

SHARP BOUNDS ON CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. The Castelnuovo-Mumford regularity is one of the most important invariants in studying the minimal free resolution of the defining ideals of the projective varieties. There are some bounds on the Castelnuovo-Mumford regularity of the projective variety in terms of the other basic measures such as dimension, codimension and degree.

In this paper we consider an upper bound on the regularity $\text{reg}(X)$ of a nondegenerate projective variety X , $\text{reg}(X) \leq \lceil (\deg(X) - 1) / \text{codim}(X) \rceil + k \cdot \dim(X)$, provided X is k -Buchsbaum for $k \geq 1$, and investigate the projective variety with its Castelnuovo-Mumford regularity having such an upper bound.

1. INTRODUCTION

Let X be a projective scheme of \mathbb{P}_K^N over a field K . Let $S = K[x_0, \dots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \dots, x_N)$ be the irrelevant ideal. Then we put $\mathbb{P}_K^N = \text{Proj}(S)$. We denote by \mathcal{I}_X the ideal sheaf of X . Let m be an integer. Then X is said to be m -regular if $H^i(\mathbb{P}_K^N, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_K^N$, introduced by Mumford by generalizing ideas of Castelnuovo, is the least such integer m and is denoted by $\text{reg}(X)$. The interest in this concept stems partly from the well-known fact that X is m -regular if and only if for every $p \geq 0$ the minimal generators of the p -th syzygy module of the defining ideal I of $X \subseteq \mathbb{P}_K^N$ occur in degree $\leq m + p$, see, e.g., [4], [5], [6].

Let k be a nonnegative integer. Then X is called k -Buchsbaum if the graded S -module $M^i(X) = \bigoplus_{\ell \in \mathbb{Z}} H^i(\mathbb{P}_K^N, \mathcal{I}_X(\ell))$, called the deficiency module of X , is annihilated by \mathfrak{m}^k for $1 \leq i \leq \dim(X)$, see, e.g., [17], [18]. Further, we call the minimal nonnegative integer k , if it exists, such that X is k -Buchsbaum, as the Ellia-Migliore-Miró Roig number of X and denote it by $k(X)$. In case X is not k -Buchsbaum for all $k \geq 0$, we put $k(X) = \infty$. It is known that the numbers $k(X)$ are invariant in a liaison class, see, e.g., [17], [24]. Note that $k(X) < \infty$ if and only if X is locally Cohen-Macaulay and equi-dimensional.

In what follows, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to ℓ , and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to ℓ .

In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety X have been given by several authors in terms of $\dim(X)$, $\deg(X)$,

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$\text{codim}(X)$ and $k(X)$, see, e.g., [13], [14], [15], [19], [22], [23]. The following bound, first obtained in [23], is the most optimal among the known results. Even so, whether such a bound is sharp is still a question.

Proposition 1.1 (see [19], [23]). *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N over an algebraically closed field K of characteristic zero. Then we have*

$$\text{reg}(X) \leq \left\lceil \frac{\text{deg}(X) - 1}{\text{codim}(X)} \right\rceil + \max\{k(X) \dim(X), 1\}.$$

The purpose of this paper is to study sharp examples which attain the upper bounds of the inequality in Proposition 1.1 and to show that a projective variety having such property must be a curve on a surface of minimal degree if its degree is large enough.

Theorem 1.2. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N over an algebraically closed field K of characteristic zero. Assume that $k(X) \geq 1$, $\text{deg}(X) \geq 2 \text{codim}(X)^2 + \text{codim}(X) + 2$ and*

$$\text{reg}(X) = \left\lceil \frac{\text{deg}(X) - 1}{\text{codim}(X)} \right\rceil + k(X) \dim(X).$$

Then $\dim(X) = 1$ and X is a curve on a rational ruled surface Y .

The results related to Theorem 1.2 are obtained in [20], [26] for arithmetically Cohen-Macaulay varieties, that is, $k(X) = 0$, especially [20] for the positive characteristic case, and in [28] for arithmetically Buchsbaum curves, that is, $k(X) = 1$ and $\dim(X) = 1$; also see [21] for arithmetically Buchsbaum varieties.

More precisely, we obtain the following classification of the projective variety with its Castelnuovo-Mumford regularity having such upper bound.

Theorem 1.3. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N satisfying the assumptions of (1.2). Then X is a divisor on a rational ruled surface Y constructed as follows:*

Let $\pi : Y = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_K^1$ be a projective bundle, see, e.g., [11, (V.2)], where $\mathcal{E} = \mathcal{O}_{\mathbb{P}_K^1} \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e)$ for some $e \geq 0$. Let Z be a minimal section of π corresponding to the natural map $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}_K^1}(-e)$ and F be a fibre corresponding to $\pi^ \mathcal{O}_{\mathbb{P}_K^1}(1)$. We have an embedding of Y in \mathbb{P}_K^N by a very ample sheaf corresponding to a divisor $H = Z + n \cdot F$ ($n > e$), where $N = 2n - e + 1$. Then X is a divisor on Y linearly equivalent to $a \cdot Z + b \cdot F$ such that $a \geq 1$ and $an + 2 \leq b \leq (a + 2)n - e + 1$.*

In this case, $\text{codim}(X) = 2n - e$, $\text{deg}(X) = a(n - e) + b$, $k(X) = \lfloor (b - an - 2)/(n - e) \rfloor + 1$ and $\text{reg}(X) = \lfloor (b - an - 2)/(n - e) \rfloor + a + 2$.

