argument applies if $L$ and $L_2$ are
independent. If \( L, L_1 \) and \( L_2 \) are all dependent, then \( H = (L^2 + C_1, L^2 + C_2) = (L^2 + C_1, C_2) \).

If \( H = (LL_1 + C_1, L^2 + C_2) \), since we must have \( f \in H \), either \( LL_1 \) divides \( f \) or \( L^2 \) divides \( f \). Suppose \( L \) does not divide \( f \). Then the associated graded ring must contain everything of order three or higher and \( f = (L + D_2)(LL_1 + C_1) - (L_1 + D_1)(L^2 + C_2) \). But everything on the right-hand side has order three or higher; hence \( L \) divides \( f \).

If \( H = (L^2 + C_1, C_2) \), then \( L^2 \) must divide \( f \). To see this, note that if \( C_2 \) divides \( f \), then \( f \) will be a minimal generator of \( H \). Since \( z^2 \in I \) and \( z^3 = -f \), if \( f \) is a minimal generator of \( H \), then \( I \) will be generated by \( z^2 \) and the other minimal generator of \( H, L^2 + C_1 \). In other words, \( I \) will be generated by parameters and we know that the socle mod \( I \) is contained in \( I^5 \) by Proposition 4.3.

\((4.14)\) Comment. The remaining cases have proved to be very challenging. In particular, even the question of whether \( xyz \in (x^2, y^2, z^2) \) is quite difficult. A. Singh has given an argument using determinants of matrices of binomial coefficients to show that indeed \( xyz \in (x^2, y^2, z^2) \) for all \( p \) and \( xyz \in (x^2, y^2, z^2)^P \) for \( p \equiv 2 \) mod 3 [Si].

5. Generalizations to Other Rings

Many of the results in this paper can be generalized to rings of the form \( K[[x, y, z]]/(z^3 - F(x, y)) \) where \( F(x, y) \) is a homogeneous polynomial of degree three, \( K \) is a field of characteristic \( p \) and \( p \neq 3 \). We first note that the maximal ideal, \( m \), is the test ideal for these rings. For \( p > 3 \) this is a consequence of a tight closure interpretation of the Kodaira Vanishing Theorem for Gorenstein rings in dimension two [HuS, (4.5) and (5.4)].

We give a proof for all positive prime characteristics here.

\((5.1)\) Proposition. Let \( R = K[[x, y, z]]/(z^3 - F(x, y)) \), where \( K \) is a field of characteristic \( p \), and \( F(x, y) \) is a homogeneous polynomial of degree three. Then \( m \) is the test ideal for \( R \).

Proof. The beginning of the proof is the same as the beginning of the proof of Proposition 1.4. We can show that it is sufficient to check that \( \lambda_1 x^{t-1} y^{t-1} z \notin (x^t, y^t)^* \) and \( \lambda_1 x^{t-2} y^{t-1} z^2 + \lambda_2 x^{t-1} y^{t-2} z^2 \notin (x^t, y^t)^* \). The proof that \( \lambda_1 x^{t-1} y^{t-1} z \notin (x^t, y^t)^* \) is also the same as the proof in Proposition 1.4.

