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SIMPLE AND SEMISIMPLE LIE ALGEBRAS AND CODIMENSION GROWTH

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Abstract. We study the exponential growth of the codimensions $c_n^L(B)$ of a finite dimensional Lie algebra B over a field of characteristic zero. In the case when B is semisimple we show that $\lim_{n\to\infty} \sqrt[n]{c_n^L(B)}$ exists and, when F is algebraically closed, is equal to the dimension of the largest simple summand of B. As a result we characterize central-simplicity: B is central simple if and only if $\dim B = \lim_{n\to\infty} \sqrt[n]{c_n^L(B)}$.

1. Introduction and the main results

Let A be an associative P.I. algebra. The identities of A, $Id(A)$, form a T-ideal in the free associative algebra $F\langle X\rangle = F\langle x_1, x_2, \dots\rangle : Id(A) \subseteq F\langle X\rangle$. V_n are the multilinear polynomials in $x_1, \dots, x_n$, FS_n is the group algebra of the symmetric group S_n, and $\sigma \longleftrightarrow x_{\sigma(1)} \cdots x_{\sigma(n)}$ gives an isomorphism of FS_n and V_n as left FS_n modules. The cocharacters $\chi_n(A)$ and the codimensions $c_n(A)$ of A are defined via the quotient modules $V_n/(Id(A) \cap V_n)$, [R1], [R2].

The generalization of these invariants to Lie algebras was first considered by Volichenko [V]. Let B be a Lie algebra. Its identities $Id^L(B)$ are elements of the free Lie algebra $L\langle X\rangle$. Let $V_n^L \subseteq L\langle X\rangle$ denote the multilinear Lie polynomials in $x_1, \dots, x_n$; then, in complete analogy to the associative case, $V_n^L/(Id^L(B) \cap V_n^L)$ defines the Lie cocharacter $\chi_n^L(B)$ with the corresponding Lie codimension $c_n^L(B) = \deg \chi_n^L(B)$. We write $\chi_n^L(B) = \sum_{\lambda\vdash n} m_\lambda^L(B)\chi_\lambda$ (see 2.4 for generalizations).

In the associative case, the basic property of the codimensions is that they are exponentially bounded:

Theorem [R1]**.** *Let A be an associative P.I. algebra. Then there exist $k > 0$ and $\alpha \geq 1$ such that*

$$c_n(A) \leq k \cdot \alpha^n$$

for all n.

In this paper we study the Lie codimensions of a finite dimensional Lie algebra over a field F of characteristic zero. It was shown by Volichenko [V] that, in general, Lie codimensions need not be exponentially bounded. However, by the

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Ado-Iwasawa theorem (see [B, 6.2.3]) every finite dimensional Lie algebra B over F has a faithful finite dimensional representation. It follows that $B \subseteq A^{(-)}$, where A is an associative finite dimensional algebra (hence is P.I.) and $A^{(-)}$ is its Lie structure. As we show below, this implies that $\{c_n^L(B)\}$ is also exponentially bounded (see [B]).

In general, $\{c_n^L(B)\}$ seems to be rather complicated, and even its asymptotic behavior is hard to describe. If $c_n^L(B)$ is exponentially bounded, it seems reasonable to conjecture that for some constants a, g and α, $c_n^L(B) \underset{n\to\infty}{\simeq} a \cdot n^g \cdot \alpha^n$. Similar remarks hold in the associative case.

To capture the exponential growth of $\{c_n^L(B)\}$ we introduce

1.1 Definition. Let B be a finite dimensional Lie algebra. Define

$$I\overline{nv}^L(B) = \limsup_{n\to\infty} \sqrt[n]{c_n^L(B)}$$

and

$$\underline{Inv}^L(B) = \liminf_{n\to\infty} \sqrt[n]{c_n^L(B)}.$$

In case of equality, denote

$$I\overline{nv}^L(B) = \underline{Inv}^L(B) = Inv^L(B).$$

Here we study these invariants when B is a simple or semisimple finite dimensional Lie algebra over a field F, $\mathrm{char}(F) = 0$. The main results of this paper are:

1.2 Theorem (7.3 below)**.** *Let F be algebraically closed. Then B is simple if and only if $I\overline{nv}^L(B) = \underline{Inv}^L(B)$ and $Inv^L(B) = \dim B$. Moreover, if B is not simple then $I\overline{nv}^L(B) \leq \dim B - 1$.*

1.3 Theorem (7.1 below)**.** *Let F be algebraically closed, $B = \bigoplus_i B_i$, B_i simple. Let $m_0 = \max_i\{\dim B_i\}$. Then*

$$I\overline{nv}^L(B) = \underline{Inv}^L(B) = Inv^L(B) = m_0.$$

1.4 Theorem (8.1 below)**.** *B is a central simple Lie algebra if and only if*

$$I\overline{nv}^L(B) = \underline{Inv}^L(B) = Inv^L(B) = \dim B.$$

1.5 Theorem (8.2 below)**.** *If B is simple, then $Inv^L(B)$ exists, is a positive integer which divides $\dim B$, and*

$$\dim B = (Inv^L(B)) \cdot [\mathit{centroid}(B) : F].$$

1.6 Theorem (8.3 below)**.** *Let B be semisimple and let $\overline{F}$ be the algebraic closure of F. It is known that $\overline{B} = B \otimes_F \overline{F}$ is also semisimple over $\overline{F}$. Let $\overline{B} = \bigoplus_i \overline{B}_i$, $\overline{B}_i$ simple over $\overline{F}$. Let $m_0 = \max_i\{\dim_{\overline{F}} \overline{B}_i\}$. Then $I\overline{nv}^L(B) = \underline{Inv}^L(B) = Inv^L(B) = m_0 (= Inv^L(\overline{B}))$.*

