RANDOM INTERSECTIONS OF THICK CANTOR SETS

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Abstract. Let $C_1$, $C_2$ be Cantor sets embedded in the real line, and let $\tau_1$, $\tau_2$ be their respective thicknesses. If $\tau_1 \tau_2 > 1$, then it is well known that the difference set $C_1 - C_2$ is a disjoint union of closed intervals. B. Williams showed that for some $t \in \text{int}(C_1 - C_2)$, it may be that $C_1 \cap (C_2 + t)$ is as small as a single point. However, the author previously showed that generically, the other extreme is true; $C_1 \cap (C_2 + t)$ contains a Cantor set for almost every $t$. This paper shows that small intersections of thick Cantor sets are also rare in the sense of Lebesgue measure; if $\tau_1 \tau_2 > 1$, then $C_1 \cap (C_2 + t)$ contains a Cantor set for almost all $t$ in $C_1 - C_2$.

If $C_1$, $C_2$ are Cantor sets embedded in the real line, then their difference set is

$$C_1 - C_2 \equiv \{ x - y \mid x \in C_1 \text{ and } y \in C_2 \}.$$ 

The difference set has another, more dynamical, definition as

$$C_1 - C_2 = \{ t \mid C_1 \cap (C_2 + t) \neq \emptyset \},$$

where $C_2 + t = \{ x + t \mid x \in C_2 \}$ is the translation of $C_2$ by the amount $t$. There are two reasons to say that the second definition is dynamical. First, it gives a dynamic way of visualizing the difference set; if we think of $C_1$ as being fixed in the real line and think of $C_2$ as sliding across $C_1$ with unit speed, then $C_1 - C_2$ can be thought of as giving those times when the moving copy of $C_2$ intersects $C_1$. Second, it has become a tool for studying dynamical systems. One Cantor set sliding over another one comes up in various studies of homoclinic phenomena, such as infinitely many sinks, [N1], antimonotonicity, [KKY], and $\Omega$-explosions, [PT1]; for an elementary explanation of this, see [GH, pp. 331–342] or [R, pp. 110–115]. This has led to a number of problems and results of the following form: Given conditions on the sizes of $C_1$ and $C_2$, what can be said of the sizes of either $C_1 - C_2$, or $C_1 \cap (C_2 + t)$ for $t \in C_1 - C_2$. A wide variety of notions of size have been used, such as cardinality, topology, measure, Hausdorff dimension, limit capacity, and thickness; see for example [HKY], [KP], [MO], [PT2], [PS], [S], and [W]. In this paper we will be concerned with the thickness of $C_1$ and $C_2$, and our conclusion will be about the topology of $C_1 \cap (C_2 + t)$ for almost every $t \in C_1 - C_2$.

It is not hard to show that the difference set of two Cantor sets $C_1$, $C_2$ is always a compact, perfect set. So the simplest structure that we can expect $C_1 - C_2$ to have is the disjoint union of closed intervals. There is a condition we can put on $C_1$ and $C_2$ that will guarantee this; if $\tau_1$, $\tau_2$ are the thicknesses of $C_1$, $C_2$, and...
if \( \tau_1 \tau_2 > 1 \), then \( C_1 - C_2 \) is a disjoint union of closed intervals. What about the size of \( C_1 \cap (C_2 + t) \) for \( t \in C_1 - C_2 \)? In [W] it was shown that even when \( \tau_1 \tau_2 > 1 \), it is possible that \( C_1 \cap (C_2 + t) \) can be as small as a single point for some \( t \in \text{int}(C_1 - C_2) \). But in [K1, Chapter 3], it was shown that this is exceptional, at least in the sense of category, and that in fact the other extreme is the case; if \( \tau_1 \tau_2 > 1 \), then \( C_1 \cap (C_2 + t) \) contains a Cantor set for all \( t \) in a generic subset of \( C_1 - C_2 \). Our main result in this paper is to prove a similar result for Lebesgue measure.

**Theorem 1.** Let \( C_1, C_2 \) be Cantor sets embedded in the real line and let \( \tau_1, \tau_2 \) be their respective thicknesses. If \( \tau_1 \tau_2 > 1 \), then \( C_1 \cap (C_2 + t) \) contains a Cantor set for almost all \( t \in C_1 - C_2 \).

It is worth mentioning here that, in [W], [HKY], and [K1], conditions are given on \( \tau_1 \) and \( \tau_2 \) so that \( C_1 \cap (C_2 + t) \) contains a Cantor set for all \( t \in \text{int}(C_1 - C_2) \).

Before proving Theorem 1, let us look at the definition of thickness and see how it is used. If \( C \) is a Cantor set embedded in the real line, then the complement of \( C \) is a disjoint union of open intervals. We call the components of the complement of \( C \) the gaps of \( C \). Let \( \{U_n\}_{n=1}^\infty \) be an ordering of the bounded gaps of \( C \) by decreasing length, so \( |U_{n+1}| \leq |U_n| \), where \( |U| \) denotes the Lebesgue measure of \( U \). Let \( I_1 \) denote the smallest closed interval containing \( C \). For \( n > 1 \), let \( I_n = I_1 \setminus (\bigcup_{i=1}^{n-1} U_i) \). Note that \( I_n \) has \( n \) components. Let \( A_n \) denote the component of \( I_n \) that contains \( U_n \). Let \( L_n \) and \( R_n \) denote the left and right components of \( A_n \setminus U_n \). Then the thickness \( \tau \) of \( C \) is defined by

\[
\tau(C) \equiv \inf_n \left\{ \min \left\{ \frac{|L_n|}{|U_n|}, \frac{|R_n|}{|U_n|} \right\} \right\}.
\]

This definition of thickness is from [W]; in both [W] and [K1, pp. 15–16] it is shown that (i) this definition does not depend on the choice of an ordering for the gaps of \( C \) in the case when \( |U_{n+1}| = |U_n| \) for some \( n \), and (ii) this definition is equivalent to the usual definition of thickness (e.g., [N2, pp. 99–100]).

