EXTENDING PARTIAL AUTOMORPHISMS
AND THE PROFINITE TOPOLOGY ON FREE GROUPS

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ABSTRACT. A class of structures $C$ is said to have the extension property for partial automorphisms (EPPA) if, whenever $C_1$ and $C_2$ are structures in $C$, $C_1$ finite, $C_1 \subseteq C_2$, and $p_1, p_2, \ldots, p_n$ are partial automorphisms of $C_1$ extending to automorphisms of $C_2$, then there exist a finite structure $C_3$ in $C$ and automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $C_3$ extending the $p_i$. We will prove that some classes of structures have the EPPA and show the equivalence of these kinds of results with problems related with the profinite topology on free groups. In particular, we will give a generalisation of the theorem, due to Ribes and Zalesski stating that a finite product of finitely generated subgroups is closed for this topology.

1. INTRODUCTION

In this paper, we will consider and relate two kinds of results. We begin by giving the basic definitions that are needed to understand these relations.

On the one hand, there will be the theorems concerning the so-called “profinite topology” on the free groups. Given a group $G$, the profinite topology on $G$ is the topology for which a basis of open subsets is

$$\{gH; g \in G \text{ and } H \text{ is a subgroup of } G \text{ of finite index}\}.$$ 

Let us recall a classical result, due to M. Hall [3]:

**Theorem 1.1.** Let $P$ be a finite set, and $F(P)$ the free group generated by $P$. Then every finitely generated subgroup of $F(P)$ is closed for the profinite topology.

This result can be rephrased as follows: let $H$ be a finitely generated subgroup of $F(P)$. Then

$$H = \bigcap \{K; K \text{ is a subgroup of finite index of } F(P) \text{ and } H \subseteq K\}.$$ 

More recently, Ribes and Zalesski [9] proved:

**Theorem 1.2.** Let $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of $F(P)$. Then

$$H_1H_2\cdots H_n = \{h_1h_2\cdots h_n; h_1 \in H_1, h_2 \in H_2, \ldots, h_n \in H_n\}$$

is closed for the profinite topology.

On the other hand, we will consider some combinatorial results concerning the extension of partial automorphisms.
Definition 1.3. Let $M, M'$ be two structures in a given finite relational language $\mathcal{L}$. A partial isomorphism from $M$ into $M'$ is an isomorphism of a substructure of $M$ onto a substructure of $M'$. We will denote by $\text{Part}(M, M')$ the set of partial isomorphisms from $M$ into $M'$. A partial automorphism of $M$ is a partial isomorphism from $M$ into $M$.

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures (containing both finite and infinite structures), $M_0$ a finite structure in $\mathcal{C}$ and $P$ a set of partial automorphisms of $M_0$. We consider the following problem (the $(M_0, P, \mathcal{C})$-extension problem): find a structure $M_1 \in \mathcal{C}$, which is an extension of the structure $M_0$ and for each $p \in P$ an automorphism $\alpha_p$ of $M_1$ extending $p$. We say $(M_1, \alpha_p)_{p \in P}$ is a solution of our problem, and we will say it is finite if $M_1$ is.

We say that $\mathcal{C}$ has the extension property for partial automorphisms (EPPA for short) if for all finite $M_0$ and $P \subseteq \text{Part}(M_0, M_0)$, if the $(M_0, P, \mathcal{C})$-extension problem has a solution, then it has a finite solution.

An example of this family of results is the following theorem of Hrushovski (§):

Theorem 1.4. Let $\Gamma_0$ be a finite graph. Then there exists a finite graph $\Gamma_1$ extending $\Gamma_0$ and such that every partial automorphism of $\Gamma_0$ can be extended to an automorphism of $\Gamma_1$.

(Here, a graph means undirected loop free graph.)

Hrushovski’s theorem just states that the class of all graphs has the EPPA (note that in the case where $\mathcal{C}$ is the class of all graphs, every extension problem has a solution, because every finite graph is embeddable in the random graph, which is homogeneous).

Herwig has generalised this result to the class of structures of a given finite relational language (§) and various other classes of graphs (see [6]). This kind of result is of importance for proving the small index property for the automorphism group of the corresponding generic structures (see [7] or [6] for more about this question).

This paper is organised as follows: in the next section, we show how to use the properties of the profinite topology to prove some EPPA-results. In particular, we give a proof of Hrushovski’s theorem (Theorem 1.4) from the theorem of Ribes and Zalesski (Theorem 1.2). This proof is not simpler than the original one. It is only given here as an illustration.

Next, we go in the other direction. First, starting from the fact that the class of $n$-partitioned cycle-free graphs has the EPPA, we show the Ribes-Zalesski theorem. Then, using the most general extension result that we have been able to prove (Theorem 3.2), we prove a property of the profinite topology (Theorem 3.3) generalising the theorem of Ribes and Zalesski.

The next two sections are devoted to proving extension results. First, we give a proof of the EPPA for the class of graphs (that is the theorem of Hrushovski). This proof has the advantage of being short and of admitting natural generalisation to the class of all structures in a given finite relational language. This last result, which had already been obtained by the first author (see [5]), will be used later. We will also give a proof of the EPPA for the class of $n$-partitioned cycle-free graphs. This proof was not necessary, since it is just a particular case of Theorem 3.2.
we wanted to spare them the complication of the proof of Theorem 3.2. Section 5 almost half of the paper, is devoted to this proof.