1.7 Theorem (8.4 below)**.** *If B is not simple, then $I\overline{nv}^L(B) \leq \dim B - 1$.*

1.8 Theorem (8.5 below)**.** *Let $\widetilde{B} = B \otimes_F F[t, t^{-1}] + \langle z \rangle$ be a non-twisted affine Kac-Moody algebra corresponding to B. If B is semisimple, then $Inv^L(\widetilde{B})$ exists and $Inv^L(\widetilde{B}) = Inv^L(B)$ (hence is an integer by 1.6).*

For Jordan algebras M we define $Inv^J(M)$ similarly. In section 10 we calculate $Inv^J(M)$ in some special cases where M are Jordan matrix algebras.

The basic technique for proving that $Inv^L(B) = \alpha$ is to find constants a_1, $a_2 \geq 0$, g_1 and g_2 such that

$$a_2 n^{g_2} \alpha^n \leq c_n^L(B) \leq a_1 n^{g_1} \alpha^n$$

for all n. Both bounds follow from studying the cocharacters $\chi_n^L(B)$: upper bounds follow height-restrictions on the Young diagrams in $\chi_n^L(B)$, as well as from the polynomial bounds on their multiplicities [BR]. An important tool for constructing the lower bounds is a theorem of Razmyslov [Ra, 12.1], which implies the existence in $\chi_n^L(B)$ of a certain rectangular diagram of arbitrary length.

2. The various codimensions

Here we look more closely at Lie codimensions and cocharacters, as well as at some generalizations.

Recall from §1 that $c_n^L(B) = \deg \chi_n^L(B)$, where $\chi_n^L(B)$ is the S_n-character of the left FS_n module $V_n^L/(Id^L(B) \cap V_n^L)$. Denote $\chi_n^L(B) = \sum_{\lambda \vdash n} m_\lambda^L(B) \cdot \chi_\lambda$. Here λ is a partition of n, χ_λ the corresponding S_n-irreducible character, with multiplicity $m_\lambda^L(B)$ in $\chi_n^L(B)$.

Call $[x_1, [x_2, \dots, [x_{n-2}, [x_{n-1}, x_n]] \dots]] = [x_1, \dots, x_n]$ "right normed Lie monomial". It easily follows from the Jacobi-identity that

$$V_{n+1}^L = V_{n+1}^L(x_1, \dots, x_n, x_0) = Span_F\{[x_{\sigma(1)}, \dots, x_{\sigma(n)}, x_0] \mid \sigma \in S_n\}.$$

Both V_n and V_{n+1}^L are left FS_n modules in an obvious way, and the map

$$\varphi : x_{\sigma(1)} \cdots x_{\sigma(n)} \to [x_{\sigma(1)}, \dots, x_{\sigma(n)}, x_0]$$

is an FS_n-module-isomorphism $V_n \simeq_{FS_n} V_{n+1}^L$. In particular, dim $V_{n+1}^L = n!$. Given $f \in V_n$, denote

$$\varphi(f) = f^L = f^L(x_1, \dots, x_n, x_0) \in V_{n+1}^L.$$

Recall the "ad" map for Lie algebras: $ad : B \to adB \subseteq End_F(B)$. We say that the associative polynomial $f(x_1, \dots, x_n) \in F\langle X \rangle$ is an identity of adB if $f(adb_1, \dots, adb_n) = 0$ for all $b_1, \dots, b_n \in B$ (i.e. f is an identity of the pair $(adB,\ End_F(B))$ [Ra]).

2.1 Lemma. *Let B be a Lie algebra and let $f(x_1, \dots, x_n) \in V_n$. Then f is an identity of adB if and only if $f^L(x_1, \dots, x_n, x_0)$ is an identity of L.*

Proof. This lemma easily follows from the equality

$$f(adx_1, \dots, adx_n)(x_0) = f^L(x_1, \dots, x_n, x_0).$$

□

Recall the decomposition $FS_n = \bigoplus_{\lambda \vdash n} I_\lambda$, where the I_λ are minimal two-sided ideals.

2.2 Lemma. *Let B be a Lie algebra. Let $\lambda \vdash n$ with the corresponding two sided ideal $I_\lambda \subseteq FS_n$. Let $f(x_1, \dots, x_n) \in I_\lambda$ be such that $f^L(x_1, \dots, x_n, x_0)$ is a non-identity of B (or equivalently, f is a non-identity of adB). Then $c_{n+1}^L(B) \geq d_\lambda$ (where $\dim I_\lambda = d_\lambda^2$).*

Proof. The FS_n left module isomorphism $V_n \cong V_{n+1}^L$ restricts to $FS_n \cdot f \cong FS_n \cdot f^L$. Since f is a non-identity, $FS_n \cdot f$ contains a minimal left ideal $0 \neq J_\lambda \subseteq I_\lambda$, all of whose non-zero elements are non-identities of adB. Hence, by 2.1, $J_\lambda^L \cap Id^L(B) = 0$, where $J_\lambda^L = \varphi(J_\lambda)$. But $\dim J_\lambda^L = \dim J_\lambda = d_\lambda$; hence $c_{n+1}^L(B) \geq d_\lambda$. □

We shall need the following extension of 2.2:

2.3 Lemma. *Let B be a Lie algebra, $\lambda \vdash n$, $f(x_1, \dots, x_n) \in I_\lambda$, and $0 \leq q \in \mathbb{Z}$, such that $f(x_1, \dots, x_n)x_{n+1} \cdots x_{n+q}$ is a non-identity of adB. Then $c_{n+q+1}^L(B) \geq d_\lambda$.*

Proof. Let $f(x) \in J_\lambda \subseteq I_\lambda$, J_λ as in the proof of 2.2. Then

$$f(x_1, \dots, x_n)x_{n+1} \cdots x_{n+q} \in FS_{n+q}J_\lambda.$$

By the branching rule for FS_n, $FS_{n+q}J_\lambda$ is a sum of irreducible left ideals J_μ (in FS_{n+q}), for partitions $\lambda \subseteq \mu \vdash n+q$. By irreducibility, it follows that there exist such μ and a left ideal J_μ all of whose non-zero elements are non-identities of adB. As in 2.2, this implies that $c_{n+q+1}^L(B) \geq d_\mu$. Finally, again by the branching rule, $d_\mu \geq d_\lambda$. □

Cocharacters and codimensions of other algebraic structures can be defined analogously. For example, we discuss these invariants for S-algebras which are embedded in the associative P.I. algebras.

2.4 S-algebras. Let A be an associative algebra over a field F. Let $\alpha, \beta \in F$ and let $*$ be the operation on A defined by setting $a*b = \alpha ab + \beta ba$, for $a,\ b \in A$. Then $*$ defines a structure of a (not necessarily associative) algebra on the underlying vector space of A. We shall assume that α and β are not both zero, and we shall say that $A(+,*)$ is an S-algebra. The following are the two most significant examples: if $[a,\ b] = ab - ba$, the bracket operation $[\ ,\ ]$ defines a structure of Lie algebra on A, denoted by $A^{(-)}$; also, we may regard A as a special Jordan algebra under the operation $\{a, b\} = ab + ba$ and denoted it by $A^{(+)}$. If B is a subspace of A invariant under $*$, we shall say that B is an S-subalgebra of A.

Let $F_S\langle X\rangle$ be the S-subalgebra of $F\langle X\rangle$ generated by X; $F_S\langle X\rangle$ is the free S-algebra on X. Let V_n^S be the space of multilinear S-polynomials in $x_1, \dots, x_n$. If $f(x_1, \dots, x_n) \in V_n^S$ is a multilinear S-polynomial and $\sigma \in S_n$, then

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

induces a structure of S_n-module on V_n^S. Let $Id^S(B) \subseteq F_S\langle X\rangle$ be the T-ideal of S-identities of B. Then $V_n^S \cap Id^S(B)$ is invariant under the S_n-action, and we denote by $\chi_n^S(B)$ the character of the left S_n-module $\frac{V_n^S}{V_n^S \cap Id^S(B)}$; $\chi_n^S(B)$ is the n-th S-cocharacter of B and $c_n^S(B) = \deg \chi_n^S(B)$ the n-th S-codimension of B. Let $\lambda \vdash n$ with χ_λ the corresponding irreducible S_n-character; then $\chi_n^S(B) = \sum_{\lambda \vdash n} m_\lambda^S(B)\chi_\lambda$, for some multiplicities $m_\lambda^S(B)$.

In section 3 we shall observe some obvious relations between these invariants.

3. Comparison between the various cocharacters

Here we study the relations between the various cocharacters. Those of associative algebras yield upper bounds, bounds that imply:

3.1 Theorem. *Let A be an associative algebra, $B \subseteq A^{(-)}$ a Lie subalgebra (or more generally, $B \subseteq A(+,*)$ an S-algebra). Let $\dim B = m$. Write $\chi_n^L(B) = \sum_{\lambda \vdash n} m_\lambda^L(B)\chi_\lambda$ and $c_n^L(B) = \deg \chi_n^L(B)$. Then:*

1. *There exists r_1, such that for all n and all $\lambda \vdash n$, $m_\lambda^L(B) \leq n^{r_1}$.*
2. *There exist constants C, r_2 such that for all n, $c_n^L(B) \leq C \cdot n^{r_2} \cdot m^n$.*

The proof will follow from

3.2 Lemma. *As in 2.4, let $B \subseteq A(+,*)$ be an S-subalgebra of the associative P.I. algebra A. Write $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda$ and $\chi_n^S(B) = \sum_{\lambda \vdash n} m_\lambda^S(B)\chi_\lambda$.*

Then $m_\lambda^S(B) \leq m_\lambda(A)$ for all λ, i.e.

$$\chi_n^S(B) \leq \chi_n(A).$$

Proof. Let $Q \subseteq F\langle X\rangle$ be the ideal of associative polynomials vanishing when evaluated in B, i.e.,

$$Q = \{f(x_1, \dots, x_k) \in F\langle X\rangle \mid f(b_1, \dots, b_k) = 0, \text{ for all } b_i \in B\}.$$