Thickness gives us a way of measuring the size of Cantor sets embedded in the real line. The larger the thickness, the “bigger” the Cantor set. So for example, as a consequence of the next lemma the condition \( \tau_1 \tau_2 > 1 \) implies that \( C_1 \) and \( C_2 \) are big enough that their difference set is large in the sense that \( C_1 - C_2 \) is a disjoint union of closed intervals.

**Lemma 2.** Let \( C_1, C_2 \) be Cantor sets embedded in the real line, with thicknesses \( \tau_1, \tau_2 \). If \( \tau_1 \tau_2 > 1 \) and neither \( C_1 \) nor \( C_2 \) is contained in a gap of the other, then \( C_1 \cap C_2 \neq \emptyset \).

This lemma is often referred to as the Gap Lemma, [PT2, p. 63]. There is a slightly stronger version of the Gap Lemma that uses the notion of an overlapped point in the intersection of two Cantor sets. This is a simple, but useful, definition from [K1, pp. 17–18]. Suppose that \( x \in C_1 \cap C_2 \). Let \( \{U_n\}_{n=1}^\infty \) and \( \{V_n\}_{n=1}^\infty \) denote the bounded gaps, and let \( I_1 \) and \( J_1 \) denote the convex hulls, of \( C_1 \) and \( C_2 \). Let \( A_n \) and \( B_n \) denote the components of \( I_1 \setminus (\bigcup_{i=1}^{n-1} U_i) \) and \( J_1 \setminus (\bigcup_{i=1}^{n-1} V_i) \), respectively, that contain \( x \). Then \( x \) is an overlapped point from \( C_1 \cap C_2 \) if \( A_n \cap B_n \) has nonempty interior for all \( n \). To put this another way, if \( x \in C_1 \cap C_2 \), then \( x \) is not an overlapped point if and only if there is an \( n \) such that \( A_n \cap B_n = \{x\} \), i.e., \( A_n \) and \( B_n \) look
like the following picture.

\[ \begin{array}{c}
A_n \\
\hline \\
x \\
\hline \\
B_n
\end{array} \]

Now we can state the slightly stronger version of the Gap Lemma.

**Lemma 3.** Let $C_1$, $C_2$ be Cantor sets embedded in the real line, with thicknesses $\tau_1$, $\tau_2$. If $\tau_1 \tau_2 > 1$ and neither $C_1$ nor $C_2$ is contained in the closure of a gap of the other, then $C_1 \cap C_2$ contains an overlapped point.

This version of the Gap Lemma implies that $C_1 - C_2$ is a disjoint union of closed intervals, and that $C_1 \cap (C_2 + t)$ contains an overlapped point for all $t \in \text{int}(C_1 - C_2)$. It is not hard to see that $C_1 \cap (C_2 + t)$ contains only non-overlapped points when $t$ is a boundary point of $C_1 - C_2$. We say that Cantor sets $C_1$ and $C_2$ are interweaved if neither $C_1$ nor $C_2$ is contained in the closure of a gap of the other.

Here is a sketch of the proof of the Gap Lemma. Let $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ denote the bounded gaps, and let $I_1$, $J_1$ denote the convex hulls, of $C_1$ and $C_2$, respectively. The key idea is that, since $\tau_1 \tau_2 > 1$, we cannot have the following picture of $I_1 \setminus U_1$ and $J_1 \setminus V_1$.

\[ \begin{array}{c}
L_1 \\
\hline \\
U_1 \\
\hline \\
R_1
\end{array} \]

So it must be that the intersection of $I_1 \setminus U_1$ and $J_1 \setminus V_1$ has nonempty interior. A careful induction argument, based on the above idea, gives that the intersection of $I_1 \setminus (\bigcup_{i=1}^n U_i)$ and $J_1 \setminus (\bigcup_{i=1}^n V_i)$ has nonempty interior for all $n > 1$; this implies that $C_1 \cap C_2$ contains an overlapped point. Notice that if the hypothesis $\tau_1 \tau_2 > 1$ is replaced with $\tau_1 \tau_2 \geq 1$, then we can still conclude that $C_1 \cap C_2 \neq \emptyset$, but we cannot conclude that $C_1 \cap C_2$ contains an overlapped point.

If $C$ is a Cantor set embedded in the real line, then the components of each $I_n$ are called the bridges of $C$; if $B$ is any bridge of $C$, then $B \cap C$ is called a segment of $C$. Clearly any segment of $C$ is also a Cantor set. As a consequence of the definition of thickness, we have the following simple lemma, [K1, p. 16], which will allow us to apply the Gap Lemma “locally.”

**Lemma 4.** Let $C$ be a Cantor set embedded in the real line with thicknesses $\tau$. If $C'$ is any segment of $C$, then the thickness of $C'$ is greater than or equal to $\tau$.

The main result we need in order to prove Theorem 1 is the following lemma, which at first glance seems to be only slightly stronger than the Gap Lemma.

**Lemma 5.** Let $C_1$, $C_2$ be Cantor sets embedded in the real line, with thicknesses $\tau_1$, $\tau_2$. If $\tau_1 \tau_2 > 1$, then $C_1 \cap (C_2 + t)$ contains at least two overlapped points for almost all $t \in C_1 - C_2$.

