SOME PROPERTIES OF PARTITIONS
IN TERMS OF CRANK

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Abstract. Let \( N(r, m, n) \) (resp. \( M(r, m, n) \)) denote the number of partitions of \( n \) whose ranks (resp. cranks) are congruent to \( r \) modulo \( m \). Atkin and Swinnerton-Dyer gave the relations between the numbers \( N(r, m, mn+k) \) when \( m = 5, 7 \) and \( 0 \leq r, k < m \). Garvan gave the relations between the numbers \( M(r, m, mn+k) \) when \( m = 5, 7, \) and \( 11, 0 \leq r, k < m \). Here, we show that the methods of Atkin and Swinnerton-Dyer can be extended to prove the relations for the crank.

1. Introduction

Let \( N(m, n) \) and \( M(m, n) \) denote the number of partitions of \( n \) with rank and crank \( m \), respectively. We change this definition of \( M(m, n) \) just a little, setting \( M(0, 1) = -1 \) and \( M(-1, 1) = 1 = M(1, 1) \), and modify \( M(r, m, n) \) accordingly. We shall also suppose that the empty partition of 0 has rank 0.

For convenience, we write \( N_1 \) for \( M \) and \( N_3 \) for \( N \). So, by (12) of [9] and (1.11) of [2] when \( k = 1 \), and by (2.12) of [3] when \( k = 3 \), we have

\[
\sum_m \sum_{n \geq 0} N_k(m, n) z^m q^n = \left( \prod_{r=1}^{\infty} \frac{1}{1-q^r} \right) (1-z) \sum_n (-1)^n q^{n(kn+1)/2} 1 - zq^n.
\]

Here, and below, \( \sum_n \) denotes a sum over all integers \( n \), while \( \sum'_n \) denotes a sum over all non-zero integers \( n \).

For odd positive integers \( k \), defining \( N_k(m, n) \) by \( (1.1) \), we observe that \( N_k(m, n) \) \( \geq 0 \) for almost all \( n \). If we put \( z = 1 \) in \( (1.1) \), we find that

\[
\sum_m N_k(m, n) = p(n),
\]

where \( p(n) \) is the number of partitions of \( n \), and, replacing \( z \) by \( z^{-1} \) in \( (1.2) \),

\[
N_k(-m, n) = N_k(m, n).
\]

Thus, one can ask whether there is a \(^{"k\text{-rank}"} \) (1-rank = crank, 3-rank = rank) such that \( N_k(m, n) \) counts the number of partitions of \( n \) with \( k\text{-rank} m \). This question has been answered by Garvan [8].

In [7], Garvan found a number of relations for the crank modulo 5, 7 and 11 analogous to those found by Dyson [5] for the rank modulo 5 and 7. These relations were proved by Garvan using various results for theta functions, including

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Winquist’s identity \[11\]. Dyson’s results were proved by Atkin and Swinnerton-Dyer \[3\] by relating certain mock-theta-like functions with elliptic-theta function identities. In this paper we show how Atkin and Swinnerton-Dyer’s method may be extended to prove Garvan’s crank relations. In the process we find some new relations for mock-theta-like functions.

Since we use the method of Atkin and Swinnerton-Dyer \[3\], we adopt their notations: \( m \) will take the values 5, 7 and 11, and the variables \( q \) and \( y \) are always related by the equation

\[ y = q^m. \]

We shall regard any power series in \( q \) as a polynomial of degree \( m - 1 \) in \( y \) whose coefficients are power series in \( y \). Thus, any identity between two power series in \( q \) can be regarded, on equating coefficients of powers of \( q \), as equivalent to \( m \) identities between power series in \( y \). We write

\[ P(z, q) := \prod_{r=1}^{\infty} (1 - z q^{r-1})(1 - z^{-1} q^r), \tag{1.4} \]

so that \( P(z, q) \) is a single-valued analytic function of \( z \) in any ring-shaped region \( 0 < z_1 \leq |z| \leq z_2 \), and satisfies

\[ P(z^{-1}, q) = P(z, q), \quad P(z q, q) = -z^{-1} P(z, q). \tag{1.5} \]

We also write

\[ P(a) := P_m(a) := P(y^a, y^m) = \prod_{r=1}^{\infty} (1 - y^m(r-1)+a)(1 - y^{mr-a}), \]

\[ P(0) := P_m(0) := \prod_{r=1}^{\infty} (1 - y^{mr}), \]

where \( a \) is not a multiple of \( m \). It should be noted that \( P(0) \) is not the expression that would be obtained by writing 0 instead of \( a \) in the definition of \( P(a) \). From \( (1.5) \), we have

\[ P(m - a) = P(a), \quad P(-a) = P(m + a) = -y^{-a} P(a), \tag{1.6} \]

which we shall use without explicit mention below.