This result indicates that the inequality

$$\text{reg}(X) \leq \left\lceil \frac{\text{deg}(X) - 1}{\text{codim}(X)} \right\rceil + \max\{k(X), 1\}$$

is sharp for a nondegenerate irreducible reduced projective curve X in \mathbb{P}_K^N over an algebraically closed field K of characteristic zero. In fact, for positive integers c and t with $2 \leq c \leq t - 2$, we take the integers q and r satisfying that $t - 2 = cq + r$ and $0 \leq r \leq c - 1$. Then we define a non-empty set

$$\mathfrak{S}(c, t) = \{1 + \lfloor 2r/\ell \rfloor \mid \ell \in \mathbb{Z}, 2 \leq \ell \leq c\}.$$

Note that every element $k \in \mathfrak{S}(c, t)$ satisfies $1 \leq k \leq r + 1 (\leq c)$.

Theorem 1.4. *Let c , t and k be positive integers with $2 \leq c \leq t - 2$. Let us put a subset $\mathfrak{S}(c, t)$ of \mathbb{Z} as above. Let K be an algebraically closed field.*

- (i) *If $k \in \mathfrak{S}(c, t)$, then there exists a nondegenerate irreducible smooth projective curve X in \mathbb{P}_K^{c+1} with $\deg(X) = t$, $k(X) = k$ and $\operatorname{reg}(X) = \lceil (\deg(X) - 1) / \operatorname{codim}(X) \rceil + k(X)$.*
- (ii) *Assume that $t \geq 2c^2 + c + 2$ and $\operatorname{char}(K) = 0$. If there exists a nondegenerate irreducible reduced projective curve X in \mathbb{P}_K^{c+1} with $\deg(X) = t$, $k(X) = k$ and $\operatorname{reg}(X) = \lceil (\deg(X) - 1) / \operatorname{codim}(X) \rceil + k(X)$, then $k \in \mathfrak{S}(c, t)$.*

Theorem 1.5. *Let K be an algebraically closed field. For any given positive integers c and k with $c \geq k$, there exists a nondegenerate irreducible smooth projective curve X in \mathbb{P}_K^{c+1} with $k(X) = k$ and $\operatorname{reg}(X) = \lceil (\deg(X) - 1) / \operatorname{codim}(X) \rceil + k(X)$.*

These results motivate us to state the following problem.

Problem 1.6. *Does the inequality $\operatorname{reg}(X) \leq \lceil (\deg(X) - 1) / \operatorname{codim}(X) \rceil + k(X)$ hold for a nondegenerate irreducible reduced projective variety X with $k(X) \geq 1$ over an algebraically closed field K ?*

For the case $\dim(X) = 1$ and $\operatorname{char}(K) = 0$, Proposition 1.1 and the theorems in this paper are answers to this problem and show that the inequality is best possible. The theorems give a classification of projective varieties with the regularity bound under the assumption $\deg(X) \gg 0$. However, the assumption is indispensable. In fact, the canonical embedding of a non-hyperelliptic curve C in \mathbb{P}_K^{g-1} with the genus of $g \geq 5$, gives the upper bound of $\operatorname{reg}(C)$, while not contained in any surface of minimal degree, see [28]. On the other hand, you can find how scarce the curves are which achieve the bound. If C is a space curve with the degree bound and the regularity bound, then C is a divisor of either type $(a, a + 2)$ or type $(a, a + 3)$ on a smooth quadric surface from Theorem 1.3. Accordingly we describe the following problem arising from our consideration.

Problem 1.7. *Classify all the nondegenerate irreducible reduced projective curves C with $\operatorname{reg}(C) = \lceil (\deg(C) - 1) / \operatorname{codim}(C) \rceil + \max\{k(C), 1\}$. Or describe the best possible condition that the curve C having the equality above is contained in a surface of minimal degree.*

Finally, we conclude this section by stating Hoa's conjectures.

Conjecture 1.8 ([12]). *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N over an algebraically closed field K . Let k be a positive integer. Assume that, for all $r \geq 0$, the variety $X \cap L$ has the Ellia-Migliore-Miró Roig number $k(X \cap L) \leq k$ for any $(N - r)$ -plane L with $\dim(X \cap L) = \dim(X) - r$, in other words, X is $(k, \dim(X))$ -Buchsbaum by using the terminology of [13], [15]. Then we have*

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + k.$$

Furthermore, assume that $\deg(X)$ is large enough. Then the equality holds only if X is a divisor on a variety of minimal degree.

Throughout this paper we only consider the characteristic zero case. However, if you apply some results of [2], [3], you might partially have the corresponding results in positive characteristic case.

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2. BOUNDS ON CASTELNUOVO-MUMFORD REGULARITY

This section is devoted to the proof of the theorems stated in §1.

First, we describe a sketch of a proof of the upper bound of the Castelnuovo-Mumford regularity, following, e.g., [19, Section 4], in order to make clear what the sharp examples should be.

