RARIFIED SUMS OF THE THUE-MORSE SEQUENCE

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Abstract. Let \( q \) be an odd number and \( S_{q,0}(n) \) the difference between the number of \( k < n, k \equiv 0 \mod q \), with an even binary digit sum and the corresponding number of \( k < n, k \equiv 0 \mod q \), with an odd binary digit sum. A remarkable theorem of Newman says that \( S_{3,0}(n) > 0 \) for all \( n \). In this paper it is proved that the same assertion holds if \( q \) is divisible by 3 or \( q = 4^n + 1 \). On the other hand, it is shown that the number of primes \( q \leq x \) with this property is \( o(x/\log x) \). Finally, analogs for “higher parities” are provided.

1. Introduction

The Thue-Morse sequence [9], [5] is defined by

\[
 t_n = (-1)^{s(n)},
\]

where \( s(n) \) denotes the number of ones in the binary representation of \( n \). For any positive integer \( q \) and \( i \in \mathbb{Z} \) we denote

\[
 S_{q,i}(n) = \sum_{0 \leq j < n, j \equiv i \mod q} t_j.
\]

In 1969 Newman [10] proved a remarkable conjecture of L. Moser saying that for any \( n \geq 1 \)

\[
 S_{3,0}(n) > 0.
\]

More precisely, he proved that

\[
 \frac{3^\alpha}{20} < \frac{S_{3,0}(n)}{n^\alpha} < 5 \cdot 3^\alpha \quad \text{with} \quad \alpha = \frac{\log 3}{\log 4}.
\]

In 1983 Coquet [1] provided an explicit precise formula for \( S_{3,0}(n) \) by the use of a continuous function \( \psi_3(x) \) with period 1 which is nowhere differentiable (\( \eta_3(n) \in \{-1, 0, 1\} \)):

\[
 S_{3,0}(n) = n^{\log_3} \cdot \psi_3 \left( \frac{\log n}{\log 4} \right) - \frac{\eta_3(n)}{3}.
\]

Furthermore, he was able to identify \( \min \psi([0, 1]) > 0 \) and \( \max \psi([0, 1]) \).

In general, (asymptotic) representations similar to (3) exist for any \( S_{q,i}(n) \) (see [5] and section 2). But it is a non-trivial problem to decide whether the continuous function \( \psi_{q,i}(x) \) has a zero or not. The only known examples where \( \psi_q(x) = \psi_{q,0}(x) \) has no zero are \( q = 3^{k5^l} \) ([6]) and \( q = 17 \) ([7]). (Note that the assertion that \( \psi_{q,i}(x) \)
Our first result provides infinitely many new examples where \( \psi_q(x) \) has no zero. 

**Theorem 1.** Suppose that \( q \) is divisible by 3 or \( q = 4^N + 1 \). Then \( S_{q,0}(n) > 0 \) for almost all \( n \).\(^1\)

However, if \( q \) is prime then we can prove that there are only a few exceptions (e.g., Fermat primes). Let \( P_t, t \geq 1 \), denote the set of those primes \( p \) where the order \( \text{ord}_p(2) \) of 2 in the multiplicative group \((\mathbb{Z}/p\mathbb{Z})^*\) equals \( \text{ord}_p(2) = (p-1)/t \).

**Theorem 2.** There exists a constant \( C > 0 \) such that for any \( t \geq 1 \) the primes \( p \in P_t \) satisfying \( S_{p,0}(n) > 0 \) for almost all \( n \) are bounded by

\[
p \leq C t^2 \log^2 t.
\]

Furthermore, the total number of primes \( p \leq x \) with \( S_{p,0}(n) > 0 \) for almost all \( n \) is \( o(x/\log x) \) as \( x \to \infty \).

The first part of Theorem 2 generalizes a result by the authors [2], where it is shown that 3 and 5 are the exceptional primes of \( P_1 \) and 17 and possibly 41 those of \( P_2 \). (In fact, \( p = 41 \) is not exceptional, see section 3.)

It is surely a very difficult problem to decide whether there are infinitely many primes \( p \) satisfying \( S_{p,0}(n) > 0 \) for almost all \( n \) or not. Unfortunately our methods are not strong enough to settle this problem. But it should be noted that if there were only finitely many primes with this property, Theorem 1 would imply that there were only finitely many Fermat primes.

However, the methods to be developed are essentially sufficient to decide this problem for any concrete value \( q \). For example, we can prove the following theorem.

**Theorem 3.** The only primes \( p \leq 1000 \) satisfying \( S_{p,0}(n) > 0 \) for almost all \( n \) are \( p = 3, 5, 17, 43, 257, 683 \).

Note that \( p = 43 \in P_3 \) and \( p = 683 \in P_{31} \) are not Fermat primes.\(^2\)

We will prove Theorems 1 and 2 in sections 4 and 5. The negative part of Theorem 3 is proved at the end of section 3 and the positive part at the end of section 4. Section 6 is devoted to the case of higher parities where similar phenomena appear. In section 2 we collect some basic facts on the fractal structure of \( S_{q,i}(n) \), and in section 3 we discuss two different kinds of positivity phenomena.

2. Basic Facts

For any fixed positive integer \( q \) and \( i \in \mathbb{Z} \), set

\[
S_{q,i}(y,n) = \sum_{j<n, j \equiv i \mod q} y^{s(j)},
\]

\(^1\)The phrase “almost all” means “all but finitely many”, i.e. there might be finitely many exceptions.

\(^2\)Note that both 43 and 684 are of the form \( (2^{2N+1} + 1)/3 \). Recently, by extending the methods of section 4, Leinfellner [8] showed that \( q \) of the form \( (2^{2N+1} + 1)/3 \) have the property that \( S_{q,0}(n) > 0 \) for almost all \( n \).
in which \( n \geq 0 \) and \( y \) is a (complex) parameter. With help of these expressions we can determine the numbers

\[
A_{q,i;r,m}(n) = |\{ j < n : j \equiv i \mod q, s(j) \equiv m \mod r \}|
\]

\[
= \frac{1}{r} \sum_{l=0}^{r-1} \zeta_r^{-ml} S_{q,i}(l^i_r, n),
\]

where \( r \) is a positive integer (which will be called a parity), \( m \in \mathbb{Z} \), and \( \zeta_r \) denotes the \( r \)-th primitive root of unity, \( \zeta_r = \exp \left( \frac{2\pi i}{r} \right) \).

Note that \( S_{q,i}(y, n) \), \( 0 \leq i < q \), satisfies a simple generating relation if \( n \) is a power of 2:

\[
q-1 \sum_{i=0}^{q-1} S_{q,i}(y, 2^k) \zeta_q^{-i} = \prod_{j=0}^{k-1} \left( 1 + y \zeta_q^{2^j} \right),
\]

in which \( \zeta_q = \exp \left( \frac{2\pi i}{q} \right) \) denotes the \( q \)-th primitive root of unity and \( l \in \mathbb{Z} \). Hence we directly obtain

\[
S_{q,i}(y, 2^k) = \frac{1}{q} \sum_{l=0}^{q-1} \zeta_q^{-l} \prod_{j=0}^{k-1} \left( 1 + y \zeta_q^{2^j} \right).
\]

Moreover, the obvious relation

\[
S_{q,i}(y, 2^k + n') = S_{q,i}(y, 2^k) + yS_{q,i-2^k}(y, n') \quad (n' < 2^k)
\]

can be used to calculate \( S_{q,i}(n) \) inductively for any integer \( n \geq 0 \).

We will further need

\[
S(y, n) = \sum_{j<n} y^{s(j)} = \sum_{i=0}^{q-1} S_{q,i}(y, n)
\]

and the numbers

\[
A_{r,m}(n) = |\{ j < n : s(j) \equiv m \mod r \}|
\]

\[
= \frac{1}{r} \sum_{l=0}^{r-1} \zeta_r^{-ml} S(l^i_r, n).
\]

\( S(y, 2^k) \) is given by

\[
S(y, 2^k) = (1 + y)^k
\]

and satisfies

\[
S(y, 2^k + n') = S(y, 2^k) + yS(y, n') \quad (n' < 2^k).
\]

Our first aim is to describe the asymptotic behaviour of \( A_{q,i;r,m}(n) \). The natural leading term is \( \frac{1}{q} A_{r,m}(n) \):

\[
A_{q,i;r,m}(n) = \frac{1}{q} A_{r,m}(n) + R_{q,i;r,m}(n).
\]

From (6), (8), (12), and (13) we obtain the representations

\[
A_{q,i;r,m}(2^k) = \frac{1}{rq} \sum_{l_1=0}^{r-1} \sum_{l_2=0}^{q-1} \sum_{j=0}^{k-1} \left( 1 + \zeta_{l_1} \zeta_{l_2}^{2^j} \right)
\]
and

\[ A_{r,m}(2^k) = \frac{1}{r} \sum_{l_1=0}^{r-1} \zeta_r^{-l_1 m} (1 + \zeta_r^{l_1})^k, \]

so that

\[ R_{q,i;r,m}(2^k) = \frac{1}{rq} \sum_{l_1=0}^{r-1} \zeta_r^{-l_1 m} \sum_{l_2=0}^{r-1} \zeta_q^{-l_2 i} \prod_{j=0}^{k-1} \left( 1 + \zeta_r^{l_1} \zeta_q^{l_2 i} \right). \]

These Fourier expansions will be frequently used in the proofs of our main results.

From now on let \( q \) be an odd positive integer and let \( s = \text{ord}_q(2) \) be the order of the multiplicative subgroup \( \langle 2 \rangle \) of \( (\mathbb{Z}/q\mathbb{Z})^* \). (Since we are mainly interested in \( A_{q,0,r,m}(n) \), it is no real restriction to assume that \( q \) is odd.) Furthermore, let \( S_q(y,n) = (S_{q,0}(y,n), \ldots, S_{q,q-1}(y,n))^t \) denote the vector of \( S_q, i(y,n) \). Let \( e_0, \ldots, e_{q-1} \) denote the canonical basis of the \( q \)-dimensional vector space \( \mathbb{C}^q \) and let \( T \) denote the matrix defined by \( T e_i = e_{i+1} \) (\( e_q = e_0 \)). The identity matrix is denoted by \( I \).

The following observations are more or less direct generalizations of [5].

**Proposition 1.** Let \( M(y) \) be defined by

\[ M(y) = \prod_{m=0}^{s-1} \left( I + yT^2^m \right). \]

Then

\[ S_q(y,2^n) = M(y)S_q(y,n). \]

**Proof.** By using the relations \( s(2j) = s(j) \) and \( s(2j+1) = s(j)+1 \) we obtain

\[ S_{q,i}(y,2n) = \sum_{j<n, j \equiv i \mod q} y^{s(j)} \]

\[ = \sum_{2j<n, 2j \equiv i \mod q} y^{s(2j)} + \sum_{2j+1<n, 2j+1 \equiv i \mod q} y^{s(2j+1)} \]

\[ = \sum_{j<n, j \equiv 2^{-1} \mod q} y^{s(j)} + y \sum_{j<n, j \equiv 2^{-1} \mod q} y^{s(j)} \]

\[ = S_{q,2^{-1}i}(y,n) + yS_{q,2^{-1}(i-1)}(y,n). \]

Hence, denoting by \( U \) the matrix defined by \( U e_i = e_{2i} \), we have

\[ S_q(y,2n) = (U + yUT)S_q(y,n). \]

By using the property \( UT = T^2U \) it follows by induction that

\[ (U + yUT)^i = \left( \prod_{m=1}^{i} \left( I + yT^2^m \right) \right) U^i. \]

Since \( T^q = U^s = I \), we directly obtain (19) by setting \( i = s \). \( \square \)
The eigenvalues of $T$ are exactly the $q$-th roots of unity $\zeta_q^l$, $0 \leq l < q$, with corresponding eigenvectors $v_l = \sum_{i=0}^{q-1} \zeta_q^{-il}e_i$ which are orthogonal. Since $M(y)$ is a polynomial in $T$, the eigenvalues of $M(y)$ are given by

$$\lambda_l(y) = \prod_{m=0}^{s-1} \left( 1 + y\zeta_q^{l2^m} \right)$$

It is clear that $\lambda_l(y) = \lambda_{l'}(y)$ if and only if $l' \cdot 2^m = l \cdot 2^n$. (Observe that $l = l' \cdot 2^n$ contains $\mid q/(q,l) \mid \cdot 2^n$ elements, where $(q,l)$ denotes the greatest common divisor of $q$ and $l$.) Appropriately we will write $\lambda_l(y)$ instead of $\lambda_l(y)$ if $l \in L$. Let $L$ denote the system of equivalence classes $l \equiv l' \pmod{2^n}$. Then a basis of the eigenspace $V_l$ corresponding to $\lambda_l(y)$, $l \in L$, is given by $v_l$, $l \in L$. All these eigenspaces are orthogonal. $P_l$, $l \in L$, will denote the orthogonal projection on $V_l$. Furthermore, let $V^{(0)}, V^{(1)}, \ldots, V^{(m)}$ denote the orthogonal projection on $V^{(0)}, V^{(1)}, \ldots, V^{(m)}$, respectively.

Using these notations and the same methods as in [5], we immediately obtain a fractal representation for $S_q(y,n)$.