2. Preparation

We define

\[ M(r) := \sum_{n \geq 0} M(r, m, n) q^n \tag{2.1} \]

and

\[ S(a) := \sum_{n} (-1)^n q^{n(n+1)/2+an} \frac{q^{mn}}{1 - q^{mn}}. \tag{2.2} \]

The power series \( S(a) \) differs only from (6.1) of \[3\] in that 3 is replaced by 1, which appears in the power of \( q \) in the numerator of (6.1) of \[3\]. As in \[3\], writing \(-n\) for \( n \) in \( (2.2) \), we find that

\[ S(a) = -S(m - 1 - a). \tag{2.3} \]
This gives
\begin{equation}
S\left(\frac{m-1}{2}\right) = 0.
\end{equation}

Replacing \( z \) by a primitive \( m \)-th root of unity in (1.1), we can see that
\begin{align}
M(0) &= F \left( S(0) - S(m-1) + 1 \right), \\
M(r) &= F \left( S(r) - S(r-1) \right) \quad (r = 1, 2, 3, \ldots, m - 1),
\end{align}
where \( F = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \) (see [10] for details). To express the power series \( S(a) \) as a polynomial in \( q \) of degree \( m - 1 \) in \( q \) whose coefficients are power series in \( y \), we define (for any complex number \( \zeta \))
\begin{equation}
T(z, \zeta, q) := \sum_n (-1)^n \frac{\zeta^n q^{n(n+1)/2}}{1 - zq^n},
\end{equation}
which is obviously an analytic function of \( z \) in every region \( 0 < r_1 \leq |z| \leq r_2 \) except for simple poles at point \( z = q^n \). Also, let
\begin{equation}
T^*(\zeta, q) := \sum_n' (-1)^n \frac{\zeta^n q^{n(n+1)/2}}{1 - q^n}.
\end{equation}

Following the similar proof of (6.6) of [3], we find
\begin{equation}
S\left(\frac{m-1}{2} - b\right) = (-1)^b b^{(m-b)/2} T(q^{mb}, 1, q^{m})
+ T^*(q^{-mb}, q^{m^2}) + q^{mmb} T(q^{2mb}, q^{mb}, q^{m^2})
+ \sum_{a \equiv b \mathrm{mod} (m)} (-1)^{a+b} q^{(a-b)(a+b-m)/2} \left\{ q^ma T(q^{m(b+a)}, q^{ma}, q^{m^2})
+ T(q^{m(b-a)}, q^{-ma}, q^{m^2}) \right\},
\end{equation}

It follows from Lemma 2 of [3] that
\begin{equation}
\zeta T(z\zeta, \zeta, q) + T(z\zeta^{-1}, \zeta^{-1}, q) = \frac{P(z; q)P(\zeta^2; q)F^{-2}}{P(z\zeta^{-1}; q)P(z\zeta; q)P(\zeta; q)},
\end{equation}
and that
\begin{equation}
T(z, 1, q) = \frac{F^{-2}}{P(z; q)}.
\end{equation}

We now write
\begin{equation}
h(z; q) := T^*(z^{-1}, q) + zT(z^2, z, q).
\end{equation}

**Lemma 1.**
\begin{enumerate}
\item[(i)] \( h(z; q) - h(zq; q) = 1 \),
\item[(ii)] \( h(z; q) + h(z^{-1}; q) = 0 \),
\item[(iii)] \( h(z; q) + h(z^{-1}; q) = -1 \),
\item[(iv)] \( 3h(z; q) - h(z^3; q) = \frac{P^3(z^2; q)F^{-2}}{P^3(z; q)P(z^2; q)} - \frac{P^3(z^4; q)F^{-2}}{P^3(z^2; q)P(z^2; q)} \).
\end{enumerate}
Proof. (i). By Jacobi's triple product identity (Thm.2.8 in [I]) we have
\[(2.12)\]
\[- P(z^{-1}; q)F^{-1} = - \sum_{n} (-1)^{n} \frac{z^{-n} q^{n(n-1)/2}}{1 - q^{n}} + \sum_{n} (-1)^{n} \frac{z^{-n} q^{n(n+1)/2}}{1 - q^{n}} - 1\]
and
\[(2.13)\]
\[P(z^{-1}; q)F^{-1} = \sum_{n} (-1)^{n} z^{-n} q^{n(n+1)/2} \frac{1}{1 - z^{2} q^{n+1}} (1 - z^{2} q^{n+1})\]
\[= \sum_{n} (-1)^{n} \frac{z^{-n} q^{n(n+1)/2}}{1 - z^{2} q^{n+1}} - \sum_{n} (-1)^{n} \frac{z^{n+2} q^{(n+1)(n+2)/2}}{1 - z^{2} q^{n+1}}\]
(replace \(n\) by \(n + 1\))
\[= - zq \sum_{n} (-1)^{n} \frac{(zq)^{n} q^{n(n+1)/2}}{1 - z^{2} q^{n+2}} + z \sum_{n} (-1)^{n} \frac{z^{n} q^{n(n+1)/2}}{1 - z^{2} q^{n}}.\]

