RARIFIED SUMS OF THE THUE-MORSE SEQUENCE

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Abstract. Let \(q\) be an odd number and \(S_{q,0}(n)\) the difference between the number of \(k<n, k \equiv 0 \mod q\), with an even binary digit sum and the corresponding number of \(k<n, k \equiv 0 \mod q\), with an odd binary digit sum. A remarkable theorem of Newman says that \(S_{3,0}(n) > 0\) for all \(n\). In this paper it is proved that the same assertion holds if \(q\) is divisible by 3 or \(q = 4N + 1\). On the other hand, it is shown that the number of primes \(q \leq x\) with this property is \(o(x/\log x)\). Finally, analogs for “higher parities” are provided.

1. Introduction

The Thue-Morse sequence [9], [5] is defined by
\[
t_n = (-1)^{s(n)},
\]
where \(s(n)\) denotes the number of ones in the binary representation of \(n\). For any positive integer \(q\) and \(i \in \mathbb{Z}\) we denote
\[
S_{q,i}(n) = \sum_{0 \leq j < n, j \equiv i \mod q} t_j.
\]
In 1969 Newman [10] proved a remarkable conjecture of L. Moser saying that for any \(n \geq 1\)
\[S_{3,0}(n) > 0.\]
More precisely, he proved that
\[
\frac{3^\alpha}{20} < \frac{S_{3,0}(n)}{n^\alpha} < 5 \cdot 3^\alpha \quad \text{with} \quad \alpha = \frac{\log 3}{\log 4}.
\]
In 1983 Coquet [1] provided an explicit precise formula for \(S_{3,0}(n)\) by the use of a continuous function \(\psi_3(x)\) with period 1 which is nowhere differentiable (\(\eta_3(n) \in \{-1,0,1\}\)):
\[
S_{3,0}(n) = n \frac{\log 3}{\log 4} \cdot \psi_3 \left( \frac{\log n}{\log 4} \right) - \frac{\eta_3(n)}{3}.
\]
Furthermore, he was able to identify \(\min \psi([0,1]) > 0\) and \(\max \psi([0,1])\).

In general, (asymptotic) representations similar to (3) exist for any \(S_{q,i}(n)\) (see [5] and section 2). But it is a non-trivial problem to decide whether the continuous function \(\psi_{q,i}(x)\) has a zero or not. The only known examples where \(\psi_q(x) = \psi_{q,0}(x)\) has no zero are \(q = 345^4 ([6])\) and \(q = 17 ([7])\). (Note that the assertion that \(\psi_{q,1}(x)\)
Our first result provides infinitely many new examples where $\psi_q(x)$ has no zero.

**Theorem 1.** Suppose that $q$ is divisible by 3 or $q = 4^N + 1$. Then $S_{q,0}(n) > 0$ for almost all $n$.

However, if $q$ is prime then we can prove that there are only a few exceptions (e.g. Fermat primes). Let $P_t$, $t \geq 1$, denote the set of those primes $p$ where the order $\text{ord}_p(2)$ of 2 in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ equals $\text{ord}_p(2) = (p-1)/t$.

**Theorem 2.** There exists a constant $C > 0$ such that for any $t \geq 1$ the primes $p \in P_t$ satisfying $S_{p,0}(n) > 0$ for almost all $n$ are bounded by

$$p \leq Ct^2 \log^2 t.$$ 

Furthermore, the total number of primes $p \leq x$ with $S_{p,0}(n) > 0$ for almost all $n$ is $o(x/\log x)$ as $x \to \infty$.

The first part of Theorem 2 generalizes a result by the authors [2], where it is shown that 3 and 5 are the exceptional primes of $P_1$ and 17 and possibly 41 those of $P_2$. (In fact, $p = 41$ is not exceptional, see section 3.)

It is surely a very difficult problem to decide whether there are infinitely many primes $p$ satisfying $S_{p,0}(n) > 0$ for almost all $n$ or not. Unfortunately our methods are not strong enough to settle this problem. But it should be noted that if there were only finitely many primes with this property, Theorem 1 would imply that there were only finitely many Fermat primes.

However, the methods to be developed are essentially sufficient to decide this problem for any concrete value $q$. For example, we can prove the following theorem.

**Theorem 3.** The only primes $p \leq 1000$ satisfying $S_{p,0}(n) > 0$ for almost all $n$ are $p = 3, 5, 17, 43, 257, 683$.

Note that $p = 43 \in P_3$ and $p = 683 \in P_{31}$ are not Fermat primes.

We will prove Theorems 1 and 2 in sections 4 and 5. The negative part of Theorem 3 is proved at the end of section 3 and the positive part at the end of section 4. Section 6 is devoted to the case of higher parities where similar phenomena appear. In section 2 we collect some basic facts on the fractal structure of $S_{q,i}(n)$, and in section 3 we discuss two different kinds of positivity phenomena.

## 2. Basic Facts

For any fixed positive integer $q$ and $i \in \mathbb{Z}$, set

$$S_{q,i}(y,n) = \sum_{j<n, j \equiv i \mod q} y^{s(j)}, \quad (4)$$

---

1. The phrase “almost all” means “all but finitely many”, i.e. there might be finitely many exceptions.

2. Note that both 43 and 684 are of the form $(2^{2^N+1} + 1)/3$. Recently, by extending the methods of section 4, Leinfellner [8] showed that $q$ of the form $(2^{2^N+1} + 1)/3$ have the property that $S_{q,0}(n) > 0$ for almost all $n$. 

in which \( n \geq 0 \) and \( y \) is a (complex) parameter. With help of these expressions we can determine the numbers

\[
A_{q,i;r,m}(n) = |\{ j < n : j \equiv i \mod q, s(j) \equiv m \mod r \}|
\]

(5)

\[
= \frac{1}{r} \sum_{l=0}^{r-1} \zeta_r^{-ml} S_{q,i}(\zeta_r^l, n),
\]

(6)

where \( r \) is a positive integer (which will be called a parity), \( m \in \mathbb{Z} \), and \( \zeta_r \) denotes the \( r \)-th primitive root of unity, \( \zeta_r = \exp \left( \frac{2\pi i}{r} \right) \).

Note that \( S_{q,i}(y,n) \), \( 0 \leq i < q \), satisfies a simple generating relation if \( n \) is a power of 2:

\[
q^{-1} \sum_{i=0}^{q-1} S_{q,i}(y,2^k)\zeta_q^i = \prod_{j=0}^{k-1} \left( 1 + y\zeta_q^{2^j} \right).
\]

(7)

in which \( \zeta_q = \exp \left( \frac{2\pi i}{q} \right) \) denotes the \( q \)-th primitive root of unity and \( l \in \mathbb{Z} \). Hence we directly obtain

\[
S_{q,i}(y,2^k) = \frac{1}{q} \sum_{l=0}^{q-1} \zeta_q^{-li} \prod_{j=0}^{k-1} \left( 1 + y\zeta_q^{2^j} \right).
\]

(8)

Moreover, the obvious relation

\[
S_{q,i}(y,2^k+n') = S_{q,i}(y,2^k) + yS_{q,i-2^k}(y,n') \quad (n' < 2^k)
\]

(9)

can be used to calculate \( S_{q,i}(n) \) inductively for any integer \( n \geq 0 \).

We will further need

\[
S(y,n) = \sum_{j<n} y^{s(j)} = \sum_{i=0}^{q-1} S_{q,i}(y,n)
\]

(10)

and the numbers

\[
A_{r,m}(n) = |\{ j < n : s(j) \equiv m \mod r \}|
\]

(11)

\[
= \frac{1}{r} \sum_{l=0}^{r-1} \zeta_r^{-ml} S(\zeta_r^l, n).
\]

(12)

\( S(y,2^k) \) is given by

\[
S(y,2^k) = (1+y)^k
\]

(13)

and satisfies

\[
S(y,2^k+n') = S(y,2^k) + yS(y,n') \quad (n' < 2^k).
\]

(14)

Our first aim is to describe the asymptotic behaviour of \( A_{q,i;r,m}(n) \). The natural leading term is \( \frac{1}{q} A_{r,m}(n) \):

\[
A_{q,i;r,m}(n) = \frac{1}{q} A_{r,m}(n) + R_{q,i;r,m}(n).
\]

(15)

From (6), (8), (12), and (13) we obtain the representations

\[
A_{q,i;r,m}(2^k) = \frac{1}{rq} \sum_{l_1=0}^{r-1} \zeta_r^{-l_1m} \sum_{l_2=0}^{q-1} \zeta_q^{-l_2l_1} \prod_{j=0}^{k-1} \left( 1 + \zeta_q^{l_1} \zeta_q^{2^{l_2}} \right)
\]

(16)
and

\[ A_{r,m}(2^k) = \frac{1}{r} \sum_{l_1=0}^{r-1} \zeta_r^{-l_1m} \left( 1 + \zeta_r^{l_1} \right)^k, \]

so that

\[ R_{q,i;r,m}(2^k) = \frac{1}{rq} \sum_{l_1=0}^{r-1} \zeta_r^{-l_1m} \sum_{l_2=1}^{q-1} \zeta_q^{-l_2i} \prod_{j=0}^{k-1} \left( 1 + \zeta_r^{l_1} \zeta_q^{l_2} \right). \]

These Fourier expansions will be frequently used in the proofs of our main results.

From now on let \( q \) be an odd positive integer and let \( s = \text{ord}_q(2) \) be the order of the multiplicative subgroup \( (\Z/q\Z)^* \). (Since we are mainly interested in \( A_{q,0,r,m}(n) \), it is no real restriction to assume that \( q \) is odd.) Furthermore, let \( S_q(y,n) = (S_{q,0}(y,n), \ldots, S_{q,q-1}(y,n))^t \) denote the vector of \( S_{q,i}(y,n) \). Let \( e_0, \ldots, e_{q-1} \) denote the canonical basis of the \( q \)-dimensional vector space \( \C^q \) and let \( T \) denote the matrix defined by \( Te_i = e_{i+1} \) (with \( e_q = e_0 \)). The identity matrix is denoted by \( I \).

The following observations are more or less direct generalizations of [5].

**Proposition 1.** Let \( M(y) \) be defined by

\[ M(y) = \prod_{m=0}^{s-1} \left( I + yT^{2^m} \right). \]

Then

\[ S_q(y,2^n) = M(y)S_q(y,n). \]

**Proof.** By using the relations \( s(2j) = s(j) \) and \( s(2j + 1) = s(j) + 1 \) we obtain

\[
\begin{align*}
S_{q,i}(y,2n) &= \sum_{j<2n, j \equiv \text{mod} q} y^{s(j)} \\
&= \sum_{2j<2n, 2j \equiv \text{mod} q} y^{s(2j)} + \sum_{2j+1<2n, 2j+1 \equiv \text{mod} q} y^{s(2j+1)} \\
&= \sum_{j<n, j \equiv 2^{-1} \text{mod} q} y^{s(j)} + y \sum_{j<n, j \equiv 2^{-1} (i-1) \text{mod} q} y^{s(j)} \\
&= S_{q,2^{-1}i}(y,n) + yS_{q,2^{-1}(i-1)}(y,n).
\end{align*}
\]

Hence, denoting by \( U \) the matrix defined by \( Ue_i = e_{2i} \), we have

\[ S_q(y,2n) = (U + yUT)S_q(y,n). \]

By using the property \( UT = T^2U \) it follows by induction that

\[ (U + yUT)^i = \left( \prod_{m=1}^{i} \left( I + yT^{2^m} \right) \right) U^i. \]

Since \( T^n = U^s = I \), we directly obtain (19) by setting \( i = s \).
The eigenvalues of $T$ are exactly the $q$-th roots of unity $\zeta_q^l$, $0 \leq l < q$, with corresponding eigenvectors $v_l = \frac{q-1}{q} \zeta_q^{-il} e_i$ which are orthogonal. Since $M(y)$ is a polynomial in $T$, the eigenvalues of $M(y)$ are given by

$$ (20) \quad \lambda_l(y) = \sum_{m=0}^{s-1} \left( 1 + y\zeta_q^{2m} \right) $$

It is clear that $\lambda_l(y) = \lambda_{l'}(y)$ if and only if $l/2 = l'/2$. (Observe that $l = l/2$ contains $\{q/(q,l)\}(2)$ elements, where $(q,l)$ denotes the greatest common divisor of $q$ and $l$.) Appropriately we will write $\lambda_l(y)$ instead of $\lambda_l(y)$ if $l \in 1$. Let $L$ denote the system of equivalence classes $1 \in L$. Then a basis of the eigenspace $V_1$ corresponding to $\lambda_l(y)$, $1 \in L$, is given by $v_l$, $l \in L$. All these eigenspaces are orthogonal. $P_l$, $l \in L$, will denote the orthogonal projection on $V_l$. Furthermore, let $V^{(0)}$ denote the eigenspace corresponding to the eigenvalue 0 (if 0 is an eigenvalue), $V^{(s)}$ the subspace corresponding to eigenvalues of modulus $< 1$, $V^{(1)}$ the subspace corresponding to those of modulus 1, $V^{(u)}$ corresponding to those with modulus $> 1$, and $V^{(m)}$ that corresponding to those eigenvalues with maximal modulus. Furthermore, let $P^{(0)}, P^{(s)}, P^{(1)}, P^{(u)}$, and $P^{(m)}$ denote the orthogonal projections on $V^{(0)}, V^{(s)}, V^{(1)}, V^{(u)}$, and $V^{(m)}$, respectively.

Using these notations and the same methods as in [5], we immediately obtain a fractal representation for $S_q(y, n)$.

