SOME PROPERTIES OF PARTITIONS
IN TERMS OF CRANK

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Abstract. Let \( N(r, m, n) \) (resp. \( M(r, m, n) \)) denote the number of partitions of \( n \) whose ranks (resp. cranks) are congruent to \( r \) modulo \( m \). Atkin and Swinnerton-Dyer gave the relations between the numbers \( N(r, m, mn+k) \) when \( m = 5, 7 \) and \( 0 \leq r, k < m \). Garvan gave the relations between the numbers \( M(r, m, mn+k) \) when \( m = 5, 7, \) and \( 11, 0 \leq r, k < m \). Here, we show that the methods of Atkin and Swinnerton-Dyer can be extended to prove the relations for the crank.

1. Introduction

Let \( N(m, n) \) and \( M(m, n) \) denote the number of partitions of \( n \) with rank and crank \( m \), respectively. We change this definition of \( M(m, n) \) just a little, setting \( M(0; 1) = 1 \) and \( M(1; 1) = 1 = M(1, 1) \), and modify \( M(r; m; n) \) accordingly. We shall also suppose that the empty partition of 0 has rank 0.

For convenience, we write \( N_1 \) for \( M_1 \) and \( N_3 \) for \( N_3 \). So, by (12) of [9] and (1.11) of [2] when \( k = 1 \), and by (2.12) of [3] when \( k = 3 \), we have

\[
\sum_{m} \sum_{n \geq 0} N_k(m, n) z^m q^n = \left( \prod_{r=1}^{\infty} \frac{1}{1 - q^r} \right) (1 - z) \sum_{n} (-1)^n q^{n(kn+1)/2} \frac{1}{1 - zq^n}.
\]

Here, and below, \( \sum_{n} \) denotes a sum over all integers \( n \), while \( \sum'_{n} \) denotes a sum over all non-zero integers \( n \).

For odd positive integers \( k \), defining \( N_k(m, n) \) by (1.1), we observe that \( N_k(m, n) \geq 0 \) for almost all \( n \). If we put \( z = 1 \) in (1.1), we find that

\[
\sum_{m} N_k(m, n) = p(n),
\]

where \( p(n) \) is the number of partitions of \( n \), and, replacing \( z \) by \( z^{-1} \) in (1.2),

\[
N_k(-m, n) = N_k(m, n).
\]

Thus, one can ask whether there is a “k-rank” (1-rank = crank, 3-rank = rank) such that \( N_k(m, n) \) counts the number of partitions of \( n \) with k-rank \( m \). This question has been answered by Garvan [8].

In [7], Garvan found a number of relations for the crank modulo 5, 7 and 11 analogous to those found by Dyson [5] for the rank modulo 5 and 7. These relations were proved by Garvan using various results for theta functions, including

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Winquist’s identity [11]. Dyson’s results were proved by Atkin and Swinnerton-Dyer [3] by relating certain mock-theta-like functions with elliptic-theta function identities. In this paper we show how Atkin and Swinnerton-Dyer’s method may be extended to prove Garvan’s crank relations. In the process we find some new relations for mock-theta-like functions.

Since we use the method of Atkin and Swinnerton-Dyer [3], we adopt their notations:

\[ m \] will take the values 5, 7, and 11, and the variables \( q \) and \( y \) are always related by the equation

\[ y = q^m. \]

We shall regard any power series in \( q \) as a polynomial of degree \( m - 1 \) in \( q \) whose coefficients are power series in \( y \). Thus, any identity between two power series in \( q \) can be regarded, on equating coefficients of powers of \( q \), as equivalent to \( m \) identities between power series in \( y \). We write

\[
P(z, q) := \prod_{r=1}^{\infty} (1 - zq^{r-1})(1 - z^{-1}q^r),
\]

so that \( P(z, q) \) is a single-valued analytic function of \( z \) in any ring-shaped region \( 0 < z_1 \leq |z| \leq z_2 \), and satisfies

\[
P(z^{-1}, q) = P(z, q), \quad P(zq, q) = -z^{-1}P(z, q).
\]

We also write

\[
P(a) := P_m(a) := P(y^a, y^m) = \prod_{r=1}^{\infty} (1 - y^{m(r-1)+a})(1 - y^{mr-a}),
\]

\[
P(0) := P_m(0) := \prod_{r=1}^{\infty} (1 - y^{mr}),
\]

where \( a \) is not a multiple of \( m \). It should be noted that \( P(0) \) is not the expression that would be obtained by writing 0 instead of \( a \) in the definition of \( P(a) \). From (1.5), we have

\[
P(m - a) = P(a), \quad P(-a) = P(m + a) = -y^{-a}P(a),
\]

which we shall use without explicit mention below.