Suppose \( \lambda_1 x^{t-2} y^{t-1} z^2 + \lambda_2 x^{t-1} y^{t-2} z^2 \in (x^t, y^t)^* \). This implies that \( \lambda_1 x^2 + \lambda_2 y z^2 \in (x^t, y^t)^* \). We know that \( (x^t, y^t)^* : x^{t-2} y^{t-2} \subseteq (x^2, y^2)^* \) by the usual colon-capturing argument [HH1, Theorem 7.15a]. If \( \lambda_1 x^2 + \lambda_2 y z^2 \in (x^t, y^t)^* \), then we can find \( c \neq 0 \) such that \( c(\lambda_1 x + \lambda_2 y)q z^{2q} \in (x^{2q}, y^{2q}) \) for all \( q \). Write \( 2q = 3h + 2 \). Using the basic relation in \( R \), this implies that \( c(\lambda_1 x + \lambda_2 y)^q F^h \in (x^{2q}, y^{2q}) \) or \( cF^h \in (x^{2q}, y^{2q}) : (\lambda_1 x + \lambda_2 y)^q \). This is equivalent to having \( cF^h \in (x^{2q}, y^{2q}, x^q y^q, (\lambda_1 x - \lambda_2 y)^q) = (x^{2q}, (\lambda_1 x - \lambda_2 y)^q) \). We can use \( F \) as a test element and then \( F^{h+1} \in (x^{2q}, (\lambda_1 x - \lambda_2 y)^q) \). Suppose \( F^{h+1} = Ax^{2q} + B(\lambda_1 x - \lambda_2 y)^q \), where \( A, B \in K[[x, y]] \). By a degree argument we must have \( F^{h+1} = (a_1 x + a_2 y)^q x^{2q} + B(\lambda_1 x - \lambda_2 y)^q \), where \( a_1, a_2 \in K \). Let \( F_x \) denote the partial derivative of \( F \) with respect to \( x \). Taking derivatives twice yields \( (h+1)h F^{h+1}_x + (h+1)F F_{xx} = B_x + (\lambda_1 x - \lambda_2 y)^q \). This implies that \( (h+1)h F^{2}_x + (h+1)FF_{xx} = 0 \), because after we divide both sides by \( F^{h+1} \), we still have a high power of \( (\lambda_1 x - \lambda_2 y) \) that must divide the left-hand side. This implies that \( h F^{2}_x + FF_{xx} = 0 \). We can
assume that $F$ has distinct linear factors, and by making a change of variable if necessary, we can assume that $F = xy(ax + by)$ with $a, b \neq 0$. Write $L$ for $(ax + by)$. Then $F = xyL$, $F_x = y(ax + L)$ and $F_{xx} = 2ay$. Substituting yields $hy^2(ax + L)^2 + xyL(2ay) = 0$ or $hy^2(a^2x^2 + 2axL + L^2) + 2axyL = 0$. If $p \neq 2$, then this implies that $L$ divides $ha^2x^2y^2$ which is not possible. If $p = 2$, then we must have $hy^2(ax + L)^2 = hy^2(by)^2 = 0$. This implies that $hb^2 = 0$, but $h$ can be chosen larger than 2 and $b$ was assumed to be non-zero.

As $m$ is the test ideal, we know that if $u \in I^* \setminus I$, then $u$ is in the socle mod $I$ (Proposition 1.5). We will combine this fact with the following analogue of Proposition 4.3 in order to make the generalizations.

(5.2) Proposition. Let $R = K[[x, y, z]]/(z^3 - F(x, y))$, where $K$ is a field of characteristic $p$, $p \equiv 2 \mod 3$, and $F(x, y)$ is a homogeneous polynomial of degree three. Let $I$ be an irreducible $m$-primary ideal of $R$ with $I \subseteq (x, y)$. Suppose $u$ represents the socle mod $I$. Then $u \in I^F$.

Proof. We know that there is an injection $R/(x, y) \hookrightarrow R/I$ (Proposition 4.1). It suffices to see that $z^{2p} \in (x^p, y^p)$. Suppose $p = 3h + 2$. Then $z^{2p} = F^{2h+1}z^2$. Now it is enough to see that $F^{2h+1} \in (x^p, y^p)$ in $K[[x, y]]$. The degree of $F^{2h+1}$ is $2p - 1$, so every term of $F^{2h+1}$ has a factor of $x^p$ or $y^p$. In other words, $F^{2h+1} \in (x^p, y^p)$. Hence $u \in I^F$ (4.1).

The classification of irreducibles (Proposition 4.4) also follows essentially unchanged. Thus the irreducible $m$-primary ideals of $R = K[[x, y, z]]/(z^3 - F(x, y))$ are exactly the ideals of the form $(H, H, H)$, $(H, H : F, H : F)$ and $(H, H, H : F)$ where $H$ is an irreducible $m$-primary ideal of $K[[x, y]]$ and $F = F(x, y)$. As before, $(H, H, H) \subseteq (x, y)$ and $(H, H : F, H : F)$ and $(H, H, H : F)$ are both contained in the ideal $(x, y)$ as long as $F \not\in H$. We know then that $I^F = I^*$ and $I \neq I^*$ in these cases. More generally, for any irreducible $m$-primary ideal of $R$ contained in $(x, y)$ we have that $I^F = I^*$.

References


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