**Proposition 2.** There exists a continuous function $F(y, \cdot) : R^+ \to V^{(u)}$ satisfying

$$ (21) \quad F(y, 2^x) = M(y)F(y, x) \quad (x > 0) $$

and $P_nS_q(y, n) = F(y, n)$. Consequently

$$ \begin{align*}
S_q(y, n) &= F(y, n) + \left\{ \begin{array}{ll}
\mathcal{O}(1) & \text{if } V^{(1)} = \{0\}, \\
\mathcal{O}(\log n) & \text{if } V^{(1)} \neq \{0\},
\end{array} \right.
\end{align*} $$

Let $|\lambda_l(y)| > 1$. Then $G_1(y, t) = \lambda_l(y)^{-1} P_lF(y, 2^x)$ is a continuous function $G_1(y, \cdot) : R \to V_1$ which satisfies $G_1(y, t + 1) = G_1(y, t)$. With $\alpha_1(y) = \log \lambda_l(y)/(s \log 2)$ we finally obtain a fractal representation for $S_q(y, n)$:

$$ (22) \quad S_q(y, n) = \sum_{|\lambda_l(y)| > 1} n^{\alpha_1(y)} G_1 \left( y, \frac{\log n}{s \log 2} \right) + \mathcal{O}(\log n). $$

We want to mention also that it is quite easy to evaluate $G_1(y, t)$ for special values of $t$ by using the representation (8):

$$ (22) \quad S_q(y, 2^k) = \sum_{l=0}^{q-1} \sum_{j=0}^{b-1} \left( 1 + y\zeta_q^{2j} \right) $$

where $k = as + b$, $0 \leq b < s$. In particular, the first component of $G_1(y, 0)$ is non-zero.

Sometimes it would be more convenient to operate with real exponents instead of in general complex exponents $\alpha_1(y)$. For example, if $\lambda_l(y)^r$ is real and positive for
some positive integer $r'$, then we can use $\tilde{G}_1(y, t) = \lambda_1(y)^{-r'}P_1y^{2r's t}$ instead of $G_1(y, t)$ and $\tilde{a}_1(y) = \Re(\alpha_1(y))$ instead of $\alpha_1(y)$. (Compare with [5].)

For the evaluation of $A_{q,i,r,m}(n)$ we will need $S_q(\zeta^m, n)$, $0 < m < r$. It is an easy exercise to show that $\arg(\lambda_i(\zeta^m)) = sm\pi/r + m'i\pi$ for some $m' \in \mathbb{Z}$. Thus $\lambda_l(\zeta^m)^r$ is real and $\lambda_l(\zeta^m)^{2r} > 0$. Hence it is always possible to operate with positive exponents.

Finally, observe that $S(y, n)$ can be treated in a similar fashion as above but much more easily. Using the relation $S(y, 2n) = (1 + y)S(y, n)$, it follows that there is a continuous function $F(y, x)$ satisfying $F(y, 2x) = (1 + y)F(y, x)$ in the case $|1 + y| > 1$ such that

$$S(y, n) = F(y, n) = n^\alpha G\left(y, \frac{\log n}{\log 2}\right),$$

where $\alpha(y) = \log(1 + y)/\log 2$ and $G(y, t) = (1 + y)^{-t}F(2^t)$. Furthermore, $S(y, n) = O(1)$ if $|1 + y| < 1$ and $S(y, n) = O(\log n)$ if $|1 + y| = 1$.

Now the fractal representations for $A_{r,m}(n)$ and $R_{q,i,r,m}(n)$ follow immediately.

**Theorem 4.** Let $q, r$ be positive integers such that $q$ is odd and $r \geq 2$. Set

$$\alpha_r = \frac{\log \left(2 \cos \frac{\pi}{r}\right)}{\log 2} \quad (r > 2),$$

$$\alpha_{q,r} = \max_{0 < m < r, 0 < i < q} \frac{\log |\lambda_i(\zeta^m)|}{s \log 2}.$$

Furthermore, let $r'$ be the least positive integer such that $\lambda_l(\zeta^m)^{r'} > 0$ for those $\lambda_l(\zeta^m)$, $0 < l < q$, $0 < m < r$, with largest modulus.

Then there exist real valued periodic continuous functions $\psi_{r,m}(x)$, $\psi_{q,i,r,m}(x)$, $0 \leq m < r$, $0 \leq i < q$, with period 1 such that

$$A_{r,m}(n) = \frac{n}{r} + \begin{cases} 
(-1)^m \eta_n/2 + n^{\alpha_r} \cdot \psi_{r,m} \left(\frac{\log n}{2r \log 2}\right) + O(n^{\beta_r}) & (r = 2), \\
\eta_n/r \cdot \psi_{q,i,r,m} \left(\frac{\log n}{2r \log 2}\right) + O(n^{\beta_{q,r}}) & (r > 2),
\end{cases}$$

$$R_{q,i,r,m}(n) = \frac{n^{\alpha_{q,r}} \cdot \psi_{q,i,r,m} \left(\frac{\log n}{2r \log 2}\right)}{r} + O(n^{\beta_{q,r}}),$$

where $\beta_r < \alpha_r$, $\beta_{q,r} < \alpha_{q,r}$, and $\eta_n = 0$ if $n \equiv 0 \mod 2$ and $\eta_n = t_n$ if $n \equiv 1 \mod 2$.

**Proof.** Since $A_{r,m}(n)$ is given by (12) and $A_{q,i,r,m}$ by (6) (compare also with (16) and (17)), it follows that the asymptotic leading term of $A_{r,m}(n) - n/r$ depends on the largest eigenvalue $\lambda_0(\zeta^m) = (1 + \zeta^m)^r$, $0 < m < r$, and the asymptotic leading term of $R_{q,i,r,m}(n)$ on the largest eigenvalue $\lambda_l(\zeta^m)$, $0 < l < q$, $0 \leq m < r$.

Since $|1 + \zeta^m| = 2|\cos(\pi m/r)|$ is maximal for $m = 1$, we immediately obtain the asymptotic expansion for $A_{r,m}(n)$. (Note that $\beta_r = \log(2 \cos \pi r/\log 2).$

Furthermore, since $\lambda_l(1) = 1 + \zeta^l + \zeta^{2l} + \cdots + \zeta^{(2^{l-1} - 1)l} = 0$ for $0 < l < q$, it is clear that $\alpha_{q,r}$ is the correct exponent in the asymptotic leading term of $R_{q,i,r,m}(n)$.

Finally, $A_{2,m}(n)$ can be directly evaluated. \hfill \Box

**Remark.** In this paper we will only discuss binary digits. But the above concept easily applies for arbitrary $b$-ary digit expansions. Let $s(j)$ be a sequence satisfying $s(bn + c) = s(n) + s(c)$ for $n \geq 0$ and $0 \leq c < b$. Let $S_q(y, n)$ be defined as above.
and assume that $b$ and $q$ are relatively prime. Then
\[ S_q(y, bn) = U_b \left( \sum_{c=0}^{b-1} y^{s(c)} T^c \right) S_q(y, n), \]
where $U_b e_i = e_{bi}$, $0 \leq i < q$, and $s = \text{ord}_q(b)$. Hence $S_q(y, b^* n) = M_b(y) S_q(y, n)$, where
\[ M_b(y) = \prod_{m=0}^{b-1} \left( \sum_{c=0}^{b-1} y^{s(c)} T^{c b^m} \right), \]
and we are in the same position as above. All eigenvalues and eigenvectors of $M_b(y)$ are known, and we immediately obtain a fractal representation for $S_q(y, n)$. (In [5] only the case $b = r$ is mentioned.)

3. **Newman-like Phenomena**

We want to discuss two kinds of positivity phenomena:

(N1) \[ A_{q,0;r,0}(n) > \max_{0 < m < r} A_{q,0;r,m}(n) \text{ for almost all } n \geq 0, \]
(N2) \[ R_{q,0;r,0}(n) > 0 \text{ for almost all } n \geq 0. \]

Newman’s theorem $S_{3,0}(n) > 0$ ($n \geq 0$) is precisely the same as
\[ A_{3,0;2,0}(n) > A_{3,0;2,1}(n). \]

Therefore (N1) is a natural generalization of this property. Recall that $R_{q,0;r,m}(n)$ is the remainder term of $A_{q,0;r,m}(n)$ if $\frac{1}{q} A_{r,m}(n)$ is considered as the “natural” leading term of $A_{q,0;r,m}(n)$ (see section 2). Hence, (N2) means that the remainder term $R_{q,0;r,0}(n)$ is positive (for almost all $n$). We will now show that (N1) implies (N2) if $\alpha_r \neq \alpha_q; r$.

The following lemma provides a necessary condition for (N1).

**Lemma 1.** If (N1) holds then $\alpha_r \leq \alpha_{q,r}$.

**Proof.** Suppose that $\alpha_r > \alpha_{q,r}$. In this case (see Theorem 4) the asymptotic behaviour of $A_{q,0;r,m}(n)$ is determined by $A_{r,m}(n)$. However, we will show that $A_{r,0}(2^{(2a+1)r}) < A_{r,m}(2^{(2a+1)r})$ for all $m \equiv 0 \mod r$ and sufficiently large $a$. Therefore (N1) cannot occur.

Combining (13) and Theorem 4, we obtain
\[ A_{r,m}(2^k) - \frac{2^k}{r} \sim 2 \Re (\zeta_r^{-m}(1 + \zeta_r)^k). \]

Since $(1 + \zeta_r)^r$ is real and negative, everything follows. \qed

Hence, if $\alpha_r \neq \alpha_{q,r}$ then (N1) implies
\[ R_{q,0;r,0}(n) > \max_{0 < m < r} R_{q,0;r,m}(n) \text{ for almost all } n \geq 0. \]

Finally, (23) always implies (N2). This follows from the following property.

**Lemma 2.**
\[ \sum_{m=0}^{r-1} R_{q,i;r,m}(n) = O(\log n) \]
for all $i = 0, \ldots, q - 1$. 

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Proof. From (17) we get
\[ \sum_{m=0}^{r-1} R_{q,i,r,m}(2^k) = \frac{1}{q} \sum_{l=1}^{q-1} \zeta_q^{-li} \prod_{j=0}^{k-1} \left( 1 + \zeta_q^{l2^j} \right). \]
This means that the asymptotic behaviour of this sum is determined by the eigenvalues \( \lambda_l(1) \), which are given by
\[ \lambda_l(1) = \prod_{j=0}^{r-1} \left( 1 + \frac{\zeta_q^{l2^j}}{q} \right) = 1. \]
Hence (24) follows. \( \square \)

Note that there are situations where (N2) holds although (N1) fails; see Theorem 8. However, in the “classical” case \( r = 2 \) it is easy to verify that (N1) and (N2) are equivalent to \( S_{q,0}(-1, n) > 0 \) (for almost all \( n \)).

Before we prove further necessary conditions for (N1) and (N2), we want to mention that “converse” phenomena of the form \( A_{q,0;r,0}(n) < \min_{0 < m < r} A_{q,0;r,m}(n) \) or \( R_{q,0;r,0}(n) < 0 \) for almost all \( n \geq 0 \) do not exist.

Lemma 3. There exist infinitely many \( n \geq 0 \) such that
\[ A_{q,0;r,0}(n) > \max_{0 < m < r} A_{q,0;r,m}(n) \]
and
\[ R_{q,0;r,0}(n) > 0. \]
Proof. Let \( s = \text{ord}_q(2) \) and let \( n = 2^{2rs}a \) for some \( a \geq 0 \). Then \( \lambda_l(\zeta_r^l)^{2ra} > 0 \) for all \( l \in L \) and \( t = 0, \ldots, q-1 \). Hence (25) and (26) follow from
\[
A_{q,0;r,m}(n) = \frac{1}{q} A_{r,m}(n) + R_{q,0;r,m}(n)
\]
\[ = \frac{1}{rq} \sum_{l=0}^{r-1} \cos \left( \frac{2\pi l m}{r} \right) \lambda_l(\zeta_r^l)^{2ra} + \frac{1}{rq} \sum_{l=0}^{r-1} \cos \left( \frac{2\pi l m}{r} \right) \sum_{0 \neq l \in L} |l| \lambda_l(\zeta_r^l)^{2ra}. \]

Theorem 5. Let \( q, r \) be positive integers such that \( q \) is odd and \( r \geq 2 \). If \( s = \text{ord}_q(2) \) and \( r \) are coprime or if there exists an integer \( r' > 0 \) such that \( \lambda_l(\zeta_r^m)^{r'} < 0 \) for those \( \lambda_l(\zeta_r^m) \), \( 0 < l < q, 0 < m < r \), with maximal modulus, then (N1) and (N2) fail.
Proof. We only prove that (N2) fails. Since \( \lambda_0(\zeta_r)^r < 0 \), the following proof can be extended to contradict (N1).

Let \( L_m \) denote the set of pairs \( (l,m), l \in L, 0 < m < r, \) such that the eigenvalues \( \lambda_l(\zeta_r^m) \) have maximal modulus \( \rho \). Then the asymptotic leading term of \( R_{q,0;m,0}(n) \) only depends on these eigenvalues. In particular, we have
\[ R_{q,0;m,0}(2^{ks}) \sim \frac{1}{rq} \sum_{(l,m) \in L_m} |l| \lambda_l(\zeta_r^m)^k. \]
If there exists an integer \( r' > 0 \) such that \( \lambda_l(\zeta_r^m)^{r'} < 0 \) for \( (l,m) \in L_m \), then \( R_{q,0;m,0}(2^{s2ra+r's}) < 0 \) for all \( a \geq 0 \).
Now suppose that $r$ and $s$ are coprime. Since \( \arg(\lambda_t(\zeta^m_r)) = ms\pi/r + \eta\pi \), where \( \eta \in \{0, 1\} \), any eigenvalue \( \lambda_t(\zeta^m_r) \) is not real. Set \( \eta_{m,t} = \lambda_t(\zeta^m_r)/\rho \) for \((1,m) \in L_m\). Then \( \eta_{m,t} \) are non-real \((2r)\)-th roots of unity. Thus

\[
\sum_{b=0}^{2r-1} \sum_{(1,m) \in L_m} |l|\eta^b_{m,t} = 0,
\]

and consequently there exists \( b_0, 0 < b_0 < 2r \), such that

\[
\sum_{(1,m) \in L_m} |l|\lambda_t(\zeta^m_r)^{b_0} = \rho^{b_0} \sum_{(1,m) \in L_m} |l|\Re(\eta^b_{m,t}) < 0.
\]

Hence \( R_{q;0,0}(2a^{2rs+b_0}s) < 0 \) for sufficiently large \( a \). $\square$

With the help of Theorem 5 we will prove the negative part of Theorem 3 saying that primes \( p \leq 1000, p \neq 3, 5, 17, 43, 257, 683 \), do not satisfy \( S_{p,0}(-1, n) > 0 \) for almost all \( n \). First, we only have to consider \( p \in \mathbb{P}_t \) with \( t > 2 \). In [2] it is shown that \( p = 3 \) and \( p = 5 \) are the only exceptional primes in \( \mathbb{P}_1 \), and \( p = 17 \) and possibly \( p = 41 \) those of \( \mathbb{P}_2 \). (We will treat the case \( p = 41 \) in a moment.) Next, it follows from Theorem 5 that we only have to pay attention to those primes \( p \in \mathbb{P}_t, t > 2 \), with even \( s = \text{ord}_p(2) \), e.g. for \( p = 109 \in \mathbb{P}_3 \) we have \( s = 36 \). Finally, if there is \( k < s \) with

\[
S_{p,0}^{(m)}(-1, 2^k) = \frac{1}{p} \sum_{j=0}^{s-1} \prod_{i=0}^{k-1} \left(1 - \zeta_p^{j+s^i} \right) < 0,
\]

in which \( \lambda_m = \lambda_{l_m}(-1) \) is the largest eigenvalue, then \( S_{p,0}(-1, 2^{s+k}) < 0 \) for sufficiently large \( a \). For example, for \( p = 109 \) we have \( l_m = 9 \) and \( S_{109,0}^{(m)}(-1, 2^0) < 0 \). Hence, for \( p = 109 \) there is no phenomenon of type (N1). Similarly it follows that \( S_{41,0}^{(m)}(-1, 2^8) < 0 \), and we really have to consider just primes \( p \in \mathbb{P}_t \) with \( t > 2 \).