We will be dealing, throughout the paper with structures in some relational language. We assume that the reader understands these words, and also the notation $M = Ra_1a_2 \cdots a_n$ (where $M$ is a structure in a language $L$, $R$ is a symbol of arity $n$ in the language $L$ and $a_1, a_2, \ldots, a_n$ are elements in $M$). If $L'$ is a language included in $L$ and $M$ an $L$-structure, $M_{L'}$ is the restriction of $M$ to $L'$, that is the $L'$-structure obtained from $M$ by just forgetting the interpretation of the symbols of $L$ which are not in $L'$. We say $M_1$ is an extension of $M_0$ (or $M_0$ is a substructure of $M_1$) if the underlying set of $M_0$ is contained in that of $M_1$ and for every symbol $R$ in $L$ and $a_1, a_2, \ldots, a_n \in M_0 : M_0 \models Ra_1a_2 \cdots a_n \Leftrightarrow M_1 \models Ra_1a_2 \cdots a_n$.

We will use the same letter, $M$, for example, for a structure and its underlying set. The sign $\cdot$ will denote the product operation in whatever group we are manipulating (but it will be often omitted, depending on the context), and $\circ$ will denote the composition of maps (which may be partial).

If $I$ is a set which is totally ordered by the relation $<$, we may define the lexicographical order on the set $I^{<\omega}$ of finite sequences of elements of $I$: given two sequences $a = (i_1,i_2,\ldots,i_n)$ and $b = (j_1,j_2,\ldots,j_m)$, then $a < b$ if and only if one of the following cases occurs

- $a$ is a proper initial segment of $b$;
- $a$ is not an initial segment of $b$, and if $k$ is the smallest integer such that $i_k \neq j_k$, then $i_k < j_k$.

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2. From the profinite topology to the extension properties

2.1. A sophisticated proof of a theorem of Hrushovski. We will give a proof (using Theorem 1.2) of the theorem of Hrushovski (Theorem 1.4)

Let $\Gamma_0$ be a finite graph and let $P$ be the (finite) semi-group of partial automorphisms of $\Gamma_0$. Let us agree that, when we write $p(a) = p'(a')$ or $p(a) \neq p'(a')$ (where $p,p'$ are elements of $P$ and $a,a'$ are elements of $\Gamma_0$) this means that both $p(a)$ and $p'(a')$ are defined and of course the equality or the inequality holds.

Choose an element $a_0$ of $\Gamma_0$ and let $H_0$ be the subgroup of $F(P)$ generated by $X_0$ where

$$X_0 = \{ p^{-1} \cdot p' ; \ p,p' \in P, p(a_0) = p'(a_0) \} \cup \{ p_3^{-1} \cdot p_1 \cdot p_2 ; \ p_1,p_2,p_3 \in P, \ p_1 \circ p_2(a_0) = p_3(a_0) \}$$

(note that $X_0$ is finite). Let $H$ be any subgroup of $F(P)$ such that

$$H_0 \subseteq H.$$  

(1)

For each $a \in \Gamma_0$, there is a partial automorphism $p \in P$ such that $p(a_0) = a$, and if $p' \in P$ is such that $p'(a_0) = a$, then $p \cdot H = p' \cdot H$; so we may define a map $\phi$ from $\Gamma_0$ into $F(P)/H$ by: for all $a \in \Gamma_0$, $\phi(a) = p \cdot H$ where $p$ is any element of $P$ such that $p(a_0) = a$.

If moreover we demand that

$$H \cap X_1 = \emptyset$$  

(2)
where:

\[ X_1 = \{ p^{-1} \cdot p'; \ p, p' \in A, p(a_0) \neq p'(a_0) \} \]

then this map \( \phi \) is injective.

We assume that this condition is satisfied. For each \( g \in F(P) \), define the permutation \( \tilde{g} \) of \( F(P)/H \) by: for all \( x \in F(P) \), \( \tilde{g}(xH) = gxH \). We remark that, for every \( p \in P \) and \( a \in \Gamma_0 \), if \( p(a) \) is defined, then \( \phi(p(a)) = \tilde{p}(\phi(a)) \): indeed for some \( q \in P \), we have \( a = q(a_0) \) and \( p(a) = p \circ q(a_0) \). Let \( p' = p \circ q \). Thus \( \phi(p(a)) = p'H \). On the other hand, \( \phi(a) = qH \), and \( \tilde{p}(\phi(a)) = pqH \). But \( p'H = pqH \), since \( p'^{-1}pq \in H \), by \( \text{(1)} \).

We will consider \( \Gamma_0 \) as a subset of \( F(P)/H \) by identifying each \( a \in \Gamma_0 \) with \( \phi(a) \). Thus, for all \( p \in P \), \( \tilde{p} \) extends \( p \). It is also clear that the map \( g \mapsto \tilde{g} \) is a group homomorphism from \( F(P) \) into the group of permutations of \( F(P)/H \).

We want to endow \( F(P)/H \) with a graph structure extending \( \Gamma_0 \) and in such a way that for every \( g \in F(P) \), \( \tilde{g} \) is an automorphism of this graph. We do that by adding the minimal number of edges: given \( \alpha \) and \( \alpha' \) in \( F(P)/H \), we decide that \( F(P)/H \models R_{\alpha \alpha'} \) if and only if there exist \( \alpha, \alpha' \in \Gamma_0 \) and \( g \in F(P) \) such that \( \Gamma_0 \models R_{\alpha \alpha'} \) and \( \tilde{g}(\alpha) = \alpha \) and \( \tilde{g}(\alpha') = \alpha' \). We denote by \( \Gamma_1 \) the graph that we get this way.

So, by construction, every \( g \in F(P) \) induces an automorphism of \( \Gamma_1 \). What is not clear is whether \( \Gamma_1 \) is an extension of \( \Gamma_0 \). We have to be careful not to add an edge between two elements of \( \Gamma_0 \). This will be true if and only if the following condition is satisfied:

- For all \( p_0, p_1, p_2, p_3 \in P \) such that \( \Gamma_0 \models R_{p_0}(a_0)p_1(a_0) \land \neg R_{p_2}(a_0)p_3(a_0) \), there is no \( g \in F(P) \) such that \( gp_0H = p_2H \) and \( gp_1H = p_3H \).

