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SHINTANI FUNCTIONS ON $GL(2, \mathbf{R})$

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Abstract. In this paper, we give a formulation and an explicit formula for Shintani function on $GL(2, \mathbf{R})$, which has been studied by Murase and Sugano in the theory of automorphic L-functions. In particular, we obtain the multiplicity of this function.

1. Introduction

A detailed study of various special functions makes the foundation for the theory of automorphic forms and automorphic representations. For example, the multiplicity of the Whittaker functionals is a problem which is very interesting and important in view of both local and global theory of automorphic L-functions.

Recently the Shintani functions, originally introduced by T. Shintani for the symplectic groups, were studied by Murase and Sugano [7] for $GL(n)$ (see also [6]). They obtained new kinds of integral formulas for the standard L-functions in terms of the global Shintani function, and proved also the uniqueness of the local one at the finite primes. However the multiplicity of the archimedean Shintani functions has not been studied.

Now we define the Shintani functions for the triple

$$G = GL(n, \mathbf{R}), \qquad H = \{\left(\begin{smallmatrix} g_0 & 0 \\ 0 & g \end{smallmatrix}\right) | (g_0, g) \in GL(n-1, \mathbf{R}) \times GL(1, \mathbf{R})\}$$

and the maximal compact subgroup $K = O(n, \mathbf{R})$ of G as follows (see section 3): Let $(\eta, \mathcal{F}_\eta)$ be an irreducible unitary representation of H and let $C^\infty_\eta(H\backslash G)$ be the space of C^∞-functions $F : G \to \mathcal{F}_\eta$ satisfying $F(hg) = \eta(h)F(g)$ for all $(h, g) \in H \times G$. Moreover take the contragredient representation π^* of an arbitrary irreducible unitary representartion π of G. We consider the intertwining space $\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}^{\mathbb{C}},K)}(\pi^*, C^\infty_\eta(H\backslash G))$ between $(\mathfrak{g}^{\mathbb{C}}, K)$-modules and its restriction

$$\mathcal{I}_{\eta,\pi} \to \mathrm{Hom}_K(\tau^*, C^\infty_\eta(H\backslash G)) \cong C^\infty_{\eta,\tau}(H\backslash G/K)$$

to the minimal K-type (τ^*, V_{τ^*}) of π^*, where $C^\infty_{\eta,\tau}(H\backslash G/K)$ is the space of C^∞-functions $F : G \to \mathcal{F}_\eta \otimes V_\tau$ satisfying $F(hgk) = (\eta(h) \otimes \tau(k)^{-1})F(g)$ for all $(h, g, k) \in H \times G \times K$. Now we define a *Shintani function* as a function which belongs to the image $\mathcal{S}_{\eta,\pi}(\tau)$ of the above map. We remark that the above definition coincides with that of [7, Section 5.5] if we take as π a class 1 unitary principal series representation and as η an irreducible unitary representation which is compatible with the central character and is trivial on $K \cap H$.

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In this paper we give explicit forms of the archimedean Shintani functions for $GL(2,\mathbf{R})$ in terms of the Gauss hypergeometric functions. In particular, we prove the following theorem.

Theorem (see Theorem 6.2 and 6.3)**.** *Let η be an irreducible unitary representation of $H \cong GL(1,\mathbf{R}) \times GL(1,\mathbf{R})$.*

1. *Let π be a unitary principal series representation of $G = GL(2,\mathbf{R})$ and τ be the minimal K-type of π. Then $\dim \mathcal{S}_{\eta,\pi}(\tau) = 0$ or 1 if $\dim \tau = 1$, and $\dim \mathcal{S}_{\eta,\pi}(\tau) = 0$ or 2 if $\dim \tau = 2$.*
2. *Let π be a discrete series representation of G and τ be the minimal K-type of π. Then $\dim \mathcal{S}_{\eta,\pi}(\tau) = 0$ or 1.*

For the trivial character η of H, our result agrees with that of Waldspurger [9, Proposition 10]. We remark that the above theorem says that the multiplicity free property is not valid in general; however, in the more limited sense of Murase and Sugano or of Waldspurger, the multiplicity one holds.

On the other hand, in the representation theoretical point of view, this result may be applicable to the Plancherel formula for $H\backslash G$ with non-trivial representations of H (cf. [3] and its references).

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2. Preliminaries

2.1. Groups. Let G be the real reductive Lie group $GL(2,\mathbf{R})$ and θ be an involution defined by $\theta(g) = {}^t g^{-1}$ $(g \in G)$, where ^{t} means the transpose.

Then the set of fixed points of θ is equal to $K = O(2,\mathbf{R})$, which is a maximal compact subgroup of G. Moreover, define an involutive automorphism σ of G by $\sigma(g) = JgJ$ $(g \in G)$, where $J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\theta\sigma = \sigma\theta$, and the set of fixed points of σ is equal to $H \cong GL(1,\mathbf{R}) \times GL(1,\mathbf{R})$, i.e.

$$H = \{g \in G \,|\, \sigma(g) = g\} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in G \;\middle|\; a_i \in \mathbf{R}^\times \right\}.$$

Let $\mathfrak{g} = \mathfrak{gl}(2,\mathbf{R})$ be the Lie algebra of G. If we denote the differentials of θ and σ again by θ and σ, then we have $\theta(X) = -{}^t X$ and $\sigma(X) = JXJ$ $(X \in \mathfrak{g})$. Let us define the eigenspaces of θ and σ by

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \,|\, \theta(X) = X\}, & \mathfrak{p} &= \{X \in \mathfrak{g} \,|\, \theta(X) = -X\}, \\ \mathfrak{h} &= \{X \in \mathfrak{g} \,|\, \sigma(X) = X\}, & \mathfrak{q} &= \{X \in \mathfrak{g} \,|\, \sigma(X) = -X\}. \end{aligned}$$

Then

$$\mathfrak{k} = \mathbf{R}X_1, \quad \mathfrak{p} = \mathbf{R}Y_1 \oplus \mathbf{R}Y_2 \oplus \mathbf{R}Z_{\mathfrak{p}}, \quad \mathfrak{h} = \mathbf{R}Y_1 \oplus \mathbf{R}Z_{\mathfrak{p}}, \quad \mathfrak{q} = \mathbf{R}X_1 \oplus \mathbf{R}Y_2,$$

with

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Z_{\mathfrak{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore we have the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Observe that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{h}$ is that of H.

Let

$$A = \left\{ a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \in G \;\middle|\; r \in \mathbf{R} \right\}, \quad \mathfrak{a} = \mathbf{R}Y_2.$$

Then $\mathfrak{a}$ is the Lie algebra of A and is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. For every integer n, set $\mathfrak{g}_n = \{X \in \mathfrak{g} \mid [Y_2, X] = nX\}$. Then

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathbf{R}Z_{\mathfrak{p}}, \quad \mathfrak{g}_2 = \mathbf{R}(X_1 - Y_1), \quad \mathfrak{g}_{-2} = \theta\mathfrak{g}_2 = \mathbf{R}(X_1 + Y_1),$$

and $\mathfrak{g}_n = \{0\}$ $(n \neq 0, \pm 2)$.