Table 1 gives a list of all primes \( p \leq 1000, p \in \mathbb{P}_t, t > 2 \), such that \( s \) is even. Furthermore the largest eigenvalue \( \lambda_m = \lambda_{l_m}(-1) \) is represented by \( l_m \), and if there is \( k < s \) such that \( S_{p,0}^{(m)}(-1, 2^k) < 0 \) then \( k \) is listed.

The only primes for which this method provides no answer are \( p = 43, 257, 683 \).

At the end of section 4 it will be shown that for these primes \( S_{p,0}(-1, n) > 0 \) for almost all \( n \). This completes the proof of the negative part of Theorem 3.

Remark. It is also an interesting problem to consider \( A_{q;i,r,m}(n) \) and \( R_{q;i,r,m}(n) \) \((0 < m < r)\) for some fixed \( i \neq 0 \text{ mod } q \). For example, it is known that \( A_{3;1,2,0}(n) < A_{3;1,2,1}(n) \) for almost all \( n \geq 0 \) (see [3]). Most of our methods can be applied in these cases too. However, for the sake of shortness we restrict ourselves to the case \( i = 0 \). $\square$
Table 1

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4. Proof of Theorem 1

In the case of the usual parity \( r = 2 \) we just have to discuss \( S_{q,i}(-1, n) \) to obtain all informations needed. For short we will write \( S_{q,i}(n), \lambda_l, \) and \( M \) instead of \( S_{q,i}(-1, n), \lambda_l(-1), \) and \( M(-1). \)

From an heuristic point of view integers of the form \( q = 4N + 1 \) or \( q = 4N - 1 \) are ‘good candidates’ for a phenomenon of type (N1). In both cases we have \( s(j) \equiv 0 \mod 2 \) for \( j \equiv 0 \mod q, j < q \), i.e. \( S_{q,0}(n) \) is as positive as possible. (The first case is trivial. For the second case see Proposition 4.) In fact, Theorem 1 says that \( S_{q,0}(n) > 0 \) (for almost all \( n \)) for these \( q \). However, an heuristic argument of this kind does not work in all cases. Suppose that \( q = 2^{2N+1} - 1 \). Then \( s(j) \equiv 1 \mod 2 \) for \( j \equiv 0 \mod q, j < q2^{N+1} + 1 \), i.e. \( S_{q,0}(n) \) is as negative as possible. Furthermore, \( s = \text{ord}_{q}(2) = 2N + 1 \) is odd. Hence, by Theorem 5 \( S_{q,0}(n) < 0 \) for infinitely many \( n \). But we know from Lemma 3 that we also have \( S_{q,0}(n) > 0 \) for infinitely many \( n \).

Let \( S_{q}^{(m)}(n) = (S_{q,0}^{(m)}(n), \ldots, S_{q,q-1}^{(m)}(n))^{t} = P^{(m)}S_{q}(n) \). According to the above considerations it is sufficient to show that

\[
S_{q,0}^{(m)}(n) \gg n^{(\log \lambda_m)/(s \log 2)},
\]

where \( \lambda_m \) denotes the maximal eigenvalue, resp. \( \min \psi_{q,0;m,0} > 0 \).

First we will discuss the case \( 3 | q \), where it is rather easy to identify \( \lambda_m \).

Lemma 4. Suppose that \( q \) is a positive odd integer. Then any eigenvalue

\[
\lambda_l = \prod_{m=0}^{s-1} \left( 1 - \zeta_q^{2^m} \right)
\]

of \( M \) is bounded by \( |\lambda_l| < 3^{s/2} \) or \( \lambda_l = 3^{s/2} \).
The case $l_1 = 3^{s/2}$ appears if and only if $q \equiv 0 \mod 3$ and $l \equiv q/3 \mod q$ or $l \equiv 2q/3 \mod q$.

Proof. It is an elementary exercise to show that

$$|1 - z^2| < \sqrt{3} \quad \text{and} \quad |(1 - z)(1 - z^2)| < 3$$

if $|z| = 1$ and $|1 - z| > \sqrt{3}$. Furthermore $|1 - z^2| = \sqrt{3}$ if $|z| = 1$ and $|1 - z| = \sqrt{3}$.

Now let $\lambda_l = \prod_{m=0}^{l-1} (1 - \zeta_q^{2m})$ be an eigenvalue of $M$. Let us consider a partition $M_0, M_1, M_2, M_3$ of the set $\{0, 1, \ldots, s-1\}$, where $M_0$ consists of those $m$ with $|1 - \zeta_q^{2m}| = \sqrt{3}$, $M_1$ of those with $|1 - \zeta_q^{2m}| > \sqrt{3}$, and $M_2 = M_1 + 1$. It is clear that either $M_0 = \emptyset$ or $M_0 = \{0, 1, \ldots, s-1\}$. Furthermore $M_1, M_2, M_3$ are pairwise disjoint. If $M_0 = \emptyset$ then

$$|\lambda_l| = \prod_{m \in M_1} |(1 - \zeta_q^{2m})(1 - \zeta_q^{2m+1})| \prod_{m \in M_3} |1 - \zeta_q^{2m}| < |3^{M_1}3^{M_3}/2| = 3^{s/2}.$$

On the other hand, if $M_0 = \{0, 1, \ldots, s-1\}$, then $s$ is even and $\lambda_l = 3^{s/2}$. Furthermore, the case $M_0 = \{0, 1, \ldots, s-1\}$ occurs only if $q \equiv 0 \mod 3$ and $l \equiv q/3 \mod q$ or $l \equiv 2q/3 \mod q$. \qed

Lemma 5. Suppose that $q$ is an odd multiple of 3. Then

$$|S_{q,i}^{(m)}(2^k)| \leq \frac{2}{q} 3^{k/2} \quad (0 \leq i < q),$$

(27)

$$S_{q,-2}^{(m)}(2^k) \leq 0 \quad (0 \leq j < s),$$

(28)

$$S_{q,0}^{(m)}(2^k) \geq \frac{\sqrt{3}}{q} j^{k/2}. \quad (0 \leq j < s).$$

(29)

Proof. Set $\omega = \zeta_3$. By (8) we have

$$S_{q,i}^{(m)}(2^k) = \frac{1}{q} \left( \omega^{-1} \prod_{j=0}^{k-1} (1 - \omega^{2^j}) + \omega^i \prod_{j=0}^{k-1} (1 - \omega^{-2^j}) \right).$$

Since $\omega^{2^j} = \omega^{(-1)^j}$ and $|1 - \omega^{2^j}| = \sqrt{3}$, we immediately obtain the estimate (27). Furthermore,

$$\prod_{j=0}^{k-1} (1 - \omega^{2^j}) = \begin{cases} 3^{k/2} & \text{if } k \text{ is even}, \\ 3^{(k-1)/2}(1 - \omega) & \text{if } k \text{ is odd}. \end{cases}$$

Hence

$$S_{q,-2}^{(m)}(2^k) = \begin{cases} -q^{-1}3^{k/2} & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd and } i \text{ is even}, \\ -q^{-1}3^{(k+1)/2} & \text{if } k \text{ and } i \text{ are odd}, \end{cases}$$

and

$$S_{q,0}^{(m)}(2^k) = \begin{cases} 2q^{-1}3^{k/2} & \text{if } k \text{ is even}, \\ q^{-1}3^{(k+1)/2} & \text{if } k \text{ is odd}, \end{cases}$$

which prove (28) and (29). \qed
Now suppose that \( n = 2^k + 2\delta^{k-1} + r \), where \( \delta \in \{0, 1\} \) and \( r < 2^{k-1} \). Then by using (9), (27), (28), and (29) we immediately obtain

\[
S_{q,0}^{(m)}(n) = S_{q,0}^{(m)}(2^k) - \delta S_{q,0}^{(m)}(2^{k-1}) + \sum_{j=0}^{k-2} \eta_j S_{q,0}^{(m)}(2^j)
\geq \left( \frac{\sqrt{3}}{2} - \frac{(1 - 3^{-1/2})^{-1/2}}{3} \right) \frac{2}{q} 3^{k/2}
> 0.077 \cdot \frac{2}{q} 3^{k/2} \gg n^{(\log \lambda_m)/(s \log 2)}.
\]

This proves Theorem 1 in the case \( 3|q \).

The case \( q = 4^N + 1 \) is a little bit more involved. The first step is to identify the largest eigenvalue \( \lambda_m \). Note that \( s = 4N \).

**Lemma 6.** If \( q = 4^N + 1 \) then \( \lambda_m \) is given by

\[
\lambda_m = \prod_{j=1}^{4N-1} \left( 1 - \zeta_q^{l_{m+2j}} \right) = c 3^{2N} \left( 1 + \mathcal{O}(2^{-2N}) \right),
\]

where \( l_m = (q + 1)/3 \) and \( c = 0.363247 \ldots > 0 \). Moreover, if \( l \not\equiv l_m \pmod{4N} \) then \( |\lambda_l| < \lambda_m \).

**Proof.** First observe that for \( 0 \leq i < N \)

\[
\begin{align*}
\arg \zeta_q^{l_{m+2i}} &\in I_1 \equiv \left( \frac{2\pi}{3}, \frac{5\pi}{6} \right), \\
\arg \zeta_q^{l_{m+2i+1}} &\in I_2 \equiv \left( -\frac{2\pi}{3}, -\frac{\pi}{3} \right), \\
\arg \zeta_q^{l_{m+2N+2i}} &\in I_3 \equiv \left( -\frac{5\pi}{6}, -\frac{2\pi}{3} \right), \\
\arg \zeta_q^{l_{m+2N+2i+1}} &\in I_4 \equiv \left( \frac{\pi}{3}, \frac{2\pi}{3} \right).
\end{align*}
\]

This means that there are exactly \( N \) elements \( \zeta_q^{l_i} \), \( 0 \leq i < 4N \), satisfying \( \arg \zeta_q^{l_i} \in I_1 \). Furthermore, the eigenvalue \( \lambda_m \) is calculated by

\[
\begin{align*}
\lambda_m &= \prod_{i=0}^{N-1} \left| 1 - \zeta_q^{l_i} \right|^2 \left| 1 - \zeta_q^{l_i+1} \right|^2 \\
&= \prod_{i=0}^{N-1} 16 \sin^2 \left( \frac{\pi}{3} + \frac{\pi 4^i}{3q} \right) \sin^2 \left( \frac{2\pi}{3} + \frac{2\pi 4^i}{3q} \right) \\
&= 3^{2N} \prod_{j=1}^{N} \frac{16}{9} \sin^2 \left( \frac{\pi}{3} + \frac{\pi 4^j}{3q} \right) \sin^2 \left( \frac{2\pi}{3} + \frac{2\pi 4^j}{3q} \right) \left( 1 + \mathcal{O}(2^{-2N-j}) \right) \\
&= 3^{2N} \left( \prod_{j=1}^{\infty} \frac{16}{9} \sin^2 \left( \frac{\pi}{3} + \frac{\pi 4^j}{3q} \right) \sin^2 \left( \frac{2\pi}{3} + \frac{2\pi 4^j}{3q} \right) \right) \left( 1 + \mathcal{O}(2^{-2N}) \right) \\
&= c 3^{2N} \left( 1 + \mathcal{O}(2^{-2N}) \right).
\end{align*}
\]

If \( \arg \zeta_q^{l_i} \in I_1 \) for some \( l \not\equiv 0 \pmod{q} \), then \( \arg \zeta_q^{2l} \in I_2 \), \( \arg \zeta_q^{2N+2l} \in I_3 \), and \( \arg \zeta_q^{2N+2l+1} \in I_4 \). Hence, the number \( N_0 \) of elements \( \zeta_q^{l_i} \), \( 0 \leq i < 4N \), satisfying \( \arg \zeta_q^{l_i} \in I_1 \) is always bounded by \( N_0 \leq N \).
The most interesting case appears if \( N_0 = N \). It is clear that this occurs if and only if \( \text{arg} \, \zeta_q^{l^i} \notin \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \) for all \( i \geq 0 \). Let us classify those \( x \in (0, 1) \) such that 
\[
z = e^{2\pi(1+x) i/3} \text{ satisfies arg } z^i \notin \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \text{ for all } i \geq 0.
\]

Since \( z \notin \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \) it follows that \( z \notin \left[ \frac{5\pi}{6}, \frac{7\pi}{6} \right] \) and consequently \( z \notin \left[ \frac{7\pi}{12}, \frac{5\pi}{12} \right] \) etc. By induction it follows that \( \text{arg } z \) must be contained in a zero set quite similar to the Cantor set. More precisely, the only possible values \( x \in (0, 1) \) are given by 
\[
x = \sum_{n \geq 1} a_n 4^{-n},
\]
where \( a_n \in \{0, 3\} \) and there exist \( n_1, n_2 \geq 1 \) with \( a_{n_1} = 0 \) and \( a_{n_2} = 3 \). If \( z \) is in addition a \( q \)-th root of unity then \( x \) must be of the form \( x = k/q \), where \( k = 1 \mod 3 \) and \( 1 \leq k \leq 4^N \). Since 
\[
\frac{1}{q} = \frac{4^N - 1}{42N - 1} = \sum_{p \geq 0} \sum_{n = 2pN + N + 1}^{2(p+1)N} \frac{3 \cdot 4^{-n}}{42N - 1},
\]
we immediately obtain 
\[
k \cdot \frac{1}{q} = k \sum_{p \geq 1} (4^N - 1)4^{-2pN} = \sum_{p \geq 1} \left( (k - 1)4^N + (4^N - 1)(k - 1) \right) 4^{-2pN},
\]
and observe that the 4-adic digits \( a_n \) of the digit expansion of \( k/q, 1 \leq k \leq 4^N \), satisfy \( a_n \in \{0, 3\} \) for all \( n \geq 1 \) if and only if the 4-adic digit expansion of \( k - 1 \) has the same property. (Evidently \( k \equiv 1 \mod 3 \) in these cases.) This means that if we choose digits \( b_n \in \{0, 3\}, 1 \leq n \leq N \), and set 
\[
k = 1 + \sum_{n = 1}^{N} b_n 4^{N-n},
\]
then 
\[
k \cdot \frac{1}{q} = \sum_{p \geq 0} \left( \sum_{n = 1}^{N} b_n 4^{-2Np-n} + \sum_{n = 1}^{N} (3 - b_n) 4^{-2Np-N-n} \right).
\]

In this way we get all \( q \)-th roots of unity \( z = \zeta_q^k \) with \( \text{arg} \, \zeta_q^k \in I_1 \cup I_3 \) such that \( N_0 = N \). Furthermore, the digits \( b_n, 1 \leq n \leq N \), encode the distribution of \( \zeta_q^{l^i} \).