Before proving this lemma, let us see how it is used to prove Theorem 1.

**Proof of Theorem 1.** Let $\{C_{1,n}\}_{n=1}^\infty$ be any ordering of all the segments of $C_1$, and let $\{C_{2,n}\}_{n=1}^\infty$ be any ordering of all the segments of $C_2$. Then, by Lemmas 4 and
5, for any \(i \) and \(j \) there is a set \( E_{ij} \subset C_{1,i} - C_{2,j} \) of measure zero, such that \( C_{1,i} \cap (C_{2,j} + t) \) contains at least two overlapped points for all \( t \in (C_{1,i} - C_{2,j}) \backslash E_{ij} \).

Let \( E = \bigcup_{i,j} E_{ij} \). So then \( E \) has measure zero, and \( E \subset C_1 - C_2 \).

Using terminology from [K1, p. 20], if \( t \in (C_1 - C_2) \backslash E \), then \( C_1 \cap (C_2 + t) \) has no isolated overlapped points. An overlapped point is isolated if there is a neighborhood of it which contains no other overlapped points. In [K1, pp. 20–21] it is shown that if the intersection of two Cantor sets does not contain isolated overlapped points, then the intersection must contain a Cantor set. But here we will sketch a proof that if \( t \in (C_1 - C_2) \backslash E \), then \( C_1 \cap (C_2 + t) \) contains a Cantor set.

Let \( t \in (C_1 - C_2) \backslash E \). Then \( C_1 \cap (C_2 + t) \) contains at least two overlapped points, so let \( x, y \) be distinct overlapped points in \( C_1 \cap (C_2 + t) \). Choose integers \( i_1, j_1 \) large enough so that \( x \) and \( y \) are in distinct components of \( I_{i_1} \) and \( J_{j_1} + t \).

Let \( K_1 = K_{1,1} \cup K_{1,2} \) denote the two components of \( I_{i_1} \) that contain \( x \) and \( y \), and let \( L_1 = L_{1,1} \cup L_{1,2} \) denote the two components of \( J_{j_1} + t \) that contain \( x \) and \( y \).

Since \( t \in (C_1 - C_2) \backslash E \), \( K_{1,1} \cap L_{1,1} \) contains at least two overlapped points from \( C_1 \cap (C_2 + t) \), and so does \( K_{1,2} \cap L_{1,2} \). Now choose integers \( i_2 > i_1 \) and \( j_2 > j_1 \) large enough so that these four overlapped points are in distinct components of \( I_{i_2} \) and \( J_{j_2} + t \), and let \( K_2 = \bigcup_{\nu=1}^1 K_{2,\nu} \) and \( L_2 = \bigcup_{\nu=1}^1 L_{2,\nu} \) denote these components.

In general, suppose we are given integers \( i_n \) and \( j_n \), and \( 2^n \) distinct components \( K_n = \bigcup_{\nu=1}^{2^n} K_{n,\nu} \) from \( I_{i_n} \), and \( 2^n \) distinct components \( L_n = \bigcup_{\nu=1}^{2^n} L_{n,\nu} \) from \( J_{j_n} + t \), such that each of \( K_{n,\nu} \cap L_{n,\nu} \) contains an overlapped point from \( C_1 \cap (C_2 + t) \). Then, since \( t \in (C_1 - C_2) \backslash E \), each of \( K_{n,\nu} \cap L_{n,\nu} \) actually contains two overlapped points from \( C_1 \cap (C_2 + t) \). So we can choose integers \( i_{n+1} > i_n \) and \( j_{n+1} > j_n \) large enough so that these \( 2^{n+1} \) overlapped points are contained in \( 2^{n+1} \) distinct components \( K_{n+1} = \bigcup_{\nu=1}^{2^{n+1}} K_{n+1,\nu} \) from \( I_{i_{n+1}} \), and \( L_{n+1} = \bigcup_{\nu=1}^{2^{n+1}} L_{n+1,\nu} \) from \( J_{j_{n+1}} + t \). So for every \( n \geq 1 \), the set \( K_n \cap L_n \) has \( 2^n \) components, \( (K_n \cap L_n) \subset (I_{i_n} \cap J_{j_n}) \), and \( (K_{n+1} \cap L_{n+1}) \subset (K_n \cap L_n) \). Finally, the set \( \bigcap_{n=1}^\infty (K_n \cap L_n) \) is a Cantor set contained in \( C_1 \cap (C_2 + t) \).

Now we shall begin working on the proof of Lemma 5. For Cantor sets \( C_1, C_2 \) with thicknesses \( \tau_1, \tau_2 \), and \( \tau_1 \tau_2 > 1 \), let

\[
O \equiv \{ t \mid C_1 \cap (C_2 + t) \text{ contains exactly one overlapped point} \},
\]

and

\[
T \equiv \{ t \mid C_1 \cap (C_2 + t) \text{ contains two or more overlapped points} \}.
\]

Notice that \( O \cap T = \emptyset \), and \( O \cup T = C_1 - C_2 \) up to a set of measure zero (in fact \( (C_1 - C_2) \backslash (O \cup T) \) is a countable set). To prove Lemma 5, we need to show that \( O \) has measure zero. To do this, it helps to make a distinction between three kinds of overlapped points. Suppose that \( x \in C_1 \cap C_2 \) is an overlapped point. Let \( A_n \) and \( B_n \) denote the components of \( I_1 \setminus (\bigcup_{i=1}^{n-1} U_i) \) and \( J_1 \setminus (\bigcup_{i=1}^{n-1} V_i) \), that contain \( x \) (where \( \{U_n\}_n \) and \( \{V_n\}_n \) denote the bounded gaps, and \( I_1, J_1 \) denote the convex hulls, of \( C_1, C_2 \)). Then \( x \) is an overlapped point of the first, second, or third kind, respectively, if one of the following three conditions holds, respectively,