If we add (2.12) and (2.13), we have (i). (ii) is trivial and (iii) is consequence of (i) and (ii).

For (iv), let \(f_{L}(z)\) and \(f_{R}(z)\) denote the left and right sides of (iv). Then (i) shows that \(f_{L}(zq) = f_{L}(z)\). By (1.5), \(f_{R}(zq) = f_{R}(z)\). Now, \(f_{L}(z) - f_{R}(z)\) is free from poles and so, by Lemma 2 of [3], either is free from zeros or is identically zero. By using (1.5), we find that \(f_{R}(z^{-1}) = -f_{R}(z)\), and by (ii) \(h(z^{-1}; q) = -h(z; q)\). Therefore \(z = 1\) is a zero of \(f_{L}(z) - f_{R}(z)\).

Consequently, with the help of (2.9)-(2.11) with \(z = q^{ma}\), \(\zeta = q^{mb}\), and \(q^{m^{2}}\) for \(q\), we have
\[(2.14)\]
\[S(\frac{m-1}{2} - b) = h(q^{mb}; q^{m^{2}}) + (-1)^{b} q^{b(m-b)/2} \frac{P_{m}^{2}(0)}{P_{m}(b)}\]
\[+ \sum_{a \neq b \mod (m) \atop a+b = \mp 1} (-1)^{a+b} q^{(a-b)(a+b-m)/2} \frac{P_{m}(b) P_{m}(2a) P_{m}^{2}(0)}{P_{m}(b-a) P_{m}(b+a) P_{m}(a)}.\]

We shall need the following, which is Lemma 6 in [3]:
\[(2.15)\]
\[F^{-1} = (-1)^{\lambda} q^{\lambda(3\lambda+\mu)/2} P_{m}(0) \left[ 1 + \sum_{c=1}^{(m-1)/2} (-1)^{c} q^{c(3c-m)/2} \frac{P_{m}(2c)}{P_{m}(c)} \right],\]
where \(m = 6\lambda + \mu\), \(\lambda\) is a positive integer and \(\mu = \mp 1\).

We are now in a position to state and prove the results for crank of ordinary partition in the cases of modulo 5, 7 and 11. For convenience, we shall write
\[(2.16)\]
\[R_{ij}(k) := \sum_{n \geq 0} (M(i, m, mn + k) - M(j, m, mn + k)) y^{n},\]
and also
\[(2.17)\]
\[h(a) := h(y^{a}, y^{m}).\]
so that Lemma 1, with \( q \) replaced by \( y^m \) and \( z \) by \( y^a \), states that
\[
\begin{align*}
(2.18) & \quad h(a) - h(m + a) = 1, \\
(2.19) & \quad h(a) + h(m - a) = -1, \\
(2.20) & \quad 3h(a) - h(3a) = \frac{P^3(2a) - P^2(6a)}{P^3(a) P(3a)} - \frac{P^3(4a) - P^2(6a)}{P^3(2a) P(6a)}.
\end{align*}
\]

3. Some results for cranks modulo 5

Taking \( m = 5 \) in (2.5), with the help of (2.3) and (2.4), we find that
\[
\begin{align*}
M(0) &= F (2S(0) + 1), \\
M(1) &= F ((S(1) - S(0)), \\
M(2) &= -F (S(1)).
\end{align*}
\]

Taking \( m = 5 \) and \( b = 2 \), 1 in (2.14), we have
\[
\begin{align*}
S(0 &= h(2) - q \frac{P^2(0) P(2)}{P^2(1)} + q^3 \frac{P^2(0)}{P(2)}, \\
S(1 &= h(1) - q^2 \frac{P^2(0)}{P(1)} + q^4 \frac{P^2(0) P(1)}{P^2(2)}.
\end{align*}
\]