Let $R = K[R_1]$ be a finitely generated graded algebra over a field K . We denote by \mathfrak{m} the irrelevant ideal of R . Let M be a finitely generated graded R -module with $\dim(M) = d + 1 > 0$. We write $[M]_n$ for the n -th graded piece of M , and $M(p)$ for the graded module with $[M(p)]_n = [M]_{p+n}$. Then, for $i = 0, \dots, d + 1$, we set

$$a_i(M) = \max\{n \mid [H_{\mathfrak{m}}^i(M)]_n \neq 0\}$$

if the max exists, and $a_i(M) = -\infty$ otherwise. In particular, we set $a(M) = a_{d+1}(M)$. The Castelnuovo-Mumford regularity of M is defined as follows:

$$\operatorname{reg}(M) = \max\{a_i(M) + i \mid i = 0, \dots, d + 1\}.$$

For an integer $k \geq 0$, the graded R -module M is called k -Buchsbaum if $\mathfrak{m}^k H_{\mathfrak{m}}^i(M) = 0$ for all $i = 0, \dots, d$. The following result is an easy consequence of the proof of [19, (2.7.2)].

Proposition 2.1. *Let M be a finitely generated graded R -module with $\dim(M) = d + 1 > 0$. Let k be a positive integer. Assume that M is k -Buchsbaum. Then*

$$a_i(M) \leq \max\{a_j(M) + j - i + k \mid j = i + 1, \dots, d + 1\}$$

for $i = 1, \dots, d - 1$, and

$$a_d(M) \leq a(M/hM) + k - 1$$

for any linear parameter $h \in R_1$ for the graded R -module M . Furthermore, the equalities hold only if $a_i(M) \neq -\infty$ and

$$[H_{\mathfrak{m}}^i(R)/hH_{\mathfrak{m}}^i(R)]_{\ell} = 0, \quad \ell \geq a_i(R) - k + 2,$$

for $i = 1, \dots, d$. Consequently, for any integer $1 \leq i \leq d$, we have

$$a_i(M) + i \leq a(M/hM) + d + k(d + 1 - i) - 1,$$

for any linear parameter $h \in R_1$ for the R -module M .

Let X be a projective scheme in $\mathbb{P}_K^N = \operatorname{Proj}(S)$, where S is the polynomial ring $K[x_0, \dots, x_N]$. Let I be the defining ideal $\bigoplus_{\ell \in \mathbb{Z}} \Gamma(\mathbb{P}_K^N, \mathcal{I}_X(\ell))$ of X and R be the coordinate ring S/I of X . Then we see that $\operatorname{reg}(X) = \operatorname{reg}(I) = \operatorname{reg}(R) + 1$. By taking $M = R$ in the above proposition, we have the following bound by using the Ellia-Migliore-Miró Roig number $k(X)$.

Lemma 2.2. *Let X be a projective scheme in \mathbb{P}_K^N . Let R be the coordinate ring of X . Then*

$$\operatorname{reg}(X) \leq a(R/hR) + \dim(X) + \max\{k(X) \dim(X), 1\}$$

for any linear parameter $h \in R_1$.

Now we state a well-known fact, see, e.g., [23, (4.6.b)].

Lemma 2.3. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N with $\dim(X) = d$ over an algebraically closed field K of characteristic zero. Let R be the coordinate ring of X . Then*

$$a(R/h_1R) + d \leq \dots \leq a(R/(h_1, \dots, h_d)R) + 1 \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil$$

for any part of linear system of parameters h_1, \dots, h_d of the graded ring R .

In this way we obtained Proposition 1.1 from Lemma 2.2 and Lemma 2.3, see [19]. Furthermore, the following result has an important role in studying the projective variety having an upper bound on the Castelnuovo-Mumford regularity in the inequality of Proposition 1.1.

Proposition 2.4. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N with $\dim(X) = d$ over an algebraically closed field K of characteristic zero. Let R be the coordinate ring of X . Assume that $k(X) \geq 1$ and the equality in Proposition 1.1 holds, that is, $\operatorname{reg}(X) = \lceil (\deg(X) - 1) / \operatorname{codim}(X) \rceil + k(X)d$. Let h_1, \dots, h_d be a part of linear system of parameters of the graded ring R .*

- (i) $a_i(R) = a_{i+1}(R) + k(X) + 1$ for $1 \leq i \leq d - 1$.
- (ii) $a_d(R) = a_d(R/h_1R) + k(X) - 1$ and $a(R) + 1 \leq a_d(R/h_1R) \leq a(R) + 2$.
- (iii) $[H_m^i(R)/h_1H_m^i(R)]_\ell = 0$ for $1 \leq i \leq d$ and $\ell \geq a_i(R) - k(X) + 2$.
- (iv) $a(R/h_1R) + d = \dots = a(R/(h_1, \dots, h_d)R) + 1 = \lceil (\deg(X) - 1) / \operatorname{codim}(X) \rceil$.

Proof. It follows immediately from (2.1), (2.2) and (2.3). □

Now let us describe a refined result of [16] and [28] on the relationship between a zero-dimensional scheme with uniform position and its h -vectors.

Lemma 2.5. *Let X be a zero-dimensional scheme in uniform position in \mathbb{P}_K^N over an algebraically closed field K . Let R be the coordinate ring of X . Assume that*

$$\deg(X) \geq N^2 + 2N + 2 \quad \text{and} \quad a(R) + 1 = \left\lceil \frac{\deg(X) - 1}{N} \right\rceil.$$

Then X lies on a rational normal curve.