1. \( x \in \text{int}(A_n) \) and \( x \in \text{int}(B_n) \) for all \( n \),
2. \( x \in \text{int}(A_n) \) for all \( n \) and there is an \( n \) such that \( x \) is an endpoint of \( B_n \), or \( x \in \text{int}(B_n) \) for all \( n \) and there is an \( n \) such that \( x \) is an endpoint of \( A_n \),
3. there is an \( n \) such that \( x \) is an endpoint of both \( A_n \) and \( B_n \), and \( A_n \cap B_n \neq \{x\} \).
Figure 1 gives an idea of what the three different kinds of overlapped points look like with respect to the bridges $A_n$ and $B_n$. For specific examples of Cantor sets whose intersection contains a single overlapped point of either the first or third kind, see [K3] and [K4].

If $t \in \mathcal{O}$, then $C_1 \cap (C_2 + t)$ contains only one overlapped point; so we can partition $\mathcal{O}$ into three subsets according to whether $C_1 \cap (C_2 + t)$ contains an overlapped point of the first, second or third kind. There are only a countable number of $t \in C_1 - C_2$ for which $C_1 \cap (C_2 + t)$ can have an overlapped point of the third kind (since there are only a countable number of “endpoints” in $C_1$ or $C_2$), so the part of $\mathcal{O}$ for which $C_1 \cap (C_2 + t)$ contains an overlapped point of the third kind has measure zero. So we need to concentrate on the part of $\mathcal{O}$ for which $C_1 \cap (C_2 + t)$ contains an overlapped point of the first or second kind. Define

$$\mathcal{O}' = \{ t \in \mathcal{O} \mid \text{the overlapped point in } C_1 \cap (C_2 + t) \text{ is not of the third kind} \}.$$  

We need to show that $\mathcal{O}'$ has measure zero. Our proof is by contradiction; we assume that $\mathcal{O}'$ has positive measure, but then show that no point of $\mathcal{O}'$ is a density point. The main part of the proof is the next lemma; it gives a lower bound on the density of $T$ in a neighborhood of any point $t \in \mathcal{O}'$.

We need two more definitions. Let us say that two bounded, closed, intervals are linked if each one contains exactly one boundary point of the other; see [PT2, pp. 63–64]. We say that two Cantor sets embedded in the real line are linked Cantor sets if their convex hulls are linked. Notice that linked Cantor sets are interwoven.

**Lemma 6.** Let $C_1$, $C_2$ be linked Cantor sets, with thicknesses $\tau_1$, $\tau_2$, such that $C_1 \cap C_2$ contains a single overlapped point which is of the first or second kind. If $\tau_1 \tau_2 > 1$, then there is a constant $\epsilon = \epsilon(\tau_1, \tau_2) > 0$, which only depends on $\tau_1$ and $\tau_2$, and a neighborhood $(a, b)$ of 0, such that

$$\frac{|T \cap (a, b)|}{b - a} \geq \epsilon.$$ 

**Proof.** Let $I$, $J$ denote the convex hulls of $C_1$, $C_2$. We are assuming that $I$ and $J$ are linked so they are positioned, relative to each other, something like the following.

```
\begin{center}
\includegraphics[width=0.5\textwidth]{linked_cantors.png}
\end{center}
```

Let $U$ denote the longest gap of $C_1$ which intersects with $J$, and let $V$ denote the longest gap of $C_2$ which intersects with $I$. Now we make the following claim: Either
the closure of $U$ contains an endpoint of $J$, or the closure of $V$ contains an endpoint of $I$. To prove the claim, suppose it is not true; suppose that the closure of $U$ does not contain an endpoint of $J$, and the closure of $V$ does not contain an endpoint of $I$. So $U$ and $V$ might be positioned, relative to each other, something like the following picture.

![Diagram](image)

But then $C_1$ and $C_2$ have (at least) two pairs of linked segments, so by Lemma 4 and the Gap Lemma, $C_1 \cap C_2$ contains at least two overlapped points, which is a contradiction, which proves the claim.

Now we have two cases to consider. The first case is when both the closure of $U$ contains an endpoint of $J$, and the closure of $V$ contains an endpoint of $I$. The second case is when either the closure of $U$ does not contain an endpoint of $J$, or the closure of $V$ does not contain an endpoint of $I$.

**Case 1.** In this case $I \setminus U$ and $J \setminus V$ are positioned, relative to each other, as in the following picture.

![Diagram](image)

Notice that we have two linked bridges, which are denoted by $A$ and $B$ (the intervals $A$ and $B$ cannot have a common endpoint, since it would have to be either an overlapped point of the third kind or a nonoverlapped point, contradicting in either case one of our hypotheses). The two nonlinked bridges are denoted by $L$ and $R$. Let $A = [a_0, a_1]$ and $B = [b_0, b_1]$. Let $c \equiv a_1 - b_1 < 0$, and let $d \equiv a_1 - b_0 > 0$ (notice that $d - c = |B|$). Now $(c, d)$ is a neighborhood of 0, and $(c, d)$ has been chosen so that the segments $A \cap C_1$ and $(B \cap C_2) + t$ are interweaved for all $t \in (c, d)$. To prove this, notice that if $|B| \leq |A|$, then $A$ and $B + t$ are in fact linked for all $t \in (c, d)$. On the other hand, if $|B| > |A|$, then for $t \in (a_0 - b_0, d)$, $A$ and $B + t$ are linked, but for $t \in (c, a_0 - b_0]$, we have $A \subset B + t$. However, when $t \in (c, a_0 - b_0]$, $C_1$ and $C_2 + t$ are linked, so in order that $C_1 \cap (C_2 + t) \neq \emptyset$, it must be that $A \cap C_1$ and $(B \cap C_2) + t$ are interweaved. So for all $t \in (c, d)$, we know that $C_1 \cap (C_2 + t)$ contains at least one overlapped point in $A \cap (B + t)$.