Putting \( a = 1 \) in (2.18) and (2.20) with \( m = 5 \), and \( a = 2 \) in (2.19) and (2.20)
with \( m = 5 \), we also obtain
\[
\begin{align*}
h(1) &= \frac{1}{5} + 1 \left\{ \frac{P^2(0) P^2(2)}{P^3(1)} + 2q \frac{P^2(0) P^2(1)}{P^3(2)} \right\}, \\
h(2) &= -\frac{2}{5} + 1 \left\{ q \frac{P^2(0) P^2(2)}{P^3(1)} - q^3 \frac{P^2(0) P^2(1)}{P^3(2)} \right\}.
\end{align*}
\]

After all these preparations the following is easily proven.

**Theorem 1.**
\[
\begin{align*}
R_{01}(0) &= \frac{P(2) P(0)}{P^2(1)}, \\
R_{01}(1) &= -2 \frac{P(0)}{P(1)}, \\
R_{12}(1) &= \frac{P(0)}{P(1)}, \\
R_{12}(2) &= -\frac{P(0)}{P(2)}, \\
R_{01}(3) &= -R_{12}(3) = \frac{P(1) P(0)}{P^2(2)}
\end{align*}
\]
and all other functions \( R_{b, b+1}(d) \), where \( b = 0 \) or 1, are zero.

By (3.1), to prove the theorem we only have to show that
\[
\begin{align*}
3S(0) - S(1) + 1 &= \left\{ \frac{P(0) P(2)}{P^2(1)} - 2q \frac{P(0)}{P(1)} + q^3 \frac{P(0) P(1)}{P^2(2)} \right\} F^{-1}, \\
2S(1) - S(0) &= \left\{ q \frac{P(0)}{P(1)} - q^2 \frac{P(0)}{P(2)} - q^3 \frac{P(0) P(1)}{P^2(2)} \right\} F^{-1}.
\end{align*}
\]
Since by (2.15) we have
\[
F^{-1} = P(0) \left\{ \frac{P(2)}{P(1)} - q^2 P(1) \right\},
\]
these are respectively equivalent to
\[
3h(2) - h(1) + 1 = \frac{P^2(0) P^2(2)}{P^3(1)} - y \frac{P^2(0) P^2(1)}{P^3(2)},
\]
\[
2h(1) - h(2) = y \frac{P^2(0) P^2(1)}{P^3(2)},
\]
which are true by (3.3). This proves the theorem.

4. SOME RESULTS FOR CRANKS MODULO 7

Here and in the next section we need the following for simplifications, which is Lemma 4 of [3],
\[
P^2(b)P(c + d)P(c - d) - P^2(c)P(b + d)P(b - d)
= y^{e-d}P^2(d)P(b + c)P(b - c) = 0,
\]
where none of \( b, c, d, b \mp c, c \pm d, b \mp d \) is divisible by \( m \). This gives, for \( (b, c, d) = (3, 2, 1), \)
\[
P(1)P^3(3) - P(3)P^3(2) + yP(2)P^3(1) = 0.
\]
As in the previous section, taking \( m = 7 \) in (2.5), with the help of (2.3) and (2.4) we find that
\[
M(0) = F(2S(0) + 1),
M(1) = F(S(1) - S(0)),
M(2) = F(S(2) - S(1)),
M(3) = -F(S(2)).
\]
Taking \( m = 7 \) and \( b = 3, 2, \) and \( 1 \) in (2.14), we have
\[
S(0) = h(3) - q \frac{P^2(0) P^2(3)}{P^2(2) P(1)} + q^3 \frac{P^2(0) P^2(1)}{P(1)} - q^6 \frac{P^2(0)}{P(3)},
\]
\[
S(1) = h(2) - q \frac{P^2(0) P^2(2)}{P^2(1) P(3)} + q^5 \frac{P^2(0) P(2)}{P(3)} + q^6 \frac{P^2(0)}{P(3)},
\]
\[
S(2) = h(1) - q^3 \frac{P^2(0) P^2(1)}{P(1)} + q^5 \frac{P^2(0) P(2)}{P(3)} - q^{11} \frac{P^2(0) P^2(1)}{P^2(3) P(2)}.
\]
Putting \( a = 1, 2 \) and \( 3 \), respectively, in (2.18)-(2.20) with \( m = 7 \), and using (1.2), we obtain
\[
h(1) = - \frac{1}{7} + \frac{1}{7} P^2(0) \left\{ \frac{P(3)}{P^2(1)} + 3y \frac{P(1)}{P^2(2)} + 2y \frac{P(2)}{P^2(3)} \right\},
\]
\[
h(2) = - \frac{2}{7} + \frac{1}{7} P^2(0) \left\{ 2 \frac{P(3)}{P^2(1)} - y \frac{P(1)}{P^2(2)} - 3y \frac{P(2)}{P^2(3)} \right\},
\]
\[
h(3) = - \frac{3}{7} + \frac{1}{7} P^2(0) \left\{ 3 \frac{P(3)}{P^2(1)} + 2y \frac{P(1)}{P^2(2)} - y \frac{P(2)}{P^2(3)} \right\}.
\]
After all these preparations the following is easily proven.