If \( \zeta_q = e^{2\pi (1+x_0)/3} \) with \( x_0 = \sum_{n \geq 1} c_n 4^{-n} \) \( c_{2Np+n} = b_n \), \( c_{2Np+N+n} = 3 - b_n \), \( 1 \leq n \leq N \), \( p \geq 0 \), then \( \zeta_q^{l^i} = e^{2\pi (1+x_i)/3} \), where \( x_i = \sum_{n \geq 1} c_{n+i} 4^{-n} \). The periodicity \( \zeta_q^{l^{2N+n+1}} = \zeta_q^{l^i} \) is reflected by the periodic digit expansion of \( x_0 \). In particular, \( \zeta_q^{l^m} \) corresponds to the digits \( b_n = 0, 1 \leq n \leq N \). This means that \( \zeta_q^{l^{2N}} = e^{2\pi (1+x_{i_0})/3} \) are the only \( q \)-th roots of unity (with \( N_0 = N \)), where one period of the digits of \( x_{i_0} \) contains just one subblock of the form 03. In other words, there is exactly one element \( \zeta_q^{l^{2N}}, 0 \leq i < N \), satisfying \( \text{arg} \, \zeta_q^{l^{2N}} \in [19\pi/24, 5\pi/6] \), namely \( \zeta_q^{l^{2N-1}} \). For any other \( \zeta_q^l, l \notin L_m \) (with \( N_0 = N \)), there are at least two subblocks of the form 03 in any period of the digit expansion of \( x_0 \). Thus there exist \( 0 \leq i_1 < i_2 < N \) with \( \text{arg} \, \zeta_q^{l^{i_1}}, \text{arg} \, \zeta_q^{l^{i_2}} \in [19\pi/24, 5\pi/6] \). Consequently 
\[
\lambda_l < 32N \frac{16^2}{9^2} \sin^4 \left( \frac{19\pi}{24} \right) \sin^4 \left( \frac{19\pi}{48} \right) = 0.34899 \cdots 32N < \lambda_m.
\]
The case \( N_0 < N \) is much easier. Let \( J_1 \) denote the set of \( j, \ 0 \leq j < 4N \), such that \( \arg \zeta_{2i}^j \in I_1 \). We assume that the elements \( j_i, \ 0 \leq i < N_0 \), of \( J_1 = \{ j_0, j_1, \ldots, j_{N_0-1} \} \) are ‘ordered’ in such a way that \( \arg \zeta_{2i}^{j_i} \leq \arg \zeta_{2i+1}^{j_i}, \ 0 \leq i < N_0 - 1 \). (Recall that \( |J_1| = N_0 \).) Our first aim is to show that for any \( i, \ 0 \leq i < N_0 \), we have

\[
\arg \zeta_{2i}^{l_{m+4i}} < \arg \zeta_{2i}^{j_i}.
\]

Let \( b_i, \ 1 \leq i < N \), denote the number of \( j \in J_1 \) satisfying \( \arg \zeta_{2i}^{j_i} \in I^{(i)} = \left( \arg \zeta_{2i}^{l_{m+4i}}, \arg \zeta_{2i}^{l_{m+4i-1}} \right) \). Furthermore set \( c_i = \sum_{1 \leq j \leq i} b_j \). Observe that

\[
c_i \leq i, \quad 1 \leq i < N,
\]

immediately implies (30). Since \( \arg \zeta_{2i}^{l_{2j}} \in I^{(i)}, \ 1 \leq i < N - 1, \) implies \( \arg \zeta_{2i}^{l_{2j+2}} \in I^{(i+1)} \), we always have \( b_{i+1} = b_i \). Set \( a_1 = b_1 \) and \( a_i = b_i - b_{i-1}, \ 2 \leq i < N \). Then \( a_1 \geq 0, \ b_i = \sum_{1 \leq j \leq i} a_j, \) and \( c_i = \sum_{1 \leq j \leq i} (i - j + 1)a_j \).

Since \( N_{N-1} = N_0 \leq N - 1, \) condition (31) is satisfied for \( i = N - 1 \). Now we show that \( c_i \leq i \) implies \( c_{i-1} \leq i - 1 \). Suppose that \( c_{i-1} \geq i; \) then we obtain \( a_1 + \cdots + a_i = c_i - c_{i-1} \leq 0 \). Thus \( a_j = 0, \ 1 \leq j \leq i \), which implies \( c_{i-1} = 0 \) and contradicts \( c_{i-1} \geq i \). This completes the proof of (31) and consequently that of (30).

Let \( J_2 \) denote the set of \( j, \ 0 \leq j < 4N \), such that \( \arg \zeta_{2j}^{l_{2j}} \in (5\pi/6, \pi) \), and \( J_3 \) the set of those \( j, \ 0 \leq j < 4N \), such that \( \arg \zeta_{2j}^{l_{2j}} \in (0, \pi/3) \). Clearly \( N_0 + |J_2| + |J_3| = N \) and

\[
|1 - \zeta_{2j}^{l_{2j}}| \cdot |1 - \zeta_{2j}^{l_{2j+1}}| < |1 - \zeta_{m^{2N-1}}^{l_{m^{2N-1}}}| \cdot |1 - \zeta_{m^{2N}}^{l_{m^{2N}}}| \quad \text{for } j \in J_2 \cup J_3.
\]

Therefore we can estimate \( \lambda_t \) by

\[
\lambda_t = \prod_{j \in J_1 \cup J_2 \cup J_3} \left( |1 - \zeta_{2j}^{l_{2j}}| \cdot |1 - \zeta_{2j}^{l_{2j+1}}| \right)^2 \sum_{i=0}^{N_0-1} \left( \frac{16 \sin^2 \left( \frac{\arg \zeta_{2i}^{j_i}}{2} \right)}{\sin^2 \left( \arg \zeta_{2i}^{j_i} \right)} \right) \cdot \prod_{j \in J_2 \cup J_3} \left( \frac{16 \sin^2 \left( \frac{\arg \zeta_{2j}^{l_{2j}}}{2} \right)}{\sin^2 \left( \arg \zeta_{2j}^{l_{2j}} \right)} \right) \lessgtr \sum_{i=0}^{N_0-1} \left( \frac{16 \sin^2 \left( \frac{\arg \zeta_{m^{2N-1}}^{l_{m^{2N-1}}}}{2} \right)}{\sin^2 \left( \arg \zeta_{m^{2N-1}}^{l_{m^{2N-1}}} \right)} \right) \cdot \left( \frac{16 \sin^2 \left( \frac{\arg \zeta_{m^{2N}}^{l_{m^{2N}}}}{2} \right)}{\sin^2 \left( \arg \zeta_{m^{2N}}^{l_{m^{2N}}} \right)} \right) |J_2| + |J_3| \left\langle \lambda_m, \right\rangle
\]

which finishes the proof of Lemma 6.

\[ \square \]

In order to complete the proof of Theorem 1 we need an analogon to Lemma 5. However, the situation is much more delicate. For the following estimates we use
the notation

\begin{equation}
(32) \quad c_j = \prod_{i > j} \left( \frac{2}{\sqrt{3}} \sin \left( \frac{\pi}{3} + (-1)^j \frac{\pi}{3} 2^{-i} \right) \right) = 1 + \mathcal{O}(2^{-j}).
\end{equation}

The proof is completely elementary and just uses the Fourier expansion (8) of $S_{q,i}(2^k)$, or its dominant term $S_{q,i}^{(m)}(2^k)$ corresponding to $\zeta_q^m$.

**Lemma 7.** Suppose that $q = 4^N + 1$ and $0 \leq k \leq 2N$. Furthermore, let $i = 0$ or $i = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$, in which $k < k_1 < k_2 < \cdots < k_l \leq 2N$, and set

\begin{align*}
w_1 & = \sum_{l'=1}^l (-1)^{k_{l'} - k}, & w_2 & = \sum_{l'=1}^l 2^{k_{l'}-k}, \\
w_3 & = \sum_{l'=1}^l (-1)^{k_{l'}-2k}, & w_4 & = \sum_{l'=1}^l 2^{k_{l'}-2k}.
\end{align*}

If $k \equiv 0 \mod 2$, then

\begin{align*}
S_{q,i}^{(m)}(2^k) & = \frac{3^{k/2}}{q} \left( 2 \sum_{j=1}^{2N-k} c_j \cos \left( \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) \\
& \quad + 2c_0 \sum_{j=1}^{k} \frac{c_j + 2N-k}{c_j} \sin \left( (-1)^j \frac{\pi}{6} + \frac{\pi}{3} 2^{-j} + \frac{2\pi}{3} 2^{-j} 2N+k \right) \\
& \quad + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 + \mathcal{O}(2^{-k}) \right) \\
& = \frac{3^{k/2}}{q} \left( 2(2N-k) \cos \left( \frac{2\pi}{3} w_1 \right) + C_1(k; k_1, \ldots, k_l) \right.
\end{align*}

\begin{align*}
& \quad + C_2(k; k_1, \ldots, k_l) + \mathcal{O}(2^{-k}) + \mathcal{O}(2^{-2N}) \right),
\end{align*}

where the constants $C_1(k; k_1, \ldots, k_l)$, $C_2(k; k_1, \ldots, k_l)$ are given by

\begin{align*}
C_1(k; k_1, \ldots, k_l) & = 2 \sum_{j \geq 1} \left( c_j \cos \left( \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) - \cos \left( \frac{2\pi}{3} w_1 \right) \right) \\
C_2(k; k_1, \ldots, k_l) & = 2c_0 \sum_{j \geq 1} \left( c_j^{-1} \sin \left( (-1)^j \frac{\pi}{6} + \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) \\
& \quad - \sin \left( (-1)^j \frac{\pi}{6} + (-1)^j \frac{2\pi}{3} w_3 \right) \right),
\end{align*}

and $C_2(k; k_1, \ldots, k_l)$ is uniformly bounded by $|C_2(k; k_1, \ldots, k_l)| \leq 3.64$. 

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If $k \equiv 1 \mod 2$, then
\[
S_{q,k}^{(m)}(2^k) = \frac{3^{k/2}}{q} \left( 2 \sum_{j=1}^{2N-k} c_j \cos \left( (-1)^j \frac{\pi}{6} + \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) 
+ 2c_0 \sum_{j=1}^{k} \frac{c_j + 2N-k}{c_j} \sin \left( \frac{\pi}{3} 2^{-j} + \frac{\pi}{3} 2^{-j-2N+k} + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) + O(2^{-k}) \right) 
= \frac{3^{k/2}}{q} \left( 2(2N-k) \cos \left( \frac{\pi}{6} + \frac{2\pi}{3} w_3 \right) + D_1(k; k_1, \ldots, k_l) 
+ D_2(k; k_1, \ldots, k_l) + O(2^{-k}) + O(2^{k-2N}) \right),
\]
where the constants $D_1(k; k_1, \ldots, k_l)$, $D_2(k; k_1, \ldots, k_l)$ are given by
\[
D_1(k; k_1, \ldots, k_l) = 2 \sum_{j \geq 0} \left( c_j \cos \left( (-1)^j \frac{\pi}{6} + \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) - \cos \left( \frac{\pi}{6} + \frac{2\pi}{3} w_3 \right) \right),
\]
\[
D_2(k; k_1, \ldots, k_l) = 2c_0 \sum_{j \geq 1} \left( c_j^{-1} \sin \left( \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) - \sin \left( (-1)^j \frac{2\pi}{3} w_3 \right) \right)
\]
and $D_2(k; k_1, \ldots, k_l)$ is uniformly bounded by $|D_2(k; k_1, \ldots, k_l)| \leq 2.22$.