**Proposition 2.** There exists a continuous function $F(y,\cdot): \mathbb{R}^+ \to V^{(u)}$ satisfying

$$F(y,2^x) = M(y)F(y,x) \quad (x > 0)$$

and $P_lS_q(y,n) = F(y,n)$. Consequently

$$S_q(y,n) = F(y,n) + \begin{cases} O(1) & \text{if } V^{(1)} = \{0\}, \\ O(\log n) & \text{if } V^{(1)} \neq \{0\}, \end{cases}$$

Let $|\lambda_l(y)| > 1$. Then $G_l(y,t) = \lambda_l(y)^{-t}P_lF(y,2^t)$ is a continuous function $G_l(y,\cdot): \mathbb{R} \to V_l$ which satisfies $G_l(y,t+1) = G_l(y,t)$. With $\alpha_l(y) = (\log \lambda_l(y))/(\log 2)$ we finally obtain a fractal representation for $S_q(y,n)$:

$$S_q(y,n) = \sum_{|\lambda_l(y)| > 1} n^\alpha_l(y)G_l \left( y, \frac{\log n}{s \log 2} \right) + O(\log n).$$

We want to mention also that it is quite easy to evaluate $G_l(y,t)$ for special values of $t$ by using the representation (8):

$$S_{q,l}(y,2^k) = \frac{1}{q} \sum_{l=0}^{q-1} \zeta_q^{-il}\lambda_l(y)^a \prod_{j=0}^{b-1} \left( 1 + y\zeta_q^{2^j} \right)$$

$$= \frac{1}{q} \sum_{l \in L} (2^k)^{\alpha_l(y)} \lambda_l^{-b/s} \prod_{l \in L} \zeta_q^{-il} \prod_{j=0}^{b-1} \left( 1 + y\zeta_q^{2^j} \right),$$

where $k = as + b$, $0 \leq b < s$. In particular, the first component of $G_l(y,0)$ is non-zero.

Sometimes it would be more convenient to operate with real exponents instead of in general complex exponents $\alpha_l(y)$. For example, if $\lambda_l(y)^{r_l}$ is real and positive for
some positive integer \( r' \), then we can use \( \tilde{G}_1(y,t) = \lambda_1(y)^{-r'}P_1F(y, 2^{st}) \) instead of \( G_1(y,t) \) and \( \lambda_1(y) = \Re(\alpha_1(y)) \) instead of \( \alpha_1(y) \). (Compare with [5].)

For the evaluation of \( A_q; i; r; m(n) \) we will need \( S_q(c_m^n, n), 0 \leq m < r \). It is an easy exercise to show that \( \arg(\lambda_l(\zeta_m^n)) = \pi n/m + n'\pi \) for some \( m' \in \mathbb{Z} \). Thus \( \lambda_l(\zeta_m^n)^r \) is real and \( \lambda_l(\zeta_m^n)^{2r} > 0 \). Hence it is always possible to operate with positive exponents.

Finally, observe that \( S(y, n) \) can be treated in a similar fashion as above but much more easily. Using the relation \( S(y, 2n) = (1 + y)S(y, n) \), it follows that there is a continuous function \( F(y, x) \) satisfying \( F(y, 2x) = (1 + y)F(y, x) \) in the case \( |1 + y| > 1 \) such that

\[
S(y, n) = F(y, n) = n^\alpha G \left( y, \frac{\log n}{\log 2} \right),
\]

where \( \alpha(y) = \log(1 + y)/\log 2 \) and \( G(y, t) = (1 + y)^{-1}F(2^t) \). Furthermore, \( S(y, n) = \mathcal{O}(1) \) if \( |1 + y| < 1 \) and \( S(y, n) = \mathcal{O}(\log n) \) if \( |1 + y| = 1 \).

Now the fractal representations for \( A_r; m(n) \) and \( R_{q; i; r; m}(n) \) follow immediately.

**Theorem 4.** Let \( q, r \) be positive integers such that \( q \) is odd and \( r \geq 2 \). Set

\[
\alpha_r = \frac{\log (2\cos \frac{\pi}{r})}{\log 2} \quad (r > 2),
\]

\[
\alpha_{q, r} = \max_{0 < m < r, 0 < l < q} \frac{\log |\lambda_l(\zeta_m^n)|}{s\log 2}.
\]

Furthermore, let \( r' \) be the least positive integer such that \( \lambda_l(\zeta_m^n)^{r'} > 0 \) for those \( \lambda_l(\zeta_m^n), 0 < l < q, 0 < m < r \), with largest modulus.

Then there exist real valued periodic continuous functions \( \psi_r; m(x), \psi_{q; i; r; m}(x), 0 \leq m < r, 0 \leq i < q \), with period 1 such that

\[
A_r; m(n) = \frac{n}{r} + \left\{ \begin{array}{ll} (-1)^m \eta_0 / 2 + n^{\alpha_r} \cdot \psi_r; m \left( \frac{\log n}{2^{r/m'\log 2}} \right) + \mathcal{O}(n^{\beta_r}) & \text{if } r = 2, \\ n^{\alpha_{q, r}} \cdot \psi_{q; i; r; m} \left( \frac{\log n}{2^{r/s'\log 2}} \right) + \mathcal{O}(n^{\beta_{q, r}}) & \text{if } r > 2, \end{array} \right.
\]

where \( \beta_r < \alpha_r, \beta_{q, r} < \alpha_{q, r}, \text{and } \eta_0 = 0 \text{ if } n \equiv 0 \text{ mod } 2 \text{ and } \eta_0 = t_n \text{ if } n \equiv 1 \text{ mod } 2 \).

**Proof.** Since \( A_r; m(n) \) is given by (12) and \( A_{q; i; r; m} \) by (6) (compare also with (16) and (17)), it follows that the asymptotic leading term of \( A_r; m(n) - n/r \) depends on the largest eigenvalue \( \lambda_0(\zeta_m^n) = (1 + \zeta_m^n)^r, 0 < m < r \), and the asymptotic leading term of \( R_{q; i; r; m}(n) \) on the largest eigenvalue \( \lambda_l(\zeta_m^n), 0 < l < q, 0 \leq m < r \).

Since \( |1 + \zeta_m^n| = 2|\cos(m\pi/r)| \) is maximal for \( m = 1 \), we immediately obtain the asymptotic expansion for \( A_r; m(n) \). (Note that \( \beta_r = \log(2\cos \frac{\pi}{r})/\log 2 \).

Furthermore, since \( \lambda_l(1) = 1 + \zeta_1^l + \zeta_2^l + \cdots + \zeta_q^{(2^r-1)}l = 0 \) for \( 0 < l < q \), it is clear that \( \alpha_{q, r} \) is the correct exponent in the asymptotic leading term of \( R_{q; i; r; m}(n) \).

Finally, \( A_2; m(n) \) can be directly evaluated.

**Remark.** In this paper we will only discuss binary digits. But the above concept easily applies for arbitrary \( b \)-ary digit expansions. Let \( s(j) \) be a sequence satisfying \( s(bn + c) = s(n) + s(c) \) for \( n \geq 0 \) and \( 0 \leq c < b \). Let \( S_q(y, n) \) be defined as above
and assume that $b$ and $q$ are relatively prime. Then
\[ S_q(y, bn) = U_b \left( \sum_{c=0}^{b-1} y^{s(c)} T^c \right) S_q(y, n), \]
where $U_b e_i = e_{bi}$, $0 \leq i < q$, and $s = \text{ord}_q(b)$. Hence $S_q(y, b^n n) = M_b(y) S_q(y, n)$, where
\[ M_b(y) = \prod_{m=0}^{b-1} \left( \sum_{c=0}^{b-1} y^{s(c)} T^{cbm} \right), \]
and we are in the same position as above. All eigenvalues and eigenvectors of $M_b(y)$ are known, and we immediately obtain a fractal representation for $S_q(y, n)$. (In [5] only the case $b = r$ is mentioned.)

3. Newman-like Phenomena

We want to discuss two kinds of positivity phenomena:

**N1** \[ A_{q,0;r,0}(n) > \max_{0 < m < r} A_{q,0;r,m}(n) \text{ for almost all } n \geq 0, \]

**N2** \[ R_{q,0;r,0}(n) > 0 \text{ for almost all } n \geq 0. \]

Newman’s theorem $S_{3,0}(n) > 0$ ($n \geq 0$) is precisely the same as
\[ A_{3,0;2,0}(n) > A_{3,0;2,1}(n). \]

Therefore **N1** is a natural generalization of this property. Recall that $R_{q,0;r,m}(n)$ is the remainder term of $A_{q,0;r,m}(n)$ if $\frac{1}{q} A_{r,m}(n)$ is considered as the “natural” leading term of $A_{q,0;r,m}(n)$ (see section 2). Hence, **N2** means that the remainder term $R_{q,0;r,0}(n)$ is positive (for almost all $n$). We will now show that **N1** implies **N2** if $\alpha_r \neq \alpha_{q,r}$.

The following lemma provides a necessary condition for **N1**.

**Lemma 1.** If **N1** holds then $\alpha_r \leq \alpha_{q,r}$.

**Proof.** Suppose that $\alpha_r > \alpha_{q,r}$. In this case (see Theorem 4) the asymptotic behaviour of $A_{q,0;r,m}(n)$ is determined by $A_{r,m}(n)$. However, we will show that $A_{r,0}(2^{(2a+1)r}) < A_{r,m}(2^{(2a+1)r})$ for all $m \neq 0 \mod r$ and sufficiently large $a$. Therefore **N1** cannot occur.

Combining (13) and Theorem 4, we obtain
\[ A_{r,m}(2^{k}) - \frac{2^k}{r} \sim 2R \left( \zeta_r^{-m}(1 + \zeta_r)^k \right). \]

Since $(1 + \zeta_r)^r$ is real and negative, everything follows. \qed

Hence, if $\alpha_r \neq \alpha_{q,r}$ then **N1** implies
\[ R_{q,0;r,0}(n) > \max_{0 < m < r} R_{q,0;r,m}(n) \text{ for almost all } n \geq 0. \]

Finally, (23) always implies **N2**. This follows from the following property.

**Lemma 2.**
\[ \sum_{m=0}^{r-1} R_{q,i;r,m}(n) = O(\log n) \]
for all $i = 0, \ldots, q - 1$. 

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Proof. From (17) we get
\[ \sum_{m=0}^{r-1} R_{q,i,r,m}(2^k) = \frac{1}{q} \sum_{l=1}^{q-1} \lambda_l^{-i} \prod_{j=0}^{k-1} \left( 1 + \zeta_q^{2^j l} \right). \]
This means that the asymptotic behaviour of this sum is determined by the eigenvalues \( \lambda_l(1) \), which are given by
\[ \lambda_l(1) = \prod_{j=0}^{r-1} \left( 1 + \zeta_q^{2^j l} \right) = 1. \]
Hence (24) follows. \( \square \)

Note that there are situations where \((\text{N}2)\) holds although \((\text{N}1)\) fails; see Theorem 8. However, in the “classical” case \( r = 2 \) it is easy to verify that \((\text{N}1)\) and \((\text{N}2)\) are equivalent to \( S_{q,0}(-1, n) > 0 \) (for almost all \( n \)).

Before we prove further necessary conditions for \((\text{N}1)\) and \((\text{N}2)\), we want to mention that “converse” phenomena of the form \( A_{q,0;r,0}(n) < \min_{0 < m < r} A_{q,0;r,m}(n) \) or \( R_{q,0;r,0}(n) < 0 \) for almost all \( n \geq 0 \) do not exist.

Lemma 3. There exist infinitely many \( n \geq 0 \) such that
\[ A_{q,0;r,0}(n) > \max_{0 < m < r} A_{q,0;r,m}(n) \tag{25} \]
and
\[ R_{q,0;r,0}(n) > 0. \tag{26} \]
Proof. Let \( s = \text{ord}_q(2) \) and let \( n = 2^{2rs+a} \) for some \( a \geq 0 \). Then \( \lambda_l(\zeta_r^l)^{2ra} > 0 \) for all \( l \in L \) and \( 0 \leq a \leq q - 1 \). Hence (25) and (26) follow from
\[
A_{q,0;r,m}(n) = \frac{1}{q} A_{r,m}(n) + R_{q,0;r,m}(n) \\
= \frac{1}{rq} \sum_{l=0}^{r-1} \cos \left( 2\pi \frac{lm}{r} \right) \lambda_l(\zeta_r^l)^{2ra} + \frac{1}{rq} \sum_{l=0}^{r-1} \cos \left( 2\pi \frac{lm}{r} \right) \sum_{0 \neq \lambda_l \in L} |l| \lambda_l(\zeta_r^l)^{2ra}. \hspace{1cm} \square
\]

Theorem 5. Let \( q, r \) be positive integers such that \( q \) is odd and \( r \geq 2 \). If \( s = \text{ord}_q(2) \) and \( r \) are coprime or if there exists an integer \( r' > 0 \) such that \( \lambda_l(\zeta_r^m)^{r'} < 0 \) for those \( \lambda_l(\zeta_r^m) \), \( 1 \leq l < q, \) \( 0 < m < r, \) with maximal modulus, then \((\text{N}1)\) and \((\text{N}2)\) fail.
Proof. We only prove that \((\text{N}2)\) fails. Since \( \lambda_0(\zeta_r)^r < 0 \), the following proof can be extended to contradict \((\text{N}1)\).

Let \( L_m \) denote the set of pairs \((l,m)\), \( 1 \in L, 0 < m < r \), such that the eigenvalues \( \lambda_l(\zeta_r^m) \) have maximal modulus \( \rho \). Then the asymptotic leading term of \( R_{q,0;0,0}(n) \) only depends on these eigenvalues. In particular, we have
\[
R_{q,0;0,0}(2^{ks}) \sim \frac{1}{rq} \sum_{(l,m) \in L_m} |l| \lambda_l(\zeta_r^m)^k.
\]
If there exists an integer \( r' > 0 \) such that \( \lambda_l(\zeta_r^m)^{r'} < 0 \) for \((l,m) \in L_m\), then \( R_{q,0;0,0}(2^{a2rs+rs'}) < 0 \) for all \( a \geq 0 \).
Hence for \( \leq A \) almost all \( n \) we have \( \frac{\lambda_1(n^m)}{\rho} \) for \( (1, m) \in L_m \). Then \( \eta_{n, m} \) are non-real \((2r)\)-th roots of unity. Thus
\[
\sum_{b=0}^{2r-1} \sum_{(1, m) \in L_m} |l|\eta_{n, m}^b = 0,
\]
and consequently there exists \( b_0, 0 < b_0 < 2r \), such that
\[
\sum_{(1, m) \in L_m} \|l|\lambda_1(n^m)^{b_0} = \rho^{b_0} \sum_{(1, m) \in L_m} |l|\Re(\eta_{n, m}^b) < 0.
\]
Hence \( R_{q, 0; m, 0}(2^{a2^{2r}+b_0s}) < 0 \) for sufficiently large \( a \).