**Theorem 2.**

(4.6) \[ R_{01}(0) = \frac{P(3)P(0)}{P(1)P(2)}, \]

(4.7) \[ R_{01}(1) = -2\frac{P(0)}{P(1)}, \]

(4.8) \[ R_{12}(1) = \frac{P(0)}{P(1)}, \]

(4.9) \[ R_{12}(2) = -R_{23}(2) = -\frac{P(2)P(0)}{P(1)P(3)}, \]

(4.10) \[ R_{01}(3) = -R_{23}(3) = \frac{P(0)}{P(2)}, \]

(4.11) \[ R_{01}(4) = -R_{12}(4) = \frac{P(0)}{P(3)}, \]

(4.12) \[ R_{01}(6) = -R_{12}(6) = R_{23}(6) = -\frac{P(1)P(0)}{P(2)P(3)}, \]

and all other functions \( R_{b,b+1}(d) \), where \( 0 \leq b \leq 2 \), are zero.

To prove the theorem we only consider the three pairs of values \((i,j) = (0,1), (1,2)\) and \((2,3)\) in (2.16). So, by (4.3), we only have to show that

(4.13) \[ 3S(0) - S(1) + 1 = \left\{ \frac{P(0)P(3)}{P(1)P(2)} - 2q\frac{P(0)}{P(1)} + q^3\frac{P(0)}{P(2)} + q^4\frac{P(0)}{P(3)} - q^6\frac{P(0)}{P(2)P(3)} \right\} F^{-1}, \]

(4.14) \[ 2S(1) - S(2) - S(0) = \left\{ q\frac{P(0)}{P(1)} - q^2\frac{P(0)P(2)}{P(1)P(3)} - q^3\frac{P(0)}{P(3)} + q^4\frac{P(0)}{P(2)} \right\} F^{-1}, \]

(4.15) \[ 2S(2) - S(1) = \left\{ q^2\frac{P(0)P(2)}{P(1)P(3)} - q^3\frac{P(0)}{P(2)} - q^6\frac{P(0)}{P(2)P(3)} \right\} F^{-1}. \]

Now by (2.15) we have

\[ F^{-1} = P(0) \left\{ \frac{P(2)}{P(1)} - \frac{P(3)}{P(2)} - q^2 + q^5\frac{P(1)}{P(3)} \right\}. \]

Substituting this in each of (4.13)–(4.15) and equating coefficients of powers of \( q \), we have 21 equations to prove. The coefficients of \( q^0 \) give us respectively

\[ 3h(3) - h(2) + 1 = P^2(0) \left\{ \frac{P(1)}{P^2(2)} + \frac{P(3)}{P^2(1)} \right\}, \]

\[ 2h(2) - h(1) - h(3) = -P^2(0) \left\{ \frac{P(2)}{P^2(3)} + \frac{P(1)}{P^2(2)} \right\}, \]

\[ 2h(1) - h(2) = -P^2(0) \left\{ \frac{P(2)}{P^2(3)} + \frac{P(1)}{P^2(2)} \right\}. \]
which are true by (4.5). All the other equations are trivially satisfied except for the coefficients of \( q^2 \) and \( q^4 \) in (4.13), of \( q \) and \( q^3 \) in (4.14), and of \( q^4 \) in (4.15). The coefficients of \( q, q^2 \) and \( q^4 \) are respectively

\[
P^2(0) \left\{ \frac{P(2)}{P^2(1)} - \frac{P^2(3)}{P(1) P^2(2)} - y \frac{P(1)}{P(2) P(3)} \right\} = 0,
\]

\[
P^2(0) \left\{ y \frac{P(1)}{P^2(3)} - \frac{P^2(2)}{P(3) P^2(1)} + \frac{P(3)}{P(1) P(2)} \right\} = 0,
\]

\[
P^2(0) \left\{ \frac{P(3)}{P^2(2)} + y \frac{P^2(1)}{P(2) P^2(3)} - \frac{P(2)}{P(1) P(3)} \right\} = 0,
\]

and each of them reduces to (4.2). This proves the theorem.