When $t = d$, the segments $A \cap C_1$ and $(B \cap C_2) + d$ are no longer interweaved, but $C_1$ and $C_2 + d$ are, so $C_1 \cap (C_2 + d)$ still contains at least one overlapped point. So when $t = d$, it must be that at least one of the originally nonlinked intervals $R$ and $L + d$ intersects with either $A$ or $B + d$. There are eight possible “geometries” of $I \setminus U$ and $(J \setminus V) + d$, depending on how either $R$ intersects with $B + d$, or $L + d$ intersects with $A$; they are listed in Figure 2. For each configuration, we want to show that there is an neighborhood $(a, b) \subset (c, d)$ of 0 such that the density of $T$ in $(a, b)$ has a lower bound that only depends on $\tau_1$ and $\tau_2$.  


Figure 2. All the subcases of Case 1.

Case 1a. In this case, when \( t = c \), we get the following picture of \( I \setminus U \) and \((J \setminus V) + c\).

\[
\begin{array}{c}
A \quad U \quad R \\
L + c \\
\end{array}
\]

\[
\begin{array}{c}
V + c \\
B + c \\
\end{array}
\]

And when \( t = d \), we get the following picture of \( I \setminus U \) and \((J \setminus V) + d\).

\[
\begin{array}{c}
A \quad U \quad R \\
L + d \\
\end{array}
\]

\[
\begin{array}{c}
V + d \\
B + d \\
\end{array}
\]
For all $t \in (c, d)$, the segments in $A$ and $B + t$ are interwoven. The intervals $R$ and $B + t$ start out nonintersecting, then they are linked, then they become nonlinked but intersecting. By the Gap Lemma, the interwoven segments in $A$, $B + t$, and the linked pair $R$, $B + t$ each guarantee us an overlapped point. However, the segments contained in the nonlinked but still intersecting pair $A$, $B + t$ need not be interwoven. So we restrict $t$ to avoid this situation. Let $a \equiv c$, and let $b \equiv (a_1 + |U| + |R|) - b_1 > 0$. When $t = b$, we get the following picture of $I \setminus U$ and $(J \setminus V) + b$.

![Diagram]

Now we can give a lower bound, for this case, on the density of those $t$ in $(a, b)$ for which $C_1 \cap (C_2 + t)$ contains at least two overlapped points. Notice that $b - a = ((a_1 + |U| + |R|) - b_1) - (a_1 - b_1) = |U| + |R|$. Then

$$\frac{|(a, b) \cap T|}{b - a} \geq \frac{|R|}{|U| + |R|} = \frac{1}{1 + \frac{|R|}{|U|}} \geq \frac{1}{1 + 1/\tau_1} = \frac{\tau_1}{1 + \tau_1}.$$  

**Case 1b.** This case is handled the same as Case 1a, since the interval $L + t$ was not used in that case, and everything else is the same.

**Case 1c.** Again, this case is the same as Case 1a.

**Case 1d.** In this case, let $a \equiv c$ and $b \equiv d$, so $b - a = |B|$. Then

$$\frac{|(a', b) \cap T|}{b - a'} \geq \frac{|B| - |U|}{|B|} = 1 - \frac{1}{(|B|/|V|)(|V|/|U|)}$$

$$\geq 1 - \frac{1}{(|B|/|V|)(|A|/|U|)} \quad \text{(since } |A| \leq |V|)$$

$$\geq 1 - \frac{1}{\tau_1 \tau_2} > 0.$$  

**Case 1e.** This is the most complicated case, and we handle it a bit differently. Let $b \equiv d$, $a' \equiv a_0 - b_0 > 0$, and $a'' \equiv a_1 - b_1 > 0$. Notice that $b - a' = |A|$, $b - a'' = |B|$, and that $A \cap C_1$ and $(B \cap C_2) + t$ are interwoven for all $t$ in either $(a', b)$ or $(a'', b)$.

The density of $T$ in $(a', b)$ is bounded from below by

$$\frac{|(a', b) \cap T|}{b - a'} \geq \frac{|A| - |V|}{|A|} = 1 - \frac{|V|}{|A|},$$

and density of $T$ in $(a'', b)$ is bounded from below by

$$\frac{|(a'', b) \cap T|}{b - a''} \geq \frac{|B| - |U|}{|B|} = 1 - \frac{|U|}{|B|}.$$  

Since $|V|$ can be arbitrarily close to $|A|$, or $|U|$ can be arbitrarily close to $|B|$, we cannot say anything more about these last two estimates other than they are greater than zero. However, since $\tau_1 \tau_2 > 1$, we cannot have both $|V|$ arbitrarily close to $|A|$, and $|U|$ arbitrarily close to $|B|$; as the lengths of $A$ and $V$ get close to each other, the lengths of $U$ and $V$ must be bounded away from each other, and vice versa. So there is a trade off between the density of $T$ in the intervals $(a', b)$.
and \((a'', b);\) as one of the densities decreases, the other one must increase. We will analyze this trade off by introducing a rescaling of the Cantor set \(C_2.\)