Since $V_n \cap Q$ is invariant under the S_n action, $\frac{V_n}{V_n \cap Q}$ is a left FS_n-module and we denote by $\chi_n^{as}(B)$ the corresponding S_n character. We shall prove that for all n

$$\chi_n^S(B) \leq \chi_n^{as}(B) \leq \chi_n(A).$$

Since $V_n^S \cap Id^S(B) = V_n^S \cap Q \subseteq V_n \cap Q$, there is a natural monomorphism of FS_n-modules $\frac{V_n^S}{V_n^S \cap Id^S(B)} \to \frac{V_n}{V_n \cap Q}$. Hence $\chi_n^S(B) \leq \chi_n^{as}(B)$. Also, since $Id(A) \subseteq Q$, it follows that there is a natural epimorphism of S_n-modules $\frac{V_n}{V_n \cap Id(A)} \to \frac{V_n}{V_n \cap Q}$. Hence $\chi_n^{as}(B) \leq \chi_n(A)$, and the lemma is proved. □

3.3 Corollary. *If $\chi_n^S(B) = \sum_{\lambda \vdash n} m_\lambda(B)\chi_\lambda$ and $\chi_n^{as}(B) = \sum_{\lambda \vdash n} m'_\lambda(B)\chi_\lambda$, then for all $\lambda \vdash n$, $m_\lambda(B) \leq |\lambda|^r$ and $m'_\lambda(B) \leq |\lambda|^r$, for some fixed r. This proves the first part of 3.1.*

Proof. This follows from the previous theorem since by [BR, Theorem 16] the multiplicities in the n-th cocharacter of an associative P.I.-algebra are polynomially bounded. □

Recall the notation

$$H(k, l; n) = \{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n \mid \lambda_{k+1} \leq l\}.$$

3.4 Lemma. *Let $B \subseteq A(+,*), \dim B = m$, and let $\chi_n^S(A)$ and $\chi_n^{as}(B)$ be as in 3.2. Then*

$$\chi_n^S(B) \leq \chi_n^{as}(B) = \sum_{\lambda \in H(m,0;n)} m'_\lambda(B)\chi_\lambda$$

(and by 3.3 $m'_\lambda(B) \leq |\lambda|^r$ for some fixed r).

Proof. Let $f = f(x_1, \dots, x_{m+1}, y_1, y_2, \dots)$ be multilinear and alternating in $x_1, \dots, x_{m+1}$. Since $\dim\ B = m$, $f = 0$ is an identity for B: $f \in Q$.

Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ with $\lambda_{m+1} \gneqq 0$ and $f(x) \in I_\lambda$. Then f is a linear combination of polynomials, each with an alternating subset of (at least) $m+1$ variables; hence $f \in Q$ (see [R3] for some details), and the proof clearly follows. □

We can now give (the S-generalization of)

3.5 Proof of 3.1.2. By 3.3 and 3.4,

$$\chi_n^S(B) = \sum_{\lambda \in H(m,0;n)} m_\lambda^S(B) \cdot \chi_\lambda,$$

and $m_\lambda^S(B) \leq |\lambda|^{r_1}$ for some fixed r_1. Taking degrees, we obtain

$$c_n^S(B) \leq n^{r_1} \sum_{\lambda \in H(m,0;n)} d_\lambda.$$

The precise asymptotics of $\sum_{\lambda \in H(m,0;n)} d_\lambda, n \to \infty$, is calculated in [R4]:

$$\sum_{\lambda \in H(m,0;n)} d_\lambda \underset{n \to \infty}{\simeq} C \cdot n^{r_2} \cdot m^n$$

for some (explicit) constants C and r_2. This clearly completes the proof. □

4. A Theorem of Razmyslov and multi-alternating polynomials

We begin with an important theorem of Razmyslov. This theorem is essential for proving, in section 6, the lower bounds for the codimensions.

4.1 Theorem [Ra, Thm. 12.1, p. 73]**.** *Let B be a semisimple Lie algebra over an algebraically closed field of characteristic zero, and let* $\dim\ B = m$. *Let $B \subseteq U$ be an associative algebra generated by B, which is simple and with a non-zero center. Then there exist k and a multilinear polynomial $f(x_1^1, \dots, x_m^1, \dots, x_1^k, \dots, x_m^k)$, alternating in each $\{x_1^i, \dots, x_m^i\}$, $1 \leq i \leq k$, such that*

1. *f does not vanish on B, and*
2. *for all $\bar{x}_j^i \in B$, $f(\bar{x}_1^1, \dots, \bar{x}_m^1, \dots, \bar{x}_1^k, \dots, \bar{x}_m^k)$ belongs to the center of U.*

We call the above $f(x)$ "multi-alternating" (or m.a.) and we introduce:

4.2 Definition. Let $m \cdot k \leq n$. Denote by $Q_{m,k,n} \subseteq V_n$ the subspace spanned by all polynomials that are alternating in k disjoint subsets of variables $\{x_1^i, \dots, x_m^i\} \subseteq \{x_1, \dots, x_n\}$, $1 \leq i \leq k$. Then denote $Q_{m,k} = \bigcup_n Q_{m,k,n}$.

Here and in section 5 we deduce some properties of such polynomials.

Let $n = mk$ and let $f \in Q_{m,k,mk}$.

Clearly such a polynomial is a linear combination of "monomial" m.a. polynomials $p_M(x)$. Here $M(x) = M(x_1^1, \dots, x_m^1, \dots, x_1^k, \dots, x_m^k)$ is a multilinear monomial, and

$$p_M(x) = \sum_{\sigma^{(i)} \in S_m} (-1)^{\sigma^{(1)}} \cdots (-1)^{\sigma^{(k)}} \sigma^{(1)} \cdots \sigma^{(k)} M(x),$$

where $\sigma^{(i)}$ permutes the variables $\{x_1^i, \cdots, x_m^i\}$.