**Corollary 1.** Suppose that $q = 4^N + 1$ and $0 \leq k \leq 2N$. Then
\[
|S_{q,k}^{(m)}(2^k)| \leq \frac{3^{k/2}}{q} (2(2N-k) + 3.65), \quad (2^{k+1} \leq i \leq 4^N + 1),
\]
\[
-S_{q,-2k+1}^{(m)}(2^k) \geq \begin{cases} 
q^{-13k/2} ((2N-k) - 2.674) & (k \equiv 0 \mod 2), \\
q^{-13k/2} \cdot 1.453 & (k \equiv 1 \mod 2), 
\end{cases}
\]
\[
-S_{q,-2k+2}^{(m)}(2^k) \geq \begin{cases} 
q^{-13k/2} ((2N-k) - 0.669) & (k \equiv 0 \mod 2), \\
q^{-13k/2} \left( \sqrt{3}(2N-k) - 5.12 \right) & (k \equiv 1 \mod 2), 
\end{cases}
\]
\[
-S_{q,-2k+3}^{(m)}(2^k) \geq \begin{cases} 
q^{-13k/2} ((2N-k) - 2.358) & (k \equiv 0 \mod 2), \\
q^{-13k/2} \cdot 4.791 & (k \equiv 1 \mod 2), 
\end{cases}
\]
\[
S_{q,-2k+1,-2k+2}^{(m)}(2^k) \geq \begin{cases} 
q^{-13k/2} (2(2N-k) - 5.984) & (k \equiv 0 \mod 2), \\
q^{-13k/2} \left( \sqrt{3}(2N-k) - 3.699 \right) & (k \equiv 1 \mod 2), 
\end{cases}
\]
\[
S_{q,0}^{(m)}(2^k) \geq \begin{cases} 
q^{-13k/2} (2(2N-k) + 0.831) & (k \equiv 0 \mod 2), \\
q^{-13k/2} \left( \sqrt{3}(2N-k) + 1.262 \right) & (k \equiv 1 \mod 2), 
\end{cases}
\]
where all error terms $O(2^{-2N})$ are neglected.
Proof. (33) follows from Lemma 7 and the fact that
\[ \sum_{i=1}^{n} c_i \leq n + 0.05 \quad (n \geq 1). \]
The constants in (34)–(38) are easy to calculate. \[ \square \]

Now, let \( 2^{4Na} \leq n \leq 2^{4Na+2N} \) for some \( a \geq 0 \). Then the binary digit expansion of \( n \) is given by
\[ n = d_0d_1 \cdots d_{4Na+k} = \sum_{j=0}^{2na+k} d_j2^{4Na+k-j}, \]
in which \( d_0 = 1 \) and \( 0 \leq k \leq 2N \). Furthermore, let \( d_j \), \( 0 \leq i < s(n) \), denote exactly those digits with \( d_j = 1 \). Then
\[ S_{q,0}^{(m)}(n) = \sum_{i=0}^{s(n)-1} (-1)^i S_{q,-n_i}^{(m)}(2^{4Na+k-j_i}) \]
\[ = S_{q,0}^{(m)}(2^{4Na+k}) - S_{q,-2^k}^{(m)}(2^{4Na+k-j_1}) + S_{q,-2^{k-1}}^{(m)}(2^{4Na+k-j_2}) + \cdots, \]
where
\[ n_i = \sum_{j<j_i} d_j2^{4Na+k-j}. \]
Since \( S_{q,i}^{(m)}(2^{4Na+k}) = \lambda_m^{a} S_{q,i}^{(m)}(2^k) \), we can use Corollary 1 in order to estimate \( S_{q,0}^{(m)}(n) \) and \( S_{q,0}(n) \).

First, suppose that \( k \equiv 0 \mod 2 \). In the case \( d_0 = 1, \ d_1 = d_2 = d_3 = 0 \) we have \( j_1 \geq 4 \), and consequently
\[ S_{q,0}^{(m)}(n) = S_{q,0}^{(m)}(2^{4Na+k}) + \sum_{i \geq 1} (-1)^i S_{q,-n_i}^{(m)}(2^{4Na+k-j_i}) \]
\[ \geq \frac{\lambda_m^{a} 3^{k/2}}{q} \left( 2(2N-k) + 0.831 - \sum_{i \geq 4} (2(2N-k) + 2i + 3.65)3^{-i/2} \right) \]
\[ \geq \frac{\lambda_m^{a} 3^{k/2}}{q} (1.474(2N-k) - 2.951). \]
Hence, if \( k \leq 2N-3 \) and \( k \equiv 0 \mod 2 \) (i.e. \( k \leq 2N-4 \)), then \( S_{q,0}^{(m)}(n) > 0 \). If \( d_0 = 1, d_1 = d_2 = 0, d_3 = 1 \), then we obtain in the same way
\[ S_{q,0}^{(m)}(n) = S_{q,0}^{(m)}(2^{4Na+k}) - S_{q,-2^k}^{(m)}(2^{4Na+k-3}) + \sum_{i \geq 2} (-1)^i S_{q,-n_i}^{(m)}(2^{4Na+k-j_i}) \]
\[ \geq \frac{\lambda_m^{a} 3^{k/2}}{q} \left( 2(2N-k) + 0.831 + 3^{-3/2}4.791 \right. \]
\[ - \sum_{i \geq 4} (2(2N-k) + 2i + 3.65)3^{-i/2} \]
\[ \geq \frac{\lambda_m^{a} 3^{k/2}}{q} (1.474(2N-k) - 2.029). \]
Thus, \( S_{q,0}^{(m)}(n) > 0 \) if \( k \leq 2N - 2 \). Next, let \( d_0 = 1, d_1 = 0, d_2 = 1 \). Here we can verify that
\[
2(2N - k) + 0.831 + 3\frac{1}{2}((2N - k) + 2) - 0.669
\]
\[
- \sum_{i \geq 3}(2(2N - k) + 2i + 3.65)3^{-i/2}
\]
\[
= 1.422(2N - k) - 4.363 > 0
\]
for \( k \leq 2N - 4 \). In the case \( d_0 = d_1 = 1, d_2 = 0 \) we have
\[
2(2N - k) + 0.831 + 3\frac{1}{2} \cdot 1.453 - 2(2N - k) \cdot 0.456 - 5.638
\]
\[
= 1.088(2N - k) - 3.979 > 0
\]
if \( k \leq 2N - 4 \). Finally, if \( d_0 = d_1 = d_2 = 1 \) we can check that
\[
2(2N - k) + 0.831 + 3\frac{1}{2} \cdot 1.453 + 3^{-1}((2N - k + 2) - 5.984)
\]
\[
- 2(2N - k) \cdot 0.456 - 5.638 = 1.754(2N - k) - 4.578 > 0
\]
for \( k \leq 2N - 3 \).

Next, suppose that \( k \equiv 1 \mod 2 \). If \( d_0 = 1, d_1 = d_2 = d_3 = 0 \), then
\[
\sqrt{3}(2N - k) + 1.262 - \sum_{i \geq 4}(2(2N - k) + 2i + 3.65)3^{-i/2}
\]
\[
= 1.206(2N - k) - 2.52 > 0
\]
for \( k \leq 2N - 3 \). If \( d_0 = 1, d_1 = d_2 = 0, d_3 = 1 \), then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-3/2}((2N - k + 3) - 2.358)
\]
\[
- 2(2N - k) \cdot 0.263 - 3.782
\]
\[
= 1.786(2N - k) - 2.397 > 0
\]
for \( k \leq 2N - 2 \). If \( d_0 = 1, d_1 = 0, d_2 = 1 \), then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-1}\sqrt{3}(2N - k + 2) - 5.12
\]
\[
- 2(2N - k) \cdot 0.456 - 5.638
\]
\[
= 1.397(2N - k) - 4.928 > 0
\]
for \( k \leq 2N - 4 \). If \( d_0 = d_1 = 1, d_2 = 0 \), then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-1/2}((2N - k + 1) - 2.674)
\]
\[
- 2(2N - k) \cdot 0.456 - 5.638
\]
\[
= 1.397(2N - k) - 5.343 > 0
\]
for \( k \leq 2N - 4 \). Finally, if \( d_0 = d_1 = d_2 = 1 \), then
\[
\sqrt{3}(2N - k) + 1.262 + 3^{-1/2}((2N - k + 1) - 5.12)
\]
\[
+ 3^{-1}(\sqrt{3}(2N - k + 2) - 3.699) - 2(2N - k) \cdot 0.456 - 5.638
\]
\[
= 1.974(2N - k) - 5.007 > 0
\]
for \( k \leq 2N - 3 \).

This implies \( S_{q,0}(2^{4N} + \cdots) > 0 \) if \( k \leq 2N - 4 \). The remaining cases \( k = 2N, k = 2N - 1, k = 2N - 2 \), and \( k = 2N - 3 \) must be treated separately.
First let $k = 2N$. By Lemma 7 it is easy to calculate $S_{q,0}^{(m)}(2^k)$, $S_{q,-2}^{(m)}(2^k)$, etc. up to an error term $O(2^{-k}) = O(2^{-2N})$. Let us consider a first example: $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $d_4 = 1$. We have

\[
S_{q,0}^{(m)}(2^{4aN+2N}) = \lambda_m^a 3^N \left( 2.20605 \ldots + O(2^{-k}) \right),
\]
\[
S_{q,-2}^{(m)}(2^{4aN+2N-2}) = \lambda_m^a 3^{N-1} \left( -4.4423 \ldots + O(2^{-k}) \right),
\]
\[
S_{q,-22}^{(m)}(2^{4aN+2N-4}) = \lambda_m^a 3^{N-2} \left( -0.1559 \ldots + O(2^{-k}) \right),
\]
and

\[
\sum_{i=3}^{s(n)-1} (-1)^i S_{q,-i}(2^{4aN+2N-j_i}) \leq \lambda_m^a 3^N \sum_{i \geq 5} (2i + 3.56)3^{-i/2} \leq 2.4865 \lambda_m^a 3^N.
\]

Hence

\[
S_{q,0}^{(m)}(n) \geq 3^{2N} \left( 2.20605 + 3^{-1} \cdot 4.4423 - 3^{-2} \cdot 0.1559 - 2.4865 + O(2^{-2N}) \right)
\]
\[
> (3.6695 - 2.4865)3^{2N},
\]
which gives $S_{q,0}(2^{4aN}(2^2N + 2^N - 4 + \ldots)) > 0$ for sufficiently large $a$.

All other cases can be treated in the same fashion. For completeness all relevant values are provided in Tables 2–5. The first column corresponds to the leading digits $d_0d_1d_2 \cdots d_j$ of $n = 2^{4aN}(d_02^k + d_12^{k-1} + \cdots + d_j2^{k-j} + \cdots)$, the second one to the (approximate) value of the constant $c$ in

\[
S_{q,0}^{(m)}(2^{4aN}(d_02^k + d_12^{k-1} + \cdots + d_j2^{k-j})) = \lambda_m^a 3^{k/2} (c + O(2^{-k}))
\]
and the third one to the error estimate

\[
d = \sum_{i \geq j+1} (2(2N - k) + 2i + 3.65)3^{-i/2}.
\]

For example, if $k = 2N$ and $d_0 \cdots d_j = 10101$, then $j = 4$, $c = 3.669508$ and $d = 2.4865$.

Since $c > d$, in any case we have proved that $S_{q,0}(n) > 0$ for $2^{4aN} \leq n \leq 2^{4aN+2N}$ if $a$ and $N$ are sufficiently large. The remaining cases $2^{4aN+2N} < n < 2^{4(a+1)N}$ can be tackled in the same fashion. We just need to find an analog to Lemma 7 and to consider several cases. Thus we have proved the second part of Theorem 1 for sufficiently large $N$. The above proof has neglected the error terms $O(2^{-2N})$. It is an easy but messy job to take these errors into account. In fact, it turns out that the above proof gives the second part of Theorem 1 for $N \geq 5$. Therefore we just have to check the two cases $N = 3$ and $N = 4$. We omit the details, but it is clear how to proceed in these cases in order to prove that $S_{q,1,0}(n) > 0$ for almost all $n$.

In the same fashion it is possible to prove $S_{43,0}(n) > 0$ and $S_{683,0}(n) > 0$ for almost all $n$. (Of course, a simple computer program assists us.) This completes the proof of Theorem 3.
Table 2. $k = N$

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Table 3. $k = N - 1$

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<td>11000</td>
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<td>2.79</td>
</tr>
</tbody>
</table>

Table 4. $k = N - 2$

<table>
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<tr>
<th>$d_0 \cdots d_j$</th>
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<th>$d$</th>
</tr>
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<tbody>
<tr>
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<tr>
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</tr>
<tr>
<td>1111</td>
<td>7.18199</td>
<td>4.832</td>
</tr>
</tbody>
</table>

Table 5. $k = N - 3$

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<td>1111</td>
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<td>5.3581</td>
</tr>
</tbody>
</table>

5. Proof of Theorem 2

The crucial step of the proof of Theorem 2 is contained in the following lemma.

**Lemma 8.** Let $p$ be an odd prime number and $s = \text{ord}_p(2)$. Then

\[
S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{l \in L} \lambda_l^{4k} \left( \frac{s}{2} - \frac{1}{4} \sum_{l \in L} \frac{1}{1 - \Re \zeta_p^l} \right).
\]  

(39)

Proof. Since $\lambda_l^4$ is real for all eigenvalues $\lambda_l = \prod_{l \in L} (1 - \zeta_p^l)$ and since

\[
S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{l \in L} \lambda_l^{4k} \sum_{l \in L} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{2l})},
\]

(39) follows from

\[
\Re \left( \frac{1}{1 - z(1 - z^2)} \right) = \frac{1}{2} - \frac{1}{4(1 - \Re z)},
\]

in which $z \in \mathbb{C}$ has modulus $|z| = 1$. 

\[ \square \]
The next lemma ensures that

$$\frac{s}{2} < \frac{1}{4} \sum_{l \in L} \frac{1}{1 - \Re_\ell}$$

for all \(l \in L\) if \(p \in \mathbf{P}_t\) is sufficiently large. Hence \(S_{p,0}(2^{4ks-2}) < 0\) for all \(k \geq 1\).