Proof. Let (h_0, \dots, h_s) be the h -vector of the one-dimensional graded ring R . In other words, we write $h_i = \dim_K(R_i) - \dim_K(R_{i-1})$ for all nonnegative integers i , and s for the maximal integer such that $h_s \neq 0$. Note that $h_0 = 1$, $h_1 = N$, $s = a(R) + 1$ and $\deg(X) = h_0 + \dots + h_s$. Suppose that X does not lie on a rational normal curve. By [27, (2.3)], we have that $h_i \geq h_1 + 1$ for all $2 \leq i \leq s - 2$, and $h_{s-1} \geq h_1$. Thus we have

$$\begin{aligned} \frac{\deg(X) - 1}{N} &= \frac{h_1 + \dots + h_s}{h_1} \\ &\geq 1 + \overbrace{\frac{N+1}{N} + \dots + \frac{N+1}{N}}^{s-3} + 1 + \frac{h_s}{N} \\ &= a(R) + \frac{a(R) - 2 + h_s}{N} \\ &\geq a(R) + \frac{a(R) - 1}{N}. \end{aligned}$$

Since $a(R) + 1 \geq (\deg(X) - 1)/N$, we see that $a(R) \leq N + 1$. Hence we have

$$\deg(X) - 1 \leq N(a(R) + 1) \leq N(N + 2),$$

which contradicts the hypothesis. □

Remark 2.6. There is a counterexample in case $\deg(X) = N^2 + 2N + 1$, namely, a complete intersection of type $(2, 2, 4)$ in \mathbb{P}_K^3 , which is pointed out by the referee. So we really need the strong condition on the degree.

Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N over an algebraically closed field K . It is well-known that $\deg(X) \geq \text{codim}(X) + 1$, and that if the equality holds, then X is either (i) a smooth hyperquadric, (ii) the Veronese surface in \mathbb{P}_K^5 , (iii) a rational normal scroll, or their cone, see [10, (3.10)] or [7]. In these cases, X is called a variety of minimal degree. Of course, a rational normal curve is a curve of minimal degree. The next lemma yields an application of Lemma 2.5 to higher dimensional cases through hyperplane section method.

Lemma 2.7. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N with $\dim(X) \geq 1$ over an algebraically closed field K . Assume that X is linearly normal, that is, $H^1(\mathbb{P}_K^N, \mathcal{I}_X(1)) = 0$. If, for infinitely many general hyperplanes H , its hyperplane section $X_0 = X \cap H$ is a divisor on a variety Y_0 of minimal degree with $\Gamma(Y_0, \mathcal{I}_{X_0/Y_0}(2)) = 0$, then X is a divisor on a variety of minimal degree.*

Proof. The defining ideal of the projective variety Y_0 in $H \cong \mathbb{P}_K^{N-1}$ is generated by quadric polynomials. Since X is nondegenerate and linearly normal, we have $\Gamma(\mathbb{P}_K^N, \mathcal{I}_X(2)) \cong \Gamma(\mathbb{P}_K^{N-1}, \mathcal{I}_{X_0}(2))$. On the other hand, $\Gamma(Y_0, \mathcal{I}_{X_0/Y_0}(2)) = 0$ gives an isomorphism $\Gamma(\mathbb{P}_K^{N-1}, \mathcal{I}_{X_0}(2)) \cong \Gamma(\mathbb{P}_K^{N-1}, \mathcal{I}_{Y_0}(2))$. So the defining equations f_1, \dots, f_r of Y_0 can be lifted to polynomials g_1, \dots, g_r with $\varphi(f_1) = g_1, \dots, \varphi(f_r) = g_r$ in $\Gamma(\mathbb{P}_K^N, \mathcal{O}_{\mathbb{P}_K^N}(2))$ through the isomorphism $\varphi : \Gamma(\mathbb{P}_K^{N-1}, \mathcal{I}_{Y_0}(2)) \cong \Gamma(\mathbb{P}_K^N, \mathcal{I}_X(2))$. Let Y be a projective scheme defined by the polynomials g_1, \dots, g_r in \mathbb{P}_K^N . Then Y is the intersection of the quadric hypersurfaces containing X . Note that $\dim(Y) = \dim(X) + 1$. Then there exists an irreducible component Y' of Y such that Y' is a variety of minimal degree with $Y' \cap H = Y_0$, and in fact $Y' = Y$ by showing $\Gamma(\mathbb{P}_K^N, \mathcal{I}_Y(2)) = \Gamma(\mathbb{P}_K^N, \mathcal{I}_{Y'}(2))$. Hence X is a divisor on the projective variety Y of minimal degree, and in this case $Y \cap H = Y_0$. □

In the following we show a useful lemma for the proof of a criterion of the linear normality.

Lemma 2.8. *Let R be a graded ring with $\dim(R) = d + 1 \geq 1$ over a field K , and \mathfrak{m} be the irrelevant ideal of R . Let h be a linear parameter of R . Then*

$$a(R/hR) = \max\{a(R) + 1, n\},$$

where $n = \max\{\ell \mid [H_{\mathfrak{m}}^d(R)/hH_{\mathfrak{m}}^d(R)]_{\ell} \neq 0\}$.

Proof. It immediately follows from the exact sequence:

$$0 \rightarrow H_{\mathfrak{m}}^d(R)/hH_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(R/hR) \rightarrow H_{\mathfrak{m}}^{d+1}(R)[-1] \xrightarrow{\cdot h} H_{\mathfrak{m}}^{d+1}(R).$$

□

Now let us show a criterion of the linear normality which is applied to give a proof of (2.10) on the dimensional induction by combining (2.7) and (2.11).