A straightforward calculation shows that this condition is equivalent to:

\[
(3) \quad p_0p_2^{-1}p_3p_1^{-1} \notin p_0Hp_0^{-1}p_1Hp_1^{-1}.
\]

Let us sum up: for every subgroup \( H \) of \( F(P) \) satisfying the conditions \( \text{(1)}, \text{(2)} \), and \( \text{(3)} \), we have an extension \( \Gamma_1 \) of \( \Gamma_0 \) whose universe is \( F(P)/H \) such that every partial automorphism of \( \Gamma_0 \) extends to an automorphism of \( \Gamma_1 \). So the problem is to find such a subgroup \( K \) of finite index.

We remark that if we drop the assumption that \( K \) is of finite index, then we can solve the problem. Indeed, we know that there exists a graph \( \Gamma \) (possibly infinite) extending \( \Gamma_0 \) and for all \( p \in P \) an automorphism \( \tilde{p} \) of \( \Gamma \) extending \( p \) (for example the random graph). Let \( \eta \) be the homomorphism of \( F(P) \) into \( \text{Aut}(\Gamma) \) such that \( \eta(p) = \tilde{p} \) for all \( p \in P \), and write \( \tilde{h} \) instead of \( \eta(h) \). Reversing all that we have just said, we see that, if we set

\[
H = \{ h \in F(P); \ \tilde{h}(a_0) = a_0 \},
\]

then \( H \) satisfies the conditions \( \text{(1)}, \text{(2)} \) and \( \text{(3)} \). Thus \( H_0 \) also satisfies the conditions \( \text{(1)}, \text{(2)} \) and \( \text{(3)} \).

Since a finite intersection of subgroups of finite index is of finite index, it suffices to prove the following facts:

- For all \( x \in X_1 \), there exists a subgroup \( K \) of \( F(P) \) of finite index, containing \( H_0 \), but not containing \( \alpha \);

This is exactly the theorem of M. Hall (Theorem \( \text{(4)} \)).
• For all \( p_0, p_1, p_2, p_3 \in P \) such that \( \Gamma_0 \models R p_0(a_0) p_1(a_0) \land \neg R p_2(a_0) p_3(a_0) \), there exists a subgroup \( K \) of \( F(P) \) of finite index containing \( H_0 \) and such that \( p_0 p_2^{-1} p_3 p_1^{-1} \notin p_0 K p_0^{-1} \cdot p_1 K p_1^{-1} \).

Here we apply the Theorem\[\ref{thm:main}\] since \( p_0 H_0 p_0^{-1} p_1 H_0 p_1^{-1} \) is closed for the profinite topology and does not contain \( p_0 p_2^{-1} p_3 p_1^{-1} \), there exist a subgroup \( N \) of \( F(P) \) of finite index such that

\[
p_0 p_2^{-1} p_3 p_1^{-1} \notin (p_0 H_0 p_0^{-1} p_1 H_0 p_1^{-1}) N.
\]

We may moreover choose \( N \) to be normal in \( F(P) \). Then \( K = H_0 \cdot N \) is a subgroup of \( F(P) \) of finite index containing \( H_0 \) and \( (p_0 H_0 p_0^{-1} p_1 H_0 p_1^{-1}) N = p_0 K p_0^{-1} p_1 K p_1^{-1} \), so \( p_0 p_2^{-1} p_3 p_1^{-1} \notin p_0 K p_0^{-1} p_1 K p_1^{-1} \).

2.2. Generalisation. In fact the same argument can be used to prove many other results of the same kind. For example, let us prove that the class of triangle free graphs has the EPPA. We start from a finite triangle free graph. We construct a graph \( \Gamma_1 \) extending \( \Gamma_0 \) as above, using a subgroup \( K \) of \( F(P) \) of finite index, and, as above, \( F(P) \) acts on \( \Gamma_1 \). We demand in addition that \( \Gamma_1 \) is triangle free. For this, it is sufficient and necessary that the following condition is satisfied:

• For all \( a_1, a_2, a_3, a_1', a_2', a_3, a_1' \in \Gamma_0 \) if \( \Gamma_0 = R a_1 a_2' \land R a_2 a_1' \land R a_3 a_1' \), then, there is no \( h_1, h_2, h_3 \in F(P) \) such that \( h_3(a_1) = h_2(a_2'), h_1(a_2) = h_3(a_2'), h_2(a_3) = h_1(a_3) \).

For \( i = 1, 2, 3 \), let \( p_i \) and \( p_i' \) be elements of \( P \) such that \( a_i = p_i(a_0) \) and \( a_i' = p_i'(a_0) \). A calculation shows that the above condition is equivalent to

• For all \( p_1, p_2, p_3, p_1', p_3, p_1' \in P \) if \( \Gamma_0 = R p_1(a_0) p_2(a_0) \land R p_2(a_0) p_3(a_0) \land R p_3(a_0) p_1'(a_0) \), then \( 1 \notin p_1' K p_1^{-1} p_3' K p_3^{-1} \).

We finish the proof as above, using the Theorem\[\ref{thm:main}\].

The case of the \( K_2 \)-free graphs seems to be more difficult and, in fact, we have not been able to deduce it from the theorem of Ribes and Zalesskii. But it has been proved by Herwig \(\cite{Herwig} \) and, as a matter of fact, is just a particular case of Theorem\[\ref{thm:main}\].

Let us phrase the above arguments in a systematic way: let \( X \) be a finite set. We consider the set

\[
\text{Part}(X) = \{ p; p \text{ is an injective map from a subset of } X \text{ into } X \}
\]

with its natural monoid structure. Let \( P \) be a subset of \( \text{Part}(X) \). Consider the set \( \Sigma \) of words on the alphabet \( P \cup P^{-1} \) (that is the free monoid generated by \( P \cup P^{-1} \)).