With the help of Theorem 5 we will prove the negative part of Theorem 3 saying that primes \( p \leq 1000, p \neq 3, 5, 17, 43, 257, 683 \), do not satisfy \( S_{p, 0}(-1, n) > 0 \) for almost all \( n \). First, we only have to consider \( p \in P_t \) with \( t > 2 \). In \([2]\) it is shown that \( p = 3 \) and \( p = 5 \) are the only exceptional primes in \( P_1 \), and \( p = 17 \) and possibly \( p = 41 \) those of \( P_2 \). (We will treat the case \( p = 41 \) in a moment.) Next, it follows from Theorem 5 that we only have to pay attention to those primes \( p \in P_t, t > 2 \), with even \( s = \text{ord}_p(2) \), e.g. for \( p = 109 \in P_3 \) we have \( s = 36 \). Finally, if there is \( k < s \) with
\[
S_{p, 0}^{(m)}(-1, 2^k) = \frac{1}{p} \sum_{j=0}^{s-1} \prod_{i=0}^{k-1} \left(1 - \zeta_p^{l_m 2^{i+1}}\right) < 0,
\]
in which \( \lambda_m = \lambda_{l_m}(-1) \) is the largest eigenvalue, then \( S_{p, 0}(-1, 2^{ks+k}) < 0 \) for sufficiently large \( a \). For example, for \( p = 109 \) we have \( l_m = 9 \) and \( S_{109, 0}^{(m)}(-1, 2^6) < 0 \). Hence, for \( p = 109 \) there is no phenomenon of type \((N1)\). Similarly it follows that \( S_{41, 0}^{(m)}(-1, 2^8) < 0 \), and we really have to consider just primes \( p \in P_t \) with \( t > 2 \).

Table 1 gives a list of all primes \( p \leq 1000, p \in P_t, t > 2 \), such that \( s \) is even. Furthermore the largest eigenvalue \( \lambda_m = \lambda_{l_m}(-1) \) is represented by \( l_m \), and if there is \( k < s \) such that \( S_{p, 0}^{(m)}(-1, 2^k) < 0 \) then \( k \) is listed.

The only primes for which this method provides no answer are \( p = 43, 257, 683 \). At the end of section 4 it will be shown that for these primes \( S_{p, 0}(-1, n) > 0 \) for almost all \( n \). This completes the proof of the negative part of Theorem 3.

Remark. It is also an interesting problem to consider \( A_{q, i, r, m}(n) \) and \( R_{q, i, r, m}(n) \) \((0 \leq n < r)\) for some fixed \( i \not\equiv 0 \mod q \). For example, it is known that \( A_{3, 1; 2, 0}(n) < A_{3, 1; 2, 1}(n) \) for almost all \( n \geq 0 \) (see \([3]\)). Most of our methods can be applied in these cases too. However, for the sake of shortness we restrict ourselves to the case \( i = 0 \). □
Table 1

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4. Proof of Theorem 1

In the case of the usual parity \( r = 2 \) we just have to discuss \( S_{q,i}(-1,n) \) to obtain all informations needed. For short we will write \( S_{q,i}(n), \lambda_l, \) and \( M \) instead of \( S_{q,i}(-1,n), \lambda_l(-1), \) and \( M(-1). \)

From an heuristic point of view integers of the form \( q = 4N + 1 \) or \( q = 4N - 1 \) are 'good candidates' for a phenomenon of type (N1). In both cases we have \( s(j) \equiv 0 \mod 2 \) for \( j \equiv 0 \mod q, j < q \), i.e. \( S_{q,0}(n) \) is as positive as possible. (The first case is trivial. For the second case see Proposition 4.) In fact, Theorem 1 says that \( S_{q,0}(n) > 0 \) (for almost all \( n \)) for these \( q \). However, an heuristic argument of this kind does not work in all cases. Suppose that \( q = 2^{2N+1} - 1 \). Then \( s(j) \equiv 1 \mod 2 \) for \( j \equiv 0 \mod q, j < 2^{2N+1} + 1, \) i.e. \( S_{q,0}(n) \) is as negative as possible. Furthermore, \( s = \text{ord}_q(2) = 2N + 1 \) is odd. Hence, by Theorem 5 \( S_{q,0}(n) < 0 \) for infinitely many \( n \). But we know from Lemma 3 that we also have \( S_{q,0}(n) > 0 \) for infinitely many \( n \).

Let \( S_q^{(m)}(n) = (S_{q,0}^{(m)}(n), \ldots, S_{q,q-1}^{(m)}(n))^t = P^{(m)}S_q(n) \). According to the above considerations it is sufficient to show that

\[
S_q^{(m)}(n) \gg n^{(\log \lambda_m)/(s \log 2)},
\]

where \( \lambda_m \) denotes the maximal eigenvalue, resp. \( \min \psi_{q,0;m,0} > 0 \).

First we will discuss the case \( 3 \mid q \), where it is rather easy to identify \( \lambda_m \).

**Lemma 4.** Suppose that \( q \) is a positive odd integer. Then any eigenvalue

\[
\lambda_l = \prod_{m=0}^{s-1} \left( 1 - \zeta_q^{2m} \right)
\]

of \( M \) is bounded by \( |\lambda_l| < 3^{s/2} \) or \( \lambda_l = 3^{s/2} \).
The case $\lambda_l = 3^{s/2}$ appears if and only if $q \equiv 0 \mod 3$ and $l \equiv q/3 \mod q$ or $l \equiv 2q/3 \mod q$.

Proof. It is an elementary exercise to show that

\[ |1 - z^2| < \sqrt{3} \quad \text{and} \quad |(1 - z)(1 - z^2)| < 3 \]

if $|z| = 1$ and $|1 - z| > \sqrt{3}$. Furthermore $|1 - z^2| = \sqrt{3}$ if $|z| = 1$ and $|1 - z| = \sqrt{3}$.

Now let $\lambda_l = \prod_{m=0}^{l-1} (1 - \zeta_q^{2^m})$ be an eigenvalue of $M$. Let us consider a partition $M_0, M_1, M_2, M_3$ of the set $\{0, 1, \ldots, s-1\}$, where $M_0$ consists of those $m$ with $|1 - \zeta_q^{2^m}| = \sqrt{3}$, $M_1$ of those with $|1 - \zeta_q^{2^m}| > \sqrt{3}$, and $M_2 = M_1 + 1$. It is clear that either $M_0 = \emptyset$ or $M_0 = \{0, 1, \ldots, s-1\}$. Furthermore $M_1, M_2, M_3$ are pairwise disjoint. If $M_0 = \emptyset$ then

\[ |\lambda_l| = \prod_{m \in M_1} |(1 - \zeta_q^{2^m})(1 - \zeta_q^{2^m+1})| \prod_{m \in M_3} |1 - \zeta_q^{2^m}| < 3^{M_1} |3^{M_3}/2| = 3^{s/2}. \]

On the other hand, if $M_0 = \{0, 1, \ldots, s-1\}$, then $s$ is even and $\lambda_l = 3^{s/2}$. Furthermore, the case $M_0 = \{0, 1, \ldots, s-1\}$ occurs only if $q \equiv 0 \mod 3$ and $l \equiv q/3 \mod q$ or $l \equiv 2q/3 \mod q$. \hfill \Box

Lemma 5. Suppose that $q$ is an odd multiple of 3. Then

\begin{align*}
(27) \quad |S_{q,i}^{(m)}(2^k)| & \leq \frac{2}{q} 3^{k/2} \quad (0 \leq i < q), \\
(28) \quad S_{q,-2}^{(m)}(2^k) & \leq 0 \quad (0 \leq j < s), \\
(29) \quad S_{q,0}^{(m)}(2^k) & \geq \frac{\sqrt{3}}{q} 3^{k/2}.
\end{align*}

Proof. Set $\omega = \zeta_3$. By (8) we have

\[ S_{q,i}^{(m)}(2^k) = \frac{1}{q} \left( \omega^{-1} \prod_{j=0}^{k-1} \left( 1 - \omega^{2j} \right) + \omega^i \prod_{j=0}^{k-1} \left( 1 - \omega^{-2j} \right) \right). \]

Since $\omega^{2j} = \omega^{(-1)^j}$ and $|1 - \omega^{2k}| = \sqrt{3}$, we immediately obtain the estimate (27). Furthermore,

\[ \prod_{j=0}^{k-1} \left( 1 - \omega^{2j} \right) = \begin{cases} 
3^{k/2} & \text{if } k \text{ is even}, \\
3^{(k-1)/2} (1 - \omega) & \text{if } k \text{ is odd}.
\end{cases} \]

Hence

\[ S_{q,-2}^{(m)}(2^k) = \begin{cases} 
-q^{-1} 3^{k/2} & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd and } i \text{ is even,} \\
-q^{-1} 3^{(k+1)/2} & \text{if } k \text{ and } i \text{ are odd,}
\end{cases} \]

and

\[ S_{q,0}^{(m)}(2^k) = \begin{cases} 
2q^{-1} 3^{k/2} & \text{if } k \text{ is even}, \\
q^{-1} 3^{(k+1)/2} & \text{if } k \text{ is odd,}
\end{cases} \]

which prove (28) and (29). \hfill \Box
Now suppose that \( n = 2^k + 2^{k-1} + r \), where \( \delta \in \{0, 1\} \) and \( r < 2^{k-1} \). Then by using (9), (27), (28), and (29) we immediately obtain

\[
S_{q,0}^{(m)}(n) = S_{q,0}^{(m)}(2^k) - \delta S_{q,2^k}(2^{k-1}) + \sum_{j=0}^{k-2} \eta_j S_{q,ij}^{(m)}(2^j)
\]

\[
\geq \left( \frac{\sqrt{3}}{2} - \frac{(1 - 3^{-1/2})^{-1/2}}{3} \right) \frac{2}{q} 3^{k/2}
\]

\[
> 0.077 \cdot \frac{2}{q} 3^{k/2} \gg n^{(\log \lambda_m)/(q \log 2)}.
\]

This proves Theorem 1 in the case \( 3|q \).

The case \( q = 4^N + 1 \) is a little bit more involved. The first step is to identify the largest eigenvalue \( \lambda_m \). Note that \( s = 4N \).

**Lemma 6.** If \( q = 4^N + 1 \) then \( \lambda_m \) is given by

\[
\lambda_m = \prod_{j=0}^{3N-1} \left( 1 - \zeta_q^{l_m 2^i} \right) = c3^{2N} \left( 1 + O(2^{-2N}) \right),
\]

where \( l_m = (q + 1)/3 \) and \( c = 0.363247 \cdots > 0 \). Moreover, if \( l \notin l_m = l_m(2) \) then \( |\lambda_l| < \lambda_m \).

**Proof.** First observe that for \( 0 \leq i < N \)

\[
\arg \zeta_q^{l_m 2^i} \in I_1 = \left( \frac{2\pi}{3}, \frac{5\pi}{6} \right), \quad \arg \zeta_q^{l_m 2^i+1} \in I_2 = \left( -\frac{2\pi}{3}, \frac{\pi}{3} \right),
\]

\[
\arg \zeta_q^{l_m 2^{N+2i}} \in I_3 = \left( -\frac{5\pi}{6}, -\frac{2\pi}{3} \right), \quad \arg \zeta_q^{l_m 2^{N+2i+1}} \in I_4 = \left( \frac{\pi}{3}, \frac{2\pi}{3} \right).
\]

This means that there are exactly \( N \) elements \( \zeta_q^{l_m 2^i} , 0 \leq i < 4N \), satisfying \( \arg \zeta_q^{l_m 2^i} \in I_1 \). Furthermore, the eigenvalue \( \lambda_m \) is calculated by

\[
\lambda_m = \prod_{i=0}^{N-1} \left| 1 - \zeta_q^{l_m 2^i} \right|^2 \left| 1 - \zeta_q^{l_m 2^i+1} \right|^2
\]

\[
= \prod_{i=0}^{N-1} 16 \sin^2 \left( \frac{\pi}{3} + \frac{\pi 4^i}{3} \right) \sin^2 \left( \frac{2\pi}{3} + \frac{2\pi 4^i}{3} \right)
\]

\[
= 3^{2N} \prod_{j=1}^{N} \frac{16}{9} \sin^2 \left( \frac{\pi}{3} + \frac{\pi 4^j}{3} \right) \sin^2 \left( \frac{2\pi}{3} + \frac{2\pi 4^j}{3} \right) \left( 1 + O(2^{-2N-j}) \right)
\]

\[
= 3^{2N} \left( \frac{\infty}{1} \frac{16}{9} \sin^2 \left( \frac{\pi}{3} + \frac{\pi 4^j}{3} \right) \sin^2 \left( \frac{2\pi}{3} + \frac{2\pi 4^j}{3} \right) \right) \left( 1 + O(2^{-2N}) \right)
\]

\[
= c3^{2N} \left( 1 + O(2^{-2N}) \right).
\]

If \( \arg \zeta_q^{l_m} \in I_1 \) for some \( l \neq 0 \) mod \( q \), then \( \arg \zeta_q^{2l} \in I_2 \), \( \arg \zeta_q^{2^N l} \in I_3 \), and \( \arg \zeta_q^{2^{N+1} l} \in I_4 \). Hence, the number \( N_0 \) of elements \( \zeta_q^{l_2^i} , 0 \leq i < 4N \), satisfying \( \arg \zeta_q^{l_2^i} \in I_1 \) is always bounded by \( N_0 \leq N \).
The most interesting case appears if \( N_0 = N \). It is clear that this occurs if and only if \( \arg \zeta_q^{2i} \not\in \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \) for all \( i \geq 0 \). Let us classify those \( x \in (0,1) \) such that \( z = e^{2\pi i x} \) satisfies \( z^{i} \not\in \left[-\frac{\pi}{3}, \frac{\pi}{3} \right] \) for all \( i \geq 0 \).