Now, the right action of a permutation on a monomial is a position rearrangement of the variables. Hence there exists $\rho \in S_{mk}$ such that

$$M(x) = (x_1^1 \cdots x_m^1 \cdot x_1^2 \cdots x_m^2 \cdots x_1^k \cdots x_m^k)\rho.$$

It follows that

$$p_M(x) = (s_m[x_1^1, \dots, x_m^1] \cdots s_m[x_1^k, \dots, x_m^k])\rho,$$

where $s_m[x_1, \dots, x_m]$ is the standard polynomial. By taking linear combinations we obtain:

4.3 Lemma. *Let $f(x) \in Q_{m,k,mk}$ as above. Then there exists $a \in FS_{km}$ such that*

$$f(x) = (s_m[x_1^1, \dots, x_m^1] \cdots s_m[x_1^k, \dots, x_m^k])a.$$

4.4 Remark. Note that $s_m[x_1, \dots, x_m] \in I_{(1^m)}$. By Young's (or Pieri's) rule, in FS_{km},

$$s_m[x_1^1, \dots, x_m^1] \cdots s_m[x_1^k, \cdots, x_m^k] \in \bigoplus_{\substack{\lambda \vdash km \\ \lambda_1 \leq k}} I_\lambda,$$

where I_λ is the two-sided ideal in FS_{km} corresponding to the partition λ.

We have thus proved

4.5 Proposition. *In FS_{km}*

$$Q_{m,k,mk} \subseteq \bigoplus_{\substack{\lambda \vdash km \\ \lambda_1 \leq k}} I_\lambda \ .$$

Note that if $\lambda \vdash km$ and $\lambda_1 \leq k$, then either $\lambda = (k^m)$, the $k \times m$ rectangle, or $\lambda_1' \geq m+1$, where λ' is the conjugate partition of λ. Thus

4.6 Corollary. *Let A be an associative algebra and let $B \subseteq A$ be a subspace with $\dim B = m$. Let $p(x) \in Q_{m,k,mk}$, and write $p(x) = g(x) + r(x)$, where $g(x) \in I_{(k^m)} = I_{(k \times m)}$ and $r(x) \in \bigoplus_{\lambda \vdash km, \lambda_1' > m} I_\lambda$. Then $p(b) = g(b)$ for all $(b) = (b_1^1, \dots, b_m^k), b_j^i \in B$.*

Proof. Let $\lambda \vdash km$ with $\lambda_1' \gneqq m$, and let $f(x) \in I_\lambda$. As in the proof of 3.4, f is a linear combination of polynomials, each with an alternating subset of at least $m+1$ variables. Hence $f(b) = 0$, and so $r(b) = 0$, for any such $(b) = (b_1^1, \dots, b_m^k)$, $b_j^i \in B$. □

4.7 Remark. Let $\lambda \vdash n \geq km$ be such that $\lambda_m \geq k$ (i.e. λ contains the $k \times m$ rectangle). By (now) standard arguments it follows that $I_\lambda \subseteq Q_{m,k,n}$. (See for example [R3]).

5. An upper bound for the Lie codimensions

By the Ado-Iwasawa theorem (see [B, 6.2.3]) every finite dimensional Lie algebra B over a field has a faithful finite dimensional representation. It follows that $B \subseteq A^{(-)}$, where A is a finite dimensional associative P.I. algebra. We now prove

5.1 Proposition. *Let B be a finite dimensional Lie algebra. Assume adB satisfies all the identities $f \equiv 0$, $f \in Q_{m+1,k}$, for some m and k. Then $c_n^L(B) \leq C \cdot n^r \cdot m^n$ for some constants C and r.*

Proof. Let $\dim B = s$. By 3.3 and 3.4, $\chi_n^L(B) = \sum_{\lambda \in H(s,0;n)} m_\lambda \chi_\lambda$ and $m_\lambda \leq |\lambda|^{r_1}$ for some r_1 and all n's and λ's. By 4.7, if $\lambda_{m+1} \geq k$ then $I_\lambda \subseteq Q_{m+1,k}$; hence I_λ are identities of adB. It follows that $\chi_n^L(B) = \sum_{\substack{\lambda \in H(s,0;n) \\ \lambda_{m+1} \leq k-1}} m_\lambda \chi_\lambda$, so

$$c_n^L(B) \leq n^{r_1} \sum_{\substack{\lambda \in H(s,0;n) \\ \lambda_{m+1} \leq k-1}} d_\lambda.$$

Denote $A(k,m,s,n) = \{\lambda \in H(s,0;n) \mid \lambda_{m+1} \leq k-1\}$. The proof of 5.1 is complete once we prove

5.2 Lemma. *There exist constants C and r such that $\sum_{\lambda\in A(k,m,s,n)} d_\lambda \le C\cdot n^r\cdot m^n$.*

Proof. Let $\lambda\in A(k,m,s,n)$ and denote $\bar\lambda=(\lambda_1,\cdots,\lambda_m)$ and $u=u(\lambda)=\lambda_{m+1}+\cdots+\lambda_s$. Since $k-1\ge\lambda_{m+1}\ge\cdots\ge\lambda_s$, $u\le(s-m)\cdot(k-1)$. By the branching rule, χ_λ appears in $\psi=\chi_{\bar\lambda}\hat\otimes\underbrace{\chi_{(1)}\hat\otimes\cdots\hat\otimes\chi_{(1)}}_{u}$; hence $d_\lambda=\deg\chi_\lambda\le\deg\psi$.