Lemma 2.9. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N over an algebraically closed field K of characteristic zero. Assume that*

$$\text{reg}(X) = \left\lceil \frac{\deg(X) - 1}{\text{codim}(X)} \right\rceil + \max\{k(X) \dim(X), 1\}$$

and

$$\deg(X) \geq 2 \text{codim}(X)^2 + \text{codim}(X) + 2.$$

Then X is linearly normal, that is, $H^1(\mathbb{P}_K^N, \mathcal{I}_X(1)) = 0$.

Proof. If $k(X) = 0$, then X is, of course, linearly normal. So we may assume that $k(X) \geq 1$. We put $\mathbb{P}_K^N = \text{Proj}(S)$, where S is the polynomial ring and \mathfrak{m} is the irrelevant ideal of S . Suppose that X is not linearly normal. Then there is a nondegenerate projective variety X' in \mathbb{P}_K^{N+1} such that X' is isomorphic to X in \mathbb{P}_K^N by a linear projection. Let R and R' be the coordinate rings of X and X' respectively.

Then we have only to prove that

$$\left\lceil \frac{\deg(X) - 1}{\text{codim}(X)} \right\rceil \leq \left\lceil \frac{\deg(X') - 1}{\text{codim}(X')} \right\rceil + 1.$$

In fact, this inequality yields $(t - 1)/c \leq (t - 1)/(c + 1) + 2 - 1/(c + 1)$, where $t = \deg(X) = \deg(X')$ and $c = \text{codim}(X) = \text{codim}(X') - 1$. Therefore $t \leq 2c^2 + c + 1$, which contradicts the hypothesis.

For the proof of $\lceil (t - 1)/c \rceil \leq \lceil (t - 1)/(c + 1) \rceil + 1$, we have only to show that

$$a(R/hR) \leq a(R'/hR') + 1,$$

where h is a linear parameter for R and R' , because $a(R'/hR') + \dim(X') \leq \lceil (t - 1)/(c + 1) \rceil$ by (2.3) and $a(R/hR) + \dim(X) = \lceil (t - 1)/c \rceil$ by (2.4), (iv).

Note that $H_{\mathfrak{m}}^i(R) \cong H_{\mathfrak{m}}^i(R')$ and $H_{\mathfrak{m}}^i(R/hR) \cong H_{\mathfrak{m}}^i(R'/hR')$ for $i \geq 2$ since R' is a finite R -algebra. In particular, we have $a(R/hR) = a(R'/hR')$ in case $\dim(X) \geq 2$. Hence the assertion is proved for the case $\dim(X) \geq 2$.

Now we may assume that $\dim(X) = 1$. Since $H_{\mathfrak{m}}^1(R')$ is a homomorphic image of $H_{\mathfrak{m}}^1(R)$, we see

$$[H_{\mathfrak{m}}^1(R)/hH_{\mathfrak{m}}^1(R)]_{\ell} = [H_{\mathfrak{m}}^1(R')/hH_{\mathfrak{m}}^1(R')]_{\ell} = 0$$

for $\ell \geq a_1(R) - k(X) + 2$ by (2.4), (iii). Therefore, by using Lemma 2.8, we have

$$a(R/hR) = a(R'/hR') \quad (= a(R) + 1)$$

in case $a(R) = a_1(R) - k(X)$, and

$$a(R/hR) = a(R'/hR') \quad \text{or} \quad a(R'/hR') + 1 \quad (= a(R) + 2)$$

in case $a(R) = a_1(R) - k(X) - 1$, see (2.4), (ii). Hence the assertion is proved. \square

Proposition 2.10. *Let X be a nondegenerate irreducible reduced projective variety in \mathbb{P}_K^N over an algebraically closed field K of characteristic zero. If*

$$\text{reg}(X) = \left\lceil \frac{\deg(X) - 1}{\text{codim}(X)} \right\rceil + \max\{k(X) \dim(X), 1\}$$

and

$$\deg(X) \geq 2 \text{codim}(X)^2 + \text{codim}(X) + 2,$$

then X is a divisor on a variety of minimal degree.

Proof. It follows immediately from (2.5), (2.7), (2.9) and (2.11) by induction on $\dim(X)$. Lemma 2.11 is proved later. \square

By Proposition 2.10 we need to study a divisor X of a variety Y of minimal degree in order to give a classification of the projective varieties having an equality in Theorem 1.2. In case Y is a cone over the projective variety Z either (i), (ii) or (iii) described in the paragraph before (2.7), the divisor X on Y is linearly equivalent to the cone over a divisor X_0 on Z , see, e.g., [11, (II.Exercise 6.3)]. Since $\text{codim}(X) = \text{codim}(X_0)$, $\text{deg}(X) = \text{deg}(X_0)$, $\text{reg}(X) = \text{reg}(X_0)$ and $k(X) = k(X_0)$, the projective variety X cannot be an extremal case. In case Y is a smooth hyperquadric, X is a complete intersection of Y and a hypersurface and so $k(X) = 0$, except the case Y a smooth quadric surface, see, e.g., [11, (II.Exercise 6.5)]. In case Y is the Veronese surface, we see $k(X) = 0$. Since we have only to consider the case $k(X) \geq 1$, the projective variety Y can be assumed to be a rational normal scroll.