2. Preparation

We define

\[
M(r) := \sum_{n \geq 0} M(r, m, n)q^n
\]

and

\[
S(a) := \sum_{n} (-1)^n \frac{q^{n(n+1)/2+an}}{1 - q^{mn}}.
\]

The power series \( S(a) \) differs only from (6.1) of [3] in that 3 is replaced by 1, which appears in the power of \( q \) in the numerator of (6.1) of [3]. As in [3], writing \( -n \) for \( n \) in (2.2), we find that

\[
S(a) = -S(m - 1 - a).
\]
This gives

\[(2.4) \quad S\left(\frac{m-1}{2}\right) = 0.\]

Replacing \(z\) by a primitive \(m\)-th root of unity in \((1.1)\), we can see that

\[(2.5) \quad M(0) = F(S(0) - S(m - 1) + 1),\]

\[M(r) = F(S(r) - S(r - 1)), \quad (r = 1, 2, 3, \ldots, m - 1),\]

where \(F = \prod_{r=1}^{\infty}(1 - q^r)^{-1}\) (see \([10]\) for details). To express the power series \(S(a)\) as a polynomial in \(q\) of degree \(m - 1\) in \(q\) whose coefficients are power series in \(y\), we define (for any complex number \(\zeta\))

\[(2.6) \quad T(z, \zeta, q) := \sum_n (-1)^n \frac{z^n q^{n(n+1)/2}}{1 - z q^n},\]

which is obviously an analytic function of \(z\) in every region \(0 < r_1 \leq |z| \leq r_2\) except for simple poles at point \(z = q^n\). Also, let

\[(2.7) \quad T^*(\zeta, q) := \sum_n (-1)^n \frac{z^n q^{n(n+1)/2}}{1 - q^n}.\]

Following the similar proof of (6.6) of \([9]\), we find

\[(2.8) \quad S\left(\frac{m-1}{2}\right) - b = (-1)^b b^{(m-b)/2} T(q^m, 1, q^2) + T^*(q^{-m}, q^2) + q^{mb} T(q^{2mb}, q^m, q^2) + \sum_{a \neq 1 \mod (m)} (-1)^{a+b} q^{(a-b)(a+b-m)/2} \left\{ q^{ma} T(q^{m(a+b)}, q^m, q^2) + T(q^{m(a-b)}, q^{-ma}, q^2) \right\}.\]

It follows from Lemma 2 of \([9]\) that

\[(2.9) \quad \zeta T(z\zeta, \zeta, q) + T(z\zeta^{-1}, \zeta^{-1}, q) = \frac{P(z; q) P(\zeta^2; q) F^{-2}}{P(z\zeta^{-1}; q) P(z\zeta; q) P(\zeta; q)},\]

and that

\[(2.10) \quad T(z, 1, q) = \frac{F^{-2}}{P(z; q)}.\]

We now write

\[(2.11) \quad h(z; q) := T^*(z^{-1}, q) + z T(z^2, z, q).\]

**Lemma 1.**

(i) \(h(z; q) - h(zq; q) = 1,\)

(ii) \(h(z; q) + h(z^{-1}; q) = 0,\)

(iii) \(h(z; q) + h(z^{-1}; q) = -1,\)

(iv) \(3h(z; q) - h(z^3; q) = \frac{P^3(z^2; q) F^{-2}}{P^3(z; q) P(z^3; q)} - \frac{P^3(z^4; q) F^{-2}}{P^3(z^2; q) P(z^6; q)}.\)
Proof. (i). By Jacobi’s triple product identity (Thm. 2.8 in [2]) we have
\begin{equation}
- P(z^{-1}; q)F^{-1} = - \sum_{n} (-1)^n \frac{z^n q^{n(n-1)/2}}{1 - q^n} + \sum_{n} (-1)^n \frac{z^n q^{n(n+1)/2}}{1 - q^n} - 1
\end{equation}
and
\begin{equation}
P(z^{-1}; q)F^{-1} = \sum_{n} (-1)^n \frac{z^n q^{n(n+1)/2}}{1 - z^2 q^{n+1}} \frac{1 - z^2 q^{n+1}}{1 - q^n}
\end{equation}
(2.13)

\begin{equation}
= \sum_{n} (-1)^n \frac{z^n q^{n(n+1)/2}}{1 - z^2 q^{n+1}} - \sum_{n} (-1)^n \frac{z^{n+2} q^{(n+1)(n+2)/2}}{1 - z^2 q^{n+1}}
\end{equation}
(replace $n$ by $n + 1$) \quad (replace $n$ by $n - 1$)

\begin{equation}
= -zq \sum_{n} (-1)^n \frac{(zq)^n q^{n(n+1)/2}}{1 - z^2 q^{n+2}} + z \sum_{n} (-1)^n \frac{z^n q^{n(n+1)/2}}{1 - z^2 q^n}.
\end{equation}