For a Lie algebra $\mathfrak{b}$, we denote by $\mathfrak{b}^{\mathbb{C}}$ the complexification $\mathfrak{b} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{b}$.

2.2. Parametrizations of representations. In this subsection, we recall the parametrizations of irreducible unitary representations of K, H, and G.

Let us denote the set of the equivalence classes of irreducible finite dimensional representations of K by $\hat{K}$. It is well known that $\hat{K}$ consists of two dimensional representations (τ_n, V_{τ_n}) $(n \in \mathbf{N})$ and one dimensional representations $(\tau_0^\varepsilon, V_{\tau_0^\varepsilon})$ $(\varepsilon \in \{0,1\})$. Then we can take the basis $\{v_n, v_{-n}\}$ of V_{τ_n} and $\{v_0^\varepsilon\}$ of $V_{\tau_0^\varepsilon}$ so that the associated representation is given by

$$\begin{cases} \tau_n(r_\theta)v_{\pm n} = e^{\pm in\theta}v_{\pm n}, \\ \tau_n(w_0)v_{\pm n} = v_{\mp n}, \end{cases} \qquad \begin{cases} \tau_0^\varepsilon(r_\theta)v_0^\varepsilon = v_0^\varepsilon, \\ \tau_0^\varepsilon(w_0)v_0^\varepsilon = (-1)^\varepsilon v_0^\varepsilon. \end{cases}$$

Here r_θ and w_0 are the elements of K defined by

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad w_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\theta \in \mathbf{R}).$$

Next let us parametrize the totality $\hat{H}$ of the equivalence classes of irreducible unitary representations of H. For $s = (s_1, s_2)$ $(s_i \in \sqrt{-1}\mathbf{R})$ and $k = (k_1, k_2)$ $(k_i \in \{0,1\})$, we define a one dimensional representation η_s^k of H by

$$\eta_s^k\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right) = h_1^{k_1}h_2^{k_2}|h_1|^{s_1-k_1}|h_2|^{s_2-k_2}, \quad \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \in H.$$

Clearly η_s^k is unitary and $\hat{H} = \{\eta_s^k \mid s_i \in \sqrt{-1}\mathbf{R}, k_i \in \{0,1\}\}$.

Let $P = M_P A_P N_P$ be the Langlands decomposition of the upper triangular group P of G. For a pair (z, l) $(z = (z_1, z_2),\ z_i \in \mathbf{C},\ l = (l_1, l_2),\ l_i \in \{0,1\})$, define σ_l on M_P and ν_z on $\mathfrak{a}_P = \mathrm{Lie}\ A_P$ by

$$\sigma_l\left(\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}\right) = \varepsilon_1^{l_1}\varepsilon_2^{l_2}, \qquad \nu_z\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) = z_1t_1 + z_2t_2,$$

with $\varepsilon_i \in \{\pm 1\}$, $t_i \in \mathbf{R}$. Then we can define the nonunitary principal series representation π_z^l of G by $\mathrm{Ind}_P^G(\sigma_l \otimes \exp\nu_z \otimes 1_{N_P})$. Here the representation space of π_z^l is given by

$$\{f \in C^\infty(G) \mid f(manx) = e^{(\nu_z+\rho)\log a}\sigma_l(m)f(x), (m,a,n) \in M_P \times A_P \times N_P, x \in G\}$$

with norm

$$\|f\|^2 = \int_K |f(k)|^2 dk,$$

and G acts by $\pi_z^l(g)f(x) = f(xg)$, where ρ is the half of the root of $(\mathfrak{g}, \mathfrak{a}_P)$ positive for N_P. If $z_i \in \sqrt{-1}\mathbf{R}$, then the representation π_z^l is unitary and is usually called the unitary principal series representation of G. The unitary principal series representation π_z^l has the following K-types;

$$\begin{array}{ll} \pi_z^l|_K = \tau_0^{l_1} \oplus \sum_{n\in\mathbf{N}} \tau_{2n}, & \text{if } l_1 + l_2 \equiv 0 \pmod 2, \\ \pi_z^l|_K = \sum_{n\in\mathbf{N}} \tau_{2n-1}, & \text{if } l_1 + l_2 \equiv 1 \pmod 2. \end{array}$$

If $z_1 + z_2 \in \sqrt{-1}\mathbf{R}$, $z_1 - z_2 = -j - 1$ for $j \in \mathbf{Z}_{\geq 0}$, and $l_1 + l_2 \equiv j \pmod 2$, then π_z^l contains the discrete series representation D^l_{j,z_1+z_2} of G as a subrepresentation. The K-types of D^l_{j,z_1+z_2} are

$$D^l_{j,z_1+z_2}|_K = \sum_{n\in\mathbf{N}} \tau_{j+2n}.$$

3. Shintani function and radial part

3.1. Shintani function. Let $\eta \in \hat{H}$. Consider the C^∞-induced module $C^\infty\mathrm{Ind}_H^G(\eta)$ with the representation space

$$C^\infty_\eta(H\backslash G) = \{F \in C^\infty(G) \,|\, F(hg) = \eta(h)F(g), (h,g) \in H \times G\}$$

on which G acts by the right translation. Then $C^\infty_\eta(H\backslash G)$ has the structure of smooth G-module and $(\mathfrak{g}^{\mathbb{C}}, K)$-module.

On the other hand, take an irreducible Harish-Chandra module Π of G. Let us consider the intertwining space

$$\mathcal{I}_{\eta,\Pi} := \mathrm{Hom}_{(\mathfrak{g}^{\mathbb{C}},K)}(\Pi^*, C^\infty\mathrm{Ind}_H^G(\eta))$$

with Π^* the contragredient $(\mathfrak{g}^{\mathbb{C}}, K)$-module of Π, and its image

$$\mathcal{S}_{\eta,\Pi} := \bigcup_{T\in\mathcal{I}_{\eta,\Pi}} \mathrm{Image}(T).$$

We call $\varphi \in \mathcal{S}_{\eta,\Pi}$ a Shintani function on G.