Since \( z \not\in \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \) it follows that \( z \not\in \left[ \frac{5\pi}{6}, \frac{7\pi}{6} \right] \) and consequently \( z \not\in \left[ \frac{7\pi}{12}, \frac{11\pi}{12} \right] \) etc. By induction it follows that \( \arg z \) must be contained in a zero set quite similar to the Cantor set. More precisely, the only possible values \( x \in (0,1) \) are given by

\[
x = \sum_{n \geq 1} a_n 4^{-n},
\]

where \( a_n \in \{0,3\} \) and there exist \( n_1, n_2 \geq 1 \) with \( a_{n_1} = 0 \) and \( a_{n_2} = 3 \). If \( z \) is in addition a \( q \)-th root of unity then \( x \) must be of the form \( x = k/q \), where \( k \equiv 1 \mod 3 \) and \( 1 \leq k \leq 4^N \). Since

\[
\frac{1}{q} = \frac{4^N - 1}{42N - 1} = \sum_{p \geq 0} \sum_{n=2pN+N+1} 2(p+1)^N 3 \cdot 4^{-n},
\]

we immediately obtain

\[
\frac{k}{q} = k \sum_{p \geq 1} (4^N - 1)4^{-2pN} = \sum_{p \geq 1} ((k-1)4^N + ((4^N - 1) - (k-1))) 4^{-2pN},
\]

and observe that the 4-adic digits \( a_n \) of the digit expansion of \( k/q \), \( 1 \leq k \leq 4^N \), satisfy \( a_n \in \{0,3\} \) for all \( n \geq 1 \) if and only if the 4-adic digit expansion of \( k-1 \) has the same property. (Evidently \( k \equiv 1 \mod 3 \) in these cases.) This means that if we choose digits \( b_n \in \{0,3\} \), \( 1 \leq n \leq N \), and set

\[
k = 1 + \sum_{n=1}^{N} b_n 4^{N-n},
\]

then

\[
\frac{k}{q} = \sum_{p \geq 0} \left( \sum_{n=1}^{N} b_n 4^{-2Np-n} + \sum_{n=1}^{N} (3 - b_n)4^{-2Np-N-n} \right).
\]

In this way we get all \( q \)-th roots of unity \( z = \zeta_q^l \) with \( \arg \zeta_q^l \in I_1 \cup I_2 \) such that \( N_0 = N \). Furthermore, the digits \( b_n \), \( 1 \leq n \leq N \), encode the distribution of \( \zeta_q^{l^i} \).

If \( \zeta_q^l = e^{2\pi i (1+x_0)/3} \) with \( x_0 = \sum_{n \geq 1} c_n 4^{-n} \), \( c_{2Np+n} = b_n \), \( c_{2Np+N+n} = 3 - b_n \), \( 1 \leq n \leq N \), \( p \geq 0 \), then \( \zeta_q^{l^i} = e^{2\pi i (1+x_i)/3} \), where \( x_i = \sum_{n \geq 1} c_{n+i} 4^{-n} \). The periodicity \( \zeta_q^{l^{2N+i}} = \zeta_q^{l^i} \) is reflected by the periodic digit expansion of \( x_0 \). In particular, \( \zeta_q^{l^m} \) corresponds to the digits \( b_n = 0 \), \( 1 \leq n \leq N \). This means that \( \zeta_q^{l^i} = e^{2\pi i (1+x_{im})/3} \) are the only \( q \)-th roots of unity (with \( N_0 = N \), where one period of the digits of \( x_{im} \) contains just one subblock of the form 03. In other words, there is exactly one element \( \zeta_q^{l^i} \), \( 0 \leq i < N \), satisfying \( \arg \zeta_q^{l^i} \in [19\pi/24, 5\pi/6] \), namely \( \zeta_q^{l^{4^{-1}}} \). For any other \( \zeta_q^{l^i} \not\in \mathbb{L}_m \) (with \( N_0 = N \)), there are at least two subblocks of the form 03 in any period of the digit expansion of \( x_0 \). Thus there exist \( 0 \leq i_1 < i_2 < N \) with \( \arg \zeta_q^{l^{i_1}} \), \( \arg \zeta_q^{l^{i_2}} \in [19\pi/24, 5\pi/6] \). Consequently

\[
\lambda_l < 32 N 16^2 \sin^4 \left( \frac{19\pi}{24} \right) \sin^4 \left( \frac{19\pi}{48} \right) = 0.34899 \ldots 32^N < \lambda_m.
\]
The case $N_0 < N$ is much easier. Let $J_1$ denote the set of $j$, $0 \leq j < 4N$, such that $\arg \zeta_q^{12j} \in I_1$. We assume that the elements $j_i$, $0 \leq i < N_0$, of $J_1 = \{j_0, j_1, \ldots, j_{N_0-1}\}$ are ‘ordered’ in such a way that $\arg \zeta_q^{12i} \leq \arg \zeta_q^{12i+1}$, $0 \leq i < N_0 - 1$. (Recall that $|J_1| = N_0$.) Our first aim is to show that for any $i$, $0 \leq i < N_0$, we have

$$\arg \zeta_q^{l_{m4i}} < \arg \zeta_q^{l_{2i}}. \tag{30}$$

Let $b_i$, $1 \leq i < N$, denote the number of $j \in J_1$ satisfying $\arg \zeta_q^{12j} \in I^{(i)} = (\arg \zeta_q^{l_{m4i-1}}, \arg \zeta_q^{l_{m4i}})$. Furthermore set $c_i = \sum_{1 \leq j \leq i} b_j$. Observe that

$$c_i \leq i, \quad 1 \leq i < N, \tag{31}$$

immediately implies (30). Since $\arg \zeta_q^{12j} \in I^{(i)}$, $1 \leq i < N - 1$, implies $\arg \zeta_q^{l_{2i+2}} \in I^{(i+1)}$, we always have $b_{i+1} \geq b_i$. Set $a_1 = b_1$ and $a_i = b_i - b_{i-1}$, $2 \leq i < N$. Then $a_i \geq 0$, $b_i = \sum_{1 \leq j \leq i} a_j$, and $c_i = \sum_{1 \leq j \leq i} (i - j + 1)a_j$.

Since $C_{N-1} = N_0 \leq N - 1$, condition (31) is satisfied for $i = N - 1$. Now we show that $c_i \leq i$ implies $c_{i-1} \leq i - 1$. Suppose that $c_{i-1} \geq i$; then we obtain $a_1 + \cdots + a_i = c_i - c_{i-1} \leq 0$. Thus $a_j = 0$, $1 \leq j \leq i$, which implies $c_{i-1} = 0$ and contradicts $c_{i-1} \geq i$. This completes the proof of (31) and consequently that of (30).

Let $J_2$ denote the set of $j$, $0 \leq j < 4N$, such that $\arg \zeta_q^{12j} \in (5\pi/6, \pi)$, and $J_3$ the set of those $j$, $0 \leq j < 4N$, such that $\arg \zeta_q^{12j} \in (0, \pi/3)$. Clearly $N_0 + |J_2| + |J_3| = N$ and

$$|1 - \zeta_q^{12j}| \cdot |1 - \zeta_q^{12j+1}| < |1 - \zeta_q^{l_{m2N-1}}| \cdot |1 - \zeta_q^{l_{m2N}}|$$

for $j \in J_2 \cup J_3$. Therefore we can estimate $\lambda_t$ by

$$\lambda_t = \prod_{j \in J_1 \cup J_2 \cup J_3} \left( |1 - \zeta_q^{12j}|^2 |1 - \zeta_q^{12j+1}|^2 \right)$$

$$= \prod_{i=0}^{N_0-1} \left( 16 \sin^2 \left( \frac{\arg \zeta_q^{12i}}{2} \right) \sin^2 \left( \arg \zeta_q^{l_{2i}} \right) \right)$$

$$\cdot \prod_{j \in J_2 \cup J_3} \left( 16 \sin^2 \left( \frac{\arg \zeta_q^{12j}}{2} \right) \sin^2 \left( \arg \zeta_q^{l_{2j}} \right) \right)$$

$$< \prod_{i=0}^{N_0-1} \left( 16 \sin^2 \left( \frac{\arg \zeta_q^{l_{m2N-1}}}{2} \right) \sin^2 \left( \arg \zeta_q^{l_{m2N-1}} \right) \right)$$

$$< \lambda_m,$$

which finishes the proof of Lemma 6. \hfill \Box

In order to complete the proof of Theorem 1 we need an analogon to Lemma 5. However, the situation is much more delicate. For the following estimates we use
the notation

\[(32) \quad c_j = \prod_{i>j} \left( \frac{2}{\sqrt{3}} \sin \left( \frac{\pi}{3} + (-1)^j \frac{\pi}{3} 2^{-j} \right) \right) = 1 + \mathcal{O}(2^{-j}). \]

The proof is completely elementary and just uses the Fourier expansion (8) of \( S_{q,i}(2^k) \), or its dominant term \( S_{q,i}^{(m)}(2^k) \) corresponding to \( \zeta_q^i \).

**Lemma 7.** Suppose that \( q = 4^N + 1 \) and \( 0 \leq k \leq 2N \). Furthermore, let \( i = 0 \) or \( i = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l} \), in which \( k < k_1 < k_2 < \cdots < k_l \leq 2N \), and set

\[
\begin{align*}
    w_1 &= \sum_{\ell=1}^{l} (-1)^{k_\ell - k}, \quad w_2 = \sum_{\ell=1}^{l} 2^{k_\ell - k}, \\
    w_3 &= \sum_{\ell=1}^{l} (-1)^{k_\ell - 2N}, \quad w_4 = \sum_{\ell=1}^{l} 2^{k_\ell - 2N}.
\end{align*}
\]

If \( k \equiv 0 \mod 2 \), then

\[
S_{q,i}^{(m)}(2^k) = \frac{3^{k/2}}{q} \left( 2 \sum_{j=1}^{2N-k} c_j \cos \left( \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) + c_0 \sum_{j=1}^{k} \frac{c_{j+2N-k}}{c_j} \sin \left( (-1)^j \frac{\pi}{6} + \frac{2\pi}{3} 2^{-j} w_3 + \frac{2\pi}{3} 2^{-j} w_4 + \mathcal{O}(2^{-k}) \right) \right)
\]

\[
= \frac{3^{k/2}}{q} \left( 2(2N-k) \cos \left( \frac{2\pi}{3} w_1 \right) + C_1(k; k_1, \ldots, k_l) + C_2(k; k_1, \ldots, k_l) + \mathcal{O}(2^{-k}) + \mathcal{O}(2^{k-2N}) \right),
\]

where the constants \( C_1(k; k_1, \ldots, k_l) \), \( C_2(k; k_1, \ldots, k_l) \) are given by

\[
C_1(k; k_1, \ldots, k_l) = 2 \sum_{j \geq 1} \left( c_j \cos \left( \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) - \cos \left( \frac{2\pi}{3} w_1 \right) \right),
\]

\[
C_2(k; k_1, \ldots, k_l) = 2c_0 \sum_{j \geq 1} \left( c_j^{-1} \sin \left( (-1)^j \frac{\pi}{6} + \frac{2\pi}{3} 2^{-j} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) - \sin \left( (-1)^j \frac{\pi}{6} + (-1)^j \frac{2\pi}{3} w_3 \right) \right),
\]

and \( C_2(k; k_1, \ldots, k_l) \) is uniformly bounded by \(|C_2(k; k_1, \ldots, k_l)| \leq 3.64\).
If \( k \equiv 1 \mod 2 \), then

\[
S_{q,-1}^{(m)}(2^k) = \frac{3^{k/2}}{q} \left( 2 \sum_{j=1}^{2N-k} c_j \cos \left( (-1)^j \frac{\pi}{6} + \frac{2\pi}{3} 2^{-j} \right) + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) + 2c_0 \sum_{j=1}^{k} \frac{c_j + 2N-k}{c_j} \sin \left( \frac{\pi}{3} 2^{-j} + \frac{\pi}{3} 2^{-j-2N+k} + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) + O(2^{-k})
\]

\[
= \frac{3^{k/2}}{q} \left( 2(2N-k) \cos \left( \frac{\pi}{6} + \frac{2\pi}{3} w_3 \right) + D_1(k; k_1, \ldots, k_1) \right.
\]

\[
+ D_2(k; k_1, \ldots, k_1) + O(2^{-k}) + O(2^{k-2N})
\]

where the constants \( D_1(k; k_1, \ldots, k_1) \), \( D_2(k; k_1, \ldots, k_1) \) are given by

\[
D_1(k; k_1, \ldots, k_1) = 2 \sum_{j \geq 0} \left( c_j \cos \left( (-1)^j \frac{\pi}{6} + \frac{2\pi}{3} 2^{-j} \right) - \cos \left( \frac{\pi}{6} + \frac{2\pi}{3} w_3 \right) \right)
\]

\[
D_2(k; k_1, \ldots, k_1) = 2c_0 \sum_{j \geq 1} \left( c_j^{-1} \sin \left( \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) - \sin \left( (-1)^j \frac{2\pi}{3} w_3 \right) \right)
\]

and \( D_2(k; k_1, \ldots, k_1) \) is uniformly bounded by \( |D_2(k; k_1, \ldots, k_1)| \leq 2.22 \).