If we add (2.12) and (2.13), we have (i). (ii) is trivial and (iii) is consequence of (i) and (ii).

For (iv), let $f_L(z)$ and $f_R(z)$ denote the left and right sides of (iv). Then (i) shows that $f_L(zq) = f_L(z)$. By (1.5), $f_R(zq) = f_R(z)$. Now, $f_L(z) - f_R(z)$ is free from poles and so, by Lemma 2 of [3], either is free from zeros or is identically zero. By using (1.5), we find that $f_R(z^{-1}) = -f_R(z)$, and by (ii) $h(z^{-1}; q) = -h(z; q)$. Therefore $z = 1$ is a zero of $f_L(z) - f_R(z)$.

Consequently, with the help of (2.9)-(2.11) with $z = q^m a$, $\zeta = q^m b$, and $q^m z$ for $q$, we have
\begin{equation}
S(\frac{m-1}{2} - b) = h(q^m b; q^m z) + (-1)^b q^{b(m-b)/2} \frac{P_{m}(0)}{P_m(b)}
\end{equation}
\begin{equation}
+ \sum_{a = 1 \atop a \neq 3 \bmod(m)}^{(m-1)/2} (-1)^{a+b} q^{(a-b)(a+b-m)/2} \frac{P_{m}(b) P_{m}(2a) P_{m}(0)}{P_m(b-a) P_m(b+a) P_m(a)}.
\end{equation}

We shall need the following, which is Lemma 6 in [3]:
\begin{equation}
F^{-1} = (-1)^{\lambda} q^{\lambda(3\lambda + \mu)/2} P_m(0) \left[ 1 + \sum_{c=1}^{(m-1)/2} (-1)^c q^{c(3c-m)/2} P_{m}(2c) \frac{P_{m}(2c)}{F_m(c)} \right],
\end{equation}
where $m = 6\lambda + \mu$ , $\lambda$ is a positive integer and $\mu = \mp 1$.

We are now in a position to state and prove the results for crank of ordinary partition in the cases of modulo 5, 7 and 11. For convenience, we shall write
\begin{equation}
R_{ij}(k) := \sum_{n \geq 0} (M(i, m, mn + k) - M(j, m, mn + k)) y^n,
\end{equation}

and also
\begin{equation}
h(a) := h(y^n ; y^m),
\end{equation}
so that Lemma 1, with $q$ replaced by $y^m$ and $z$ by $y^a$, states that

\begin{align}
(2.18) \quad & h(a) - h(m + a) = 1, \\
(2.19) \quad & h(a) + h(m - a) = -1, \\
(2.20) \quad & 3h(a) - h(3a) = \frac{P^3(2a) P^2(0)}{P^3(a) P(3a)} - \frac{P^3(4a) P^2(0)}{P^3(2a) P(6a)}.
\end{align}

3. Some results for cranks modulo 5

Taking $m = 5$ in (2.5), with the help of (2.3) and (2.4), we find that

\begin{align}
M(0) &= F(2S(0) + 1), \\
M(1) &= F(S(1) - S(0)), \\
M(2) &= -F(S(1)).
\end{align}

Taking $m = 5$ and $b = 2, 1$ in (2.14), we have

\begin{align}
S(0) &= h(2) - q \frac{P^2(0) P(2)}{P^2(1)} + q^3 \frac{P^2(0)}{P(2)}, \\
S(1) &= h(1) - q^2 \frac{P^2(0)}{P(1)} + q^4 \frac{P^2(0) P(1)}{P^2(2)}.
\end{align}

Putting $a = 1$ in (2.18) and (2.20) with $m = 5$, and $a = 2$ in (2.19) and (2.20) with $m = 5$, we also obtain

\begin{align}
(3.3) \quad & h(1) = \frac{1}{5} + \frac{1}{5} \left\{ \frac{P^2(0) P^2(2)}{P^3(1)} + 2q \frac{P^2(0) P^2(1)}{P^3(2)} \right\}, \\
(3.4) \quad & h(2) = -\frac{2}{5} + \frac{1}{5} \left\{ 2 \frac{P^2(0) P^2(2)}{P^3(1)} - q \frac{P^2(0) P^2(1)}{P^3(2)} \right\}.
\end{align}

After all these preparations the following is easily proven.