To simplify the notation, make a couple of simple changes of variable so that \(d = 0\) and \(a_1 = b_0 = 0.\) Case 1e then looks like the following picture:

\[
\begin{array}{cccc}
  L & V & 0 & B \\
  A & U & R \\
\end{array}
\]

where now \(A = [-|A|, 0], B = [0, |B|], (a', b) = (-|A|, 0),\) and \((a'', b) = (-|B|, 0).\)

We shall apply a linear “rescaling” transformation

\[
T(x) = \lambda x \quad \text{with} \quad \frac{|U|}{|B|} < \lambda < \frac{|A|}{|V|},
\]

to the Cantor set \(C_2,\) and then compute the density of \(T(C_1, \lambda C_2)\) in each of the intervals \((a', b)\) and \((\lambda a'', b).\) (We do not need to consider \(\lambda \geq |A|/|V|\) and \(\lambda \leq |U|/|B|,\) since these are covered by Cases 1a or 1d, and Cases 1g or 1h.)

A lower bound for the density of \(T(C_1, \lambda C_2)\) in the interval \((a', b)\) is given by

\[
\frac{|(a', b) \cap T(C_1, \lambda C_2)|}{b - a'} \geq \frac{|A| - \lambda|V|}{|A|} = 1 - \frac{\lambda|V|}{|A|},
\]

and a lower bound for the density of \(T(C_1, \lambda C_2)\) in the interval \((\lambda a'', b)\) is given by

\[
\frac{|(a'', b) \cap T(C_1, \lambda C_2)|}{b - a''} \geq \frac{\lambda|B| - |U|}{\lambda|B|} = 1 - \frac{1}{\lambda} \frac{|U|}{|B|}.
\]

What we want now is

\[
\min_{|U|/|B| < \lambda < |A|/|V|} \left\{ \max \left\{ 1 - \lambda \frac{|V|}{|A|}, 1 - \frac{1}{\lambda} \frac{|U|}{|B|} \right\} \right\}.
\]

Since \(1 - (\lambda|V|/|A|)\) decreases and \(1 - (|U|/\lambda|B|)\) increases with \(\lambda,\) it suffices to solve for \(\lambda\) so that \(1 - (\lambda|V|/|A|) = 1 - (|U|/\lambda|B|).\) This is solved by

\[
\lambda = \sqrt{|A||U|/|B||V|}.
\]

If we plug this value of \(\lambda\) into our previous lower bounds, we get

\[
\max \left\{ \frac{|(a', b) \cap T|}{b - a'}, \frac{|(a'', b) \cap T|}{b - a''} \right\} \geq 1 - \frac{|V|}{|A|} \sqrt{\frac{|A||U|}{|B||V|}}
\]

\[
= 1 - \left( \frac{|A||B|}{|U||V|} \right)^{-1/2} \geq 1 - \frac{1}{\sqrt{T_1 T_2}} > 0.
\]

This is our lower bound for the density of \(T\) in one of the intervals \((a', b)\) or \((a'', b),\)

though we cannot say which one.

**Case 1f.** This case is the same as Case 1b, if we reverse the roles of \(C_1\) and \(C_2.\)

**Case 1g.** This case is the same as Case 1d, if we reverse the roles of \(C_1\) and \(C_2.\)

**Case 1h.** This case is the same as Case 1a, if we reverse the roles of \(C_1\) and \(C_2.\)
Case 2. Suppose that the closure of \( U \) contains an endpoint of \( J \), but the closure of \( V \) does not contain an endpoint of \( I \). So we might have \( I \setminus U \) and \( J \setminus V \) positioned, relative to each other, as in the following picture.

\[
\begin{array}{c}
A \quad U \quad R_1 \\
B + c \quad V + c \quad R_2 + c
\end{array}
\]

**Case 2a.**

Figure 3. The two subcases for Case 2.

However, in order that \( C_1 \) and \( C_2 \) not have two pairs of linked segments, \( V \) must contain an endpoint of \( U \). Thus, we in fact have \( U \) and \( V \) positioned as in the following picture.

\[
\begin{array}{c}
A \quad U \quad R_1 \\
B + c \quad V + c \quad R_2 + c
\end{array}
\]

Notice that we have two linked bridges, which are denoted by \( A \) and \( B \), and two nonlinked bridges, which are denoted by \( R_1 \) and \( R_2 \). Let \( A = [a_0, a_1] \) and \( B = [b_0, b_1] \). Let \( c \equiv a_0 - b_1 < 0 \), and let \( d \equiv \min\{a_0 - b_0, a_1 - b_1\} > 0 \). Notice that \( d - c = |B| \) if \( |B| \leq |A| \), and \( d - c = |A| \) if \( |A| < |B| \), and in either case \( d - c \leq |A| \).

So \( (c, d) \) is a neighborhood of 0, and \( (c, d) \) has been chosen so that the intervals \( A \) and \( B + t \) are linked for all \( t \in (c, d) \). So for all \( t \in (c, d) \), we know that \( C_1 \cap (C_2 + t) \) contains at least one overlapped point in \( A \cap (B + t) \).

When \( t = c \), \( A \) and \( B + c \) are no longer linked, but \( C_1 \) and \( C_2 + c \) are linked, so \( C_1 \cap (C_2 + c) \) contains at least one overlapped point. So when \( t = c \), it must be that the interval \( R_2 + c \) intersects with \( A \). There are two possible “geometries” of \( I \setminus U \) and \( (J \setminus V) + c \), depending on how \( R_2 + c \) intersects with \( A \); see Figure 3.