Let C be the projective line \mathbb{P}_K^1 . Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}_K^1} \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e_r)$. Let $\pi : Y = \mathbb{P}(\mathcal{E}) \rightarrow C$ be a projective bundle. Let Z be the divisor corresponding to the natural map $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}_K^1}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e_r)$. Then we see $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{O}_Y(Z)$ and $\text{Pic}(Y)$ is a free Abelian group of rank 2 generated by Z and F , where F is a fibre corresponding to $\pi^*\mathcal{O}_{\mathbb{P}_K^1}(1)$. Then we easily have intersection numbers $Z^{r+1} = -e_1 - \cdots - e_r$, $Z^r \cdot F = 1$ and $Z^i \cdot F^{r+1-i} = 0$ for $0 \leq i \leq r - 1$. We consider an embedding of Y in \mathbb{P}_K^N by a very ample divisor $H = Z + n \cdot F$ ($n > e_r$), where $N = rn + r + n - e_1 - \cdots - e_r$. Then Y is called a rational normal scroll.

Let X be an irreducible reduced effective divisor on Y linearly equivalent to $a \cdot Z + b \cdot F$. Since X is nondegenerate, in other words,

$$\Gamma(Y, \mathcal{I}_{X/Y}(1)) = \Gamma(Y, \mathcal{O}_Y((1 - a) \cdot Z + (n - b) \cdot F)) = 0,$$

we may assume that $a = 1$ and $b \geq n + 1$, or $a \geq 2$ and $b \geq 1$. Thus X is a nondegenerate projective variety in \mathbb{P}_K^N , where $N = rn + r + n - e_1 - \cdots - e_r$. Also, we have $\text{codim}(X) = rn + n - e_1 - \cdots - e_r$ and $\text{deg}(X) = (a \cdot Z + b \cdot F) \cdot (Z + n \cdot F)^r = a(rn - e_1 - \cdots - e_r) + b$.

Now let us show the following lemma to finish the proof of Proposition 2.10.

Lemma 2.11. *Let X be an effective divisor of a rational normal scroll Y with the ideal sheaf $\mathcal{I}_{X/Y}$ as the notation above.*

- (i) $\Gamma(Y, \mathcal{I}_{X/Y}(2)) \neq 0$ if and only if $a \leq 2$ and $b \leq 2n$.
- (ii) If $\text{deg}(X) \geq 2 \text{codim}(X) + 1$, then $\Gamma(Y, \mathcal{I}_{X/Y}(2)) = 0$.

Proof. Part (i) follows from isomorphisms

$$\begin{aligned} \Gamma(Y, \mathcal{I}_{X/Y}(2)) &\cong \Gamma(Y, \mathcal{O}_Y((2 - a) \cdot Z + (2n - b) \cdot F)) \\ &\cong \Gamma(C, \pi_* \mathcal{O}_Y((2 - a) \cdot Z + (2n - b) \cdot F)) \\ &\cong \Gamma(C, \text{Sym}^{2-a}(\mathcal{E}) \otimes \mathcal{O}_C(2n - b)). \end{aligned}$$

Part (ii) is an easy consequence of (i). \square

Now we are in the position to get the Castelnuovo-Mumford regularity and the Ellia-Migliore-Miró Roig number of the projective variety. Let S be the polynomial ring $K[x_0, \dots, x_N]$ and \mathfrak{m} be the irrelevant ideal (x_0, \dots, x_N) . Then we put $\mathbb{P}_K^N = \text{Proj}(S)$. Since Y is arithmetically Cohen-Macaulay, the deficiency module $M^i(X)$

of X in \mathbb{P}_K^N , $1 \leq i \leq r$, is isomorphic to $\bigoplus_{\ell \in \mathbb{Z}} H^i(Y, \mathcal{I}_{X/Y}(\ell))$ as graded S -modules. Thus we have

$$M^i(X) \cong \bigoplus_{\ell \in \mathbb{Z}} H^i(Y, \mathcal{O}_Y((-a + \ell) \cdot Z + (-b + \ell n) \cdot F))$$

for $1 \leq i \leq r$. In Lemma 2.12 and Lemma 2.13 we calculate the intermediate cohomologies $H^i(Y, \mathcal{O}_Y((-a + \ell) \cdot Z + (-b + \ell n) \cdot F))$, $1 \leq i \leq r$, and get the number $k(X)$ by considering the structure of the graded S -module $M^i(X)$.

Lemma 2.12. *Under the above condition, assume that $r = 1$.*

- (i) $H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if either $\alpha \geq 0$ and $\beta \leq e_1\alpha - 2$, or $\alpha \leq -2$ and $\beta \geq e_1\alpha + e_1$.
- (ii) X is arithmetically Cohen-Macaulay, that is, $k(X) = 0$ if and only if $an - 2n + e_1 < b < an + 2$.
- (iii) If $b \geq an + 2$, then $k(X) = \lfloor (b - an - 2)/(n - e_1) \rfloor + 1$.
- (iv) If $b \leq an - 2n + e_1$, then $k(X) = \lfloor (an - 2n + e_1 - b)/(n - e_1) \rfloor + 1$.