Theorem 1.

\begin{align}
(3.4) \quad & R_{01}(0) = \frac{P(2) P(0)}{P^2(1)}, \\
(3.5) \quad & R_{01}(1) = -2 \frac{P(0)}{P(1)}, \\
(3.6) \quad & R_{12}(1) = \frac{P(0)}{P(1)}, \\
(3.7) \quad & R_{12}(2) = -\frac{P(0)}{P(2)}, \\
(3.8) \quad & R_{01}(3) = -R_{12}(3) = \frac{P(1) P(0)}{P^2(2)}
\end{align}

and all other functions $R_{b,b+1}(d)$, where $b = 0$ or $1$, are zero.

By (3.1), to prove the theorem we only have to show that

\begin{align}
3S(0) - S(1) + 1 &= \left\{ \frac{P(0) P(2)}{P^2(1)} - 2q \frac{P(0)}{P(1)} + q^3 \frac{P(0) P(1)}{P^2(2)} \right\} F^{-1}, \\
2S(1) - S(0) &= \left\{ q \frac{P(0)}{P(1)} - q^2 \frac{P(0)}{P(2)} - q^3 \frac{P(0) P(1)}{P^2(2)} \right\} F^{-1}.
\end{align}
Since by (2.15) we have
\[ F^{-1} = P(0) \left\{ \frac{P(2)}{P(1)} - q - q^2 \frac{P(1)}{P(2)} \right\}, \]
these are respectively equivalent to
\[
3h(2) - h(1) + 1 = \frac{P^2(0)P^2(2)}{P^3(1)} - y \frac{P^2(0)P^2(1)}{P^3(2)},
\]
\[
2h(1) - h(2) = y \frac{P^2(0)P^2(1)}{P^3(2)},
\]
which are true by (3.3). This proves the theorem.

4. Some results for cranks modulo 7

Here and in the next section we need the following for simplifications, which is Lemma 4 of [3].

\[ P^2(b)P(c + d)P(c - d) - P^2(c)P(b + d)P(b - d) \]
\[ + y^{c-d}P^2(d)P(b + c)P(b - c) = 0, \]
where none of \( b, c, d, b \div c, c \div d, b \div d \) is divisible by \( m \). This gives, for \( (b, c, d) = (3, 2, 1) \),

\[ P(1)P^3(3) - P(3)P^3(2) + yP(2)P^3(1) = 0. \]

As in the previous section, taking \( m = 7 \) in (2.5), with the help of (2.3) and (2.4) we find that

\[ M(0) = F(2S(0) + 1), \]
\[ M(1) = F(S(1) - S(0)), \]
\[ M(2) = F(S(2) - S(1)), \]
\[ M(3) = -F(S(2)). \]

Taking \( m = 7 \) and \( b = 3, 2, \) and 1 in (2.14), we have

\[ S(0) = h(3) - q \frac{P^2(0)P^2(3)}{P^2(2)P^1(1)} + q^3 \frac{P^2(0)}{P^1(1)} - q^6 \frac{P^2(0)}{P^1(1)}, \]

\[ S(1) = h(2) - q^2 \frac{P^2(0)P^2(2)}{P^2(1)P^3(3)} + q^5 \frac{P^2(0)}{P^1(2)} + q^6 \frac{P^2(0)}{P^1(2)}, \]

\[ S(2) = h(1) - q^3 \frac{P^2(0)}{P^1(1)} + q^5 \frac{P^2(0)P^2(1)}{P^2(3)P^1(2)} - q^{11} \frac{P^2(0)P^2(1)}{P^2(3)P^1(2)}. \]

Putting \( a = 1, 2 \) and 3, respectively, in (2.18) - (2.20) with \( m = 7 \), and using (4.2), we obtain

\[ h(1) = -\frac{1}{7} + \frac{1}{7}P^2(0) \left\{ \frac{P(3)}{P^2(1)} + 3y \frac{P(1)}{P^2(2)} + 2y \frac{P(2)}{P^2(3)} \right\}, \]

\[ h(2) = -\frac{2}{7} + \frac{1}{7}P^2(0) \left\{ \frac{2P(3)}{P^2(1)} - y \frac{P(1)}{P^2(2)} - 3y \frac{P(2)}{P^2(3)} \right\}, \]

\[ h(3) = -\frac{3}{7} + \frac{1}{7}P^2(0) \left\{ \frac{3P(3)}{P^2(1)} + 2y \frac{P(1)}{P^2(2)} - y \frac{P(2)}{P^2(3)} \right\}. \]
After all these preparations the following is easily proven.