To a given word \( w \in \Sigma \) we may naturally associate a partial automorphism of \( Y \). It is \( \zeta(w) \), where \( \zeta \) is the homomorphism from \( \Sigma \) into the monoid of partial automorphisms of \( Y \) defined by: for \( p \in P \cup P^{-1} \), \( \zeta(p) = p \). Let \( (X_i; i = 1, 2, \ldots, n) \) be the partition of \( X \) into orbits relatively to \( P \) (that is two elements \( x \) and \( y \) of \( X \) lie in the same \( X_i \) if and only if there exists \( w \in \Sigma \) such that \( \zeta(w)(x) = y \) and for each \( i = 1, 2, \ldots, n \), choose an element \( x_i \) in \( X_i \). Furthermore choose for every \( x \in X_i \) a word \( w_x \in \Sigma \) such that \( x = \zeta(w_x)(x_i) \). Then there is a correspondence between

• the \( n \)-tuples \( (H_1, H_2, \ldots, H_n) \) of subgroups of \( F(P) \) such that for all \( p \in P \) and \( i = 1, 2, \ldots, n \):

  1. \( w_y^{-1} \cdot p \cdot w_x \in H_i \) if \( y, x \in X_i \) and \( p(x) = y \),
  2. \( w_y^{-1} \cdot w_x \notin H_i \) if \( x, y \in X_i \), \( x \neq y \).
on one hand, and

- the tuples \((Y, (\tilde{p}; p \in P))\), where \(X \subseteq Y\), and, for all \(p \in P\), \(\tilde{p}\) is a permutation of \(Y\) extending \(p\)

on the other hand.

Indeed, let \((H_1, H_2, \ldots, H_n)\) be a sequence subgroups of \(F(P)\) satisfying the conditions (1) and (2). Let \(Y\) be the disjoint union of the sets \(F(P)/H_i\). If \(x \in X_i\), set \(\phi(x) = w_x H_i\). Hereby we consider \(w_x\) in a natural way as an element of \(F(P)\).

Condition (2) insures that the map \(\phi\) is injective. We will identify \(x\) with \(\phi(x)\), so that \(X\) will be viewed as a subset of \(Y\).

For each \(p \in P\), the left multiplication defines a permutation \(\tilde{p}\) on \(Y\) which extends by condition (1) the map \(p\).

In the reverse direction, assume that \(Y\) is a set including \(X\), and for each \(p \in A\), \(\tilde{p}\) a permutation of \(Y\) extending \(p\). Let \(\varphi\) be the group homomorphism from \(F(P)\) into \(\text{Perm}(Y)\), the group of permutation of \(Y\), defined by: for all \(p \in P\), \(\varphi(p) = \tilde{p}\).

Set, for \(i = 1, 2, \ldots, n\),

\[H_i = \{h \in F(P); \varphi(h)(x_i) = x_i\}\]

Then, the sequence \((H_1, H_2, \ldots, H_n)\) satisfies the conditions (1) and (2).

Now assume that \(X\) is the universe of a structure, also denoted \(X\), in some finite relational language \(\mathcal{L}\). Assume moreover that the maps \(p \in P\) are partial automorphisms of \(X\). We want to find the solutions to the following problem (subsequently referred to as \(\mathcal{P}\)): find an \(\mathcal{L}\)-structure \(Y\) extending \(X\) and, for each \(p \in P\), an automorphism \(\tilde{p}\) of \(Y\) extending \(p\).

Suppose that \((Y, (\tilde{p}; p \in P))\) is such a solution of \(\mathcal{P}\). Let again \(\varphi\) be the homomorphism from \(F(P)\) into \(\text{Aut}(Y)\), which is defined by: \(\varphi(p) = \tilde{p}\), and, for \(h \in F(P)\), write \(\tilde{h}\) instead of \(\varphi(h)\). Now, if \(R\) is a symbol of the language \(\mathcal{L}\) of arity \(k\), and if \(y_1, y_2, \ldots, y_k\) are elements in \(Y\) such that there exist \(x_1, x_2, \ldots, x_k\) in \(X\) and \(h \in F(P)\) such that, for all \(j = 1, 2, \ldots, k\), \(h(x_j) = y_j\) and \(X \models R x_1 x_2 \cdots x_k\), then necessarily, \(Y \models R y_1 y_2 \cdots y_k\). This proves that the following condition is satisfied:

If \(R\) is a symbol of the language \(\mathcal{L}\) of arity \(k\), if \(z_1, z_2, \ldots, z_k, t_1, t_2, \ldots, t_k\) are elements in \(X\) and if

\[X \models R z_1 z_2 \cdots z_k \land \neg R t_1 t_2 \cdots t_k\]

then, there is no \(h \in F(P)\) such that, for all \(i = 1, 2, \ldots, k\), \(\tilde{h}(z_i) = t_i\).

Setting \(H_i = \{h \in F(P); \tilde{h}(x_i) = x_i\}\) as above, an easy computation shows that this condition is equivalent to

(3) If \(R\) is a symbol of the language \(\mathcal{L}\) of arity \(k\), if \(i_1, i_2, \ldots, i_k\) are elements of \(\{1, 2, \ldots, n\}\), if \(x_1, x'_1 \in X_{i_1}, x_2, x'_2 \in X_{i_2}, \ldots, x_k, x'_k \in X_{i_k}\) and if \(X \models R x_1 x_2 \cdots x_k \land \neg R x'_1 x'_2 \cdots x'_k\), then there is no \(h \in F(P)\) such that, for all \(j = 1, 2, \ldots, k\), \(h \cdot w_{x_i} \equiv w_{x'_j} \mod(H_{i_j})\).