For any $(\tau, V_\tau) \in \hat{K}$ we define $C^\infty_{\eta,\tau}(H\backslash G/K)$ to be the space of smooth functions $F : G \to V_\tau$ with the property

$$F(hgk) = \eta(h)\tau(k)^{-1}F(g), \qquad (h,g,k) \in H \times G \times K.$$

Let τ^* be the contragredient representation of τ, and $i : \tau^* \to \Pi^*|_K$ be a K-equivariant map. Moreover let i^* be the pullback via i. Then the map

$$\mathcal{I}_{\eta,\Pi} \xrightarrow{i^*} \mathrm{Hom}_K(\tau^*, C^\infty_\eta(H\backslash G)) \cong C^\infty_{\eta,\tau}(H\backslash G/K)$$

gives the restriction of $T \in \mathcal{I}_{\eta,\Pi}$ to τ^* via i, which we denote by T_i. If we set

$$\mathcal{S}_{\eta,\Pi}(\tau) := \bigcup_i T_i, \quad T \in \mathcal{I}_{\eta,\Pi},$$

then we call $\varphi \in \mathcal{S}_{\eta,\Pi}(\tau)$ a Shintani function on G with K-type τ.

3.2. Radial part. Let us denote the centralizer and the normalizer of $\mathfrak{a}$ in $K \cap H$ by $Z_{K\cap H}(\mathfrak{a})$ and $N_{K\cap H}(\mathfrak{a})$, respectively. Then

$$K \cap H = \{\pm I,\ \pm w_0\}, \quad Z_{K\cap H}(\mathfrak{a}) = \{\pm I\}, \quad N_{K\cap H}(\mathfrak{a}) = K \cap H.$$

Here $w_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $w_0 Z_{K\cap H}(\mathfrak{a})$ is the unique nontrivial element of the quotient group $W = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$.

For $\eta \in \hat{H}$, $(\tau, V_\tau) \in \hat{K}$, let us denote by $C^\infty_W(A;\eta,\tau)$ the space of smooth functions $\varphi : A \to V_\tau$ satisfying the following conditions;

$$\begin{cases} (1) & (\eta(m)\tau(m))\varphi(a) = \varphi(a), & m \in Z_{K\cap H}(\mathfrak{a}),\ a \in A, \\ (2) & (\eta(w_0)\tau(w_0))\varphi(a) = \varphi(a^{-1}), & a \in A, \\ (3) & (\eta(l)\tau(l))\varphi(1) = \varphi(1), & l \in K \cap H. \end{cases} \tag{3.1}$$

Lemma 3.1 (Flensted-Jensen [2, Theorem 4.1])**.** 1. *$G = HAK = HA^+K$, where $A^+ = \{a_r \in A \,|\, r > 0\}$.*

2. *The set $C^\infty_{\eta,\tau}(H\backslash G/K)$ is in bijective correspondance, via the restriction A, with the set $C^\infty_W(A;\eta,\tau)$.*

For (τ, V_τ), $(\tau', V_{\tau'}) \in \hat{K}$, any $\mathbf{C}$-linear map $u : C^\infty_{\eta,\tau}(H\backslash G/K) \to C^\infty_{\eta,\tau'}(H\backslash G/K)$ defines a unique $\mathbf{C}$-linear map $\mathcal{R}(u) = \mathcal{R}_{(\eta;\tau,\tau')}(u)$ on A^+ with variable cefficients such that $uf(a_r) = \mathcal{R}(u)(f|_{A^+})(a_r)$ for $f \in C^\infty_{\eta,\tau}(H\backslash G/K)$, where $|_{A^+}$ means the restriction to A^+. We call $\mathcal{R}(u)$ the radial part of u.

4. The case of the principal series representations

4.1. Radial part of the elements of $Z(\mathfrak{g}^{\mathbb{C}})$. In this subsection, we shall calculate the radial part of the elements of the center $Z(\mathfrak{g}^{\mathbb{C}})$ of the universal enveloping algebra $U(\mathfrak{g}^{\mathbb{C}})$ of $\mathfrak{g}^{\mathbb{C}}$ acting on $C^\infty_{\eta,\tau}(H\backslash G/K)$. First of all, the following two lemmas are obvious.

Lemma 4.1. *For $a_r \in A^+$, we have*

$$\mathfrak{g} = \mathrm{Ad}(a_r^{-1})\mathfrak{h} + \mathfrak{a} + \mathfrak{k}.$$

Lemma 4.2. *Let $f \in C^\infty_{\eta,\tau}(H\backslash G/K)$. For $X \in \mathfrak{k}$, $Y \in \mathfrak{h}$, $Z \in \mathfrak{a}$ and $a_r \in A^+$, we have*

$$(\mathrm{Ad}(a_r^{-1})Y)ZXf(a_r) = \eta(Y)\tau(-X)(Zf)(a_r).$$

It is well known that $Z(\mathfrak{g}^{\mathbb{C}})$ is generated by $\Omega = \frac{1}{2}(-X_1^2 + Y_1^2 + Y_2^2)$ and $Z_{\mathfrak{p}}$ (cf. [4, §5]).

Proposition 4.3. *Put $\xi = e^{2r}$, and let $\eta \in \hat{H}$, $\tau \in \hat{K}$. For $\varphi \in C^\infty_W(A;\eta,\tau)$ and $a_r \in A$, we have*

$$\begin{aligned}\mathcal{R}(\Omega)\varphi(a_r) = {} & 2\xi^2 \frac{d^2}{d\xi^2}\varphi(a_r) + \frac{4\xi^3}{\xi^2+1}\frac{d}{d\xi}\varphi(a_r) \\ & + \frac{1}{(\xi^2+1)^2}\left\{2\xi^2\eta(Y_1)^2 + 2\xi(\xi^2-1)\eta(Y_1)\tau(X_1) - 2\xi^2\tau(X_1)^2\right\}\varphi(a_r),\end{aligned}$$

$$\mathcal{R}(Z_{\mathfrak{p}})\varphi(a_r) = \eta(Z_{\mathfrak{p}})\varphi(a_r).$$

Proof. It is obvious for $Z_{\mathfrak{p}}$. Thus we prove it for Ω. From Lemma 4.1 and the definition of Ω, we have

$$\begin{aligned}\Omega = {} & \frac{1}{(\xi^2+1)^2}\left\{2\xi^2(\mathrm{Ad}(a_r^{-1})Y_1)^2 - 2\xi(\xi^2-1)(\mathrm{Ad}(a_r^{-1})Y_1)X_1 - 2\xi^2X_1^2\right\} \\ & + \frac{\xi^2-1}{\xi^2+1}Y_2 + \frac{1}{2}Y_2^2.\end{aligned}$$

Since $Y_2\varphi(a_r) = \frac{d}{dr}\varphi(a_r) = 2\xi\frac{d}{d\xi}\varphi(a_r)$, we get this proposition from Lemma 4.2. $\square$

4.2. Differential equations and their solutions. Throughout this subsection, let $\eta = \eta_s^k \in \hat{H}$, $(\tau, V_\tau) \in \hat{K}$, and let $\Pi^* = \pi_z^l$ be a unitary principal series representation of G with K-type τ^*. If $u \in Z(\mathfrak{g}^{\mathbb{C}})$, then u acts on Π^*, hence on $C^\infty_{\eta,\tau}(H\backslash G/K)$, as a scalar operator χ_u. Therefore for $\varphi \in C^\infty_W(A;\eta,\tau)$ and $u \in Z(\mathfrak{g}^{\mathbb{C}})$ we have a differential equation with A-radial part;

$$\mathcal{R}(u)\varphi(a_r) = \chi_u \varphi(a_r). \tag{4.1}$$

We want to find C^∞-solutions of this differential equation.