**Theorem 2.**

\[
R_{01}(0) = \frac{P(3)P(0)}{P(1)P(2)},
\]
\[
R_{01}(1) = -2\frac{P(0)}{P(1)},
\]
\[
R_{12}(1) = \frac{P(0)}{P(1)},
\]
\[
R_{12}(2) = -R_{23}(2) = -\frac{P(2)P(0)}{P(1)P(3)},
\]
\[
R_{01}(3) = -R_{23}(3) = \frac{P(0)}{P(2)},
\]
\[
R_{01}(4) = -R_{12}(4) = \frac{P(0)}{P(3)},
\]
\[
R_{01}(6) = -R_{12}(6) = R_{23}(6) = -\frac{P(1)P(0)}{P(2)P(3)},
\]
and all other functions \(R_{b,b+1}(d)\), where \(0 \leq b \leq 2\), are zero.

To prove the theorem we only consider the three pairs of values \((i,j) = (0,1), (1,2) and (2,3)\) in (2.10). So, by (4.6), we only have to show that

\[
3S(0) - S(1) + 1 = \left\{ \frac{P(0)P(3)}{P(1)P(2)} - 2q\frac{P(0)}{P(1)} + q^3\frac{P(0)}{P(2)} + q^4\frac{P(0)}{P(3)} - q^6\frac{P(0)P(1)}{P(2)P(3)} \right\} F^{-1},
\]
\[
2S(1) - S(2) - S(0) = \left\{ q\frac{P(0)}{P(1)} - q^2\frac{P(0)P(2)}{P(1)P(3)} - q^4\frac{P(0)}{P(2)} + q^6\frac{P(0)P(1)}{P(2)P(3)} \right\} F^{-1},
\]
\[
2S(2) - S(1) = \left\{ q^2\frac{P(0)P(2)}{P(1)P(3)} - q^3\frac{P(0)}{P(2)} - q^6\frac{P(0)P(1)}{P(2)P(3)} \right\} F^{-1}.
\]

Now by (2.10) we have

\[
F^{-1} = P(0)\left\{ \frac{P(2)}{P(1)} - q\frac{P(3)}{P(2)} - q^2 + q^5\frac{P(1)}{P(3)} \right\}.
\]

Substituting this in each of (4.13)\textendash(4.15) and equating coefficients of powers of \(q\), we have 21 equations to prove. The coefficients of \(q^b\) give us respectively

\[
3h(3) - h(2) + 1 = P^2(0)\left\{ y\frac{P(1)}{P^2(2)} + \frac{P(3)}{P^2(1)} \right\},
\]
\[
2h(2) - h(1) - h(3) = -P^2(0)\left\{ \frac{P(2)}{P^2(3)} + \frac{P(1)}{P^2(2)} \right\},
\]
\[
2h(1) - h(2) = -P^2(0)\left\{ \frac{P(2)}{P^2(3)} + \frac{P(1)}{P^2(2)} \right\},
\]
which are true by (4.5). All the other equations are trivially satisfied except for the coefficients of $q$, $q^2$ and $q^4$ in (4.13), of $q$ and $q^2$ in (4.14), and of $q^4$ in (4.15). The coefficients of $q$, $q^2$ and $q^4$ are respectively

$$P^2(0) \left\{ \frac{P(2)}{P^2(1)} - \frac{P^2(3)}{P(1) P^2(2)} - y \frac{P(1)}{P(2) P(3)} \right\} = 0,$$

$$P^2(0) \left\{ \frac{y P(1)}{P^2(3)} - \frac{P^2(2)}{P(3) P^2(1)} + \frac{P(3)}{P(1) P(2)} \right\} = 0,$$

$$P^2(0) \left\{ \frac{P(3)}{P^2(2)} + y \frac{P^2(1)}{P(2) P^2(3)} - \frac{P(2)}{P(1) P(3)} \right\} = 0,$$

and each of them reduces to (4.2). This proves the theorem.