This condition (3) (taking for granted that the conditions (1) and (2) are satisfied) is sufficient: it suffices to define on the disjoint union, say \(Y\), of the sets \(F(P)/H_i\) considered as an extension of \(X\), the \(\mathcal{L}\)-structure by setting: for all \(R\), symbol of the language \(\mathcal{L}\) of arity \(k\), and for all \(y_1, y_2, \ldots, y_k\) in \(Y\),

\[Y \models R y_1 y_2 \cdots y_k\]

if and only if there exist \(x_1, x_2, \ldots, x_k\) in \(X\) and \(h \in F(P)\) such that, for all \(j = 1, 2, \ldots, k\), \(\tilde{h}(x_j) = y_j\) and \(X \models R x_1 x_2 \cdots x_k\).
We remark that if a sequence of subgroups \((H_1, H_2, \ldots, H_n)\) satisfies conditions \(2\) and \(3\) and if for all \(i = 1, 2, \ldots, n\), \(K_i\) is a subgroup of \(H_i\), then the sequence \((K_1, K_2, \ldots, K_n)\) satisfies also \(2\) and \(3\). We will express this fact by saying that \(2\) and \(3\) are negative conditions.

The correspondence that we have been speaking about is certainly not one-to-one in general. There may be several solutions corresponding to a given sequence \((H_1, H_2, \ldots, H_n)\). The solution that we have constructed enjoys the following property of “slimness”:

**Definition 2.1.** Let \(X\) be a finite \(\mathcal{L}\)-structure, \(P\) a set of partial automorphism of \(X\). A solution \((Y, (\hat{p}; p \in P))\) of the problem \(P\) is slim if: 1) for all \(y \in Y\), there exist \(x \in X\) and \(h \in F(P)\) such that \(y = h(x)\); 2) for all \(R\), symbol of the language \(\mathcal{L}\) of arity \(k\), and \(y_1, y_2, \ldots, y_k\) elements in \(Y\), \(Y \models R\{y_1, y_2, \ldots, y_k\}\) if and only if there exist \(x_1, x_2, \ldots, x_k\) in \(X\) and \(h \in F(P)\) such that \(X \models R\{x_1, x_2, \ldots, x_k\}\) and \(y_i = h(x_i)\) for all \(i = 1, 2, \ldots, k\).

It is easy to get a slim solution from any solution: if \((Y, (\hat{p}; p \in P))\) is a solution, throw away from \(Y\) the elements which are not image by an element of the group generated by \(\{\hat{p}; p \in P\}\) of an element of \(X\), and do the same for links. There is one further condition our solutions satisfy. Namely, for \(x, y \in X\), if there exists \(h \in F(P)\) such that \(h(x) = y\), then \(x\) and \(y\) are in the same orbit relative to \(P\).

If we restrict ourself to slim solutions satisfying this further condition, we do get a one-to-one correspondence.

We will need solutions which satisfy a stronger condition. Consider again the free monoid \(\Sigma\) over \(P \cup P^{-1}\) and the homomorphism \(\zeta\) from \(\Sigma\) to \(\text{Part}(X)\). We may consider every \(w \in \Sigma\) as an element of \(F(P)\) and we write again \(\hat{w}\) for \(\varphi(w)\), where \(\varphi\) is the group homomorphism from \(F(P)\) into \(\text{Aut}(Y)\) defined by: for all \(p \in P\), \(\varphi(p) = \hat{p}\). Of course \(\hat{w}\) extends \(\zeta(w)\).

**Definition 2.2.** The solution \((Y, (\hat{p}; p \in P))\) is special if it is slim and, for all \(t_1, t_2\) in \(X\) and \(h \in F(P)\), if \(h(t_1) = t_2\), then there exists a word \(w \in \Sigma\) such that \(\zeta(w)(t_1) = t_2\) and \(\hat{w} = h\).

We show now how to get a special solution from any solution. Let \((Y, (\hat{p}; p \in P))\) a solution. Set

- \(H_i = \{h \in F(P); \varphi(h)(x_i) = x_i\}\);
- \(K\) the kernel of \(\varphi\);
- \(L_i\) the subgroup of \(F(P)\) generated by \(w_i^{-1} \cdot p \cdot w_i; x, y \in X_i\) and \(p(x) = y\);
- \(K_i = K \cdot L_i\) (it is also the subgroup of \(F(P)\) generated by \(K \cup L_i\)).

First of all, we see that the sequence \((K_1, K_2, \ldots, K_n)\) satisfies the conditions \(1\) to \(3\). Condition \(1\) is insured by the fact that, for all \(i = 1, 2, \ldots, n\), \(L_i \subset K_i\). Conditions \(2\) and \(3\) are negative conditions and are satisfied by \((H_1, H_2, \ldots, H_n)\). Moreover, for all \(i = 1, 2, \ldots, n\), \(K_i \subset H_i\). So, \((K_1, K_2, \ldots, K_n)\) satisfies the conditions \(2\) to \(3\).

It remains to see that the solution corresponding to \((K_1, K_2, \ldots, K_n)\) is special. So, let \(t_1, t_2\) in \(X\) and \(h \in F(P)\) such that \(h(t_1) = t_2\). This implies that \(t_1\) and \(t_2\) belong to the same set \(F(P)/K_i\), say \(F(P)/K_1\), and we have \(t_1 = w_{t_1} x_1 = w_{t_1} K_1\) and \(t_2 = w_{t_2} x_1 = w_{t_2} K_1\). Thus, we have \(w_{t_2}^{-1} h w_{t_1} \in K_1\), and there exist \(k \in K\).
and \( l \in L_1 \) such that \( w_{t_2}^{-1}hw_{t_1} = kl \). So, \( w_{t_2}^{-1}hw_{t_1}l^{-1} \in K \), and, since \( K \) is normal \( w_{t_2}^{-1}hw_{t_1}l^{-1}h \in K \). Since \( l \in L_1 \), it is the product of elements and inverses of the set \( \{ w_y^{-1}p_w; x, y \in X_i \text{ and } p(x) = y \} \). So \( l \) is equal to \( u \) for some word \( u \in \Sigma \) such that \( \zeta(u)(x_1) = x_1 \). Set \( w = w_{t_2}w_{t_1}^{-1} \). Then \( \zeta(w)(t_1) = t_2 \) and \( w^{-1}h \in K \) and \( \hat{w} = \hat{h} \).