**Case 2a.** Let \( a \equiv a_1 - (b_1 + |V| + |R_2|) \), so \( c < a < 0 \), and let \( b \equiv d \). Notice that if \( |A| < |B| \), then \( b - a = (a_0 - b_0) - (a_1 - (b_1 + |V| + |R_2|)) = |V| + |R_2| \), and if \( |B| \leq |A| \), then

\[
\begin{align*}
b - a &= (a_0 - b_0) - (a_1 - (b_1 + |V| + |R_2|)) \\
&= |B| + |V| + |R_2| - |A| \\
&\leq |A| + |V| + |R_2| - |A| \quad \text{(since } |B| \leq |A|) \\
&= |V| + |R_2|.
\end{align*}
\]
In either case, a lower bound on the density of $T$ in $(a, b)$ is given by
\[
\frac{|(a, b) \cap T|}{b - a} \geq \frac{|R_2|}{|V| + |R_2|} = \frac{1}{1 + \frac{1}{|R_2|/|V|}} \geq \frac{1}{1 + 1/\tau_2} = \frac{\tau_2}{1 + \tau_2}.
\]

Case 2b. Notice that, by using both the fact that $|R_2|/|U| \leq 1$ and the definition of thickness, we have
\[
\frac{|A|}{|V|} \geq \frac{|R_2|}{|V|} \geq \frac{\tau_2}{1 + \tau_2}.
\]

Now let $a \equiv c$, and $b \equiv d$, so $b - a = d - c \leq |A|$. Using inequality (1), a lower bound on the density of $T$ in $(a, b)$ is given by
\[
\frac{|(a, b) \cap T|}{b - a} \geq \frac{|A| - |V|}{|A|} = 1 - \frac{|V|}{|A|} \geq 1 - \frac{1}{\tau_1 \tau_2}.
\]

This concludes Case 2b, and also Case 2. Now that we have analyzed all the possible cases, let
\[
\epsilon_1 = \frac{\tau_1}{1 + \tau_1}, \quad \epsilon_2 = \frac{\tau_2}{1 + \tau_2}, \quad \epsilon_3 = 1 - \frac{1}{\tau_1 \tau_2}, \quad \epsilon_4 = 1 - \frac{1}{\sqrt{\tau_1 \tau_2}},
\]
and let $\epsilon \equiv \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} > 0$. Then $\epsilon$ only depends on $\tau_1$ and $\tau_2$.

**Lemma 7.** Let $C_1$, $C_2$ be linked Cantor sets, with thicknesses $\tau_1$, $\tau_2$, such that $C_1 \cap C_2$ contains a single overlapped point which is of the first or second kind. If $\tau_1 \tau_2 > 1$, then there is a constant $\epsilon = \epsilon(\tau_1, \tau_2) > 0$, which only depends on $\tau_1$ and $\tau_2$, and neighborhoods $(a_n, b_n)$ of 0 with $\lim_{n \to \infty} b_n - a_n = 0$, such that for all $n$
\[
\frac{|T \cap (a_n, b_n)|}{b_n - a_n} \geq \epsilon.
\]

**Proof.** In both Cases 1 and 2 of Lemma 6, after we removed the open intervals $U$ and $V$ from the closed intervals $I$ and $J$, we were left with a pair of linked bridges which were denoted by $A$ and $B$. The segments of $C_1$ and $C_2$ contained in $A$ and $B$ satisfy the hypotheses of Lemma 6. So we can apply Lemma 6 to these new Cantor sets, and get new linked bridges $A_2$, $B_2$, and another open neighborhood $(a_2, b_2)$ of zero where the density of $T$ is bounded from below by $\epsilon$.

By induction, given linked Cantor sets $C_1 \cap A_n$ and $C_2 \cap B_n$, we can apply Lemma 6 to get linked bridges $A_{n+1}$ and $B_{n+1}$, and an open neighborhood $(a_{n+1}, b_{n+1})$ of zero where the density of $T$ is bounded from below by $\epsilon$. Since $\tau_1$, $\tau_2$ are lower bounds on the thicknesses of $C_1 \cap A_n$, $C_2 \cap B_n$, and $\epsilon$ depends only on $\tau_1$ and $\tau_2$, the same value of $\epsilon$ works for all $n$.

To show that $\lim_{n \to \infty} b_n - a_n = 0$, it suffices to show that $|A_n| \to 0$ and $|B_n| \to 0$ as $n \to \infty$, since $(a_n, b_n) \subset A_n - B_n$ (recall that $A_n$ and $B_n + t$ are interwoven for all $t \in (a_n, b_n)$). But $\{A_n \}_{n=1}^{\infty}$ is a sequence of bridges from $C_1$ that each contain the overlapped point $x$, so it must be that $|A_n| \to 0$, since $C_1$ is a Cantor set; similarly for the $B_n$.

Now we can give the proof of Lemma 5.

**Proof of Lemma 5.** We need to show that $\mathcal{O}'$ has measure zero. Suppose that it has positive measure. By the Lebesgue density theorem, [WZ, pp. 107–109],
\[
\lim_{n \to \infty} \frac{|\mathcal{O}' \cap (a_n, b_n)|}{b_n - a_n} = 1,
\]

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for almost all $t$ in $O'$, where $\{(a_n, b_n)\}_{n=1}^{\infty}$ is any sequence of intervals that shrink regularly to $t$. (The intervals $(a_n, b_n)$ shrink regularly to $t$ if (i) $\lim_{n \to \infty} b_n - a_n = 0$, (ii) if $D_n$ is the smallest disk centered at $t$ containing $(a_n, b_n)$, then there is a constant $k$ independent of $n$ such that $|D_n| \leq k(b_n - a_n)$.)