**Lemma 9.** Suppose that \(p \in \mathbf{P}_t\) and that \(p \geq (2t \log p)^2\). Then

$$\sum_{l \in L} \frac{1}{1 - \Re_\ell} > \frac{p^{3/2}}{8\pi^2 t^2 \log p}.$$  

**Proof.** By assumption \(p \geq 2tp^{1/2} \log p\). Hence by the Polya-Vinogradov inequality [12, p. 86, Aufgabe 12 b]

$$|\{k \in 1: 0 < k < 2tp^{1/2} \log p\}| > p^{1/2} \log p$$

for all \(l \in L \setminus \{0\}\). Consequently

$$\sum_{l \in L} \frac{1}{1 - \Re_\ell} = \sum_{l \in L} \frac{1}{2 \sin^2 \left(\frac{\pi}{p}\right)} > \frac{p^2}{2\pi^2} \sum_{l \in L} \frac{1}{l^2} > \frac{p^2}{2\pi^2} \left(\frac{p^{1/2} \log p}{(2tp^{1/2} \log p)^2}\right)^2 = \frac{1}{8\pi^2 t^2 \log p}.$$ ⊓⊔

Now the first part of Theorem 2 follows from the next proposition.

**Proposition 3.** Suppose that \(p \in \mathbf{P}_t\) satisfies \(S_{p,0}(n) > 0\) for almost all \(n\). Then

$$p^{1/2} \leq 16\pi^2 t \log p,$$

i.e., if \(S_{p,0}(n) > 0\) for almost all \(n\), then \(s = \text{ord}_p(2) \leq 16\pi^2 p^{1/2} \log p\).

**Proof.** It is clear that we just have to consider primes \(p\) with \(p^{1/2} \geq 2t \log p\). If \(p^{1/2} > 16\pi^2 t \log p\), then Lemma 9 would imply

$$\frac{s}{2} - \frac{1}{4} \sum_{l \in L} \frac{1}{1 - \Re_\ell} < \frac{p}{2t} - \frac{1}{32\pi^2 t^2 \log p} < 0,$$

and by using Lemma 8 we would obtain \(S_{p,0}(2^{4ks-2}) < 0\) for all \(k \geq 1\). ⊓⊔

In order to finish the proof of Theorem 2 we just have to mention a result by Erdős [4] saying that for any sequence \(\varepsilon_p \rightarrow 0\) (as \(p \rightarrow \infty\))

$$|\{p \leq x: s = \text{ord}_p(2) < p^{1/2 + \varepsilon_p}\}| = o \left(\frac{x}{\log x}\right).$$

**Remark.** Theorem 2 also says that the number \(A_t\) of primes \(p \in \mathbf{P}_t\) satisfying \(S_{p,0}(n) > 0\) for almost all \(n\) is bounded by \(A_t \leq C p^2 \log^2 p\). However, this bound can be essentially sharpened. A theorem of Titchmarsh [11, p. 147] says that for all \(a, 0 < a < 1\), there exists a constant \(C = C(a)\) such that

$$\pi(x; k, l) < C \frac{x}{\varphi(k) \log x}$$

for all \(1 \leq k \leq x^a\) and \(0 \leq l < k\) with \(\text{gcd}(l, k) = 1\). Since \(p \in \mathbf{P}_t\) satisfies \(p \equiv 1 \mod t\), we get

$$A_t = O(t^2 (\log t)/\varphi(t)).$$
Furthermore, $\varphi(t) > ct/(\log \log t)$ for some constant $c > 0$ (see [11, p. 24]). Hence

$$A_t = O(t \log t \log \log t).$$

Comparing the above properties with Theorem 4, we find that the fractal function $\psi_p(x) = \psi_{p,0}(x)$ has a zero near $x = 1$. It is also an interesting problem to determine other zeroes and sign changes of $\psi_p(x)$. In [2] it is shown that for almost all primes $p \in \mathbb{P}_1$ the fractal function $\psi_p(x)$ has a zero near $x = 1/2$. Furthermore, a similar result may be expected for $\mathbb{P}_2$. If $|\Im(L(2, \chi))| > 40\pi^2 p^{-3/2}$, where $\chi$ denotes the biquadratic character mod $p \in \mathbb{P}_2$, then $\psi_p(x)$ has a zero near $x = 1/2$. Hence there is a connection between zeroes of $\psi_p(x)$ and properties of Dirichlet $L$-series. In what follows we will extend this connection to arbitrary $t$. However, we are unable to prove the properties of $L$-series. Nevertheless by numerical evidence (see [2]) the zeroes of $\psi_p$ seem to be very well dispersed. Therefore we conjecture that the $L$-series in question satisfy the proposed properties (43) and (44).

Let $p \in \mathbb{P}_t$, and denote by $\lambda_m$ the eigenvalue of largest modulus. If $s = \text{ord}_p(2)$ is odd, then all eigenvalues $\lambda_l$ are imaginary and $r' = 4$, which means that $\psi_p(1/2) < 0$ corresponds to $S^{(m)}(\zeta^{(4a+2)s}) < 0$. Hence the same arguments as in the proof of Theorem 2 give

$$S^{(m)}(\zeta^{4(4a+2)s-2}) > 0,$$

providing a sign change of $\psi_p(x)$ near $x = 1/2$ for sufficiently large $p$. If $s = \text{ord}_p(2)$ is even, then $2s/2 \equiv -1 \bmod p$, and consequently all eigenvalues $\lambda_l$ are real and positive. Hence $\lambda_m > 0$ and $r' = 1$. Let $\lambda_m = \prod_{i=0}^{s-1}(1 - \zeta_m^{2i+1})$ and set

$$a_j = \prod_{i=0}^{s-1}(1 - \zeta_m^{2i+1}).$$

Then

$$S^{(m)}(\zeta^{2as+s/2}) = \frac{\lambda_m}{p} \sum_{j=0}^{s-1} a_j,$$

$$S^{(m)}(\zeta^{2as+s/2-1}) = \frac{\lambda_m}{p} \sum_{j=0}^{s-1} \frac{a_j}{1 - \zeta_m^{2i+1}},$$

$$S^{(m)}(\zeta^{2as+s/2-2}) = \frac{\lambda_m}{p} \sum_{j=0}^{s-1} \frac{a_j}{(1 - \zeta_m^{2i+1})(1 - \zeta_m^{2i+2})}.$$

Since $2s/2 \equiv -1 \bmod p$ it follows that $\zeta_m^{2s+2} = \zeta_m^{-1}$. Hence $a_{j+1} = -a_j \zeta_m^{-2j}$ and

$$\sum_{j=0}^{s-1} a_j = a_0 \zeta_m^{s-1} \sum_{j=0}^{s-1} (-1)^j \zeta_m^{-2j},$$

$$\sum_{j=0}^{s-1} \frac{a_j}{1 - \zeta_m^{2i+2}} = a_0 \zeta_m^{s-1} \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_m^{-2j}}{1 - \zeta_m^{2i+2}},$$

$$\sum_{j=0}^{s-1} \frac{a_j}{(1 - \zeta_m^{2i+1})(1 - \zeta_m^{2i+2})} = a_0 \zeta_m^{s-1} \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_m^{-2j}}{(1 - \zeta_m^{2i+1})(1 - \zeta_m^{2i+2})}.$$
First, suppose that \( s \equiv 2 \mod 4 \), i.e. \( s/2 \) is odd. Then \( \pi_0 = a_{s/2} = (-1)^{s/2}a_0\zeta_p^{2l_0} \) implies that \( a_0\zeta_p^{l_0} \) is imaginary. Since

\[
\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}
\]

and

\[
\Im\left( \frac{1}{1-z} \right) = \frac{3z}{2(1-\Re z)}
\]

for \( |z| = 1 \), we directly get

\[
\sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-l_m2^j}}{1 - \zeta_p^{-l_m2^j}} = \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-l_m2^j}}{1 - \zeta_p^{-l_m2^j}} + \frac{1}{2} \sum_{j=0}^{s-1} (-1)^j \frac{i\Im\zeta_p^{-l_m2^j}}{1 - \Re\zeta_p^{-l_m2^j}}.
\]

Let \( b \) be a generator of \( G = (\mathbb{Z}/p\mathbb{Z})^*/\langle 4 \rangle \), i.e. all residue classes mod \( p \) are parameterized by \( b^i 4^j \), \( 0 \leq i \leq 2t - 1 \), \( 0 \leq j \leq s/2 - 1 \), and \( \chi_k \), \( 1 \leq k \leq 2t \), Dirichlet characters defined by \( \chi_k(b^i 4^j) = \zeta_p^{ik} \). (Obviously the \( \chi_k \), \( 1 \leq k \leq 2t \), constitute the character group of \( G \).) If

\[
g_{\chi_k} = \sum_{n=0}^{p-1} \chi_k(n)\zeta_p^n,
\]

denote the corresponding Gauss sums

\[
S_1 = \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-l_m2^j}}{1 - \zeta_p^{-l_m2^j}} = \frac{1}{2i} \sum_{k=1}^{2t} \zeta_p^{k_i m} (1 - (-1)^k)g_{\chi_k},
\]

in which \( b^i m \equiv l_m \mod p \). Furthermore, its absolute value can be estimated by \( |S_1| \leq \sqrt{p} \). Now set

\[
h_{\chi_k} = \sum_{n=0}^{p-1} \chi_k(n)\frac{\Im\zeta_p^n}{1 - \Re\zeta_p^n} = \sum_{n=0}^{p-1} \chi_k(n)\cot \frac{n\pi}{p} = \frac{p}{\pi} (1 - (-1)^k)L(1, \chi_k).
\]

Then

\[
S_2 = \frac{1}{2} \sum_{j=0}^{s-1} (-1)^j \frac{i\Im\zeta_p^{-l_m2^j}}{1 - \Re\zeta_p^{-l_m2^j}} = \frac{i}{2\pi t} \sum_{k=1}^{2t} \zeta_p^{k_i m} (1 - (-1)^k)^2L(1, \chi_k).
\]

Note that \( S_1 \) and \( S_2 \) are imaginary. This representation is interesting if \( |S_2| > \sqrt{p} \). If \( \operatorname{sgn}(iS_1) \neq \operatorname{sgn}(iS_2) \), then it is clear that there is a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \). If \( \operatorname{sgn}(iS_1) = \operatorname{sgn}(iS_2) \), then it is an easy exercise to show that \( S_{p,0}^{(m)}(2^a + s/2) \) and \( S_{p,0}^{(m)}(2^a (2^s/2 + 2 s/2 - 1)) \) have different signs. Therefore, if \( p \in \mathbb{P}_1 \), \( s \equiv 2 \mod 4 \), and

\[
\left| \frac{\sqrt{p}}{4\pi t} \sum_{k=1}^{2t} \zeta_p^{k_i m} (1 - (-1)^k)^2L(1, \chi_k) \right| > 1,
\]

then there is a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \). For example, if \( p \in \mathbb{P}_1 \) and \( p > 163 \), then Dirichlet’s class number formula and the fact that the class number \( h \) of the corresponding quadratic field satisfies \( h > 1 \) show that this case appears (see [2]).
Finally, suppose that \( p \in \mathbf{P} \), and that \( s/2 \) is even, i.e. \( s \equiv 0 \mod 4 \). Here \( a_0c_p^lm \) is real and consequently \( S_1 \) is real, too. Furthermore, \( \Re(1/(1-z)) = \frac{1}{2} \) for \( |z| = 1 \). Hence

\[
\sum_{j=0}^{s-1} (-1)^j \frac{c_p^{-lm}2^j}{1-c_p^{-lm}2^j} = \sum_{j=0}^{s-1} (-1)^j c_p^{-lm}2^j
\]

and \( S_{p,0}^{(m)}(2^{as}/s^2) = S_{p,0}^{(m)}(2^{as}/s^2-1) \). Since

\[
\Re \left( \frac{1}{z(1-z)(1-z^2)} \right) = \Re z + \frac{1}{4} - \frac{1}{4(1-\Re z)},
\]

for \( |z| = 1 \) we obtain as above

\[
\sum_{j=0}^{s-1} (-1)^j \frac{c_p^{-lm}2^j}{(1-c_p^{-lm}2^j)(1-c_p^{-lm}2^{j+1})} = S_1 - \frac{1}{4} \sum_{j=0}^{s-1} (-1)^j \frac{1}{1-\Re c_p^{-lm}2^j}
\]

\[
= S_1 - \frac{p^2}{8\pi^2 t} \sum_{k=1}^{\infty} c_{2t}^{k} (1-(-1)^k) \Lambda(2, \chi_k)
\]

\[
= S_1 - S_3.
\]

Again, if

\[
\left| \frac{p^{3/2}}{8\pi^2 t} \sum_{k=1}^{\infty} c_{2t}^{k} (1-(-1)^k) \Lambda(2, \chi_k) \right| > 1
\]

the above representation yields a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \) if \( |S_3| > \sqrt{p} \). (If \( \text{sgn}(S_1) \neq \text{sgn}(S_3) \), then consider \( S_{p,0}^{(m)}(2^{as}/(2s^2 - 2) \).) If \( p \in \mathbf{P} \) and \( p \geq 17 \), this concept can be used to prove a sign change of \( \psi_p(x) \) near \( x = \frac{1}{2} \) (see [2]).

However, if \( t > 1 \) we do not know a general concept to decide whether (43) or (44) are satisfied or not. Nevertheless, it seems to be an interesting problem to consider linear combinations of values of Dirichlet \( L \)-series (with coefficients in a proper number field) and to quantify lower bounds in terms of \( p \) and not only in terms of the heights of coefficients. We conjecture that (43) and (44) are true for sufficiently large \( p \geq c(t) \).

6. Higher Parities

The purpose of this section is to show that Newman’s phenomenon \( S_{q,0}(-1, n) > 0 \) (which is the same as \( A_{q,0,2,0}(n) > A_{q,0,2,1}(n) \) has generalizations for higher parities \( r > 2 \). However, the situation is more difficult than in the case \( r = 2 \). We show that direct analoga of Newman’s theorem appear just for \( r \leq 6 \) (Theorem 6).

For \( r > 6 \) we do not know whether a phenomenon of type \((\text{N1})\) occurs or not. But Theorem 2 has a direct analogon (Theorem 10).

Our first observation suggest that \( q = 2^r - 1 \) is a good choice for a phenomenon of type \((\text{N1})\) for a parity \( r \).

Proposition 4. Let \( q = 2^r - 1 \), \( r \geq 2 \). Then \( s(kq) = r \) for \( k \leq 2^r \), i.e. \( A_{q,0,r,m}(n) = 0 \) for \( n < 2^{2r} \) and \( m \neq 0 \mod r \).