Lemma 4.4.

$$\eta(Y_1) = s_1 - s_2, \quad \eta(Z_{\mathfrak{p}}) = s_1 + s_2, \quad \begin{cases} \tau_n(X_1)v_{\pm n} = \pm\sqrt{-1}nv_{\pm n}, \\ \tau_0^\varepsilon(X_1)v_0^\varepsilon = 0. \end{cases}$$

Proof. This assertion is trivial from the definitions. □

Lemma 4.5.

$$\chi_\Omega = \frac{1}{2}(z_1 - z_2 + 1)(z_1 - z_2 - 1), \qquad \chi_{Z_{\mathfrak{p}}} = z_1 + z_2.$$

Proof. See Jacquet-Langlands [4, Lemma 5.6]. □

Now we express a C^∞-function $\varphi : A \to V_\tau$ as

$$\varphi(a_r) = \begin{cases} c_n(r)v_n + c_{-n}(r)v_{-n}, & \text{if } \tau = \tau_n, \\ c_0^\varepsilon(r)v_0^\varepsilon, & \text{if } \tau = \tau_0^\varepsilon, \end{cases} \tag{4.2}$$

with C^∞-functions $c_{\pm n}(r)$ $(n \in \mathbf{N})$, $c_0^\varepsilon(r)$ $(\varepsilon \in \{0,1\})$.

Lemma 4.6. *A C^∞-function $\varphi : A \to V_\tau$ expressed as (4.2) belongs to the space $C^\infty_W(A;\eta,\tau)$ if and only if the following conditions are satisfied; if $\tau = \tau_n$, $n \in \mathbf{N}$,*

1. $k_1 + k_2 \equiv n \pmod 2$,
2. $(-1)^{k_2}c_{\pm n}(r) = c_{\mp n}(-r)$;

if $\tau = \tau_0^\varepsilon$, $\varepsilon \in \{0,1\}$,

1. $k_1 + k_2 \equiv 0 \pmod 2$,
2. $(-1)^{k_2+\varepsilon}c_0^\varepsilon(r) = c_0^\varepsilon(-r)$.

Proof. Since $K \cap H = N_{K\cap H}(\mathfrak{a})$, it suffices to consider (1) and (2) of (3.1). Let $\tau = \tau_n$. Since $\eta(-I) = (-1)^{k_1+k_2}$ and $\tau_n(-I)v_{\pm n} = (-1)^n v_{\pm n}$, we have

$$(\eta(-I)\tau_n(-I))\varphi(a_r) = (-1)^{k_1+k_2+n}.$$

Thus (1) of (3.1) is equivalent to $k_1 + k_2 \equiv n \pmod 2$. Because $\eta(w_0) = (-1)^{k_2}$ and $\tau_n(w_0)v_{\pm n} = v_{\mp n}$, we have

$$(\eta(w_0)\tau_n(w_0))\varphi(a_r) = (-1)^{k_2}(c_{-n}(r)v_n + c_n(r)v_{-n}).$$

Hence we get the equivalence of (2) of (3.1) and (2) above, since $a_r^{-1} = a_{-r}$. The case of $\tau = \tau_0^\varepsilon$ is proved similarly. □

Let us assume that $c_0(r)$ means either $c_0^0(r)$ or $c_0^1(r)$. Using Proposition 4.3, the differential equations (4.1) for Ω and $Z_{\mathfrak{p}}$ can be written as the following differential

equations in terms of $c_m(r)$ ($m \in \mathbf{Z}$):

$$
\begin{aligned}
(4.3)\quad & 2\xi^2 \frac{d^2}{d\xi^2} c_m(r) + \frac{4\xi^3}{\xi^2+1} \frac{d}{d\xi} c_m(r) \\
& + \frac{1}{(\xi^2+1)^2} \left\{ 2\xi^2 (s_1 - s_2)^2 + 2\xi(\xi^2-1)(s_1-s_2) m\sqrt{-1} + 2\xi^2 m^2 \right\} c_m(r) \\
& = \frac{1}{2}(z_1 - z_2 + 1)(z_1 - z_2 - 1) c_m(r),
\end{aligned}
$$

$$
(4.4) \qquad (s_1 + s_2) c_m(r) = (z_1 + z_2) c_m(r).
$$

Now we deal with the case that τ^* is the minimal K-type of Π^*, that is, $\tau = \tau_0^\varepsilon$ or τ_1.

Proposition 4.7. *If $m = 0$, then two linearly independent C^∞-solutions of the differential equation (4.3) are given by*

$$
\begin{aligned}
w_z^{s,0}(r) =& (1-x)^{\frac{z'+1}{4}} {}_2F_1\left(\frac{z'+s'+1}{4}, \frac{z'-s'+1}{4}; \frac{1}{2}; x \right), \\
w_z^{s,1}(r) =& (1-x)^{\frac{z'+1}{4}} x^{\frac{1}{2}} {}_2F_1\left(\frac{z'+s'+3}{4}, \frac{z'-s'+3}{4}; \frac{3}{2}; x \right),
\end{aligned}
$$

with $z' = z_1 - z_2$, $s' = s_1 - s_2$ and $x = \tanh^2 2r$. Here ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function of variable z with parameters a, b, c. For $m = \pm 1$, we may take a system of two linearly independent C^∞-solutions given by $u_z^{s,m}(r)$ and $u_z^{-s,-m}(r)$, where

$$
u_z^{s,m}(r) = (-y)^{\frac{s'-m}{4}} (1-y)^{\frac{z'+1}{2}} {}_2F_1\left(\frac{z'+s'+1}{2}, \frac{z'-m+1}{2}; 1 + \frac{s'-m}{2}; y \right),
$$

with

$$
y = \left(\frac{e^{2r} - \sqrt{-1}}{e^{2r} + \sqrt{-1}} \right)^2.
$$

Here $-s$ means $(-s_1, -s_2)$.