5. Some results for cranks modulo 11

As in the previous section, taking \( m = 11 \) in (2.5), with the help of (2.6) and (2.7) we find that

\[
M(0) = \text{F} (2S(0) + 1), \quad M(1) = \text{F} (S(1) - S(0)), \quad M(2) = \text{F} (S(2) - S(1)), \\
M(3) = \text{F} (S(3) - S(2)), \quad M(4) = \text{F} (S(4) - S(3)), \quad M(5) = -\text{F} (S(4)).
\]

Taking \( m = 11 \) and \( b = 5, 4, 3, 2 \) and 1 respectively in (2.14), we have

\[
S(0) = h(5) - q y \frac{P^2(0) P(1) P(2) P(3) P(5)}{P(1) P(2) P(3) P(4) P(5)} + q^3 \frac{P^2(0) P^2(5)}{P(2) P^2(3)} - q^4 \frac{P^2(0)}{P(1) P(2) P(3) P(4) P(5)},
\]

\[
S(1) = h(4) - q^2 \frac{P^2(0) P(5)}{P(1) P(3)} + q^3 y \frac{P^2(0) P(4)}{P(4)} + q^5 \frac{P^2(0) P(3) P(4)}{P(5) P^2(2)},
\]

\[
S(2) = h(3) - q y \frac{P^2(0) P(2) P(4)}{P(3) P(5)} - q^4 \frac{P^2(0) P(3) P(4)}{P(1) P(2) P(5)} + q^6 \frac{P^2(0) P(3)}{P(1) P(2) P(4)},
\]

\[
S(3) = h(2) + q^4 \frac{P^2(0) P^2(2)}{P(3) P^2(1)} + q^5 y^2 \frac{P^2(0) P(1) P(3) P(4)}{P(3) P(4) P(5)} - q^6 \frac{P^2(0) P(3)}{P(4) P(5)},
\]

\[
S(4) = h(1) - q y \frac{P^2(0) P(2)}{P(4) P(5)} + q^2 \frac{P^2(0) P(1) P(3) P(5)}{P(4) P(5)} - q^4 \frac{P^2(0) P(1) P(5)}{P(2) P(3) P(4)},
\]

\[
- q^6 \frac{P^2(0) P(4) P(5)}{P(2) P(3) P(4)}.
\]
Theorem 3.

\[(5.3) \quad R_{01}(0) = \frac{P(0)}{P(1)},\]
\[(5.4) \quad -\frac{1}{2} R_{01}(1) = R_{12}(1) = \frac{P(5) P(0)}{P(2) P(3)},\]
\[(5.5) \quad R_{12}(2) = -R_{23}(2) = -\frac{P(3) P(0)}{P(1) P(4)},\]
\[(5.6) \quad R_{01}(3) = -R_{23}(3) = R_{34}(3) = \frac{P(2) P(0)}{P(1) P(3)},\]
\[(5.7) \quad R_{01}(4) = -R_{12}(4) = R_{23}(4) = -R_{34}(4) = R_{45}(4) = \frac{P(0)}{P(2)},\]
\[(5.8) \quad R_{12}(5) = -R_{23}(5) = R_{34}(5) = -R_{45}(5) = \frac{P(4) P(0)}{P(2) P(5)},\]
\[(5.9) \quad R_{01}(7) = -R_{12}(7) = R_{34}(7) = -R_{45}(7) = -\frac{P(0)}{P(3)},\]
\[(5.10) \quad R_{01}(8) = -R_{12}(8) = R_{23}(8) = -R_{45}(8) = -y P(1) P(0) P(4) P(5),\]
\[(5.11) \quad R_{01}(9) = -R_{34}(9) = R_{45}(9) = -\frac{P(0)}{P(4)},\]
\[(5.12) \quad R_{23}(10) = -R_{34}(10) = \frac{P(0)}{P(5)},\]

and all other functions \(R_{i,b+1}(d),\) where \(0 \leq b \leq 4,\) are zero.