Suppose that $t_0 \in O'$ is a density point. By a simple change of variable, we can assume that $t_0 = 0$. Let $I, J$ denote the smallest closed interval containing $C_1, C_2$.

**Claim.** *Without loss of generality, we can assume that $I$ and $J$ are linked.*

**Proof.** To prove this claim, first notice that $I$ and $J$ cannot have a common endpoint; for if they did, the common endpoint would have to be either an overlapped point of the third kind, or a nonoverlapped point, which contradicts our assumption that $0 \in O'$. Since $I \cap J \neq \emptyset$ and $I, J$ cannot have a common endpoint, it must be that either they are linked, in which case we are done, or one of $I$ or $J$ is contained in the interior of the other. Suppose that $J$ is contained in the interior of $I$, so $I$ and $J$ are positioned, relative to each other, as in the following picture.

![Diagram](image)

Let $U$ be the longest gap of $C_1$ that intersects with $J$. So $I \setminus U$ and $J$ might be positioned, relative to each other, as in the following picture.

![Diagram](image)

But in order that $C_1$ and $C_2$ not have two linked segments, and hence two overlapped points in $C_1 \cap C_2$, it must be that $U$ contains an endpoint of $J$, i.e., $I \setminus U$ and $J$ are in fact positioned, relative to each other, as in the following picture.

![Diagram](image)

The interval to the left of $U$, which is denoted by $A$, is linked with $J$. The segment $C_1 \cap A$ has thickness at least $\tau_1$, and $(C_1 \cap A) \cap C_2$ contains a single overlapped point, which is still of the first or second kind. So, without loss of generality, we can replace $C_1$ with $C_1 \cap A$, and also $I$ with $A$, and then $I$ and $J$ are linked.

So $C_1$ and $C_2$ are linked Cantor sets such that $0 \in O'$, and their thicknesses satisfy $\tau_1 \tau_2 > 1$. By Lemma 7, we have neighborhoods $(a_n, b_n)$ of $0$ with $\lim_{n \to \infty} b_n - a_n = 0$, such that for all $n$

$$\frac{|T \cap (a_n, b_n)|}{b_n - a_n} \geq \epsilon,$$

for some constant $\epsilon > 0$ which is independent of $n$. Since $0 \in (a_n, b_n)$ for all $n$, the intervals $(a_n, b_n)$ shrink regularly to $0$ (let $k = 2$ in the definition of shrink
regularly). Since 0 is a density point of $O'$, we can choose an $n$ so that 
$$\left|O' \cap (a_n, b_n)\right| > 1 - \epsilon.$$ 
Since $T$ and $O'$ are disjoint, these last two inequalities contradict each other, so it must be that $O'$ has measure zero.

For some intuition on what $O'$ can look like see [K2], where the structure of $O'$ is examined in detail using symbolic dynamics for the special case where $C_1 = C_2$ is a middle-$\alpha$ Cantor set with $\alpha \leq 1/3$.

We end this paper with a couple of conjectures. Since the proofs of both the Gap Lemma and Theorem 1 are essentially renormalization arguments, and since renormalization often leads to critical phenomena, we can conjecture that the condition $1 \tau_2 = 1$ on thicknesses is some kind of critical boundary for difference sets of Cantor sets. Since $1 \tau_2 > 1$ implies both that $C_1 - C_2$ is a union of intervals and that $C_1 \cap (C_2 + t)$ contains a Cantor set for almost all $t \in C_1 - C_2$, we can conjecture the following phenomena for the condition $1 \tau_2 < 1$.

**Conjecture 1.** For any positive real numbers $\tau_1$ and $\tau_2$ with $\tau_1\tau_2 < 1$, there exist Cantor sets $C_1$, $C_2$ with thicknesses $\tau_1$, $\tau_2$ such that $C_1 - C_2$ does not contain any intervals (and hence it is a Cantor set).

**Conjecture 2.** For any positive real numbers $\tau_1$ and $\tau_2$ with $\tau_1\tau_2 < 1$, there exist Cantor sets $C_1$, $C_2$ with thicknesses $\tau_1$, $\tau_2$ such that $C_1 \cap (C_2 + t)$ does not contain a Cantor set for almost all real numbers $t$.

Notice that neither of these conjectures implies the other.

For any $\alpha \in (0, 1)$, let $C_\alpha$ denote the middle-$\alpha$ Cantor set in the interval $[0, 1]$. Since a middle-$\alpha$ Cantor set will minimize Hausdorff dimension among all Cantor sets of a given thickness ([PT2, pp. 77–78] and [K1, p. 23]) it would seem reasonable to expect them to be good candidates for solving the above conjectures. So we can make the following more specific conjectures.

**Conjecture 1'.** For any real numbers $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + 3\alpha_1\alpha_2 + \alpha_2 > 1$, there exists a real number $\lambda > 0$ such that $C_{\alpha_1} - \lambda C_{\alpha_2}$ does not contain any intervals.

**Conjecture 2'.** For any real numbers $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + 3\alpha_1\alpha_2 + \alpha_2 > 1$, there exists a real number $\lambda > 0$ such that $C_{\alpha_1} \cap (\lambda C_{\alpha_2} + t)$ does not contain a Cantor set for almost all real numbers $t$.

These conjectures are related to Problem 7 from [PT2, p. 151]. These conjectures are very easy to prove when $\tau_1 = \tau_2$; see [K3].

**References**


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