Proof. We recall the hypergeometric equation

$$
(4.5) \qquad z(1-z) \frac{d^2 u}{dz^2} + \{c - (1 + a + b) z\} \frac{du}{dz} - abu = 0
$$

with complex parameters a, b, c. If c is not an integer, then we may choose

$$
{}_2F_1(a, b; c; z), \qquad z^{1-c} {}_2F_1(a - c + 1, b - c + 1; 2 - c; z)
$$

as two linearly independent solutions of (4.5) in the neighborhood of $z = 0$ (cf. [1, Chapter 2]). Now we reduce (4.3) to the hypergeometric equation (4.5). For $m = 0$, we put

$$
x = \left(\frac{\xi^2 - 1}{\xi^2 + 1} \right)^2 = \tanh^2 2r \quad \text{and} \quad c_0(r) = (1-x)^{\frac{z'+1}{4}} u(x).
$$

Then $u(x)$ satisfies (4.5) on x with

$$
a = \frac{z'+s'+1}{4}, \qquad b = \frac{z'-s'+1}{4}, \qquad c = \frac{1}{2}.
$$

Similarly putting

$$y = \left(\frac{\xi - \sqrt{-1}}{\xi + \sqrt{-1}}\right)^2 \quad \text{and} \quad c_m(r) = (-y)^{\frac{s'-m}{4}}(1-y)^{\frac{z'+1}{2}} v(y)$$

for $m = \pm 1$, then $v(y)$ satisfies (4.5) on y with

$$a = \frac{z' + s' + 1}{2}, \qquad b = \frac{z' - m + 1}{2}, \qquad c = 1 + \frac{s' - m}{2}.$$

Solving these reduced differential equations, we obtain these assertions, since the smoothness on r is clear. □

5. The case of the discrete series representations

5.1. Schmid operator and shift operator. Let us consider the vector space $\mathfrak{p}^{\mathbb{C}}$ as a K-module via the adjoint representation. Put $\mathfrak{p}_S = \mathbb{C}Y_1 \oplus \mathbb{C}Y_2$; then $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_S \oplus \mathbb{C}Z_{\mathfrak{p}}$ gives an irreducible decomposition. Clearly we have an isomorphism $\mathfrak{p}_S \cong V_{\tau_2}$ via the correspondence of the basis

$$\left\{\frac{Y_1 + \sqrt{-1}Y_2}{2}, \frac{Y_1 - \sqrt{-1}Y_2}{2}\right\} \cong \{v_2, v_{-2}\},$$

and also $\mathbb{C}Z_{\mathfrak{p}} \cong V_{\tau_0^0}$.

For a given irreducible K-module V_τ, the tensor pruduct $V_\tau \otimes \mathfrak{p}_S$ has an irreducible decomposition

$$V_{\tau_n} \otimes \mathfrak{p}_S \cong V_{\tau_{|n-2|}} \oplus V_{\tau_{n+2}}, \quad \text{for } n \in \mathbf{N}_{\neq 2},$$

$$V_{\tau_2} \otimes \mathfrak{p}_S \cong V_{\tau_0^0} \oplus V_{\tau_0^1} \oplus V_{\tau_4}, \qquad V_{\tau_0^\varepsilon} \otimes \mathfrak{p}_S \cong V_{\tau_2}.$$

Now we denote the projectors induced from the above decompositions as follows;

$$P^-(n) : V_{\tau_n} \otimes \mathfrak{p}_S \to V_{\tau_{|n-2|}}, \quad P^+(n) : V_{\tau_n} \otimes \mathfrak{p}_S \to V_{\tau_{n+2}}, \qquad \text{for } n \in \mathbf{N}_{\neq 2},$$

$$P^{-\varepsilon}(2) : V_{\tau_2} \otimes \mathfrak{p}_S \to V_{\tau_0^\varepsilon}, \quad P^+(2) : V_{\tau_2} \otimes \mathfrak{p}_S \to V_{\tau_4}, \qquad \text{for } \varepsilon \in \{0,1\},$$

$$P^{+\varepsilon}(0) : V_{\tau_0^\varepsilon} \otimes \mathfrak{p}_S \to V_{\tau_2}, \qquad \text{for } \varepsilon \in \{0,1\}.$$

Lemma 5.1. *Let us denote the basis* $\frac{Y_1+\sqrt{-1}Y_2}{2}$, $\frac{Y_1-\sqrt{-1}Y_2}{2}$ *of* $\mathfrak{p}_S$ *by* w_2, w_{-2} *respectively. For* $n \in \mathbf{N}_{\neq 2}$, *we have*

$$P^-(n)(v_n \otimes w_{-2}) = v_{n-2}, \quad P^-(n)(v_{-n} \otimes w_2) = v_{-n+2},$$

$$P^+(n)(v_n \otimes w_2) = v_{n+2}, \quad P^+(n)(v_{-n} \otimes w_{-2}) = v_{-n-2},$$

$$\begin{aligned} P^-(n)(v_n \otimes w_2) &= P^-(n)(v_{-n} \otimes w_{-2}) = P^+(n)(v_n \otimes w_{-2}) \\ &= P^+(n)(v_{-n} \otimes w_2) = 0. \end{aligned}$$

For $n = 2$, $\varepsilon \in \{0,1\}$, *we have*

$$P^{-0}(2)(v_2 \otimes w_{-2} + v_{-2} \otimes w_2) = v_0^0, \quad P^{-1}(2)(v_2 \otimes w_{-2} - v_{-2} \otimes w_2) = v_0^1,$$

$$P^+(2)(v_2 \otimes w_2) = v_4, \quad P^+(2)(v_{-2} \otimes w_{-2}) = v_{-4},$$

$$\begin{aligned}&P^{-0}(2)(v_2\otimes w_{-2}-v_{-2}\otimes w_2)=P^{-1}(2)(v_2\otimes w_{-2}+v_{-2}\otimes w_2)\\&=P^{-\varepsilon}(2)(v_2\otimes w_2)=P^{-\varepsilon}(2)(v_{-2}\otimes w_{-2})\\&=P^{+}(2)(v_2\otimes w_{-2})=P^{+}(2)(v_{-2}\otimes w_2)=0.\end{aligned}$$

For $\varepsilon\in\{0,1\}$, we have

$$P^{+\varepsilon}(0)(v_0^\varepsilon\otimes w_2)=v_2,\quad P^{+\varepsilon}(0)(v_0^\varepsilon\otimes w_{-2})=(-1)^\varepsilon v_{-2}.$$

Proof. Omitted. □

Now we define a first order gradient type differential operator

$$\nabla^S_{\eta,\tau}:C^\infty_{\eta,\tau}(H\backslash G/K)\to C^\infty_{\eta,\tau\otimes\mathfrak{p}_S}(H\backslash G/K)$$

by

$$\nabla^S_{\eta,\tau}F=R_{\frac{Y_1+\sqrt{-1}Y_2}{2}}F\otimes\frac{Y_1-\sqrt{-1}Y_2}{2}+R_{\frac{Y_1-\sqrt{-1}Y_2}{2}}F\otimes\frac{Y_1+\sqrt{-1}Y_2}{2},$$

where

$$R_XF(g)=\tfrac{d}{dt}F(g\cdot\exp(tX))\big|_{t=0},\qquad\text{for } X\in\mathfrak{g}^{\mathbb{C}},g\in G.$$