**Corollary 1.** Suppose that \( q = 4^N + 1 \) and \( 0 \leq k \leq 2N \). Then

\[
|S_{q,-1}^{(m)}(2^k)| \leq \frac{3^{k/2}}{q} (2(2N-k) + 3.65), \quad (2^{k+1} \leq i \leq 4^N + 1),
\]

\[
-S_{q,-2^{k+1}}^{(m)}(2^k) \geq \begin{cases} q^{-13k/2} ((2N-k) - 2.674) & (k \equiv 0 \mod 2), \\ q^{-13k/2} \cdot 1.453 & (k \equiv 1 \mod 2), \end{cases}
\]

\[
-S_{q,-2^k+2}^{(m)}(2^k) \geq \begin{cases} q^{-13k/2} ((2N-k) - 0.669) & (k \equiv 0 \mod 2), \\ q^{-13k/2} \left( \sqrt{3}(2N-k) - 5.12 \right) & (k \equiv 1 \mod 2), \end{cases}
\]

\[
-S_{q,-2^k-3}^{(m)}(2^k) \geq \begin{cases} q^{-13k/2} ((2N-k) - 2.358) & (k \equiv 0 \mod 2), \\ q^{-13k/2} \cdot 4.791 & (k \equiv 1 \mod 2), \end{cases}
\]

\[
S_{q,-2^{k+1}-2^{k+1}}^{(m)}(2^k) \geq \begin{cases} q^{-13k/2} (2(2N-k) - 5.984) & (k \equiv 0 \mod 2), \\ q^{-13k/2} \left( \sqrt{3}(2N-k) - 3.699 \right) & (k \equiv 1 \mod 2), \end{cases}
\]

\[
S_{q,0}^{(m)}(2^k) \geq \begin{cases} q^{-13k/2} (2(2N-k) + 0.831) & (k \equiv 0 \mod 2), \\ q^{-13k/2} \left( \sqrt{3}(2N-k) + 1.262 \right) & (k \equiv 1 \mod 2), \end{cases}
\]

where all error terms \( O(2^{-2N}) \) are neglected.
Proof. (33) follows from Lemma 7 and the fact that
\[ \sum_{i=1}^{n} c_i \leq n + 0.05 \quad (n \geq 1). \]
The constants in (34)–(38) are easy to calculate. \(\square\)

Now, let \(2^4Na \leq n \leq 2^4Na+2^N\) for some \(a \geq 0\). Then the binary digit expansion of \(n\) is given by
\[ n = d_0d_1 \cdots d_{4Na+k} = \sum_{j=0}^{2na+k} d_j 2^{4Na+k-j}, \]
in which \(d_0 = 1\) and \(0 \leq k \leq 2^N\). Furthermore, let \(d_j\), \(0 \leq i < s(n)\), denote exactly those digits with \(d_j = 1\). Then
\[ S^{(m)}_{q,0}(n) = \sum_{i=0}^{s(n)-1} (-1)^i S^{(m)}_{q,-n_i}(2^{4Na+k-j_i}) \]
\[ = S^{(m)}_{q,0}(2^{4Na+k}) - S^{(m)}_{q,-2^k}(2^{4Na+k-j_1}) + S^{(m)}_{q,-2^k-2^k-j_1}(2^{4Na+k-j_2}) + \cdots, \]
where
\[ n_i = \sum_{j < j_i} d_j 2^{4Na+k-j}. \]

Since \(S^{(m)}_{q,i}(2^{4Na+k}) = \lambda_{m,q,i} S^{(m)}_{q,i}(2^k)\), we can use Corollary 1 in order to estimate \(S^{(m)}_{q,0}(n)\) and \(S_{q,0}(n)\).

First, suppose that \(k \equiv 0 \mod 2\). In the case \(d_0 = 1, d_1 = d_2 = d_3 = 0\) we have \(j_1 \geq 4\), and consequently
\[ S^{(m)}_{q,0}(n) = S^{(m)}_{q,0}(2^{4Na+k}) + \sum_{i \geq 1} (-1)^i S^{(m)}_{q,-n_j}(2^{4Na+k-j_i}) \]
\[ \geq \frac{\lambda_{m} 3^{k/2}}{q} \left( 2(2N-k) + 0.831 - \sum_{i \geq 4} (2(2N-k) + 2i + 3.65)3^{-i/2} \right) \]
\[ \geq \frac{\lambda_{m} 3^{k/2}}{q} (1.474(2N-k) - 2.951). \]

Hence, if \(k \leq 2N-3\) and \(k \equiv 0 \mod 2\) (i.e. \(k \leq 2N-4\)), then \(S^{(m)}_{q,0}(n) > 0\). If \(d_0 = 1, d_1 = d_2 = 0, d_3 = 1\), then we obtain in the same way
\[ S^{(m)}_{q,0}(n) = S^{(m)}_{q,0}(2^{4Na+k}) - S^{(m)}_{q,-2^k}(2^{4Na+k-3}) + \sum_{i \geq 2} (-1)^i S^{(m)}_{q,-n_j}(2^{4Na+k-j_i}) \]
\[ \geq \frac{\lambda_{m} 3^{k/2}}{q} \left( 2(2N-k) + 0.831 + 3^{-3/2}4.791 \right. \]
\[ - \sum_{i \geq 4} (2(2N-k) + 2i + 3.65)3^{-i/2} \]
\[ \geq \frac{\lambda_{m} 3^{k/2}}{q} (1.474(2N-k) - 2.029). \]
Thus, $S_{q,0}^{(m)}(n) > 0$ if $k \leq 2N - 2$. Next, let $d_0 = 1$, $d_1 = 0$, $d_2 = 1$. Here we can verify that
\[
2(2N - k) + 0.831 + 3^{-1}((2N - k + 2) - 0.669)
- \sum_{i \geq 3}(2(2N - k) + 2i + 3.65)3^{-i/2}
= 1.422(2N - k) - 4.363 > 0
\]
for $k \leq 2N - 4$. In the case $d_0 = d_1 = 1$, $d_2 = 0$ we have
\[
2(2N - k) + 0.831 + 3^{-1/2} \cdot 1.453 - 2(2N - k) \cdot 0.456 - 5.638
= 1.088(2N - k) - 3.979 > 0
\]
if $k \leq 2N - 4$. Finally, if $d_0 = d_1 = d_2 = 1$ we can check that
\[
2(2N - k) + 0.831 + 3^{-1/2} \cdot 1.453 + 3^{-1}(2(2N - k + 2) - 5.984)
- 2(2N - k) \cdot 0.456 - 5.638 = 1.754(2N - k) - 4.578 > 0
\]
for $k \leq 2N - 3$.

Next, suppose that $k \equiv 1 \mod 2$. If $d_0 = 1$, $d_1 = d_2 = d_3 = 0$, then
\[
\sqrt{3}(2N - k) + 1.262 - \sum_{i \geq 4}(2(2N - k) + 2i + 3.65)3^{-i/2}
= 1.206(2N - k) - 2.52 > 0
\]
for $k \leq 2N - 3$. If $d_0 = 1$, $d_1 = d_2 = 0$, $d_3 = 1$, then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-3/2}((2N - k + 3) - 2.358)
- 2(2N - k) \cdot 0.263 - 3.782
= 1.786(2N - k) - 2.397 > 0
\]
for $k \leq 2N - 2$. If $d_0 = 1$, $d_1 = 0$, $d_1 = 1$, then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-1}((2N - k) - 5.12)
- 2(2N - k) \cdot 0.456 - 5.638
= 1.397(2N - k) - 4.928 > 0
\]
for $k \leq 2N - 4$. If $d_0 = d_1 = 1$, $d_2 = 0$, then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-1/2}((2N - k + 1) - 2.674)
- 2(2N - k) \cdot 0.456 - 5.638
= 1.397(2N - k) - 5.343 > 0
\]
for $k \leq 2N - 4$. Finally, if $d_0 = d_1 = d_2 = 1$, then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-1/2}((2N - k + 1) - 5.12)
+ 3^{-1}((\sqrt{3}(2N - k) - 3.699) - 2(2N - k) \cdot 0.456 - 5.638
= 1.974(2N - k) - 5.007 > 0
\]
for $k \leq 2N - 3$.

This implies $S_{q,0}(2^{4Na+k} + \cdots) > 0$ if $k \leq 2N - 4$. The remaining cases $k = 2N$, $k = 2N - 1$, $k = 2N - 2$, and $k = 2N - 3$ must be treated separately.
First let $k = 2N$. By Lemma 7 it is easy to calculate $S_{q,0}^{(m)}(2k)$, $S_{q,2k}^{(m)}(2k-1)$, etc. up to an error term $O(2^{-k}) = O(2^{-2N})$. Let us consider a first example: $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $d_4 = 1$. We have

$$S_{q,0}^{(m)}(2^{4aN}+2N) = \lambda_m^n 3^N (2.20605 \cdots + O(2^{-k})),$$

$$S_{q,-2N}^{(m)}(2^{4aN}+2N-2) = \lambda_m^n 3^{N-1} (-4.4423 \cdots + O(2^{-k})),$$

$$S_{q,-2N-2}^{(m)}(2^{4aN}+2N-4) = \lambda_m^n 3^{N-2} (-0.1559 \cdots + O(2^{-k})),$$

and

$$s(n-1) \sum_{i=3}^{s(n)-1} (-1)^i S_{q,-n}(2^{4aN}+2N-j_i) \leq \lambda_m^n 3^N \sum_{i \geq 5} (2i + 3.56) 3^{-i/2} \leq 2.4865 \lambda_m^n 3^N.$$

Hence

$$S_{q,0}^{(m)}(n) \geq 3^{2N} (2.20605 + 3^{-1} 4.4423 - 3^{-2} 0.1559 - 2.4865 + O(2^{-2N}))$$

$$> (3.6695 - 2.4865) 3^{2N},$$

which gives $S_{q,0}(2^{4aN}+2N+2N-2+2N-4+\cdots) > 0$ for sufficiently large $a$.

All other cases can be treated in the same fashion. For completeness all relevant values are provided in Tables 2–5. The first column corresponds to the leading digits $d_0 d_1 d_2 \cdots d_j$ of $n = 2^{4aN}(d_0 2^k + d_1 2^{k-1} + \cdots + d_j 2^{k-j} + \cdots)$, the second one to the (approximate) value of the constant $c$ in

$$S_{q,0}^{(m)}(2^{4aN}(d_0 2^k + d_1 2^{k-1} + \cdots + d_j 2^{k-j})) = \lambda_m^n 3^{k/2} (c + O(2^{-k}))$$

and the third one to the error estimate

$$d = \sum_{i \geq j+1} (2(2N-k) + 2i + 3.65) 3^{-i/2}.$$

For example, if $k = 2N$ and $d_0 \cdots d_j = 10101$, then $j = 4$, $c = 3.669508$ and $d = 2.4865$.

Since $c > d$, in any case we have proved that $S_{q,0}(n) > 0$ for $2^{4aN} \leq n \leq 2^{4aN+2N}$ if $a$ and $N$ are sufficiently large. The remaining cases $2^{4aN+2N} < n < 2^{4(a+1)N}$ can be tackled in the same fashion. We just need to find an analoge to Lemma 7 and to consider several cases. Thus we have proved the second part of Theorem 1 for sufficiently large $N$. The above proof has neglected the error terms $O(2^{-2N})$. It is an easy but messy job to take these errors into account. In fact, it turns out that the above proof gives the second part of Theorem 1 for $N \geq 5$. Therefore we just have to check the two cases $N = 3$ and $N = 4$. We omit the details, but it is clear how to proceed in these cases in order to prove that $S_{q,N+1,0}(n) > 0$ for almost all $n$.

In the same fashion it is possible to prove $S_{43,0}(n) > 0$ and $S_{683,0}(n) > 0$ for almost all $n$. (Of course, a simple computer program assists us.) This completes the proof of Theorem 3. □
Proof of Theorem 2

The crucial step of the proof of Theorem 2 is contained in the following lemma.

Lemma 8. Let $p$ be an odd prime number and $s = \text{ord}_p(2)$. Then

\begin{equation}
S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{l \in L} \lambda_l^{4k} \left( \frac{s}{2} - \frac{1}{4} \sum_{l \in L} \frac{1}{1 - \Re(\zeta_p^l)} \right).
\end{equation}

Proof. Since $\lambda_l^4$ is real for all eigenvalues $\lambda_l = \prod_{i \in L} (1 - \zeta_p^i)$ and since

\begin{equation}
S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{l \in L} \lambda_l^{4k} \sum_{l \in L} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{2l})},
\end{equation}

(39) follows from

\begin{equation}
\Re \left( \frac{1}{(1 - z)(1 - z^2)} \right) = \frac{1}{2} - \frac{1}{4(1 - \Re(z))},
\end{equation}

in which $z \in \mathbb{C}$ has modulus $|z| = 1$. \hfill $\Box$

5. Proof of Theorem 2

The crucial step of the proof of Theorem 2 is contained in the following lemma.

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\end{equation}

Proof. Since $\lambda_l^4$ is real for all eigenvalues $\lambda_l = \prod_{i \in L} (1 - \zeta_p^i)$ and since

\begin{equation}
S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{l \in L} \lambda_l^{4k} \sum_{l \in L} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{2l})},
\end{equation}

(39) follows from

\begin{equation}
\Re \left( \frac{1}{(1 - z)(1 - z^2)} \right) = \frac{1}{2} - \frac{1}{4(1 - \Re(z))},
\end{equation}

in which $z \in \mathbb{C}$ has modulus $|z| = 1$. \hfill $\Box$

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The next lemma ensures that
\[ \frac{s}{2} < \frac{1}{4} \sum_{l \in L} \frac{1}{1 - R_{c_{lp}}} \]
for all \( l \in L \) if \( p \in P_l \) is sufficiently large. Hence \( S_{p,0}(2^{4k^{-2}}) < 0 \) for all \( k \geq 1 \).