Proof. In case $\alpha \geq 0$, by isomorphisms

$$\begin{aligned} H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) &\cong H^1(C, \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \\ &\cong H^1(C, \text{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta)) \\ &\cong H^1(\mathbb{P}_K^1, \mathcal{O}_{\mathbb{P}_K^1}(\beta) \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e_1 + \beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_K^1}(-\alpha e_1 + \beta)), \end{aligned}$$

we see that $H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\beta \leq e_1\alpha - 2$. In case $\alpha \leq -2$, by isomorphisms

$$\begin{aligned} H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) &\cong H^0(C, R^1\pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \\ &\cong H^0(C, (\text{Sym}^{-\alpha-2}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1) \otimes \mathcal{O}_C(\beta)) \\ &\cong H^0(\mathbb{P}_K^1, \mathcal{O}_{\mathbb{P}_K^1}(e_1 + \beta) \oplus \mathcal{O}_{\mathbb{P}_K^1}(2e_1 + \beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_K^1}((-\alpha - 1)e_1 + \beta)), \end{aligned}$$

we see that $H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\beta \geq e_1\alpha + e_1$. Similarly, we have $H^1(Y, \mathcal{O}_Y(-Z + \beta \cdot F)) = 0$ for all β . Thus we proved part (i). Part (ii) is an easy consequence of (i). By virtue of these results, the rest of the assertion, (iii) and (iv), immediately follows from a study of the structure of the graded S -module $\bigoplus_{\ell \in \mathbb{Z}} H^1(Y, \mathcal{O}_Y((-a + \ell) \cdot Z + (-b + \ell n) \cdot F))$. In fact, through the surjective homomorphism $S \cong \bigoplus_{\ell \in \mathbb{Z}} \Gamma(\mathbb{P}_K^N, \mathcal{O}_{\mathbb{P}_K^N}(\ell)) \rightarrow \bigoplus_{\ell \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(\ell \cdot Z + \ell n \cdot F))$, the structure of $M^1(X)$ as graded S -module, that is, $S_1 \otimes M^1(X)_\ell \rightarrow M^1(X)_{\ell+1}$ is given by the natural map

$$\begin{aligned} \Gamma(Y, \mathcal{O}_Y(Z + n \cdot F)) \otimes_K H^1(Y, \mathcal{O}_Y((-a + \ell) \cdot Z + (-b + \ell n) \cdot F)) \\ \rightarrow H^1(Y, \mathcal{O}_Y((-a + \ell + 1) \cdot Z + (-b + (\ell + 1)n) \cdot F)). \end{aligned}$$

This K -linear map is a zero map if and only if either of the cohomologies vanishes, by considering the isomorphisms

$$H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \cong H^1(C, \text{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta))$$

for $\alpha \geq 0$ and $H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \cong H^0(C, (\text{Sym}^{-\alpha-2}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1) \otimes \mathcal{O}_C(\beta))$ for $\alpha \leq -2$. In other words, $k(X)$ equals the diameter of $M^1(X)$ in this case, see, e.g., [17] for the definition. Thus, by using (i), we have (iii) and (iv). Therefore the assertion is proved \square

The proof of (2.12) shows that $k(X)$ equals the diameter of $M^1(X)$ for a divisor X on a rational normal scroll, while the corresponding results were shown for a curve on a smooth quadric surface in [18] and for a curve on a smooth cubic surface in [9], although there are lots of curves X with $k(X) < \text{diam}(M^1(X))$ constructed, say, by liaison addition, see [8], [17].

Lemma 2.13. *Under the above condition, assume that $r > 1$.*

- (i) $H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\alpha \geq 0$ and $\beta \leq e_r \alpha - 2$.
- (ii) $H^i(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) = 0$ for $1 < i < r$.
- (iii) $H^r(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\alpha \leq -r - 1$ and $\beta \geq e_r \alpha + re_r - e_1 - \cdots - e_{r-1}$.

Consequently, $a_i(R) = -\infty$ for $1 \leq i \leq r$ unless either $i = 1$ and $b \geq an + 2$, or $i = r$ and $b \leq an - (r + 1)n + e_1 + \cdots + e_r$, where R is the coordinate ring of X .

Proof. First, we note $R^i \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F) = 0$ for $i \neq 0, r$ and

$$H^j(C, R^i \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) = 0 \quad \text{for } j \geq 2.$$

Thus we obtain (ii). In order to prove (i), we have isomorphisms

$$\begin{aligned} H^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) &\cong H^1(C, \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \\ &\cong H^1(C, \text{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta)). \end{aligned}$$

Hence we obtain (i) from an isomorphism $\text{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta) \cong \mathcal{O}_{\mathbb{P}_K^1}(\beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_K^1}(-\alpha e_r + \beta)$ for $\alpha \geq 0$. Finally, for the proof of (iii), we have isomorphisms

$$\begin{aligned} H^r(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) &\cong H^0(C, R^r \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \\ &\cong H^0(C, (\text{Sym}^{-\alpha-r-1}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1 + \cdots + e_r) \otimes \mathcal{O}_C(\beta)). \end{aligned}$$

Hence we obtain (iii) from an isomorphism $(\text{Sym}^{-\alpha-r-1}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1 + \cdots + e_r) \otimes \mathcal{O}_C(\beta) \cong \mathcal{O}_{\mathbb{P}_K^1}(e_1 + \cdots + e_r + \beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_K^1}(e_1 + \cdots + e_{r-1} + (-\alpha - r)e_r + \beta)$ for $\alpha \leq -r - 1$. Therefore the assertion is proved. \square

Furthermore, we need the following lemma to get the Castelnuovo-Mumford regularity of the divisor X on the rational normal scroll Y in \mathbb{P}_K^N .