Now $\deg(\chi_\mu\hat\otimes\chi_{(1)})=(|\mu|+1)\deg\chi_\mu$, so $d_\lambda\le\deg\psi\le n^u d_{\bar\lambda}$. Thus

$$\sum_{\lambda\in A(k,m,s;n)} d_\lambda \le \sum_{u=0}^{(s-m)(k-1)} n^u \sum_{\mu\in H(m,0;n-u)} d_\mu\ .$$

By [R4] we have $\sum_{\mu\in H(m,0;n)} d_\mu\simeq C_2 n^{r_2}m^n$ for some (explicit) constants C_2 and r_2, and the proof of 5.2 follows. □

This also completes the proof of Proposition 5.1. □

6. $Inv^L(B)=\dim B$ when $F=\overline{F}$ and B is simple

Here we need to apply 4.1; hence we assume that F is algebraically closed. We prove:

6.1 Theorem. *Let $F=\bar F$ (i.e., F is algebraically closed) and let B be a simple Lie algebra,* $\dim B=m$. *Then there exist constants $C_1,\ C_2>0,\ r_1$ and r_2 such that*

$$C_2\cdot n^{r_2}\cdot m^n\le c_n^L(B)\le C_1\cdot n^{r_1}\cdot m^n$$

for all n. In particular, $\lim_{n\to\infty}\sqrt[n]{c_n^L(B)}=Inv^L(B)=m=\dim B$.

Proof. Note that by simplicity $B\cong adB\subseteq End(B)^{(-)}$; hence Theorem 3.1.2 implies the upper bound.

To prove the lower bound we apply 4.1. Note that by simplicity adB generates the associative (enveloping) algebra $End(B)=U$, and U has a non-trivial center. By 4.1 (with adB replacing B) there exists a multilinear polynomial $f(x_1^1,\dots,x_m^1,\dots,x_1^k,\dots,x_m^k)$, alternating in each $\{x_1^i,\dots,x_m^i\}$, $1\le i\le k$, and f is a central-non-identity of adB. Thus there exist b_j^i, $a\in B$ such that

$$f(adb_1^1,\dots,adb_m^1,\dots,adb_1^k,\dots,adb_m^k)(a)=\alpha a\ne 0,\qquad 0\ne\alpha\in F.$$

Given n, write $n-1=tkm+q$, $0\le q\lneqq km$. Also denote

$$f(x_1^1,\dots,x_m^1,\dots,x_1^k,\dots,x_m^k)=f(x_1,\dots,x_{km}).$$

Let

$$g(x_1,\dots,x_{n-1})=g_{tkm}(x)\cdot x_{tkm+1}\cdots x_{n-1},$$

where $g_{tkm}(x)=f(x_1,\dots,x_{km})f(x_{km+1},\dots,x_{2km})\cdots f(x_{(t-1)km+1},\dots,x_{tkm})$. It easily follows that $g(x_1,\dots,x_{n-1})$ is not an identity of adB.

Clearly, $g_{tkm}(x)$ is alternating in tk disjoint sets of m variables each. As in 4.6, we can decompose $g_{tkm}=g_R+g^*$, where $g_R\in I_{((tk)^m)}$ and $g_{tkm}(x)=g_R(x)$ on adB. Thus g_R is a non-identity on adB. Therefore, by 2.3, $c_n^L(B)\ge d_R=d_{((tk)^m)}$.

Note that, as $n\to\infty$, $n\simeq tkm=|R|$, where R is the partition $R=((tk)^m)$. The proof of 6.1 now follows, since by [R4, F.1.1], $d_R\underset{n\to\infty}{\simeq} u\cdot n^v\cdot m^n$ for some (explicit) constants u and v. □

We remark that 6.1 can be generalized to the following:

6.2 Theorem. *Let B be a finite dimensional Lie algebra. Let H be a semisimple Lie subalgebra, $H \subseteq B$, such that, as adH modules, B has at least one faithful irreducible submodule $M \subseteq B$. Let* $\dim\ H = m$. *Then $c_n^L(B) \geq C \cdot n^r \cdot m^n$ for some constants $C > 0$ and r.*

Proof. Denote $ad : H \to ad_M(H) \subseteq End(M)$. By faithfulness, $H \cong ad_M(H)$, and by simplicity, $ad_M(H)$ generates $End(M)$. The rest of the proof is identical to that of 6.1. □

7. A characterization of simple Lie algebras, $F = \overline{F}$

Theorem 6.1 generalizes to:

7.1 Theorem. *Let $F = \bar{F}$. Let B be a finite dimensional semisimple Lie algebra: $B = \bigoplus_i B_i$, B_i simple. Let $m_0 = \max_i\{m_i\}$, $m_i = \dim B_i$. Then*

$$C_2 n^{r_2} m_0^n \leq c_n^L(B) \leq C_1 n^{r_1} m_0^n$$

for some $C_1,\ C_2 > 0,\ r_1$ and r_2. In particular, $Inv^L(B) = m_0$.

Proof. First, $B_i \subseteq B$; hence $c_n^L(B_i) \leq c_n^L(B)$, and 6.1 (for B_i) implies the lower bound.