Proof. Since \( k(2^r-1) = (k-1)2^r + ((2^r-1) - (k-1)) \) it is clear that \( s(k(2^r-1)) = r \) if \( k-1 < 2^r \). \( \square \)
However, we will prove the following theorem, showing that \((N1)\) holds just for \(r \leq 6\).

**Theorem 6.** The equality
\[
A_{2r-1,0;r,0}(n) > \max_{0<m<r} A_{2r-1,0;r,m}(n)
\]
holds exactly for \(2 \leq r \leq 6\).

If \(r > 6\) it is very easy to disprove (45).

**Proposition 5.** Suppose that \(r > 6\). Then (45) fails.

**Proof.** We show that \(\alpha_r > \alpha_{q,r}\). By Lemma 1 this contradicts (45).

The largest eigenvalue \(\lambda_0(\zeta^m)\), \(0 < m < r\), corresponding to \(\alpha_r\) is given by
\[
\lambda_0(\zeta^r) = \left(2 \cos \frac{\pi}{m}\right)^r = -2^r \left(1 - \frac{\pi^2}{2r} + O(r^{-2})\right).
\]

Now consider any \(q\)-th root of unity \(\zeta^l_q = e^{2\pi i q l}/q\), \(0 < l < q\) \((q = 2^r - 1)\). Then
\[
x_0 = \frac{l}{q} = \sum_{j \geq 1} 2^{-jr} = \sum_{k \geq 1} c_k 2^{-k}
\]
has a periodic digit expansion \(c_{k+r} = c_k\), and for \(\zeta^l_q = e^{2\pi i q m}\) we have
\[
x_m = \sum_{k \geq 1} c_{k+m} 2^{-k}.
\]
Furthermore there exists a \(k_0\) with \(c_{k_0} = 1\) and \(c_{k_0+1} = 0\). Hence \(1/2 \leq x_{k_0} \leq 3/4\), and consequently \(|x_{k_0} - x_{k_0+1}| \geq 1/4\). Thus, for any \(m\)
\[
\min \left(\left|1 + \zeta^r_q \zeta^{l_{k_0}} \right|, \left|1 + \zeta^r_q \zeta^{l_{k_0+1}} \right|\right) \leq 2 \cos \frac{\pi}{8},
\]
which implies
\[
|\lambda_l(\zeta^m)| \leq 2^r \cos \frac{\pi}{8}.
\]
Hence there are only finitely many \(r \geq 2\) such that \(\alpha_r \leq \alpha_{q,r}\). It is an easy task to verify that this occurs exactly for \(r \leq 6\).

First, consider the case \(r = 3\) and set \(\omega = \zeta_3 = e^{2\pi i/3}\). Since
\[
S_{7,0}(\omega, n) = \sum_{m=0}^{2} A_{7,0;3,m}(n) \omega^m
\]
(45) is equivalent to the following proposition.

**Proposition 6.** We have
\[
\arg \left(S_{7,0}(\omega, n)\right) \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)
\]
for almost all \(n \geq 0\).

**Proof.** First, let us determine the corresponding eigenvalues \(\lambda_1 = \lambda_{\{1,2,4\}}(\omega)\), \(\lambda_2 = \lambda_{\{3,5,6\}}(\omega)\), and \(\lambda_3 = \lambda_{\{0\}}(\omega)\). Set \(R = \zeta_7 + \zeta^2_7 + \zeta^4_7\) and \(N = \zeta^3_7 + \zeta^6_7 + \zeta^8_7\). Since \(R + N = -1\) and
\[
R - N = \sum_{i=1}^{6} \left(\frac{i}{7}\right) \zeta^i_7 = i \sqrt{7},
\]
we have $R = (-1 + i\sqrt{7})/2$ and $N = (-1 - i\sqrt{7})/2$. Hence
\[
\lambda_1 = (1 + \omega\zeta_7)(1 + \omega^2\zeta_7) = 2 + \omega R + \omega^2 N = \frac{5 - \sqrt{21}}{2} = 0.20871 \ldots
\]
Similarly we obtain $\lambda_2 = (1 + \omega^3\zeta_7)(1 + \omega^6\zeta_7) = (5 + \sqrt{21})/2 = 4.79128 \ldots$ and $\lambda_3 = (1 + \omega)^3 = -1$. Thus, $\lambda_m = \lambda_2$ is the largest eigenvalue.

Next we will estimate $S_{T,0}^{(m)}(\omega, n) = c_{n0} + \omega d_{n0}$. Clearly it is sufficient to prove that $c_{n0} > |d_{n0}|$ for almost all $n \geq 0$. For this purpose we define $c'_{jk}$ and $d'_{jk}$ by
\[
S_{T,0}^{(m)}(2^k) = \frac{\lambda_j^2}{42} (c'_{jk} + \omega d'_{jk}).
\]
Observe that $c'_{jk}$ and $d'_{jk}$ are periodic in $k$ with period 3. We use (8) in order to calculate their values. First we have
\[
S_{T,0}^{(m)}(2^3) = \frac{3}{7} \lambda_2^3.
\]
Next we obtain
\[
S_{T,0}^{(m)}(2^{3l+1}) = \frac{\lambda_2^3}{7} ((1 + \omega\zeta_7^3) + (1 + \omega^2\zeta_7^3) + (1 + \omega^6\zeta_7^3)) = \frac{\lambda_2^3}{7} (3 + \omega N)
\]
\[
= \frac{\lambda_2^3}{42} \left(18 + 2\sqrt{21}\omega + (-3 + \sqrt{21})\omega\right).
\]
Here and in what follows we use the representations
\[
R = \frac{(-3 + \sqrt{21}) + 2\sqrt{21}\omega}{6}, \quad L = \frac{(-3 - \sqrt{21}) + 2\sqrt{21}\omega}{6}.
\]
Similarly,
\[
S_{T,0}^{(m)}(2^{3l+2})
\]
\[
= \frac{\lambda_2^3}{7} ((1 + \omega\zeta_7^3)(1 + \omega^2\zeta_7^3) + (1 + \omega^6\zeta_7^3)(1 + \omega^3\zeta_7^3) + (1 + \omega^3\zeta_7^3)(1 + \omega^6\zeta_7^3))
\]
\[
= \frac{\lambda_2^3}{42} \left(21 + 5\sqrt{21}\omega + (-3 + \sqrt{21})\omega\right).
\]
The cases $j \neq 0$ can be treated in the same way. Table 6 lists the corresponding values.

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<td>-3 - 3\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>-21 - 5\sqrt{21}</td>
<td>-24 - 4\sqrt{21}</td>
</tr>
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<td>2\sqrt{21}</td>
<td>-3 + 3\sqrt{21}</td>
<td>-3 + 3\sqrt{21}</td>
<td>2\sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
</tr>
<tr>
<td>5</td>
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<td>-2\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>-3 - 3\sqrt{21}</td>
<td>0</td>
<td>-3 - 3\sqrt{21}</td>
</tr>
<tr>
<td>6</td>
<td>-3 - \sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>-3 + \sqrt{21}</td>
<td>-3 - \sqrt{21}</td>
<td>-2\sqrt{21}</td>
<td>-3 - \sqrt{21}</td>
</tr>
</tbody>
</table>
Now set \( \beta = \lambda_2^{1/3} \). Then

\[
\max_{0 \leq j < 7, 0 \leq k < 3} \beta_k |c'_{jk}| + |d'_{jk}| \leq \frac{3}{7}.
\]

Hence, if \( n = 2^k + \cdots \) and \( S_{7,j}^{(m)}(\omega, n) = c_{nj} + \omega d_{nj} \), then

\[
|c_{nj}| + |d_{nj}| \leq \frac{3}{7} \beta^k = 1.0534 \cdots \beta^k.
\]

If \( n = 2^{3l} + 0 \cdot 2^{3l-1} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3l}}{42} (c'_{00} - |d'_{00}|) - 1.0535\beta^{3l-2} \geq 0.357 \beta^{3l}.
\]

Similarly, if \( n = 2^{3l} + 2^{3l-1} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3l}}{42} (c'_{01} - |d'_{01}|) - 1.0535\beta^{3l-2} \geq 0.23 \beta^{3l}.
\]

If \( n = 2^{3l+1} + \cdots \), then we have to distinguish more cases. In the case \( n = 2^{3l+1} + 0 \cdot 2^{3l} + 0 \cdot 2^{3l-1} + \cdots \) we immediately obtain

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3l}}{42} (c'_{01} - |d'_{01}|) - 1.0535\beta^{3l-2} \geq 0.357 \beta^{3l}.
\]

If \( n = 2^{3l+1} + 2^{3l} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3l}}{42} (c'_{01} - |d'_{01}| - d'_{50} - |d'_{50}|) - 1.0535\beta^{3l-2} \geq 0.16 \beta^{3l}.
\]

Furthermore, if \( n = 2^{3l+1} + 2^{3l} + \cdots \), then

\[
c_{n0} - |d_{n0}| \geq \frac{\beta^{3l}}{42} (c'_{01} - |d'_{01}| - d'_{50} - |d'_{50}|) - 1.0535\beta^{3l-1} \geq 0.16 \beta^{3l}.
\]

Finally, the case \( n = 2^{3l+2} + \cdots \) can be treated in the same way. Hence

\[
c_{n0} - |d_{n0}| \geq c_2^{(\log n)/(3 \log 2)},
\]

and consequently \((46)\). \(\square\)

Similarly to the first part of Theorem 1, we are also able to provide infinitely many examples for phenomena of type \((\text{N1})\) for parity \( r = 3 \).

**Theorem 7.** Suppose that \( r = 3 \) and that \( q \) is an odd multiple of 7. Then \((\text{N1})\) and \((\text{N2})\) hold.

The essential part of the proof is to identify the largest eigenvalue. This will be done in the following lemma.

**Lemma 10.** Suppose that \( q \) is a positive odd integer. Then any eigenvalue

\[
\lambda_l(\omega) = \prod_{m=0}^{s-1} (1 + \omega |q|^{2m})
\]

of \( M(\omega) \) is bounded by \(|\lambda_l(\omega)| < ((5 + \sqrt{21})/2)^{s/3} \) or \( \lambda_l(\omega) = ((5 + \sqrt{21})/2)^{s/3} \).

The case \( \lambda_l(\omega) = ((5 + \sqrt{21})/2)^{s/3} \) appears if and only if \( q \equiv 0 \mod 7 \) and either \( l \equiv 3q/7 \mod q \) or \( l \equiv 5q/7 \mod q \) or \( l \equiv 6q/7 \mod q \).
Proof. Let \( \lambda_l(\omega) = \prod_{m=0}^{s-1} (1 + \omega \zeta_q^{2m}) \) be an eigenvalue of \( M(\omega) \). If \( l \equiv 3q/7 \) mod \( q \) or \( l \equiv 5q/7 \) mod \( q \) or \( l \equiv 6q/7 \) mod \( q \), then \( \lambda_l(\omega) = ((5 + \sqrt{21})/2)^{s/3} \).

In the remaining cases we use the following partition: \( M_1, M_2 = M_1 + 1, M_3 = M_1 + 2, M_4, M_5 = M_1 + 1, M_6 = M_4 - 1, M_7 \) of \( 0, 1, \ldots, s - 1 \). \( M_1 \) consists of those \( m \) such that \( \arg(\zeta_q^{2m}) \in (-4\pi/7, -2\pi/7) \) and \( M_4 \) of those \( m \) which are not contained in \( M_2 \) and satisfy \( \arg(\zeta_q^{2m}) \in (-8\pi/7, -4\pi/7) \). Set

\[
 f(x) = 8 \left| \cos\left( \frac{x}{2} + \frac{\pi}{3} \right) \cos\left( x + \frac{\pi}{3} \right) \cos\left( 2x + \frac{\pi}{3} \right) \right|,
 g(x) = 8 \left| \cos\left( \frac{x}{2} + \frac{\pi}{3} \right) \cos\left( x + \frac{\pi}{3} \right) \cos\left( \frac{x}{4} - \frac{\pi}{3} \right) \right|.
\]

Then \( f(-2\pi/7) = (5 + \sqrt{21})/2 \) and

\[
 f(x) = |(1 + \omega e^{ix})(1 + \omega e^{2ix})(1 + \omega e^{3ix})| < f(-2\pi/7)
\]

for \( x \in (-4\pi/7, -2\pi/7) \). Hence

\[
 \prod_{m \in M_1 \cup M_2 \cup M_3} \left| 1 + \omega \zeta_q^{2m} \right| < \left( \frac{5 + \sqrt{21}}{2} \right)^{|M_1|}.
\]

Similarly, \( g(x) < f(-2\pi/7), x \in (-8\pi/7, -4\pi/7), \) implies

\[
 \prod_{m \in M_4 \cup M_5 \cup M_6} \left| 1 + \omega \zeta_q^{2m} \right| < \left( \frac{5 + \sqrt{21}}{2} \right)^{|M_4|}.
\]

Finally, \( |1 + \omega e^{ix}| < f(-2\pi/7)^{1/3}, x \in (-4\pi/7, 6\pi/7), \) provides

\[
 |1 + \omega \zeta_q^{2m}| < \left( \frac{5 + \sqrt{21}}{2} \right)^{1/3}
\]

for all \( m \in M_7 \), which completes the proof of Lemma 10.

Now the proof of Theorem 7 is almost the same as the proof of Proposition 6. Therefore we will not give the details here.

Next, let \( r = 4 \). Here we prove.