This differential operator $\nabla^S_{\eta,\tau}$ is called the Schmid operator. Moreover we define the shift operators as the compositions of $\nabla^S_{\eta,\tau}$ with the projectors from $\tau_n\otimes\mathfrak{p}_S$ into its irreducible component; for $n\in\mathbf{N}_{\neq 2}$,

$$\nabla^\pm_{\eta,\tau_n}:C^\infty_{\eta,\tau_n}(H\backslash G/K)\to C^\infty_{\eta,\tau_{n\pm2}}(H\backslash G/K),\quad\nabla^\pm_{\eta,\tau_n}F=P^\pm(n)\nabla^S_{\eta,\tau_n}F,$$

for $n=2$, $\varepsilon\in\{0,1\}$,

$$\nabla^+_{\eta,\tau_2}:C^\infty_{\eta,\tau_2}(H\backslash G/K)\to C^\infty_{\eta,\tau_4}(H\backslash G/K),\quad\nabla^+_{\eta,\tau_2}F=P^+(2)\nabla^S_{\eta,\tau_2}F,$$

$$\nabla^{-\varepsilon}_{\eta,\tau_2}:C^\infty_{\eta,\tau_2}(H\backslash G/K)\to C^\infty_{\eta,\tau_0^\varepsilon}(H\backslash G/K),\quad\nabla^{-\varepsilon}_{\eta,\tau_2}F=P^{-\varepsilon}(2)\nabla^S_{\eta,\tau_2}F,$$

and for $\varepsilon\in\{0,1\}$,

$$\nabla^+_{\eta,\tau_0^\varepsilon}:C^\infty_{\eta,\tau_0^\varepsilon}(H\backslash G/K)\to C^\infty_{\eta,\tau_2}(H\backslash G/K),\quad\nabla^+_{\eta,\tau_0^\varepsilon}F=P^{+\varepsilon}(0)\nabla^S_{\eta,\tau_0^\varepsilon}F.$$

5.2. Radial part of the shift operator. In this subsection, we compute the radial part of the shift operators defined in the last section.

Proposition 5.2. *Put $\xi=e^{2r}$, $r\in\mathbf{R}$, and let $\eta\in\hat{H}$, $\tau\in\hat{K}$. For $\varphi\in C^\infty_W(A;\eta,\tau)$ and $a_r\in A$, we have*

$$\begin{aligned}&\mathcal{R}(\nabla^S_{\eta,\tau})\varphi\\&=\left(\sqrt{-1}\xi\tfrac{d}{d\xi}+\tfrac{\xi}{\xi^2+1}\eta(Y_1)+\tfrac12\tfrac{\xi^2-1}{\xi^2+1}((\tau\otimes\mathrm{Ad}_{\mathfrak{p}})(X_1)+2\sqrt{-1})\right)(\varphi\otimes\tfrac{Y_1-\sqrt{-1}Y_2}{2})\\&\quad-\left(\sqrt{-1}\xi\tfrac{d}{d\xi}-\tfrac{\xi}{\xi^2+1}\eta(Y_1)-\tfrac12\tfrac{\xi^2-1}{\xi^2+1}((\tau\otimes\mathrm{Ad}_{\mathfrak{p}})(X_1)-2\sqrt{-1})\right)(\varphi\otimes\tfrac{Y_1+\sqrt{-1}Y_2}{2}).\end{aligned}$$

Moreover if we write $\varphi\in C^\infty_W(A;\eta,\tau)$ as (4.2), then we have the negative shift operators

$$\mathcal{R}(\nabla^-_{\eta,\tau_n})\varphi(a_r)=D^+c_n(r)v_{n-2}-D^-c_{-n}(r)v_{-n+2}\quad\text{for } n\in\mathbf{N}_{\neq2},$$
$$2\mathcal{R}(\nabla^{-\varepsilon}_{\eta,\tau_2})\varphi(a_r)=D^+c_2(r)v_0^\varepsilon-(-1)^\varepsilon D^-c_{-2}(r)v_0^\varepsilon,\quad\text{for } n=2,\varepsilon\in\{0,1\},$$

and the positive shift operators

$$\mathcal{R}(\nabla^+_{\eta,\tau_n})\varphi(a_r)=D^+c_{-n}(r)v_{-n-2}-D^-c_n(r)v_{n+2},\quad\text{for } n\in\mathbf{N},$$
$$\mathcal{R}(\nabla^+_{\eta,\tau_0^\varepsilon})\varphi(a_r)=(-1)^\varepsilon D^+c_0^\varepsilon(r)v_{-2}-D^-c_0^\varepsilon(r)v_2,\quad\text{for } \varepsilon\in\{0,1\},$$

where

$$D^{\pm} = \sqrt{-1}\xi\tfrac{d}{d\xi} \pm \tfrac{\xi}{\xi^2+1}\eta(Y_1) \pm \tfrac{1}{2}\tfrac{\xi^2-1}{\xi^2+1}(\tau(X_1) \pm 2\sqrt{-1}).$$

Proof. In view of Lemma 4.1, we express $R_{\frac{Y_1 \pm \sqrt{-1}Y_2}{2}}$ as

$$\frac{1}{2}\left(\frac{2\xi}{\xi^2+1}\mathrm{Ad}(a_r^{-1})Y_1 - \frac{\xi^2-1}{\xi^2+1}X_1 \pm \sqrt{-1}Y_2\right).$$

Moreover we have the relation

$$\tau(X_1)\varphi \otimes \frac{Y_1 \pm \sqrt{-1}Y_2}{2} = \left((\tau \otimes \mathrm{Ad}_{\mathfrak{p}})(X_1) \mp 2\sqrt{-1}\right)\left(\varphi \otimes \frac{Y_1 \pm \sqrt{-1}Y_2}{2}\right).$$

Using this relation, we obtain the radial part of the Schmid operator via direct computation. Moreover, we get the assertions for the shift operators by combining this with the projectors in Lemma 5.1. □

5.3. Differential equations and their solutions. Let us take $\eta = \eta_s^k \in \hat{H}$ and consider the case when $\tau^* = \tau_{j+2} \in \hat{K}$ is the minimal K-type of $\Pi^* = D^l_{j,z_1+z_2}$, a discrete series representation of G. Since τ_j does not occur in $D^l_{j,z_1+z_2}|_K$, each $\varphi \in C^\infty_W(A;\eta,\tau)$ satisfies the following differential equation: for $j \in \mathbf{Z}_{>0}$,

$$\mathcal{R}(\nabla^-_{\eta,\tau_{j+2}})\varphi(a_r) = 0, \tag{5.1}$$

and for $j = 0$, $\varepsilon \in \{0,1\}$,

$$\mathcal{R}(\nabla^{-\varepsilon}_{\eta,\tau_2})\varphi(a_r) = 0. \tag{5.2}$$

Putting $y = (\frac{\xi-\sqrt{-1}}{\xi+\sqrt{-1}})^2$ and using Proposition 5.2 and Lemma 4.4, we have the following differential equations in terms of $c_{\pm(j+2)}$ ($j \in \mathbf{Z}_{\geq 0}$) from (5.1) and (5.2):

$$\left((1-y)\tfrac{d}{dy} \mp \tfrac{1-y}{4y}(s_1-s_2) + \tfrac{1+y}{4y}(j+2)\right)c_{\pm(j+2)}(r) = 0. \tag{5.3}$$

Then we have the following proposition via direct computation.