5. Some results for cranks modulo 11

As in the previous section, taking $m = 11$ in (2.5), with the help of (2.3) and (2.4) we find that

$$M(0) = F (2S(0) + 1),$$
$$M(1) = F (S(1) - S(0)),$$
$$M(2) = F (S(2) - S(1)),$$
$$M(3) = F (S(3) - S(2)),$$
$$M(4) = F (S(4) - S(3)),$$
$$M(5) = -F (S(4)).$$

Taking $m = 11$ and $b = 5$, 4, 3, 2 and 1 respectively in (2.11), we have

$$S(0) = h(5) - q^2 \frac{P^2(0) P(3) P(5)}{P(1) P(2) P(4)} + q^3 \frac{P^2(0) P^2(5)}{P(2) P^2(3)} - q^4 y \frac{P^2(0)}{P(5)} - q^6 \frac{P^2(0) P(5)}{P(2) P(3)} + q^{10} \frac{P^2(0) P(2)}{P(1) P(4)},$$

$$S(1) = h(4) - q^2 \frac{P^2(0) P(5)}{P(1) P(3)} + q^3 y \frac{P^2(0) P(4)}{P(4)} + q^5 \frac{P^2(0) P^2(2)}{P(5) P^2(2)},$$

$$S(2) = h(3) - q y \frac{P^2(0) P(5)}{P(3)} - q^3 \frac{P^2(0) P(3) P(4)}{P(1) P(2) P(5)} + q^7 \frac{P^2(0) P(3)}{P(1) P(4)},$$

$$S(3) = h(2) + q^4 \frac{P^2(0) P^2(2)}{P(3) P^2(1)} + q^5 y^2 \frac{P^2(0) P(1) P(2)}{P(3) P(4) P(5)} - q^6 y \frac{P^2(0) P(3)}{P(4) P(5)} + q^8 \frac{P^2(0) P(2)}{P(1) P(3)} + q^9 \frac{P^2(0)}{P(2)},$$

$$S(4) = h(1) - q^3 \frac{P^2(0) P^2(1)}{P(4) P^2(5)} + q^2 y^2 \frac{P^2(0) P(1)}{P(4) P(5)} - q^4 y \frac{P^2(0) P(1) P(5)}{P(2) P(3) P(4)} - q^5 \frac{P^2(0)}{P(1)} + q^7 \frac{P^2(0) P(4)}{P(2) P(3)}. $$
Theorem 3.

(5.3) \[ R_{01}(0) = \frac{P(0)}{P(1)}, \]

(5.4) \[ -\frac{1}{2} R_{01}(1) = R_{12}(1) = \frac{P(5) P(0)}{P(2) P(3)}. \]

(5.5) \[ R_{12}(2) = -R_{23}(2) = -\frac{P(3) P(0)}{P(1) P(4)}. \]

(5.6) \[ R_{01}(3) = -R_{23}(3) = R_{34}(3) = \frac{P(2) P(0)}{P(1) P(3)}. \]

(5.7) \[ R_{01}(4) = -R_{12}(4) = R_{23}(4) = -R_{34}(4) = R_{45}(4) = \frac{P(0)}{P(2)}, \]

(5.8) \[ R_{12}(5) = -R_{23}(5) = R_{34}(5) = -R_{45}(5) = \frac{P(4) P(0)}{P(2) P(5)}, \]

(5.9) \[ R_{01}(7) = -R_{12}(7) = R_{34}(7) = -R_{45}(7) = -\frac{P(0)}{P(3)}. \]

(5.10) \[ R_{01}(8) = -R_{12}(8) = R_{23}(8) = -R_{45}(8) = -y P(1) P(0) \]

(5.11) \[ R_{01}(9) = -R_{34}(9) = R_{45}(9) = -\frac{P(0)}{P(4)}. \]

(5.12) \[ R_{23}(10) = -R_{34}(10) = \frac{P(0)}{P(5)}. \]

and all other functions \( R_{b,b+1}(d) \), where \( 0 \leq b \leq 4 \), are zero.