It is clear that, if we start from a finite solution \( Y \), then the special solution constructed above is also finite (since, in this case \( K \) has finite index). So we have proved:

**Proposition 2.3.** If the problem \( \mathcal{P} \) has a finite solution, then it has a finite special solution.

In fact, we have proved (and will use) more than that. We will want to solve the extension problem, not in the class \( \mathcal{C} \) of all \( \mathcal{L} \)-structures, but in a narrower class \( \mathcal{C}_1 \). Everything will go through, provided that there we can find a condition, denote it by \( (\ast) \), which is such that:

- if the solution is in \( \mathcal{C}_1 \), then the corresponding sequence \( (H_1, H_2, \ldots, H_n) \) satisfies \( (\ast) \);
- if \( (H_1, H_2, \ldots, H_n) \) satisfies \((1)\) to \((3)\) and \( (\ast) \), then the corresponding solution is in \( \mathcal{C}_1 \);
- \( (\ast) \) is a negative condition.

The classes of triangle free graphs and \( K_4 \)-free graphs are typical examples of such classes.

It is easy to see that our special solution has the following further property: If the problem \( \mathcal{P} \) has a finite solution \( (Y, (\bar{p}; p \in P)) \), then there is a finite special solution \( (Z, (p^*; p \in P)) \) and a weak homomorphism \( \rho : Z \rightarrow Y \) such that for every \( a \in Z \) and \( p \in P \) \( \rho(p^*(a)) = \tilde{p}(\rho(a)) \). For the definition of the notion of weak homomorphism see section 3.2. to define \( \rho \) use the equality \( \rho(hK_i) = \varphi(h)\langle x_i \rangle \).

### 3. From the EPPA to the profinite topology

In this section, we will show how to use the theorem concerning the extension property for automorphisms (to be proved in the next sections) to prove theorems about the profinite topology on free groups.

#### 3.1. A proof of the theorem of Ribes and Zalesskiï

Let \( \mathcal{L} \) be the language containing \( n \) unary predicate symbols \( U_1, U_2, \ldots, U_n \) and one binary predicate symbol, \( R \). Let \( \mathcal{C} \) be the class of \( \mathcal{L} \)-structures \( M \) where:

1. the universe is the disjoint union of the sets \( U_i^M \), for \( i = 1, 2, \ldots, n \);
2. \( Rxy \) implies that, for some \( i = 1, 2, \ldots, n - 1 \), \( U_i x \land U_{i+1} y \) or \( U_n x \land U_1 y \);
3. there are no \( x_1, x_2, \ldots, x_n \) in \( M \) such that
   \[
   M \models Rx_1x_2 \land Rxfx_3 \land \cdots \land Rxfx_nx_n \land Rxnx_1.
   \]

The class \( \mathcal{C} \) is called the class of cycle-free \( n \)-partitioned graphs.

**Theorem 3.1.** The class \( \mathcal{C} \) has the extension property for partial automorphisms.

This theorem will be proved in the next section. We give a proof of the Ribes-Zalesskiï theorem from Theorem 3.1 (using the techniques of the preceding section, one could also prove Theorem 3.1 from the theorem of Ribes and Zalesskiï).

Let \( H_1, H_2, \ldots, H_n \) be finitely generated subgroups of \( F(P) \), and \( g \) an element of \( F(P) \) not belonging to \( H_1 \cdot H_2 \cdot \cdots \cdot H_n \). Let \( M \) be the following structure,
Thus, \( M \). We see in a similar way that, for \( x \) in \( M \), \( y \) in \( M \), \( M \models Rx y \) if and only if: either for some \( i = 1, 2, \ldots, n - 1 \), \( x \in F(P)/H_i \), \( y \in F(P)/H_{i+1} \), and \( x \cap y \neq \emptyset \) or \( x \in F(P)/H_n \), \( y \in F(P)/H_1 \), and \( x g^{-1} \cap y \neq \emptyset \).

The fact that \( g \notin H_1 H_2 \cdots H_n \), implies that \( M \) is in \( C \). Indeed, assume, toward a contradiction, that we can find \( h_1, h_2, \ldots, h_n \) in \( F(P) \) such that, for all \( i = 1, 2, \ldots, n - 1 \) \( M \models Rh_i H_i H_{i+1} \) and \( M \models Rh_n H_i \). This implies that \( h_i H_i \cap h_{i+1} H_{i+1} = \emptyset \), which means that \( h_i^{-1} h_{i+1} \notin H_i H_{i+1} \). Similarly, we see that \( h_i^{-1} h_{i+1} \in H_i H_{i+1} \). At last, \( M \models Rh_n H_i \) implies \( g \in H_1 h_i^{-1} h_n H_n \). We deduce that \( g \in H_1 H_2 \cdots H_n \), a contradiction.