**Lemma 9.** Suppose that \( p \in P_l \) and that \( p \geq (2t \log p)^2 \). Then

\[ \sum_{l \in L} \frac{1}{1 - R_{c_{lp}}} > \frac{1}{8\pi^2 t^2 \log p} p^{3/2} \]

**Proof.** By assumption \( p \geq 2tp^{1/2} \log p \). Hence by the Polya-Vinogradov inequality [12, p. 86, Aufgabe 12 b]
\[ |\{k \in L : 0 < k \leq 2tp^{1/2} \log p\}| > p^{1/2} \log p \]
for all \( l \in L \setminus \{0\} \). Consequently
\[ \sum_{l \in L} \frac{1}{1 - R_{c_{lp}}} = \sum_{l \in L} \frac{1}{2 \sin^2 \left( \frac{1}{p} \right)} > \frac{p^2}{2\pi^2} \sum_{l \in L} \frac{1}{l^2} \]
\[ > \frac{p^2}{2\pi^2} \frac{p^{1/2} \log p}{(2tp^{1/2} \log p)^2} = \frac{1}{8\pi^2 t^2 \log p} p^{3/2}. \]

Now the first part of Theorem 2 follows from the next proposition.

**Proposition 3.** Suppose that \( p \in P_l \) satisfies \( S_{p,0}(n) > 0 \) for almost all \( n \). Then
\[ p^{1/2} \leq 16\pi^2 t \log p, \]
i.e., if \( S_{p,0}(n) > 0 \) for almost all \( n \), then \( s = \text{ord}_p(2) \leq 16\pi^2 p^{1/2} \log p \).

**Proof.** It is clear that we just have to consider primes \( p \) with \( p^{1/2} \geq 2t \log p \). If \( p^{1/2} > 16\pi^2 t \log p \), then Lemma 9 would imply
\[ \frac{s}{2} - \frac{1}{4} \sum_{l \in L} \frac{1}{1 - R_{c_{lp}}} < \frac{p}{2t} - \frac{1}{32\pi^2 t^2 \log p} p^{3/2} < 0, \]
and by using Lemma 8 we would obtain \( S_{p,0}(2^{4k^{-2}}) < 0 \) for all \( k \geq 1 \).

In order to finish the proof of Theorem 2 we just have to mention a result by Erdős [4] saying that for any sequence \( \varepsilon_p \to 0 \) (as \( p \to \infty \))
\[ \frac{\{p \leq x : s = \text{ord}_p(2) < p^{1/2 + \varepsilon_p} \}}{x} = o \left( \frac{x}{\log x} \right). \]

**Remark.** Theorem 2 also says that the number \( A_t \) of primes \( p \in P_l \) satisfying \( S_{p,0}(n) > 0 \) for almost all \( n \) is bounded by \( A_t \leq C p^2 \log^2 p \). However, this bound can be essentially sharpened. A theorem of Titchmarsh [11, p. 147] says that for all \( a, 0 < a < 1 \), there exists a constant \( C = C(a) \) such that
\[ \pi(x; k, l) < C \frac{x}{\varphi(k) \log x} \]
for all \( 1 \leq k \leq x^a \) and \( 0 \leq l < k \) with \( \gcd(l, k) = 1 \). Since \( p \in P_l \) satisfies \( p \equiv 1 \mod t \), we get
\[ A_t = O(t^2(\log t)/\varphi(t)). \]
Furthermore, \( \varphi(t) > ct/(\log \log t) \) for some constant \( c > 0 \) (see [11, p. 24]). Hence \( A_t = \mathcal{O}(t \log t \log \log t) \).

Comparing the above properties with Theorem 4, we find that the fractal function \( \psi_p(x) = \psi_{p,0}(x) \) has a zero near \( x = 1. \) It is also an interesting problem to determine other zeroes and sign changes of \( \psi_p(x) \). In [2] it is shown that for almost all primes \( p \in \mathbf{P} \) the fractal function \( \psi_p(x) \) has a zero near \( x = 1/2. \) Furthermore, a similar result may be expected for \( \mathbf{P}_2. \) If \( |\Im(L(2, \chi))| > 40 \pi^2 p^{-3/2} \), where \( \chi \) denotes the biquadratic character mod \( p \in \mathbf{P}_2, \) then \( \psi_p(x) \) has a zero near \( x = 1/2. \)

Hence there is a connection between zeroes of \( \psi_p(x) \) and properties of Dirichlet \( L \)-series. In what follows we will extend this connection to arbitrary \( t. \) However, we are unable to prove the properties of \( L \)-series. Nevertheless by numerical evidence (see [2]) the zeroes of \( \psi_p \) seem to be very well dispersed. Therefore we conjecture that the \( L \)-series in question satisfy the proposed properties (43) and (44).

Let \( p \in \mathbf{P}_t, \) and denote by \( \lambda_m \) the eigenvalue of largest modulus. If \( s = \text{ord}_p(2) \) is odd, then all eigenvalues \( \lambda_l \) are imaginary and \( r^t = 4, \) which means that \( \psi_p(\frac{1}{2}) < 0 \) corresponds to \( S_{p,0}^{(m)}(2^{(4a+2)s}) < 0. \) Hence the same arguments as in the proof of Theorem 2 give

\[
S_{p,0}^{(m)}(2^{(4a+2)s}) > 0,
\]

providing a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \) for sufficiently large \( p. \) If \( s = \text{ord}_p(2) \) is even, then \( 2^{s/2} \equiv -1 \) mod \( p, \) and consequently all eigenvalues \( \lambda_l \) are real and positive. Hence \( \lambda_m > 0 \) and \( r^t = 1. \) Let \( \lambda_m = \prod_{i=0}^{s-1}(1 - \zeta_p^{2^{i}m}) \) and set

\[
a_j = \prod_{i=0}^{s/2-1} (1 - \zeta_p^{2^{i}m}).
\]

Then

\[
S_{p,0}^{(m)}(2^{as+s/2}) = \frac{\lambda_m}{p} \sum_{j=0}^{s-1} a_j,
\]

\[
S_{p,0}^{(m)}(2^{as+s/2-1}) = \frac{\lambda_m}{p} \sum_{j=0}^{s-1} \frac{a_j}{1 - \zeta_p^{2^{i}m}},
\]

\[
S_{p,0}^{(m)}(2^{as+s/2-2}) = \frac{\lambda_m}{p} \sum_{j=0}^{s-1} \frac{a_j}{(1 - \zeta_p^{2^{i}m})(1 - \zeta_p^{2^{i+1}m})}.
\]

Since \( 2^{s/2} \equiv -1 \) mod \( p \) it follows that \( \zeta_p^{2^{i}m/2} = \zeta_p^{-lm}. \) Hence \( a_{j+1} = -a_j \zeta_p^{-lm} \) and

\[
\sum_{j=0}^{s-1} a_j = a_0 \zeta_p^{lm} \sum_{j=0}^{s-1} (-1)^j \zeta_p^{-lm},
\]

\[
\frac{a_j}{1 - \zeta_p^{2^{i}m}} = a_0 \zeta_p^{lm} \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-lm}}{1 - \zeta_p^{2^{i}m}},
\]

\[
\frac{a_j}{(1 - \zeta_p^{2^{i}m})(1 - \zeta_p^{2^{i+1}m})} = a_0 \zeta_p^{lm} \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-lm}}{(1 - \zeta_p^{2^{i}m})(1 - \zeta_p^{2^{i+1}m})}.
\]
First, suppose that $s \equiv 2 \mod 4$, i.e. $s/2$ is odd. Then $a_{s/2} = (-1)^{s/2} a_0 \zeta_p^{2t_m}$ implies that $a_0 \zeta_p^{t_m}$ is imaginary. Since

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

and

$$\Im \left( \frac{1}{1-z} \right) = \frac{3z}{2(1-\Re z)}$$

for $|z| = 1$, we directly get

$$\sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{l_m2^j}}{1-\zeta_p^{l_m2^j}} = \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{l_m2^j}}{1-\zeta_p^{l_m2^j}} + \frac{1}{2} \sum_{j=0}^{s-1} (-1)^j \frac{i\Im \zeta_p^{l_m2^j}}{1-\Re \zeta_p^{l_m2^j}}.$$

Let $b$ be a generator of $G = (\mathbb{Z}/p\mathbb{Z})^*/\langle 4 \rangle$, i.e. all residue classes mod $p$ are parameterized by $b^i4^j$, $0 \leq i \leq 2t-1$, $0 \leq j \leq s/2 - 1$, and $\chi_k$, $1 \leq k \leq 2t$, Dirichlet characters defined by $\chi_k(b^i4^j) = \zeta_p^{ik}$. (Obviously the $\chi_k$, $1 \leq k \leq 2t$, constitute the character group of $G$.) If

$$g_{\chi_k} = \sum_{n=0}^{p-1} \chi_k(n) \zeta_p^n,$$

denote the corresponding Gauss sums

$$S_1 = \sum_{j=0}^{s-1} (-1)^j \zeta_p^{l_m2^j} = \frac{1}{2t} \sum_{k=1}^{2t} \zeta_p^{l_{2t}} (1-(-1)^k) g_{\chi_k},$$

in which $b^m \equiv l_m \mod p$. Furthermore, its absolute value can be estimated by $|S_1| \leq \sqrt{p}$. Now set

$$h_{\chi_k} = \sum_{n=0}^{p-1} \chi_k(n) \frac{3\zeta_p^n}{1-\Re \zeta_p^n} = \sum_{n=0}^{p-1} \chi_k(n) \cot \frac{n\pi}{p} = \frac{p}{\pi} (1-(-1)^k) L(1, \chi_k).$$

Then

$$S_2 = \frac{1}{2} \sum_{j=0}^{s-1} (-1)^j \frac{i\Im \zeta_p^{l_m2^j}}{1-\Re \zeta_p^{l_m2^j}} = \frac{i}{2\pi t} \sum_{k=1}^{2t} \zeta_p^{l_{2t}} (1-(-1)^k)^2 L(1, \chi_k).$$

Note that $S_1$ and $S_2$ are imaginary. This representation is interesting if $|S_2| > \sqrt{p}$. If $\operatorname{sgn}(iS_1) \neq \operatorname{sgn}(iS_2)$, then it is clear that there is a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$. If $\operatorname{sgn}(iS_1) = \operatorname{sgn}(iS_2)$, then it is an easy exercise to show that $S_{p,0}^{(m)}(2^{a+s/s/2})$ and $S_{p,0}^{(m)}(2^{a}(2^s/2 + 2s/2-1))$ have different signs. Therefore, if $p \in P_1$, $s \equiv 2 \mod 4$, and

$$(43) \quad \left| \frac{\sqrt{p}}{4\pi t} \sum_{k=1}^{2t} \zeta_p^{l_{2t}} (1-(-1)^k)^2 L(1, \chi_k) \right| > 1,$$

then there is a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$. For example, if $p \in P_1$ and $p > 163$, then Dirichlet’s class number formula and the fact that the class number $h$ of the corresponding quadratic field satisfies $h > 1$ show that this case appears (see [2]).
Finally, suppose that \( p \in \mathbb{P}_t \) and that \( s/2 \) is even, i.e. \( s \equiv 0 \mod 4 \). Here \( a_0 c^{l_m}_p \) is real and consequently \( S_1 \) is real, too. Furthermore, \( \Re(1/(1-z)) = \frac{1}{2} \) for \( |z| = 1 \). Hence

\[
\sum_{j=0}^{s-1} (-1)^j \frac{c^{l_m 2^j}_p}{1 - c^{l_m 2^j}_p} = \sum_{j=0}^{s-1} (-1)^j c^{l_m 2^j}_p
\]

and \( S^{(m)}_{p,0} (2a^s/s^2) = S^{(m)}_{p,0} (2a^s/s^2-1) \). Since

\[
\Re \left( \frac{1}{z(1-z)(1-z^2)} \right) = \Re z + \frac{1}{4} - \frac{1}{4(1-\Re z)}
\]

for \( |z| = 1 \) we obtain as above

\[
\sum_{j=0}^{s-1} (-1)^j \frac{c^{l_m 2^j}_p}{(1 - c^{l_m 2^j}_p)(1 - c^{l_m 2^j+1}_p)} = S_1 - \frac{1}{4} \sum_{j=0}^{s-1} (-1)^j \frac{1}{1 - \Re c^{l_m 2^j}_p}
\]

\[
= S_1 - \frac{p^2}{8\pi^2 t} \sum_{k=1}^{2t} \zeta_{2^{l_m}}(1 - (-1)^k) L(2, \chi_k)
\]

\[
= S_1 - S_3.
\]

Again, if

\[
(44) \quad \left| \frac{p^{3/2}}{8\pi^2 t} \sum_{k=1}^{2t} \zeta_{2^{l_m}}(1 - (-1)^k) L(2, \chi_k) \right| > 1
\]

the above representation yields a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \) if \( |S_3| > \sqrt{p} \). (If \( \text{sgn}(S_1) \neq \text{sgn}(S_3) \), then consider \( S^{(m)}_{p,0} (2a^s(2s^2/2 + 2s^2/2-2)). \) If \( p \in \mathbb{P}_1 \) and \( p \geq 17 \), this concept can be used to prove a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \) (see [2]).

However, if \( t > 1 \) we do not know a general concept to decide whether (43) or (44) are satisfied or not. Nevertheless, it seems to be an interesting problem to consider linear combinations of values of Dirichlet \( L \)-series (with coefficients in a proper number field) and to quantify lower bounds in terms of \( p \) and not only in terms of the heights of coefficients. We conjecture that (43) and (44) are true for sufficiently large \( p \geq c(t) \).

6. Higher Parities

The purpose of this section is to show that Newman’s phenomenon \( S_{q,0}(-1, n) > 0 \) (which is the same as \( A_{q,0;2,0}(n) > A_{q,0;2,1}(n) \)) has generalizations for higher parities \( r > 2 \). However, the situation is more difficult than in the case \( r = 2 \). We show that direct analogs of Newman’s theorem appear just for \( r \leq 6 \) (Theorem 6). For \( r > 6 \) we do not know whether a phenomenon of type (N1) occurs or not. But Theorem 2 has a direct analogon (Theorem 10).

Our first observation suggest that \( q = 2^r - 1 \) is a good choice for a phenomenon of type (N1) for a parity \( r \).