Since \( R_{ij}(k) = -R_{ji}(k) \) and \( R_{i,k}(k) + R_{j,k}(k) = R_{ij}(k) \), to prove the theorem it is sufficient to consider the five pairs of values \( (i, j) = (0, 5), (1, 5), (2, 5), (3, 5) \) and \( (4, 5) \), so we have to prove

(5.13) \[ 2S(0) + S(4) + 1 = \left\{ \frac{P(0)}{P(1)} - q^6 \frac{P(0) P(4)}{P(2) P(3)} + q^7 \frac{P(0) P(2)}{P(1) P(3)} + q^8 \frac{P(0) P(1)}{P(2) P(4)} - q^9 \frac{P(0)}{P(5)} \right\} F^{-1}, \]

(5.14) \[ S(1) + S(4) - S(0) = \left\{ q^7 \frac{P(0) P(5)}{P(2) P(3)} + q^8 \frac{P(0) P(2)}{P(1) P(3)} + q^9 \frac{P(0) P(1)}{P(2) P(4)} \right\} F^{-1}, \]

(5.15) \[ S(2) + S(4) - S(1) = \left\{ q^8 \frac{P(0) P(3)}{P(1) P(4)} + q^9 \frac{P(0) P(2)}{P(2) P(4)} - q^{10} \frac{P(0) P(4)}{P(5)} \right\} F^{-1}, \]

(5.16) \[ S(3) + S(4) - S(2) = \left\{ q^9 \frac{P(0) P(2)}{P(1) P(3)} + q^{10} \frac{P(0) P(1)}{P(4) P(5)} - q^{11} \frac{P(0)}{P(5)} \right\} F^{-1}, \]

(5.17) \[ 2S(4) - S(3) = \left\{ q^4 \frac{P(0)}{P(2)} - q^5 \frac{P(0) P(4)}{P(2) P(3)} + q^7 \frac{P(0) P(2)}{P(1) P(3)} + q^8 \frac{P(0) P(1)}{P(4) P(5)} - q^9 \frac{P(0)}{P(5)} \right\} F^{-1}. \]
By (2.15) with $m = 11$ ($\lambda = 2$ and $\mu = -1$), we have

\[(5.18) \quad \mathbf{F}^{-1} = P(0) \begin{pmatrix} \frac{P(4)}{P(2)} - q \frac{P(2)}{P(1)} - q^2 \frac{P(5)}{P(3)} - q^3 \frac{P(1)}{P(5)} + q^4 \frac{P(3)}{P(4)} \end{pmatrix}.\]

Substituting (5.18) in each of (5.13), (5.17) and equating the coefficients of powers of $q$, we have 55 equations to prove. To do this we need the following ten identities, which can be found by taking $(b, c, d) = (5, 4, 1), (5, 4, 2), (4, 3, 1), (5, 3, 2), (3, 2, 1), (5, 3, 1), (5, 4, 3), (5, 2, 1), (4, 3, 2)$ and $(4, 2, 1)$ in (11):

\[
\begin{align*}
P(3)P^3(5) - P(5)P^3(4) + y^3P(2)P^3(1) &= 0, \quad (a1) \\
P(2)P^3(5) - P(3)P^3(4) + y^2P(1)P^3(2) &= 0, \quad (a2) \\
P(2)P^3(4) - P(5)P^3(3) + y^2P(4)P^3(1) &= 0, \quad (a3) \\
P(1)P^3(5) - P(4)P^3(3) + yP(3)P^3(2) &= 0, \quad (a4) \\
P(1)P^3(3) - P(4)P^3(2) + yP(5)P^3(1) &= 0. \quad (a5)
\end{align*}
\]

\[
\begin{align*}
P(2)P(4)P^2(5) - P(4)P(5)P^2(3) + y^2P(2)P(3)P^2(1) &= 0, \quad (b1) \\
P(1)P(4)P^2(5) - P(2)P(3)P^2(4) + yP(1)P(2)P^2(3) &= 0, \quad (b2) \\
P(1)P(3)P^2(5) - P(4)P(5)P^2(2) + yP(3)P(4)P^2(1) &= 0, \quad (b3) \\
P(1)P(5)P^2(4) - P(2)P(5)P^2(3) + yP(1)P(4)P^2(2) &= 0, \quad (b4) \\
P(1)P(3)P^2(4) - P(3)P(5)P^2(2) + yP(2)P(5)P^2(1) &= 0. \quad (b5)
\end{align*}
\]

Putting $a = 1, 2, 3, 4$ and $5$, respectively, in (2.18)-(2.20) with $m = 11$, and using (4.2), we obtain

\[
\begin{align*}
3h(1) - h(3) &= B_1, \\
3h(2) + h(5) &= B_2 - 1, \\
3h(3) + h(2) &= B_3 - 1, \\
3h(4) - h(1) &= B_4 - 1, \\
3h(5) - h(4) &= B_5 - 1,
\end{align*}
\]

where

\[
B_i = \frac{P^3(2i) P^2(0)}{P^3(i) P^3(3i)} - \frac{P^3(4i) P^2(0)}{P^3(2i) P(6i)} \quad (i = 1, 2, \ldots, 5).
\]