Lemma 2.14. *Under the above condition, $H^{r+1}(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\alpha \leq -r - 1$ and $\beta \leq -2 - e_1 - \cdots - e_{r-1}$.*

Proof. Since $R^{r+1} \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F) = 0$ and $H^i(C, R^{r+1-i} \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) = 0$ for $i \geq 2$, we have an isomorphism

$$H^{r+1}(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \cong H^1(C, R^r \pi_* \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)).$$

Hence we have the assertion. \square

Now let us prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.10, X is a divisor on a rational normal scroll Y . If we assume that $\dim X = r \geq 2$, then $a_i(R) = -\infty$ for some $1 \leq i \leq r$ by Lemma 2.13, which contradicts Proposition 2.4. Thus we see that X is one-dimensional. Hence the assertion is proved. \square

Accordingly, by Theorem 1.2, we may assume that X is one-dimensional, that is, $r = 1$, and put $e_1 = e$ to finish the proofs of the theorems in §1.

Then we state the following lemmas, (2.15) and (2.16), which are immediate consequences of Lemma 2.12 and Lemma 2.14.

Lemma 2.15. *Under the above condition, assume that $b \geq an + 2$. Then we have $\text{reg}(X) = \lfloor (b - an - 2)/(n - e) \rfloor + a + 2$ and $k(X) = \lfloor (b - an - 2)/(n - e) \rfloor + 1$.*

Lemma 2.16. *Under the above condition, assume that $b \leq an - 2n + e$. Then we have $a_1(R) = a_2(R)$, where R is the coordinate ring of X .*

We also need the following lemma.

Lemma 2.17. *Under the above condition, assume that $b \geq an + 2$. Then $\text{reg}(X) = \lceil (\text{deg}(X) - 1)/\text{codim}(X) \rceil + k(X)$ if and only if $an + 2 \leq b \leq (a + 2)n - e + 1$.*

Proof. By Lemma 2.15, $\text{reg}(X) = \lceil (\text{deg}(X) - 1)/\text{codim}(X) \rceil + k(X)$ if and only if $a + 1 = \lceil (a(n - e) + b - 1)/(2n - e) \rceil$. Since $(a(n - e) + b - 1)/(2n - e) = a + (b - an - 1)/(2n - e)$, we have the assertion. \square

Now let us prove Theorem 1.3, Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.3. By virtue of Theorem 1.2, as in the notation above, X is a divisor linearly equivalent to $a \cdot Z + b \cdot F$ on a rational ruled surface $Y = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}_K^1} \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e)$ on \mathbb{P}_K^1 . Then we see $b \geq an + 2$. In fact, if $an - 2n - e < b < an + 2$, then $k(X) = 0$ by (2.12). If $b \leq an - 2n - e$, then $a(R/hR) = a_1(R) + 1$ by (2.8) and (2.16), which contradicts (2.4), (ii), where R is the coordinate ring of X and h is a linear parameter of R . So we exclude both cases and have only to consider the case $b \geq an + 2$. By Lemma 2.15 and Lemma 2.17, we have $a \geq 1$ and $an + 2 \leq b \leq (a + 2)n - e + 1$. Hence the assertion is proved. \square

Proof of Theorem 1.4. We have only to consider a curve on a rational ruled surface by Theorem 1.2, and follow the notation in Theorem 1.3. By putting $c = 2n - e$ and $t = a(n - e) + b$, we have $n = (c + e)/2$ and $b = t - a(c - e)/2$. By substituting them, we have $ac + 2 \leq t \leq ac + c + 1$ and $e \leq c - 2$ from $an + 2 \leq b \leq (a + 2)n - e + 1$ and $n \geq e + 1$. In particular, $a = \lfloor (t - 2)/c \rfloor$. In order to prove (i), we take the integers q and r such that $t - 2 = qc + r$ and $0 \leq r \leq c - 1$ for given integers c and t . Note that q must be equal to a . Then we can take an integer e such that $k = 1 + \lfloor 2(t - 2 - ac)/(c - e) \rfloor = 1 + \lfloor 2r/(c - e) \rfloor$ if k is an element of $\mathfrak{S}(c, t)$. On the other hand, the linear system $|a \cdot Z + b \cdot F|$ on Y contains an irreducible smooth curve for $a \geq 1$ and $b \geq an + 2$ by [11, (V.2.18)]. Thus there exists a nondegenerate smooth projective curve X with $\text{codim}(X) = c$, $\text{deg}(X) = t$ and $k(X) = k$ such that $\text{reg}(X) = \lceil (\text{deg}(X) - 1)/\text{codim}(X) \rceil + k(X)$. Hence we proved (i). The proof of (ii) is similar to that of (i) and is left to the readers. \square

Proof of Theorem 1.5. For given positive integers c and k with $c \geq k$, we take $e = c - 2$, $n = c - 1$, $a = 1$ and $b = c + k$ and construct a nondegenerate smooth projective curve X as a divisor linearly equivalent to $a \cdot Z + b \cdot F$ on a rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}_K^1} \oplus \mathcal{O}_{\mathbb{P}_K^1}(-e))$ embedded by a very ample divisor $Z + n \cdot F$ to the projective space, as in the notation of Theorem 1.3. Then we have $\text{codim}(X) = c$, $\text{deg}(X) = c + 1 + k$, $k(X) = k$ and $\text{reg}(X) = k + 2$. Hence the assertion is proved. \square

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