**Proposition 7.** We have

\[
 \arg(S_{15,0}(i, n)) \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \text{ for almost all } n \geq 0.
\]

It is easy to verify that Proposition 7 implies Theorem 6 for \( r = 4 \). Since (47) is equivalent to

\[
 A_{15,0,4,0}(n) - A_{15,0,4,2}(n) > |A_{15,0,4,1}(n) - A_{15,0,4,3}(n)|,
\]

we have \( A_{15,0,4,0}(n) > A_{15,0,4,2}(n) \). By Theorem 1 \( (q = 15) \) we also know that

\[
 A_{15,0,4,0}(n) + A_{15,0,4,2}(n) > A_{15,0,4,1}(n) + A_{15,0,4,3}(n).
\]

Let \( \{k, l\} = \{1, 3\} \) and suppose that \( A_{15,0,4,k}(n) \geq A_{15,0,4,l}(n) \). Then (48) and (49) imply

\[
 A_{15,0,4,0}(n) > A_{15,0,4,k}(n) \geq A_{15,0,4,l}(n),
\]

and consequently (45).
It should also be mentioned that \( \Re(S_{15,0}(i, n)) > 0 \) for almost all \( n \) is also sufficient to prove (45). By (6) we have

\[
A_{15,0;4,0}(n) = \frac{1}{4} \sum_{l=0}^{3} i^{-lm} S_{15,0}(i^l, n).
\]

Hence \( \Re(S_{15,0}(i, n)) > 0 \) implies \( A_{15,0;4,0}(n) > A_{15,0;4,2}(n) \). Furthermore, by Theorem 1 \( S_{15,0}(-1, n) \gg n \frac{\log 3}{\log 4} \). Consequently we also have

\[
A_{15,0;4,0}(n) > \max(A_{15,0;4,1}(n), A_{15,0;4,3}(n))
\]

for sufficiently large \( n \).

**Proof of Proposition 7.** The computation of the eigenvalues of \( M(i) \) can be worked out explicitly:

\[
\lambda_1 = \lambda_{\{1,2,4,8\}} = (1 + i\zeta_5^1)(1 + i\zeta_5^2)(1 + i\zeta_5^3)(1 + i\zeta_5^4) = 2 - (\zeta_5^1 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4) + \zeta_5^1 + \zeta_5^2 - (\zeta_5^5 + \zeta_5^6 + \zeta_5^7 + \zeta_5^8) + i \sum_{j=1}^{14} \left( \frac{j}{15} \right) \zeta_5^j = 4 - \sqrt{15},
\]

where \( \left( \frac{\cdot}{15} \right) \) denotes the Jacobi-Kronecker symbol. The other eigenvalues are given by \( \lambda_2 = \lambda_{\{14,7,11,13\}} = 4 + \sqrt{15}, \lambda_3 = \lambda_{\{3,6,9,12\}} = 1, \lambda_4 = \lambda_{\{5,10\}} = -1, \) and by \( \lambda_5 = \lambda_{\{0\}} = -4 \). Hence the largest eigenvalue is \( \lambda_2 \). Now we can proceed as in the proof of Proposition 6. We just reproduce a table (Tables 7.1 and 7.2) for \( c'_{jk} \) and \( d'_{jk} \) defined by

\[
S_{15,j}^{(m)}(i, 2^k) = \frac{\lambda_2^k}{30}(c'_{jk} + id'_{jk}).
\]

\[\square\]

**Table 7.1**

<table>
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<tr>
<th>( j )</th>
<th>( c'_{j0} )</th>
<th>( d'_{j0} )</th>
<th>( c'_{j1} )</th>
<th>( d'_{j1} )</th>
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<tr>
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<td>1</td>
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<td>\sqrt{15}</td>
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<td>1 + \sqrt{15}</td>
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<td>1</td>
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<td>-2 + \sqrt{15}</td>
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</tr>
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<td>1 + \sqrt{15}</td>
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<td>-4 - \sqrt{15}</td>
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<td>0</td>
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<td>-\sqrt{15}</td>
<td>1</td>
<td>-2 - \sqrt{15}</td>
</tr>
<tr>
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<td>1</td>
<td>-\sqrt{15}</td>
<td>1 + \sqrt{15}</td>
<td>1 - \sqrt{15}</td>
</tr>
</tbody>
</table>
Finally, let us consider the cases \( r = 5 \) and \( r = 6 \). In the case \( r = 5 \) it suffices to show that
\[
\Re(S_{31,0}(\zeta_5, n)) > \Re(\zeta_5^{-m}S_{31,0}(\zeta_5, n)) \quad (m \neq 0 \mod 5),
\]
which can be checked by considering the largest eigenvalue \( \lambda_1(\zeta_5) \) and similar calculations as above. (Again a simple computer program assists us.)

The case \( r = 6 \) is interesting because (45) can be deduced from Theorems 1 and 7. By (6)
\[
A_{63,0,6,m}(n) = \frac{1}{6} \sum_{i=0}^{5} c_{6^{-i}m} S_{63,0}(\zeta_6^i, n).
\]
By Theorem 7, \( \arg(S_{63,0}(\zeta_6^2, n)) \in (-\pi/3, \pi/3) \). Thus, for sufficiently large \( n \),
\[
A_{63,0,6,0}(n) > \max(A_{63,0,6,2}(n), A_{63,0,6,4}(n)),
\]
since the largest eigenvalue of \( M(\zeta_6^2) \) is larger than the largest eigenvalue of \( M(\zeta_6) \).
Furthermore, by Theorem 1 \( S_{63,0}(-1, n) \gg n^{\log 4} \), and consequently
\[
A_{63,0,6,0}(n) > \max(A_{63,0,6,1}(n), A_{63,0,6,3}(n), A_{63,0,6,5}(n))
\]
for sufficiently large \( n \).

Therefore we have provided a complete answer for the case \( q = 2^r - 1 \) with respect to (N1). However, the situation is much more delicate when we consider (N2) instead of (N1).

**Theorem 8.** We have
\[
R_{127,0;7,0}(n) > 0 \quad \text{for almost all } n \geq 0.
\]
This means that (N2) holds for \( r = 7 \) although (N1) fails. (We do not give a detailed proof. We only want to mention that it suffices to show that \( \Re(S_{127,0}(\zeta_7, n)) > 0 \).) Therefore it might be possible that (N2) holds for all \( r \geq 2 \). But again the answer is negative.
Theorem 9. There are infinitely many \( r \geq 2 \) such that
\[
R_{2^r-1, 0, 0}(n) < 0 \quad \text{for infinitely many } n \geq 0.
\]

Sketch of the Proof. It is sufficient to show that there are infinitely many \( r \geq 2 \) such that the eigenvalue \( \lambda_l(\zeta^m_r), 0 < l < 2^r - 1, 0 < m < r \), of largest modulus \( |\lambda_m| \) is negative. In what follows we will indicate that if there exists a positive integer \( m_r \) such that \( |\sqrt{r}/\pi + C - m_r| < 1/4 \) (where \( C \) a real constant and \( r \geq r_0 \) is sufficiently large), then \( \lambda_1(\zeta^{-m_r}_r) \) is the eigenvalue of largest modulus
\[
|\lambda_m| = |\lambda_1(\zeta^{-m_r}_r)| \sim \frac{2^re^{-1/2}}{2\pi\sqrt{r}}.
\]
Since
\[
\arg(\lambda_1(\zeta^{-m_r}_r)) = \sum_{j=0}^{r-1} \pi \left( \frac{2^j}{2^r - 1} - \frac{m_r}{r} \right) = \pi(1 - m_r),
\]
we have \( \text{sgn}(\lambda_1(\zeta^{-m_r}_r)) < 0 \) if \( m_r \) is even. Obviously, this case occurs infinitely many times.

We use the fact that the digit expansion of \( x_0 = l/(2^r - 1) = \sum_{k \geq 1} c_k2^{-k} \) is periodic, i.e. \( c_{k+r} = c_k \), and that \( x_j = \sum_{k \geq 1} c_{k+j}2^{-k} \) satisfies \( \zeta^j_{2^r-1} = e^{2\pi i x_j} \).

(Compare with the proof of Proposition 5.) By considering several subcases it turns out that if \( l \) is unbounded, then
\[
|\lambda_l(\zeta^m_r)| = o(2^r r^{-1/2}).
\]
Conversely, if \( l \) is bounded, then
\[
\max_m |\lambda_l(\zeta^m_r)| \sim \frac{2^r-1e^{-1/2}}{l\pi\sqrt{r}},
\]
in which the maximum is attained for \( |m| \sim \sqrt{r}/\pi \). Therefore \( l = 1, |m| \sim \sqrt{r}/\pi \) is the only relevant case. (Since \( |1 + \zeta^m_q\zeta^2_q| < |1 + \zeta^{-m}_q\zeta^2_q| \) \( (m > 0) \), we may also assume that \( m \sim -\sqrt{r}/\pi \).) A more detailed analysis shows that the maximum value of \( |\lambda_l(\zeta^m_r)| \) is attained for
\[
m = -\frac{\sqrt{r}}{\pi} - C + \mathcal{O}(r^{-1/2}),
\]
in which \( m \) is assumed to be a continuous real parameter and \( C \) is a computable constant. Furthermore, if \( \sqrt{r}/\pi + C \) is near to an integer \( m_r \), e.g. \( |\sqrt{r}/\pi + C - m_r| < 1/4 \), and if \( r \) is sufficiently large, then \( |\lambda_m| = |\lambda_1(\zeta^{-m_r}_r)| \).

We finish this section on higher parities with an analogue to Theorem 2.

Theorem 10. For any \( r > 1 \) there exists a constant \( C_r > 0 \) such that for any \( t \geq 1 \) primes \( q \in \mathbb{P}_t \) satisfying (N1) or (N2) are bounded by
\[
q \leq C_r t^4 \log^4 t.
\]
For the proof we can use a similar procedure as above. Instead of (40) we need the following formula.
Proposition 8. Suppose that \( p \) is an odd prime and \( s = \text{ord}_p(2) \). If \( y \in \mathbb{C} \) has modulus \( |y| = 1 \), then for any \( l \in L \)

\[
\Re \left( \sum_{l \in L} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} \right) = \frac{s}{2} - \frac{1}{4} \sum_{l \in L} \frac{1}{1 - \Re \zeta_p^{2l}} \left( 1 + 2 \frac{\cos(\arg y/2)}{\cos((\arg y)/2 + \arg \zeta_p^l)} \right).
\]

(50)

Proof. From

\[
s = \sum_{l \in L} \frac{1 + y \zeta_p^l + y^2 \zeta_p^{2l} + y^2 \zeta_p^{3l}}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} = \sum_{l \in L} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} + \sum_{l \in L} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} + \sum_{l \in L} \frac{y \zeta_p^l(1 + \zeta_p^l)}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})}
\]

we obtain

\[
\Re \left( \sum_{l \in L} \frac{1}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} \right) = \frac{s}{2} - \frac{1}{2} \sum_{l \in L} \frac{y \zeta_p^l(1 + \zeta_p^l)}{(1 + y \zeta_p^l)(1 + y \zeta_p^{2l})} = \frac{s}{2} - \frac{1}{2} S(y),
\]

where the mapping \( y \mapsto S(y), y \neq -\zeta_p^{-l} \), is continuous. In particular,

\[
S(-1) = -\sum_{l \in L} \frac{\zeta_p^l}{(1 - \zeta_p^l)^2} = \frac{1}{2} \sum_{l \in L} \frac{1}{1 - \Re \zeta_p^l}.
\]

By using a partial fraction expansion it follows that \( S(y), y \neq 1 \), can be represented by

\[
S(y) = \frac{1 - y}{1 + y} \sum_{l \in L} \frac{1}{(1 + y \zeta_p^l)} - \frac{1 - y}{1 + y} \sum_{l \in L} \frac{1}{1 + y \zeta_p^{2l}} + \frac{2y}{1 + y} \sum_{l \in L} \frac{\zeta_p^l}{1 + y \zeta_p^{2l}}
\]

\[
= \frac{2y}{1 + y} \sum_{l \in L} \frac{1}{\zeta_p^{-l} + y \zeta_p^l}.
\]

Since \( S(-1) \) is finite, it follows that

\[
\sum_{l \in L} \frac{1}{\zeta_p^{-l} - \zeta_p^l} = 0
\]
and consequently

\[
S(y) = \frac{2y}{1 + y} \sum_{l \in \mathcal{I}} \left( \frac{1}{\zeta_p^l + y\zeta_p^l} - \frac{1}{\zeta_p^l - y\zeta_p^l} \right)
\]

\[
= -\Re \left( \sum_{l \in \mathcal{I}} \frac{y\zeta_p^l}{(\zeta_p^l + y\zeta_p^l)(\zeta_p^l - \zeta_p^l)} \right)
\]

\[
= -\sum_{l \in \mathcal{I}} \frac{\zeta_p^l - y\zeta_p^l}{(\zeta_p^l + y\zeta_p^l)(\zeta_p^l - \zeta_p^l)}
\]

\[
= S(-1) - \sum_{l \in \mathcal{I}} \left( \frac{\zeta_p^l - y\zeta_p^l}{(\zeta_p^l + y\zeta_p^l)(\zeta_p^l - \zeta_p^l)} - \frac{\zeta_p^l + y\zeta_p^l}{(\zeta_p^l - \zeta_p^l)^2} \right)
\]

\[
= S(-1) - \sum_{l \in \mathcal{I}} \frac{1 + y}{(\zeta_p^l - \zeta_p^l)^2} \left( \frac{1}{\zeta_p^l + y\zeta_p^l} \right)
\]

\[
= \frac{1}{2} \sum_{l \in \mathcal{I}} \frac{1}{1 - \Re \zeta_p^l} + \sum_{l \in \mathcal{I}} \frac{1}{1 - \Re \zeta_p^l} \zeta_p^l + y\zeta_p^l,
\]

which proves (50).

The essential difference between the proofs of Theorem 2 and Theorem 10 is that you have to take into account the sign of

\[
\cos\left(\frac{m\pi}{r} + \arg \zeta_p^l\right).
\]

Let \(\mathcal{I}^\circ\) denote the set of those \(l \in \mathcal{I}\) such that this sign is negative. Then it is an easy exercise to show that

\[
\sum_{l \in \mathcal{I}^\circ} \frac{1}{1 - \Re \zeta_p^l} \cos\left(\frac{m\pi}{r} \right. = O_r(p \log p).
\]

You only have to verify that \(1 - \Re \zeta_p^l > c_r\) for \(l \in \mathcal{I}^\circ\) and that \(\arg \zeta_p^l\) is different for different \(l \in \mathcal{I}\). Hence, if \(p > C_r(t \log p)^4\) (for a sufficiently large constant \(C_r > 0\)), then

\[
\frac{1}{8\pi^2 t^2 \log^2 p} > \frac{p}{2t} + O_r(p \log p),
\]

which implies that \(\Re(S_{p,0}(\zeta_p^m, 2^{2\alpha-2})) < 0\) for sufficiently large \(a\).

**References**


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