The solution of (5.21) is

\[
\begin{align*}
h(1) &= -\frac{1}{11} + \frac{1}{242}(81B_1 - 9B_2 + 27B_3 + B_4 + 3B_5), \\
h(2) &= -\frac{2}{11} + \frac{1}{242}(-3B_1 + 81B_2 - B_3 - 9B_4 - 27B_5), \\
h(3) &= -\frac{3}{11} + \frac{1}{242}(B_1 - 27B_2 + 81B_3 + 3B_4 + 9B_5), \\
h(4) &= -\frac{4}{11} + \frac{1}{242}(27B_1 - 3B_2 + 9B_3 + 81B_4 + B_5), \\
h(5) &= -\frac{5}{11} + \frac{1}{242}(9B_1 - B_2 + 3B_3 + 27B_4 + 81B_5).
\end{align*}
\]
We simplify (5.23) by using some results of [4] as follows: Write

\[ r = -y^2 \frac{P(1)}{P(3) P(5)}, \quad s = -y \frac{P(2)}{P(1) P(5)}, \quad t = \frac{P(4)}{P(1) P(2)}, \]

\[ u = y \frac{P(3)}{P(2) P(4)}, \quad v = y \frac{P(5)}{P(3) P(4)}. \]

Now, dividing (b1)–(b5) respectively by

\[ y^{-1} P(2) P(3) P(5) P^3(4), \quad P(1) P(3) P(4) P^2(2), \quad y^{-1} P(1) P(4) P(5) P^2(3), \]

\[ P(2) P(4) P(5) P^2(1), \quad y^{-1} P(1) P(2) P(3) P^2(5) \]

respectively, and dividing (a1)–(a5) by

\[ y^{-1} P(2) P(4) P^3(5), \quad P(3) P(5) P^3(2), \quad y^{-1} P(1) P(5) P^3(4), \]

\[ P(1) P(2) P^3(3), \quad P(3) P(4) P^3(1) \]

respectively, we find that

\[ B_1 = (r + u + v - s) P^2(0), \]
\[ B_2 = (t - r - s - v) P^2(0), \]
\[ B_3 = (t + u + s - v) P^2(0), \]
\[ B_4 = (t + r + s - u) P^2(0), \]
\[ B_5 = (u + v + t - r) P^2(0). \]

Thus, (5.24) becomes

\[ h(1) = -\frac{1}{11} + \frac{1}{11} (4r - 2s + t + 5u + 3v) P^2(0), \]
\[ h(2) = -\frac{2}{11} + \frac{1}{11} (-3r - 4s + 2t - u - 5v) P^2(0), \]
\[ h(3) = -\frac{3}{11} + \frac{1}{11} (r + 5s + 3t + 4u - 2v) P^2(0), \]
\[ h(4) = -\frac{4}{11} + \frac{1}{11} (5r + 3s + 4t - 2u + v) P^2(0), \]
\[ h(5) = -\frac{5}{11} + \frac{1}{11} (-2r + s + 5t + 3u + 4v) P^2(0). \]

Therefore,

\[ 2h(5) + h(1) + 1 = \left\{ \frac{P(4)}{P(2) P(1)} + y \frac{P(3)}{P(2) P(4)} + y \frac{P(5)}{P(3) P(4)} \right\} P^2(0), \]
\[ h(4) + h(1) - h(5) = y^2 \frac{P(1) P^2(0)}{P(3) P(5)}, \]
\[ h(3) + h(1) - h(4) = y \frac{P(3) P^2(0)}{P(2) P(4)}, \]
\[ h(2) + h(1) - h(3) = y \frac{P(2) P^2(0)}{P(1) P(5)}, \]
\[ h(1) - h(2) = \left\{ \frac{P(5)}{P(3) P(4)} - y \frac{P(1)}{P(3) P(5)} + y \frac{P(3)}{P(2) P(4)} \right\} P^2(0). \]
which are the coefficients of $q^0$ in each of (5.13)-(5.17). For the other coefficients, we subtract the left-hand sides from the right-hand sides in each of (5.13)-(5.17), and see that some coefficients are zero directly, others, by the help of (b1)-(b5).

References


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