Let \( X_0 \) be a finite subset of \( F(P) \) which contains \( g \), a set of generators of \( H_i \), for \( i = 1, 2, \ldots, n \), and—assuming that these elements have been written as words of \( P \cup P^{-1} \)—all final segments of these words. Let \( M_0 \) be the finite substructure of \( M \) whose universe is

\[
\{xH_i; x \in X_0, i = 1, 2, \ldots, n\}.
\]

For each \( p \in P \), let \( \overline{p} \) be the partial automorphism of \( M_0 \) defined by: for all \( x \in M_0 \), if \( px \in M_0 \), then \( \overline{p}(x) = px \). If \( px \notin M_0 \), then \( \overline{p}(x) \) is not defined. These partial automorphisms can obviously be extended to automorphisms of \( M \), so by Theorem 3.1, we know that there exist a finite extension \( M_1 \) of \( M_0 \) in \( C_0 \) and automorphisms \( \overline{p} \) of \( M_1 \) extending \( \overline{p} \). Let \( \varphi \) be the homomorphism from \( F(P) \) into \( \text{Aut}(M_1) \) such that \( \varphi(p) = \overline{p} \). We remark that, if \( h \) is one of the generators of one of the \( H_i \), then \( \varphi(h) H_i = h H_i \) (thanks to our precaution to have included in \( M_1 \) all the final segments of \( h \)). Similarly, \( \varphi(g) H_i = g H_i \). Set, for \( i = 1, 2, \ldots, n \), \( K_i = \{ h \in F(P); \varphi(h)(H_i) = H_i \} \). Obviously, for \( i = 1, 2, \ldots, n \), the subgroup \( K_i \) has a finite index in \( F(P) \), and we have already pointed out that it contains \( H_i \).

We conclude by showing that \( g \notin K_1 K_2 \cdots K_n \). Assume, toward a contradiction, that \( g = k_1 k_2 \cdots k_n \). Set \( x_1 = H_1, x_2 = \varphi(k_1)(H_2), \ldots, x_n = \varphi(k_1 k_2 \cdots k_{n-1})(H_n) \). Obviously, \( M_1 \models Rh_1 H_2 \), thus, since \( \varphi(k_1) \) is an automorphism of \( M_1 \), \( M_1 \models Rx_1 x_2 \). We see in a similar way that, for \( i = 1, 2, \ldots, n - 1 \), \( M_1 \models Rx_i x_{i+1} \). Finally, from the fact that \( M_1 \models Rh H_1 \), we deduce that \( M_1 \models R\varphi(g)(H_n) H_1 \), that is \( M_1 \models Rx_n x_1 \). Thus, \( M_1 \) is not cycle-free, a contradiction.

3.2. Statement of the main combinatorial theorem. Before going further, we will need to set up some more notation. In this subsection we will consider a finite relational language \( L \).

If \( M \) and \( M' \) are \( L \)-structures, a weak homomorphism from \( M \) to \( M' \) is a map \( h \) from \( M \) to \( M' \) which is such that: if \( n \) is an integer, \( R \) an \( n \)-ary predicate symbol of \( L \) and \( a_1, a_2, \ldots, a_n \) are elements of \( M \) such that \( M \models Ra_1 a_2 \cdots a_n \), then \( M' \models Rh(a_1) h(a_2) \cdots h(a_n) \). If \( A \) is a substructure of both \( M \) and \( M' \), a weak \( A \)-homomorphism is a weak homomorphism which leaves the elements of \( A \) fixed.

To denote that \( h \) is a weak homomorphism from \( M \) into \( M' \), we write: \( h : M \rightarrow w M' \). To denote that it is a weak \( A \)-homomorphism, we write: \( h : M \rightarrow w A M' \).

If \( M \) is a structure, a link of \( M \) is a tuple \((R, a_1, a_2, \ldots, a_n)\) where \( R \) is a \( n \)-ary predicate symbol of the language \( L \), \( a_1, a_2, \ldots, a_n \) are elements of \( M \) and \( M \models Ra_1 a_2 \cdots a_n \). We say that an element \( a \) belongs to or is contained in a link \((R, a_1, a_2, \ldots, a_n)\) if \( a \) is one of the \( a_i \).
If $M$ is an $\mathcal{L}$-structure and $\mathcal{T}$ a set of $\mathcal{L}$-structures, we say that $M$ is $\mathcal{T}$-free if there is no structure $T \in \mathcal{T}$ and weak homomorphism $h : T \to M$.

We can now state a general combinatorial theorem, that will be proved in section 3.

**Theorem 3.2.** Let $\mathcal{L}$ be a finite relational language and $\mathcal{T}$ a finite set of finite $\mathcal{L}$-structures. Then the class of $\mathcal{T}$-free $\mathcal{L}$-structures has the EPPA.

3.3. **Back to the free groups.** A natural question is: is there a generalisation of the theorem of Ribes-Zalesskii that can be proved using Theorem 3.2 or even which is “equivalent” to it. The answer is yes for both questions, and that is what we are going to expose now.

If $H$ is a subgroup of $F(P)$ and $x$ and $y$ are two elements of $F(P)$, we write $x \equiv y \mod H$ for $xH = yH$.

Let $n \in \omega$ and $X$ be a finite set (the set of unknowns). A left-system is a finite set of equations of the form

$$x \equiv_i y \cdot g \text{ where } i \in \{1, 2, \ldots, n\}, x, y \in X \text{ and } g \in F(P)$$

or of the form

$$x \equiv_i g \text{ where } i \in \{1, 2, \ldots, n\}, x \in X \text{ and } g \in F(P).$$

Let $\mathcal{H} = (H_1, H_2, \ldots, H_n)$ be a sequence of subgroups of $F(P)$. A solution of a left-system $(E)$ in $F(P)$ modulo $\mathcal{H}$ is a family $(g_x : x \in X)$ of elements of $F(A)$ such that, for each equation $x \equiv_i y \cdot g$ in $(E)$, $g_x \equiv g_y \mod H_i$, and for every $x \equiv_i g$ in $(E)$, $g_x \equiv g \mod H_i$.