Since \(R_{i,j}(k) = -R_{j,i}(k)\) and \(R_{i,a}(k) + R_{a,j}(k) = R_{i,j}(k),\) to prove the theorem it is sufficient to consider the five pairs of values \((i, j) = (0, 5), (1, 5), (2, 5), (3, 5)\) and \((4, 5),\) so we have to prove

\[(5.13) \quad 2S(0) + S(4) + 1
= \left\{ \frac{P(0)}{P(1)} - q^5 \frac{P(0) P(5)}{P(2) P(3)} + q^7 \frac{P(0) P(2)}{P(1) P(3)} + q^4 \frac{P(0) P(2)}{P(2)} - q^9 \frac{P(0)}{P(4)} \right\} F^{-1},\]
\[(5.14) \quad S(1) + S(4) - S(0)
= \left\{ q^2 \frac{P(0) P(5)}{P(2) P(3)} + q^9 \frac{P(0) P(1)}{P(3)} + q^8 y \frac{P(0) P(1)}{P(4) P(5)} \right\} F^{-1},\]
\[(5.15) \quad S(2) + S(4) - S(1)
= \left\{ q^2 \frac{P(0) P(3)}{P(1) P(4)} + q^4 \frac{P(0) P(2)}{P(2)} - q^5 \frac{P(0) P(4)}{P(2) P(5)} \right\} F^{-1},\]
\[(5.16) \quad S(3) + S(4) - S(2)
= \left\{ q^5 \frac{P(0) P(2)}{P(1) P(3)} + q^8 y \frac{P(0) P(1)}{P(4) P(5)} - q^{10} y \frac{P(0)}{P(5)} \right\} F^{-1},\]
\[(5.17) \quad 2S(4) - S(3)
= \left\{ q^4 \frac{P(0) P(4)}{P(2) - q^5 \frac{P(0) P(4)}{P(2) P(3)} + q^7 \frac{P(0) P(1)}{P(3)} + q^8 y \frac{P(0) P(1)}{P(4) P(5)} - q^9 \frac{P(0)}{P(4)} \right\} F^{-1}.\]
By (2.15) with \( m = 11 \) (\( \lambda = 2 \) and \( \mu = -1 \)), we have

\[
(5.18) \quad F^{-1} = P(0) \left\{ \frac{P(4)}{P(2)} - q \frac{P(2)}{P(1)} - q^2 \frac{P(5)}{P(3)} - q^3 \frac{P(1)}{P(5)} + q^5 \frac{P(3)}{P(4)} \right\}.
\]

Substituting (5.18) in each of (5.13)-(5.17) and equating the coefficients of powers of \( q \), we have 55 equations to prove. To do this we need the following ten identities, which can be found by taking \( (b, c, d) = (5, 4, 1), (5, 4, 2), (4, 3, 1), (5, 3, 2), (3, 2, 1), (5, 3, 1), (5, 4, 3), (5, 2, 1), (4, 3, 2) \) and \( (4, 2, 1) \) in (11):

\[
\begin{align*}
P(3)P^3(5) - P(5)P^3(4) + y^3P(2)P^3(1) &= 0, \quad (a1) \\
P(2)P^4(5) - P(3)P^4(4) + y^2P(1)P^4(2) &= 0, \quad (a2) \\
P(2)P^3(4) - P(5)P^3(3) + y^2P(4)P^3(1) &= 0, \quad (a3) \\
P(1)P^3(5) - P(4)P^3(3) + yP(3)P^3(2) &= 0, \quad (a4) \\
P(1)P^3(3) - P(4)P^3(2) + yP(5)P^3(1) &= 0. \quad (a5)
\end{align*}
\]

\[
\begin{align*}
P(2)P(4)P^2(5) - P(4)P(5)P^2(3) + y^2P(2)P(3)P^2(1) &= 0, \quad (b1) \\
P(1)P(4)P^2(5) - P(2)P(3)P^2(4) + yP(1)P(2)P^2(3) &= 0, \quad (b2) \\
P(1)P(3)P^2(5) - P(4)P(5)P^2(2) + yP(3)P(4)P^2(1) &= 0, \quad (b3) \\
P(1)P(5)P^2(4) - P(2)P(5)P^2(3) + yP(1)P(4)P^2(2) &= 0, \quad (b4) \\
P(1)P(3)P^2(4) - P(3)P(5)P^2(2) + yP(2)P(5)P^2(1) &= 0. \quad (b5)
\end{align*}
\]