**Proposition 4.** Let \( q = 2^r - 1, \ r \geq 2 \). Then \( s(kq) = r \) for \( k \leq 2^r \), i.e. \( A_{q,0;r,m}(n) = 0 \) for \( n < 2^{2^r} \) and \( m \not\equiv 0 \mod r \).

**Proof.** Since \( k(2^r - 1) = (k-1)2^r + ((2^r - 1) - (k-1)) \) it is clear that \( s(k(2^r - 1)) = r \) if \( k - 1 < 2^r \).
However, we will prove the following theorem, showing that (N1) holds just for \( r \leq 6 \).

**Theorem 6.** The equality
\[
A_{2r-1,0;r,0}(n) > \max_{0<m<r} A_{2r-1,0;r,m}(n)
\]
holds exactly for \( 2 \leq r \leq 6 \).

If \( r > 6 \) it is very easy to disprove (45).

**Proposition 5.** Suppose that \( r > 6 \). Then (45) fails.

**Proof.** We show that \( \alpha_r > \alpha_{q,r} \). By Lemma 1 this contradicts (45).

The largest eigenvalue \( \lambda_0(\zeta^m) \), \( 0 < m < r \), corresponding to \( \alpha_r \) is given by
\[
\lambda_0(\zeta^r) = \left( 2 \cos \frac{\pi}{m} \right)^r = -2^r \left( 1 - \frac{\pi^2}{2r} + O(r^{-2}) \right).
\]

Now consider any \( q \)-th root of unity \( \zeta_q^l = e^{2\pi i q l} \), \( 0 < l < q \) \( (q = 2^r - 1) \). Then
\[
x_0 = \frac{l}{q} = \sum_{j \geq 1} q^{-j r} = \sum_{k \geq 1} c_k 2^{-k}
\]
has a periodic digit expansion \( c_{k+r} = c_k \), and for \( \zeta_q^{2m} = e^{2\pi i m} \) we have
\[
x_m = \sum_{k \geq 1} c_{k+m} 2^{-k}.
\]
Furthermore there exists a \( k_0 \) with \( c_{k_0} = 1 \) and \( c_{k_0+1} = 0 \). Hence \( 1/2 \leq x_{k_0} \leq 3/4 \), and consequently \( |x_{k_0} - x_{k_0+1}| \geq 1/4 \). Thus, for any \( m \)
\[
\min \left( |1 + \zeta_q^{m \zeta_r^k}||1 + \zeta_q^{m \zeta_r^{2k+1}}| \right) \leq 2 \cos \frac{\pi}{8},
\]
which implies
\[
|\lambda_l(\zeta^m)| \leq 2^r \cos \frac{\pi}{8}.
\]

Hence there are only finitely many \( r \geq 2 \) such that \( \alpha_r \leq \alpha_{q,r} \). It is an easy task to verify that this occurs exactly for \( r \leq 6 \).

First, consider the case \( r = 3 \) and set \( \omega = \zeta_3 = e^{2\pi i/3} \). Since
\[
S_{7,0}(\omega, n) = \sum_{m=0}^2 A_{7,0;3,m}(n)\omega^m
\]
(45) is equivalent to the following proposition.

**Proposition 6.** We have
\[
\arg (S_{7,0}(\omega, n)) \in \left( -\frac{\pi}{3}, \frac{\pi}{3} \right)
\]
for almost all \( n \geq 0 \).

**Proof.** First, let us determine the corresponding eigenvalues \( \lambda_1 = \lambda_{\{1,2,4\}}(\omega) \), \( \lambda_2 = \lambda_{\{3,5,6\}}(\omega) \), and \( \lambda_3 = \lambda_{\{0\}}(\omega) \). Set \( R = \zeta_7 + \zeta_7^2 + \zeta_7^4 \) and \( N = \zeta_7^3 + \zeta_7^5 + \zeta_7^6 \). Since \( R + N = -1 \) and
\[
R - N = \sum_{i=1}^6 \left( \frac{i}{7} \right) \zeta_7^i = i\sqrt{7},
\]
we have \( R = (-1 + i \sqrt{7})/2 \) and \( N = (-1 - i \sqrt{7})/2 \). Hence
\[
\lambda_1 = (1 + \omega \zeta_2)(1 + \omega \zeta_2^2)(1 + \omega \zeta_2^3) = 2 + \omega R + \omega^2 N
\]
\[
= \frac{5 - \sqrt{42}}{2} = 0.20871 \ldots
\]
Similarly we obtain \( \lambda_2 = (1 + \omega \zeta_2^2)(1 + \omega \zeta_2^3)(1 + \omega \zeta_2^5) = (5 + \sqrt{42})/2 = 4.79128 \ldots \) and \( \lambda_3 = (1 + \omega)^2 = -1 \). Thus, \( \lambda_m = \lambda_2 \) is the largest eigenvalue.

Next we will estimate \( S_{7,0}^{(m)}(\omega, n) = c_{n0} + \omega \omega d_{n0} \). Clearly it is sufficient to prove that \( c_{n0} > |d_{n0}| \) for almost all \( n \geq 0 \). For this purpose we define \( c_{jk}' \) and \( d_{jk}' \) by
\[
S_{7,0}^{(m)}(2^k) = \frac{\lambda_2^{(k)}}{42} (c_{jk}' + \omega d_{jk}').
\]
Observe that \( c_{jk}' \) and \( d_{jk}' \) are periodic in \( k \) with period 3. We use (8) in order to calculate their values. First we have
\[
S_{7,0}^{(m)}(2^{3l}) = \frac{3}{7} \lambda_2^l.
\]
Next we obtain
\[
S_{7,0}^{(m)}(2^{3l+1}) = \frac{\lambda_2^l}{7} \left( 1 + \omega \zeta_2^3 \right) \left( 1 + \omega \zeta_2^6 \right) \left( 1 + \omega \zeta_2^9 \right) = \frac{\lambda_2^l}{7} (3 + \omega N)
\]
\[
= \frac{\lambda_2^l}{42} \left( 18 + 2 \sqrt{42} \right) \left( -3 + \sqrt{42} \right) \omega.
\]
Here and in what follows we use the representations
\[
R = \frac{(-3 + \sqrt{42}) + 2 \sqrt{21} \omega}{6}, \quad L = \frac{(-3 - \sqrt{42}) - 2 \sqrt{21} \omega}{6}.
\]
Similarly,
\[
S_{7,0}^{(m)}(2^{3l+2}) = \frac{\lambda_2^l}{7} \left( 1 + \omega \zeta_2^3 \right) \left( 1 + \omega \zeta_2^6 \right) \left( 1 + \omega \zeta_2^9 \right) \left( 1 + \omega \zeta_2^3 \right) \left( 1 + \omega \zeta_2^6 \right) = \frac{\lambda_2^l}{42} \left( 21 + 5 \sqrt{21} \right) \left( -3 + \sqrt{21} \right) \omega.
\]
The cases \( j \neq 0 \) can be treated in the same way. Table 6 lists the corresponding values.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( c_{j0} )</th>
<th>( d_{j0} )</th>
<th>( c_{j1} )</th>
<th>( d_{j1} )</th>
<th>( c_{j2} )</th>
<th>( d_{j2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18</td>
<td>0</td>
<td>18 + 2\sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
<td>21 + 5\sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
</tr>
<tr>
<td>1</td>
<td>-3 + \sqrt{21}</td>
<td>2\sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
<td>18 + 2\sqrt{21}</td>
<td>2\sqrt{21}</td>
<td>18 + 4\sqrt{21}</td>
</tr>
<tr>
<td>2</td>
<td>-3 + \sqrt{21}</td>
<td>2\sqrt{21}</td>
<td>-3 - \sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>18 + 2\sqrt{21}</td>
</tr>
<tr>
<td>3</td>
<td>-3 - \sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>-21 - 5\sqrt{21}</td>
<td>-24 - 4\sqrt{21}</td>
</tr>
<tr>
<td>4</td>
<td>-3 + \sqrt{21}</td>
<td>2\sqrt{21}</td>
<td>-3 + 3\sqrt{21}</td>
<td>-3 + 3\sqrt{21}</td>
<td>2\sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
</tr>
<tr>
<td>5</td>
<td>-3 - \sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>0</td>
<td>-3 - 3\sqrt{21}</td>
</tr>
<tr>
<td>6</td>
<td>-3 - \sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>-3 + 2\sqrt{21}</td>
<td>-3 - 2\sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>-3 - \sqrt{21}</td>
</tr>
</tbody>
</table>
Hence, if \( n = 2^k + \cdots \) and \( S_{7,j}^{(m)}(\omega, n) = c_{nj} + \omega d_{nj} \), then

\[
|c_{nj}| + |d_{nj}| \leq \frac{3}{7} \frac{\beta^k}{1 - \beta^{-1}} = 1.0534 \ldots \beta^k.
\]

If \( n = 2^{3k} + 0 \cdot 2^{3k-1} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3k}}{42} (\epsilon'_{00} - |d'_{00}|) - 1.0535 \beta^{3k-2} \geq 0.23 \beta^{3k}.
\]

Similarly, if \( n = 2^{3k} + 2^{3k-1} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3k}}{42} (\epsilon'_{01} - |d'_{01}|) - 1.0535 \beta^{3k-2} \geq 0.23 \beta^{3k}.
\]

Furthermore, if \( n = 2^{3k+1} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3k}}{42} (\epsilon'_{01} - |d'_{01}| - \beta^{-3}(-d'_{52} - |c'_{52} - d'_{52}|)) - 1.0535 \beta^{3k-2} \geq 0.16 \beta^{3k}.
\]

Finally, the case \( n = 2^{3k+2} + \cdots \) can be treated in the same way. Hence

\[
c_{n0} - |d_{n0}| \geq c_{N2}^{(\log n)/(3 \log 2)},
\]

and consequently (46).

Similarly to the first part of Theorem 1, we are also able to provide infinitely many examples for phenomena of type (N1) for parity \( r = 3 \).

**Theorem 7.** Suppose that \( r = 3 \) and that \( q \) is an odd multiple of 7. Then (N1) and (N2) hold.

The essential part of the proof is to identify the largest eigenvalue. This will be done in the following lemma.

**Lemma 10.** Suppose that \( q \) is a positive odd integer. Then any eigenvalue

\[
\lambda_l(\omega) = \prod_{m=0}^{s-1} \left(1 + \omega \epsilon_m q^{2^m}\right)
\]

of \( M(\omega) \) is bounded by \( |\lambda_l(\omega)| < ((5 + \sqrt{21})/2)^{s/3} \) or \( \lambda_l(\omega) = ((5 + \sqrt{21})/2)^{s/2} \).

The case \( \lambda_l(\omega) = ((5 + \sqrt{21})/2)^{s/3} \) appears if and only if \( q \equiv 0 \mod 7 \) and either \( l \equiv 3q/7 \mod q \) or \( l \equiv 5q/7 \mod q \) or \( l \equiv 6q/7 \mod q \).
Proof. Let $\lambda_l(\omega) = \prod_{m=0}^{s-1} (1 + \omega^{12m})$ be an eigenvalue of $M(\omega)$. If $l \equiv 3q/7 \bmod q$ or $l \equiv 5q/7 \bmod q$ or $l \equiv 6q/7 \bmod q$, then $\lambda_l(\omega) = ((5 + \sqrt{21})/2)^{s/3}$.

In the remaining cases we use the following partition: $M_1, M_2 = M_1 + 1, M_3 = M_1 + 2, M_4 = M_1 + 1, M_6 = M_4 - 1, M_7$ of $\{0, 1, \ldots, s - 1\}$. $M_1$ consists of those $m$ such that $\arg(\zeta_q^{12m}) \in (-4\pi/7, -2\pi/7)$ and $M_4$ of those $m$ which are not contained in $M_2$ and satisfy $\arg(\zeta_q^{12m}) \in (-8\pi/7, -4\pi/7)$. Set

$$f(x) = 8 \left| \cos \left( \frac{x}{2} + \frac{\pi}{3} \right) \cos \left( x + \frac{\pi}{3} \right) \cos \left( 2x + \frac{\pi}{3} \right) \right|,$$

$$g(x) = 8 \left| \cos \left( \frac{x}{2} + \frac{\pi}{3} \right) \cos \left( x + \frac{\pi}{3} \right) \cos \left( \frac{x}{4} - \frac{\pi}{3} \right) \right|.$$ 

Then $f(-2\pi/7) = (5 + \sqrt{21})/2$ and

$$f(x) = \left| (1 + \omega e^{ix})(1 + \omega e^{2ix})(1 + \omega e^{3ix}) \right| < f(-2\pi/7)$$

for $x \in (-4\pi/7, -2\pi/7)$. Hence

$$\prod_{m \in M_1 \cup M_2 \cup M_3} \left| 1 + \omega e^{12m} \right| < \left( \frac{5 + \sqrt{21}}{2} \right)^{|M_1|}.$$ 

Similarly, $g(x) < f(-2\pi/7)$, $x \in (-8\pi/7, -4\pi/7)$, implies

$$\prod_{m \in M_4 \cup M_5 \cup M_6} \left| 1 + \omega e^{12m} \right| < \left( \frac{5 + \sqrt{21}}{2} \right)^{|M_4|}.$$ 

Finally, $|1 + \omega e^{ix}| < f(-2\pi/7)^{1/3}, x \in (-4\pi/7, 6\pi/7)$, provides

$$|1 + \omega e^{12m}| < \left( \frac{5 + \sqrt{21}}{2} \right)^{1/3}$$

for all $m \in M_7$, which completes the proof of Lemma 10.

Now the proof of Theorem 7 is almost the same as the proof of Proposition 6. Therefore we will not give the details here.

Next, let $r = 4$. Here we prove.