Proposition 5.3. *Up to a constant multiple, the solution of the differential equation (5.3) is given by*

$$u^s_{\pm(j+2)}(r) = (-y)^{\frac{\pm(s_1-s_2)-j-2}{4}}(1-y)^{\frac{j+2}{2}}.$$

6. Main results

Before giving our main theorems, we need a proposition:

Proposition 6.1. *Let $\eta \in \hat{H}$. The system of differential equations (4.1) characterizes the space of Shintani functions $\mathcal{S}_{\eta,\Pi}(\tau)$ for a unitary principal series representation Π^* and its minimal K-type τ^*. Moreover, (5.1) or (5.2), and (4.4), characterize $\mathcal{S}_{\eta,\Pi}(\tau)$ for a discrete series representation Π^* and its minimal K-type τ^*.*

Proof. For a discrete series representation Π^* and its minimal K-type τ^*, this assertion follows from the theorem of Yamashita [10, Theorem 2.4].

Thus we consider the principal series case. If we denote the C^∞-solution space of (4.1) by $S^\infty_Z(\tau) = S^\infty_{Z(\mathfrak{g}^{\mathbb{C}})}(\eta,\Pi;\tau)$, then the inclusion $S^\infty_Z(\tau) \supset \mathcal{S}_{\eta,\Pi}(\tau)$ holds obviously. Thus we prove the inverse inclusion for the minimal K-type $\tau = \tau_j$ ($j = 0,1$) of Π, where τ_0 means either τ_0^0 or τ_0^1.

Now we define a map $\epsilon_\tau : C^\infty_{\eta,\tau}(H\backslash G/K) \otimes V_{\tau^*} \to C^\infty_\eta(H\backslash G)$ by

$$\epsilon_\tau(F \otimes v^*)(g) = \langle v^*, F(g)\rangle, \qquad g \in G.$$

Here $\langle , \rangle$ is the canonical pairing on $V_{\tau^*} \times V_\tau$. Let us take a nonzero $F \in S^\infty_Z(\tau_j)$ and consider the smallest $(\mathfrak{g}^{\mathbb{C}}, K)$-submodule Π_F of $C^\infty_\eta(H\backslash G)$ which contains $\epsilon_{\tau_j}(F \otimes V_{\tau_j^*})$. Then we have

$$\Pi_F = \sum_{n\geq 0} \epsilon_{\tau_{j+2n}}(\nabla^{+n}_{\eta,\tau_j} F \otimes V_{\tau^*_{j+2n}}).$$

where $\nabla^{+n}_{\eta,\tau_j}$ is the n-th composition of the positive shift operators and $\nabla^{+n}_{\eta,\tau_j} F \in \mathrm{C}^\infty_{\eta,\tau_{j+2n}}(H\backslash G/K)$. This follows from the fact that the compositions

$$\begin{aligned}
&\mathcal{R}(\nabla^-_{\eta,\tau_{n+2}})\mathcal{R}(\nabla^+_{\eta,\tau_n}) = \tfrac{1}{2}\mathcal{R}(\Omega) - \tfrac{1}{4}n(n+2), && n \in \mathbf{N},\\
&\mathcal{R}(\nabla^+_{\eta,\tau_{n-2}})\mathcal{R}(\nabla^-_{\eta,\tau_n}) = \tfrac{1}{2}\mathcal{R}(\Omega) - \tfrac{1}{4}n(n-2), && n \in \mathbf{N}_{\neq 1,2},\\
&\mathcal{R}(\nabla^{-\varepsilon'}_{\eta,\tau_2})\mathcal{R}(\nabla^+_{\eta,\tau_0^\varepsilon}) = \tfrac{1}{4}\{1+(-1)^{\varepsilon+\varepsilon'}\}\mathcal{R}(\Omega), && \varepsilon, \varepsilon' \in \{0,1\},
\end{aligned}$$

and

$$\mathcal{R}(\nabla^-_{\eta,\tau_1})\mathcal{R}(\nabla^-_{\eta,\tau_1}) = \tfrac{1}{2}\mathcal{R}(\Omega) + \tfrac{1}{4}$$

are in $Z(\mathfrak{g}^{\mathbb{C}})$, and that

$$\epsilon_{\tau_1}(\nabla^-_{\eta,\tau_1} F \otimes V_{\tau_1^*}) = \epsilon_{\tau_1}(F \otimes V_{\tau_1^*}).$$

In particular, we have

$$\Pi_F|_K = \Pi^*|_K,$$

since $\nabla^{+n}_{\eta,\tau_j} F \not\equiv 0$ for any nonzero $F \in S^\infty_Z(\tau_j)$ by the expression of F in Proposition 4.7. On the other hand, the infinitesimal character of Π_F coincides with that of Π^* by the definition. Therefore $\Pi_F \cong \Pi^*$. This means the inclusion $S^\infty_Z(\tau) \subset \mathcal{S}_{\eta,\Pi}(\tau)$. □

Now we can state explicit forms of the Shintani functions on G, which is our main goal in this paper.

Theorem 6.2. *Let $\eta = \eta^k_s \in \hat{H}$, and let $\Pi = \pi^l_z$ be a unitary principal series representation. Moreover let $(\tau, V_\tau) \in \hat{K}$ be the minimal K-type of Π, i.e. $\tau = \tau_0^\varepsilon$ or τ_1, $\varepsilon \in \{0,1\}$. Then the space $\mathcal{S}_{\eta,\Pi}(\tau)$ of the Shintani functions is nonzero if and only if the following conditions are satisfied:*

1. $s_1 + s_2 = z_1 + z_2$,
2. $k_1 + k_2 \equiv 0 \pmod 2$, $l_1 = l_2 = \varepsilon$ *if* $\tau = \tau_0^\varepsilon$,
3. $k_1 + k_2 = l_1 + l_2 = 1$ *if* $\tau = \tau_1$.