Conversely, adB_i satisfies $f \equiv 0$ for all $f \in Q_{m_i+1,1}$, hence for all $f \in Q_{m_0+1,1}$. Therefore B satisfies all $f \in Q_{m_0+1,1}$, and the upper bound follows from 5.1 (note that $B \cong adB$ here). □

7.2 Lemma. *Let B be a finite dimensional Lie algebra which is not semisimple. Denote* $\dim B = m$. *Then $c_n^L(B) \leq C \cdot n^r \cdot (m-1)^n$ for some $c,\ r$.*

Proof. It is well known that such a B contains an abelian ideal $0 \neq N \subset B$ with $[N, N] = 0$. Denote $\bar{B} = B/N$. We clearly have $\dim(ad\bar{B}) \leq \dim\ \bar{B} = \bar{m} \leq m - 1$. This implies that $ad\bar{B}$ satisfies all $f \in Q_{\bar{m}+1,1}$, and we show that adB satisfies all $f \in Q_{\bar{m}+1,2}$.

Indeed, complete a basis of adN ($\subseteq adB$) to a basis of adB. In any $\bar{m} + 1$ distinct elements from that basis there is at least one element from adN. Let $f(x_1^1, \dots, x_{\bar{m}+1}^1, x_1^2, \dots, x_{\bar{m}+1}^2, y_1, y_2, \dots) \in Q_{\bar{m}+1,2}$ be alternating in the two subsets $\{x_1^i, \dots, x_{\bar{m}+1}^i\}$, $i = 1, 2$. By multilinearity, substitutions from adB can be taken from that basis of adB. A repetition in an alternating subset yields zero. Thus, in any "basis" substitution with no such repetitions, there are at least two elements from adN, and hence the corresponding value of f is in $[N, N] = 0$. The proof now follows from 5.1. □

Thus, we obtain the following characterization of the simplicity of B in terms of $Inv^L(B)$:

7.3 Theorem. *Let $F = \bar{F}$, and let B be a finite dimensional Lie algebra. Then B is simple if and only if $Inv^L(B)$ exists (i.e. $\overline{Inv}^L(B) = \underline{Inv}^L(B)$) and $Inv^L(B) =$* $\dim B$. *Moreover, if B is not simple then $\overline{Inv}^L(B) \leq \dim B - 1$.*

Proof. If B is simple, $Inv^L(B) = \dim B$ by 6.1. If B is not simple, then either it is semisimple or B contains an abelian ideal $0 \neq N \subseteq B$.

Case 1: B is semisimple (but not simple); hence $B = \bigoplus_i B_i$ as in 7.1, and $\dim B_i \lneqq \dim B$ for all i. By 7.1, $\overline{Inv}^L(B) \lneqq \dim B - 1$.

Case 2: Let $0 \neq N \subseteq B$ be an abelian ideal: $[N, N] = 0$. By 7.2, $\overline{Inv}^L(B) \lneqq \dim B - 1$ again. □

8. The general case, $\mathrm{Char}(F) = 0$

Simple Lie algebras in characteristic zero are studied and classified, for example, in [Jac, Chap.10]. Here is a short summary of the most basic facts:

If B is a simple Lie algebra over F, then its centroid (i.e., the centralizer of the adjoint action) is a field $E \supseteq F$, and B is central if $E = F$. Let $\bar{F}$ denote the algebraic closure of F; then $B \otimes_F \bar{F}$ is a direct sum of $[E : F]$ copies of a fixed simple Lie $\bar{F}$-algebra.

Let $F \subseteq K$ be a field extension, W an F vector space, and $W_K = W \otimes_F K$.

If B is a semisimple Lie algebra over F, it follows that $B_{\bar{F}}$ is also semisimple.

It is well known that field extensions do not affect multilinear identities; hence the codimensions: $c_n^L(B) = c_n^L(B_K)$. Thus $\overline{Inv}^L(B) = \overline{Inv}^L(B_K), \underline{Inv}^L(B) = \underline{Inv}^L(B_K)$, and hence $Inv^L(B) = Inv^L(B_K)$.

Now let $K = \bar{F}$. The above remarks, together with 7.3, imply:

8.1 Theorem. *Let F be a field of characteristic zero and let B be a finite dimensional Lie algebra over F. Then B is central simple if and only if $Inv^L(B)$ exists and $Inv^L(B) = \dim B$.*

Similarly:

8.2 Theorem. *Let $char(F) = 0$, and let B be a finite dimensional Lie algebra over F. If B is simple, then $Inv^L(B)$ exists, is a positive integer which divides $\dim B$, and*

$$\frac{\dim B}{Inv^L(B)} = [centroid\ (B) : F].$$

Proof. We have $B \otimes \bar{F} = B_1 \oplus \cdots \oplus B_j$, where $j = [Centroid\ (B) : F]$ and $B_1 \cong \cdots \cong B_j$ are $\bar{F}$-simple. Clearly, $\dim B_1 = \frac{\dim B}{j}$, and by 7.1, $Inv^L(B) = Inv^L(B_{\bar{F}}) = \dim B_1$. □

In the semisimple case, the above remarks and 7.1 imply

8.3 Theorem. *Let $char(F) = 0$, and let B be a finite dimensional semisimple Lie algebra over F. Then $Inv^L(B)$ exists and is an integer.*

In fact, let $\overline{F}$ be the algebraic closure of F and write $B_{\overline{F}} = \bigoplus_i \widetilde{B}_i$, where the $\widetilde{B}_i$'s are simple (over $\overline{F}$). Let $m_0 = \max_i\{\dim \widetilde{B}_i\}$.

Then $Inv^L(B) = Inv^L(B_{\overline{F}}) = m_0$.