Putting \( a = 1, 2, 3, 4 \) and 5, respectively, in (2.18) - (2.20) with \( m = 11 \), and using (4.12), we obtain

\[
\begin{align*}
3h(1) - h(3) &= B_1, \\
3h(2) + h(5) &= B_2 - 1, \\
3h(3) + h(2) &= B_3 - 1, \\
3h(4) - h(1) &= B_4 - 1, \\
3h(5) - h(4) &= B_5 - 1,
\end{align*}
\]

where

\[
(5.21) \quad B_i = \frac{P^3(2i)P^2(0)}{P^3(4i)P(3i)} - \frac{P^3(4i)P^2(0)}{P^3(2i)P(6i)} \quad (i = 1, 2, \ldots, 5).
\]

The solution of (5.21) is

\[
\begin{align*}
h(1) &= -\frac{1}{11} + \frac{1}{242}(81B_1 - 9B_2 + 27B_3 + B_4 + 3B_5), \\
h(2) &= -\frac{2}{11} + \frac{1}{242}(-3B_1 + 81B_2 - B_3 - 9B_4 - 27B_5), \\
h(3) &= -\frac{3}{11} + \frac{1}{242}(B_1 - 27B_2 + 81B_3 + 3B_4 + 9B_5), \\
h(4) &= -\frac{4}{11} + \frac{1}{242}(27B_1 - 3B_2 + 9B_3 + 81B_4 + B_5), \\
h(5) &= -\frac{5}{11} + \frac{1}{242}(9B_1 - B_2 + 3B_3 + 27B_4 + 81B_5).
\end{align*}
\]
We simplify (5.23) by using some results of [4] as follows: Write
\[ r = y P(3) P(5), \quad s = y P(1) P(5), \quad t = P(4), \]
\[ u = y P(3) P(4), \quad v = y P(5). \]

(5.24)

Now, dividing (b1)–(b5) respectively by
\[ P(2) P(4) P(5) P^2(1), \quad y P(1) P(2) P(3) P^2(5) \]
respectively, and dividing (a1)–(a5) by
\[ y P(2) P(4) P^3(5), \quad P(3) P(5) P^3(2), \quad P(1) P(2) P^3(3), \quad P(3) P(4) P^3(1), \]

(5.25)

respectively, we find that
\[ B_1 = (r + u + v - s) P^2(0), \]
\[ B_2 = (t - r - s - v) P^2(0), \]
\[ B_3 = (t + u + s - v) P^2(0), \]
\[ B_4 = (t + r - s - u) P^2(0), \]
\[ B_5 = (u + v - t - r) P^2(0). \]

Thus, (5.24) becomes
\[ h(1) = -\frac{1}{11} + \frac{1}{11} (4r - 2s + t + 5u + 3v) P^2(0), \]
\[ h(2) = -\frac{2}{11} + \frac{1}{11} (-3r - 4s + 2t - u - 5v) P^2(0), \]
\[ h(3) = -\frac{3}{11} + \frac{1}{11} (r + 5s + 3t + 4u - 2v) P^2(0), \]
\[ h(4) = -\frac{4}{11} + \frac{1}{11} (5r + 3s + 4t - 2u + v) P^2(0), \]
\[ h(5) = -\frac{5}{11} + \frac{1}{11} (-2r + s + 5t + 3u + 4v) P^2(0). \]

(5.26)

Therefore,
\[ 2h(5) + h(1) + 1 = \left\{ \frac{P(4)}{P(2) P(1)} + y \frac{P(3)}{P(2) P(4)} + y \frac{P(5)}{P(3) P(4)} \right\} P^2(0), \]
\[ h(4) + h(1) - h(5) = y^2 \frac{P(1) P^2(0)}{P(3) P(5)}, \]
\[ h(3) + h(1) - h(4) = y \frac{P(3) P^2(0)}{P(2) P(4)}, \]
\[ h(2) + h(1) - h(3) = y \frac{P(2) P^2(0)}{P(1) P(5)}, \]
\[ h(1) - h(2) = \left\{ \frac{P(5)}{P(3) P(4)} - y \frac{P(1)}{P(3) P(5)} + y \frac{P(3)}{P(2) P(4)} \right\} P^2(0). \]

(5.27)
which are the coefficients of $q^0$ in each of (5.13)–(5.17). For the other coefficients, we subtract the left-hand sides from the right-hand sides in each of (5.13)–(5.17), and see that some coefficients are zero directly, others, by the help of (b1)–(b5).

References


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