Under these conditions, $\mathcal{S}_{\eta,\Pi}(\tau)$ has the following basis: If $\tau = \tau_0^\varepsilon$, then the radial part of the base is given by

$$\begin{aligned}
&w^{s,0}_z(r)v^\varepsilon_0, && \text{if } k_2 + \varepsilon \equiv 0 \pmod 2,\\
&w^{s,1}_z(r)v^\varepsilon_0, && \text{if } k_2 + \varepsilon = 1.
\end{aligned}$$

Here

$$\begin{aligned}
w^{s,0}_z(r) =& (1-x)^{\frac{z'+1}{4}}\, {}_2F_1\left(\frac{z'+s'+1}{4}, \frac{z'-s'+1}{4}; \frac{1}{2}; x\right),\\
w^{s,1}_z(r) =& (1-x)^{\frac{z'+1}{4}} x^{\frac{1}{2}}\, {}_2F_1\left(\frac{z'+s'+3}{4}, \frac{z'-s'+3}{4}; \frac{3}{2}; x\right),
\end{aligned}$$

with $z' = z_1 - z_2$, $s' = s_1 - s_2$ *and* $x = \tanh^2 2r$. *In particular,* $\dim \mathcal{S}_{\eta,\Pi}(\tau) = 1$.

For $\tau = \tau_1$, *the radial part of the functions in* $\mathcal{S}_{\eta,\Pi}(\tau)$ *is given by*

$$(\gamma_1 u_z^{s,1}(r) + \gamma_2 u_z^{-s,-1}(r))v_1 + (\gamma_3 u_z^{s,-1}(r) + \gamma_4 u_z^{-s,1}(r))v_{-1}$$

up to a constant multiple, where the γ_i*'s satisfy the relations*

$$(-1)^{k_2}\gamma_1 = B(1, s'; z')\gamma_3 + B(-s', -1; z')\gamma_4,$$
$$(-1)^{k_2}\gamma_2 = B(s', 1; z')\gamma_3 + B(-1, -s'; z')\gamma_4,$$

with

$$B(a, b; z') = \frac{\Gamma(1 + \frac{a+b}{2})\Gamma(\frac{a-b}{2})}{\Gamma(\frac{a+1+z'}{2})\Gamma(\frac{a+1-z'}{2})}.$$

Here

$$u_z^{s,m}(r) = (-y)^{\frac{s'-m}{4}}(1-y)^{\frac{z'+1}{2}}{}_2F_1\left(\tfrac{z'+s'+1}{2}, \tfrac{z'-m+1}{2}; 1 + \tfrac{s'-m}{2}; y\right),$$

with

$$y = \left(\frac{e^{2r} - \sqrt{-1}}{e^{2r} + \sqrt{-1}}\right)^2$$

and $-s = (-s_1, -s_2)$. *In particular,* $\dim \mathcal{S}_{\eta,\Pi}(\tau) = 2$.

Proof. The nonzero conditions of the space $\mathcal{S}_{\eta,\Pi}(\tau)$ are obvious from Lemma 4.6 (1) and (4.4).

Let us consider the case $\tau = \tau_0^\varepsilon$. By Proposition 4.7, we can express the radial part of the elements in $\mathcal{S}_{\eta,\Pi}(\tau)$ as

$$(\gamma_1 w_z^{s,0}(r) + \gamma_2 w_z^{s,1}(r))v_0^\varepsilon.$$

Note that $w_z^{s,i}(-r) = (-1)^i w_z^{s,i}(r)$ $(i = 0, 1)$ holds via direct computation. Thus Lemma 4.6 (2) shows that $\gamma_2 = 0$ for $k_2 + \varepsilon \equiv 0 \pmod 2$ and $\gamma_1 = 0$ for $k_2 + \varepsilon = 1$.

Next let $\tau = \tau_1$. Then, up to a constant, the radial parts of the functions in $\mathcal{S}_{\eta,\Pi}(\tau)$ can be expressed as

$$(\gamma_1 u_z^{s,1}(r) + \gamma_2 u_z^{-s,-1}(r))v_1 + (\gamma_3 u_z^{s,-1}(r) + \gamma_4 u_z^{-s,1}(r))v_{-1},$$

by Proposition 4.7. Using the formula ([1], p. 108, 2.10(2))

$$\begin{aligned}{}_2F_1(a, b; c; z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a, 1-c+a; 1-b+a; z^{-1}) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b, 1-c+b; 1-a+b; z^{-1})\end{aligned}$$

with $|\arg(-z)| < \pi$ and $b - a \in \mathbf{C}\backslash\mathbf{Z}$, we have the equation

$$u_z^{s,m}(-r) = B(-m, s'; z')u_z^{s,-m}(r) + B(s', -m; z')u_z^{-s,m}(r).$$

From Lemma 4.6 (2) and this equation, we get the relations of the γ_i's in the theorem. □

Theorem 6.3. *Let* $\eta = \eta_s^k \in \hat{H}$, *and let* $\Pi = D^l_{j,z_1+z_2}$ *be a discrete series representation. Moreover let* $(\tau, V_\tau) \in \hat{K}$ *be the minimal* K*-type of* Π, *i.e.* $\tau = \tau_{j+2}$. *Then the space* $\mathcal{S}_{\eta,\Pi}(\tau)$ *of the Shintani functions is nonzero if and only if the following conditions are satisfied;*

1. $s_1 + s_2 = z_1 + z_2$,
2. $k_1 + k_2 \equiv l_1 + l_2 \equiv j \pmod 2$.

Under these conditions, $\mathcal{S}_{\eta,\Pi}(\tau)$ has the base whose radial part is given by

$$u^s_{j+2}(r)v_{j+2} + (-1)^{k_2}u^s_{-j-2}(r)v_{-j-2}.$$

Here

$$u^s_{\pm(j+2)}(r) = (-y)^{\frac{\pm(s_1-s_2)-j-2}{4}}(1-y)^{\frac{j+2}{2}},$$

with

$$y = \left(\frac{e^{2r}-\sqrt{-1}}{e^{2r}+\sqrt{-1}}\right)^2.$$

In particular, $\dim \mathcal{S}_{\eta,\Pi}(\tau) = 1$.

Proof. The nonzero conditions of the space $\mathcal{S}_{\eta,\Pi}(\tau)$ are obvious from Lemma 4.6 (1) and (4.4). By Proposition 5.3 we can express the radial part of the elements of $\mathcal{S}_{\eta,\Pi}(\tau)$ as

$$\gamma_1 u^s_{j+2}(r)v_{j+2} + \gamma_2 u^s_{-j-2}(r)v_{-j-2}.$$

Note that $u^s_{j+2}(-r) = u^s_{-j-2}(r)$ holds via direct computation. Thus Lemma 4.6 (2) shows that $(-1)^{k_2}\gamma_1 = \gamma_2$. □

Remark. When $\eta = \eta^k_s \in \hat{H}$ is the trivial one, i.e. $s_1 = s_2 = k_1 = k_2 = 0$, the space $\mathcal{S}_{\eta,\Pi}(\tau)$ is one dimensional by the above theorems, if it exists. This agrees with the result of Waldspurger [9, Proposition 10].

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