We also have

8.4 Proposition. *Let B be a finite dimensional Lie algebra over F. If B is not simple, then $\overline{Inv}^L(B) \leq \dim_F B - 1$.*

Proof. If B is not simple, it easily follows that $B_{\bar{F}}$ is not simple. Hence, by 7.3,

$$\overline{Inv}^L(B) = \overline{Inv}^L(B_{\bar{F}}) \leq \dim_{\bar{F}}(B_{\bar{F}}) - 1 = \dim_F B - 1.$$

□

We conclude this section with the following corollary.

Let G be a finite-dimensional semisimple Lie algebra, $F[t,t^{-1}]$ the ring of Laurent polynomials and $\tilde{G} = G \otimes F[t,t^{-1}] + \langle z \rangle$ a non-twisted affine Kac-Moody algebra corresponding to G.

8.5 Corollary. *If G is a finite-dimensional semisimple Lie algebra and $\tilde{G}$ a non-twisted affine Kac-Moody algebra corresponding to G, then $Inv^L(\tilde{G}) = Inv^L(G) = k$, where k is a positive integer.*

The proof follows immediately from the previous theorem and from the construction of the finite-dimensional Lie algebra $\bar{G}$ such that var $\tilde{G}$ = var $\bar{G}$ (see [Z]). The corollary may also be proved directly, since $\tilde{G}$ is a central extension of $G \otimes F[t,t^{-1}]$ and this last has the same identities of G. Also, it is not difficult to show that if A is a central extension of B, then $Inv^L(A) = Inv^L(B)$.

9. Some special Jordan algebras

In this section we compute the exponential growth of some special Jordan algebras of small dimension. Define $\{x_1, \{x_2, \dots, \{x_{n-1}, x_n\}, \dots\}\} = \{x_1, \dots, x_n\}$, right normed Jordan monomials. In this case the map

$$\phi : x_{\sigma(1)} \cdots x_{\sigma(n)} \to \{x_{\sigma(1)} \cdots x_{\sigma(n)}, x_0\}$$

induces an FS_n-module monomorphism $V_n \to V_{n+1}^S$, and the analogue of 2.2 still holds, where the V_{n+1}^S are now multilinear Jordan polynomials in $x_0, x_1, \dots, x_n$.

Let $M_n(F)$ be the algebra of $n \times n$ matrices over the field F. We denote by $M_n(t)^+$ and $M_n(s)^+$ the Jordan subalgebras of symmetric elements of $M_n(F)$ under the transpose and symplectic involution respectively. We have

9.1 Theorem. 1. *For $k = 2, 3$, $Inv^J(M_k(t)^+) = \dim M_k(t)^+ = k(k+1)/2$.*

2. *For $k = 2, 4$, $Inv^J(M_k(s)^+) = \dim M_k(s)^+ = k(k-1)/2$.*

Proof. By 3.1, $\overline{Inv}^J(M_k(*)^+) \leq \dim M_k(*)^+$ where $* = t$ or s.

Let L_x denote the left Jordan multiplication by x. For $k = 2$ take $a_1 = e_{11} - e_{22}$, $a_2 = e_{11} + e_{22}$, $a_3 = e_{12} + e_{21}$ as basis of $M_2(t)^+$, let $n = 3l + t$, $0 \leq t < 3$, and let

$$f(x_1, \dots, x_n) = s_3(x_1, x_2, x_3) s_3(x_4, x_5, x_6) \cdots s_3(x_{3l-2}, x_{3l-1}, x_{3l}) x_{3l+1} \cdots x_{3l+t},$$

where s_3 denotes the standard polynomial of degree three. It can be checked that $s_3(L_{a_1}, L_{a_2}, L_{a_3})^k L_{a_1}^t(a_1) \neq 0$. Thus $f(x_1, \dots, x_n)$ is not an identity on $\{L_a \mid a \in M_2(t)\}$ and, by 2.3, $c_{n+1}^J(M_2(t)^+) \geq d_\lambda$, where $\lambda = (k^3)$. It follows that $\underline{Inv}^J(M_2(t)^+) \geq 3$, and this, combined with the above, gives $Inv^J(M_2(t)^+) = 3$.

To prove (1) for $k = 3$, let $\mu = (2^6) \vdash 12$ and let T_μ be the following tableau:

1	2
3	6
4	7
5	8
9	11
10	12

Let $f(x_1, \dots, x_{12}) = e_{T_\mu}(x_1, \dots, x_{12})$ be the corresponding (associative) polynomial.

For every natural n write $n = 12l + t$, where $0 \le t < 12$, and let

$$g(x_1, \dots, x_n) = f(x_1, \dots, x_{12}) \cdots f(x_{12(l-1)+1}, \dots, x_{12l}) x_{12l+1} \cdots x_n.$$

Consider the basis $\{a_1 = e_{11},\ a_2 = e_{22},\ a_3 = e_{33},\ a_4 = e_{12} + e_{21},\ a_5 = e_{13} + e_{31},\ a_6 = e_{23} + e_{32}\}$ of $M_3(t)^+$ and the substitution $\phi : x_1, x_2 \to L_{a_1},\ x_3, x_6 \to L_{a_2},\ x_4, x_7 \to L_{a_3},\ x_5, x_8 \to L_{a_4},\ x_9, x_{11} \to L_{a_5},\ x_{10}, x_{12} \to L_{a_6}$. By a computer calculation it can be checked that $\phi(f)(a_4) = \alpha a_4$ for some non-zero integer α. It follows that $g(x_1, \dots, x_n)$ is not an identity on $\{L_a \mid a \in M_3(t)\}$, and the conclusion follows as before.

2) is proved similarly. □

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