Proposition 7. We have

$$\arg(S_{15,0}(i, n)) \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \text{ for almost all } n \geq 0.$$ 

It is easy to verify that Proposition 7 implies Theorem 6 for $r = 4$. Since (47) is equivalent to

$$A_{15,0,4,0}(n) - A_{15,0,4,2}(n) > |A_{15,0,4,1}(n) - A_{15,0,4,3}(n)|,$$

we have $A_{15,0,4,0}(n) > A_{15,0,4,2}(n)$. By Theorem 1 ($q = 15$) we also know that

$$A_{15,0,4,0}(n) + A_{15,0,4,2}(n) > A_{15,0,4,1}(n) + A_{15,0,4,3}(n).$$

Let $\{k, l\} = \{1, 3\}$ and suppose that $A_{15,0,4,k}(n) \geq A_{15,0,4,l}(n)$. Then (48) and (49) imply

$$A_{15,0,4,0}(n) > A_{15,0,4,k}(n) \geq A_{15,0,4,l}(n),$$

and consequently (45).
Table 7.1

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It should also be mentioned that $\Re(S_{15,0}(i, n)) > 0$ for almost all $n$ is also sufficient to prove (45). By (6) we have

$$A_{15,0;4,0}(n) = \frac{1}{4} \sum_{l=0}^{3} i^{-l} m S_{15,0}(i^l, n).$$

Hence $\Re(S_{15,0}(i, n)) > 0$ implies $A_{15,0;4,0}(n) > A_{15,0;4,2}(n)$. Furthermore, by Theorem 1 $S_{15,0}(-1, n) \gg n^{\frac{\log 3}{\log 4}}$. Consequently we also have

$$A_{15,0;4,0}(n) > \max(A_{15,0;4,1}(n), A_{15,0;4,3}(n))$$

for sufficiently large $n$.

**Proof of Proposition 7.** The computation of the eigenvalues of $M(i)$ can be worked out explicitly:

$$\lambda_1 = \lambda_{1,2,4,8} = (1 + i\zeta_5)(1 + i\zeta_5^2)(1 + i\zeta_5^4)(1 + i\zeta_5^8)$$

$$= 2 - (\zeta_5^3 + \zeta_5^6 + \zeta_5^9 + \zeta_5^{12}) - (\zeta_5^5 + \zeta_5^{10}) + i \sum_{j=1}^{14} \left( \frac{j}{15} \right) \zeta_5^j$$

$$= 4 - \sqrt{15},$$

where $\left( \frac{\cdot}{15} \right)$ denotes the Jacobi-Kronecker symbol. The other eigenvalues are given by $\lambda_2 = \lambda_{14,7,11,13} = 4 + \sqrt{15}$, $\lambda_3 = \lambda_{3,6,9,12} = 1$, $\lambda_4 = \lambda_{5,10} = -1$, and by $\lambda_5 = \lambda_{0} = -4$. Hence the largest eigenvalue is $\lambda_2$. Now we can proceed as in the proof of Proposition 6. We just reproduce a table (Tables 7.1 and 7.2) for $c'_{jk}$ and $d'_{jk}$ defined by

$$S_{15,0}(i, 2^k) = \frac{\lambda_{2}^{[k]}_{j}}{30}(c'_{jk} + id'_{jk}).$$
Finally, let us consider the cases \( r = 5 \) and \( r = 6 \). In the case \( r = 5 \) it suffices to show that

\[
\Re(S_{31,0}(\zeta_5, n)) > \Re(\zeta_5^{-m} S_{31,0}(\zeta_5, n)) \quad (m \not\equiv 0 \mod 5),
\]

which can be checked by considering the largest eigenvalue \( \lambda_{-1}(\zeta_5) \) and similar calculations as above. (Again a simple computer program assists us.)

The case \( r = 6 \) is interesting because (45) can be deduced from Theorems 1 and 7. By (6)

\[
A_{63,0;6,m}(n) = \frac{1}{6} \sum_{i=0}^{5} c_{6}^{-i m} S_{63,0}(\zeta_6^i, n).
\]

By Theorem 7, \( \arg(S_{63,0}(\zeta_6^2, n)) \in (-\pi/3, \pi/3) \). Thus, for sufficiently large \( n \),

\[
A_{63,0;6,0}(n) > \max(A_{63,0;6,2}(n), A_{63,0;6,4}(n)),
\]

since the largest eigenvalue of \( \mathbf{M}(\zeta_6^2) \) is larger than the largest eigenvalue of \( \mathbf{M}(\zeta_6) \). Furthermore, by Theorem 1 \( S_{63,0}(-1, n) \gg n^{\frac{163}{144}} \), and consequently

\[
A_{63,0;6,0}(n) > \max(A_{63,0;6,1}(n), A_{63,0;6,3}(n), A_{63,0;6,5}(n))
\]

for sufficiently large \( n \).

Therefore we have provided a complete answer for the case \( q = 2^r - 1 \) with respect to (N1). However, the situation is much more delicate when we consider (N2) instead of (N1).

**Theorem 8.** We have

\[
R_{127,0;7,0}(n) > 0 \quad \text{for almost all } n \geq 0.
\]

This means that (N2) holds for \( r = 7 \) although (N1) fails. (We do not give a detailed proof. We only want to mention that it suffices to show that \( \Re(S_{127,0}(\zeta_7, n)) > 0 \).) Therefore it might be possible that (N2) holds for all \( r \geq 2 \). But again the answer is negative.

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**Theorem 9.** There are infinitely many \( r \geq 2 \) such that

\[
R_{2^{r-1},0,r,0}(n) < 0 \quad \text{for infinitely many } n \geq 0.
\]

**Sketch of the Proof.** It is sufficient to show that there are infinitely many \( r \geq 2 \) such that the eigenvalue \( \lambda_l(\zeta^m_r) \), \( 0 < l < 2^r - 1 \), \( 0 < m < r \), of largest modulus \( |\lambda_m| \) is negative. In what follows we will indicate that if there exists a positive integer \( m_r \) such that \( |\sqrt{r}/\pi + C - m_r| < 1/4 \) (where \( C \) a real constant and \( r \geq r_0 \) is sufficiently large), then \( \lambda_1(\zeta^{-m_r}) \) is the eigenvalue of largest modulus

\[
|\lambda_r| = |\lambda_1(\zeta^{-m_r})| \sim \frac{2^re^{-1/2}}{2\pi \sqrt{r}}.
\]

Since

\[
\arg(\lambda_1(\zeta^{-m_r})) = \sum_{j=0}^{r-1} \pi \left( \frac{2^j}{2^r - 1} - \frac{m_r}{r} \right) = \pi(1 - m_r),
\]

we have \( \text{sgn}(\lambda_1(\zeta^{-m_r})) < 0 \) if \( m_r \) is even. Obviously, this case occurs infinitely many times.

We use the fact that the digit expansion of \( x_0 = l/(2^r - 1) = \sum_{k \geq 1} c_k 2^{-k} \) is periodic, i.e. \( c_{k+r} = c_k \), and that \( x_j = \sum_{k \geq 1} c_k 2^{-j-k} \) satisfies \( \zeta_{2^j-1}^2 = e^{2\pi i x_j} \). (Compare with the proof of Proposition 5.) By considering several subcases it turns out that if \( l \) is unbounded, then

\[
|\lambda_l(\zeta^m_r)| = o(2^r r^{-1/2}).
\]

Conversely, if \( l \) is bounded, then

\[
\max_m |\lambda_l(\zeta^m_r)| \sim \frac{2^r - 1 e^{-1/2}}{l \pi \sqrt{r}},
\]

in which the maximum is attained for \( |m| \sim \sqrt{r}/\pi \). Therefore \( l = 1, |m| \sim \sqrt{r}/\pi \) is the only relevant case. (Since \( |1 + \zeta^m_q \zeta^2_q| < |1 + \zeta^{-m_r} \zeta^2_q| \) \( (m > 0) \), we may also assume that \( m \sim -\sqrt{r}/\pi \).) A more detailed analysis shows that the maximum value of \( |\lambda_l(\zeta^m_r)| \) is attained for

\[
m = -\frac{\sqrt{r}}{\pi} - C + O(r^{-1/2}),
\]

in which \( m \) is assumed to be a continuous real parameter and \( C \) is a computable constant. Furthermore, if \( \sqrt{r}/\pi + C \) is near to an integer \( m_r \), e.g. \( |\sqrt{r}/\pi + C - m_r| < 1/4 \), and if \( r \) is sufficiently large, then \( |\lambda_m| = |\lambda_1(\zeta^{-m_r})| \). \( \square \)

We finish this section on higher parities with an analogue to Theorem 2.

**Theorem 10.** For any \( r > 1 \) there exists a constant \( C_r > 0 \) such that for any \( t \geq 1 \) primes \( q \in \mathcal{P}_r \) satisfying (N1) or (N2) are bounded by

\[
q \leq C_r t^4 \log^4 t.
\]

For the proof we can use a similar procedure as above. Instead of (40) we need the following formula.
Proposition 8. Suppose that $p$ is an odd prime and $s = \text{ord}_p(2)$. If $y \in \mathbb{C}$ has modulus $|y| = 1$, then for any $l \in L$

$$\Re \left( \sum_{l \in 1} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} \right) = \frac{s}{2} - \frac{1}{4} \sum_{l \in 1} \frac{1}{1 - \Re \zeta_p^{2l}} \left( 1 + 2 \frac{\cos(\arg y/2)}{\cos((\arg y)/2 + \arg \zeta_p^l)} \right).$$

(50)

Proof. From

$$s = \sum_{l \in 1} \frac{1 + y \zeta_p^l + y^2 \zeta_p^{2l}}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} = \sum_{l \in 1} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} + \sum_{l \in 1} \frac{1}{(1 + y \zeta_p^{-l})(1 + y \zeta_p^{-2l})} + \sum_{l \in 1} \frac{y \zeta_p^l(1 + \zeta_p^l)}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})}$$

we obtain

$$\Re \left( \sum_{l \in 1} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} \right) = \frac{s}{2} - \frac{1}{2} \sum_{l \in 1} \frac{y \zeta_p^l(1 + \zeta_p^l)}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} = \frac{s}{2} - \frac{1}{2} S(y),$$

where the mapping $y \mapsto S(y)$, $y \neq -\zeta_p^{-l}$, is continuous. In particular,

$$S(-1) = -\sum_{l \in 1} \frac{\zeta_p^l}{(1 - \zeta_p^l)^2} = \frac{1}{2} \sum_{l \in 1} \frac{1}{1 - \Re \zeta_p^l}.$$ 

By using a partial fraction expansion it follows that $S(y)$, $y \neq 1$, can be represented by

$$S(y) = \frac{1 - y}{1 + y} \sum_{l \in 1} \frac{1}{1 + y \zeta_p^l} - \frac{1 - y}{1 + y} \sum_{l \in 1} \frac{1}{1 + y \zeta_p^{2l}} + \frac{2y}{1 + y} \sum_{l \in 1} \frac{\zeta_p^l}{1 + y \zeta_p^{2l}}$$

$$= \frac{2y}{1 + y} \sum_{l \in 1} \frac{1}{\zeta_p^{-l} + y \zeta_p^l}.$$ 

Since $S(-1)$ is finite, it follows that

$$\sum_{l \in 1} \frac{1}{\zeta_p^{-l} - \zeta_p^l} = 0.$$
and consequently
\[ S(y) = \frac{2y}{1 + y} \sum_{l \in \mathbb{I}} \left( \frac{1}{\zeta_p^{-l} + y\zeta_p^l} - \frac{1}{\zeta_p^{-l} - \zeta_p^l} \right) \]
\[ = -\Re \left( \sum_{l \in \mathbb{I}} \frac{y\zeta_p^l}{(\zeta_p^{-l} + y\zeta_p^l)(\zeta_p^{-l} - \zeta_p^l)} \right) \]
\[ = -\sum_{l \in \mathbb{I}} \frac{\zeta_p^{-l} - y\zeta_p^l}{(\zeta_p^{-l} + y\zeta_p^l)(\zeta_p^{-l} - \zeta_p^l)} \]
\[ = S(-1) - \sum_{l \in \mathbb{I}} \left( \frac{\zeta_p^{-l} - y\zeta_p^l}{(\zeta_p^{-l} + y\zeta_p^l)(\zeta_p^{-l} - \zeta_p^l)} - \frac{\zeta_p^{-l} + \zeta_p^l}{(\zeta_p^{-l} - \zeta_p^l)^2} \right) \]
\[ = S(-1) - \sum_{l \in \mathbb{I}} \frac{1 + y}{(\zeta_p^{-l} - \zeta_p^l)^2(\zeta_p^{-l} + y\zeta_p^l)} \]
\[ = \frac{1}{2} \sum_{l \in \mathbb{I}} \frac{1}{1 - \Re \zeta_p^l} + \sum_{l \in \mathbb{I}} \frac{1}{1 - \Re \zeta_p^{2l}} \frac{1 + y}{\zeta_p^{-l} + y\zeta_p^l}, \]
which proves (50).

The essential difference between the proofs of Theorem 2 and Theorem 10 is that you have to take into account the sign of
\[ \cos(m\pi/r) \cos(m\pi/r + \arg \zeta_p^l). \]
Let \( \mathbb{I}^- \) denote the set of those \( l \in \mathbb{I} \) such that this sign is negative. Then it is an easy exercise to show that
\[ \sum_{l \in \mathbb{I}^-} \frac{1}{1 - \Re \zeta_p^{2l}} \frac{\cos(m\pi/r)}{\cos(m\pi/r + \arg \zeta_p)} = O_r(p \log p). \]
You only have to verify that \( 1 - \Re \zeta_p^{2l} > c_r \) for \( l \in \mathbb{I}^- \) and that \( \arg \zeta_p \) is different for different \( l \in \mathbb{I} \). Hence, if \( p > C_r(t \log p)^4 \) (for a sufficiently large constant \( C_r > 0 \)), then
\[ \frac{1}{8\pi^2 t^2 \log p} > \frac{p}{2t} + O_r(p \log p), \]
which implies that \( \Re(S_{p,0}(\zeta_p^m, 2^{2a+2}))) < 0 \) for sufficiently large \( a \).

REFERENCES


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