**Theorem 3.3.** Let $n \in \omega, \mathcal{H} = (H_1, H_2, \ldots, H_n)$ be a sequence of finitely generated subgroups of $F(P)$ and $(E)$ be a left-system. Assume that $(E)$ has no solution in $F(P)$ modulo $\mathcal{H}$. Then there exist subgroups $K_1, K_2, \ldots, K_n$ of finite index in $F(P)$ such that $H_i \subseteq K_i$ for all $i$, $1 \leq i \leq n$ and such that $(E)$ has no solution in $F(P)$ modulo $\mathcal{K} = (K_1, K_2, \ldots, K_n)$.

We remark that this theorem immediately implies the theorem of Ribes and Zalesskii: the fact that an element $g$ of $F(P)$ does not belong to $H_1 \cdot H_2 \cdots \cdot H_n$ means exactly that the left-system

$$\begin{cases}
x_n \equiv_n g \\
x_{n-1} \equiv_{n-1} x_n \\
\cdots \\
x_2 \equiv_2 x_3 \\
x_2 \equiv_1 e
\end{cases}$$

has no solution in $F(P)$ modulo $(H_1, H_2, \ldots, H_n)$.

If $H$ is a subgroup of $F(P)$ and $x$ and $y$ are elements of $F(P)$, we will write $x \sim_H y$ for $HxH = HyH$. We notice that the relation $\sim_H$ is an equivalence relation. We first want to replace the left-system by another kind of system, easier to manage for our purpose. A double-system is a finite set of equations of the form

$$x^{-1} \cdot y \sim_i g \text{ where } i = 1, 2, \ldots, n, x, y \in X \text{ and } g \in F(P)$$

or of the form

$$x \sim_i g \text{ where } i = 1, 2, \ldots, n, x \in X \text{ and } g \in F(P).$$
Let $\mathcal{H} = (H_1, H_2, \ldots, H_n)$ be a sequence of subgroups of $F(P)$. A solution of a double-system $(E)$ in $F(P)$ modulo $\mathcal{H}$ is a family $(g_x : x \in X)$ of elements of $F(P)$ such that, for every $x^{-1} y \sim_i g$ in $(E)$, $g_x^{-1} \cdot g_y \sim_{H_i} g$, and for every $x \sim_i g$ in $(E)$, $g_x \sim_{H_i} g$.

We will prove

**Proposition 3.4.** Let $(F)$ be a double-system, $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of $F(P)$. If the double-system $(F)$ has no solution in $F(P)$ modulo $(H_1, H_2, \ldots, H_n)$, then there exist subgroups $K_1, K_2, \ldots, K_n$ of finite index in $F(P)$ such that $H_i \subseteq K_i$ for all $i$, $1 \leq i \leq n$ and such that $(F)$ has no solution in $F(P)$ modulo $(K_1, K_2, \ldots, K_n)$.

We show how to prove Theorem 3.3 from Proposition 3.4. We see that the left-system $(E)$ can be translated into a double-system. Let $(F)$ be the double-system obtained by replacing each equation of the form $x \equiv_i y \cdot g$ of $(E)$ by the two equations:

\[
\begin{align*}
  z^{-1} \cdot x & \sim_i e \\
y^{-1} \cdot z & \sim_0 g
\end{align*}
\]

where $z$ is a new unknown (of course, different $z$ should be taken for differential equations). In the same way, $x \equiv_i g$ should be replaced by

\[
\begin{align*}
  z^{-1} x & \sim_i e \\
z & \sim_0 g.
\end{align*}
\]

In this translation, we have to introduce a new subgroup $H_0$ which will be the trivial subgroup.

It is clear that the double-system $(F)$ has a solution modulo $(H_0, H_1, \ldots, H_n)$ if and only if the original left-system $(E)$ has a solution modulo $(H_1, H_2, \ldots, H_n)$; thus $(F)$ has no solution. By Proposition 3.4 there exist subgroups $K_0, K_1, \ldots, K_n$ of finite index such that $H_i \subseteq K_i$ for all $i$, $0 \leq i \leq n$, and such that $(F)$ has no solution modulo $(K_0, K_1, \ldots, K_n)$. Again, this implies that the system $(E)$ has no solution modulo $(K_1, K_2, \ldots, K_n)$.

Incidentally, we notice that a double-system can also easily be translated into a left-system, so that Theorem 3.3 and Proposition 3.4 have exactly the same content.

It remains to prove Proposition 3.4.

Let $(F)$ be a double-system. Write $X$ for its set of unknowns, and let $\mathcal{H} = (H_1, H_2, \ldots, H_n)$ be a sequence of finitely generated subgroups such that $(F)$ has no solution modulo $(H_1, H_2, \ldots, H_n)$. We first remark that we can assume that there is no equation of the form $x \sim_i g$ (so that there will remain only “homogeneous” equations, of the form $y^{-1} \cdot x \sim_i g$). Indeed, add one new unknown $z$ (only one all together), and replace each equation of the form $x \sim_i g$ by $z^{-1} \cdot x \sim_i g$. If the new system had a solution $(g_z, g_x : x \in X)$, then $(g_z^{-1} \cdot g_x : x \in X)$ would be a solution of the original system.

We consider the following relational language $\mathcal{L}$: it contains:

- $n + 1$ unary predicate symbols $U_0, U_1, \ldots, U_n$;
- $n$ binary predicate symbols $T_i$;
- for each equation $E : x^{-1} y \sim_i g$ in $(F)$, a binary predicate symbol $R_E$.

We now define a structure $M$:

- its base set is the disjoint union of the sets $U_i^M$, the interpretations of $U_i$ in $M$, and $U_1^M$ is the set $F(P)/H_i$ for $i = 1, 2, \ldots, n$, and $U_0^M = F(P)$.
for $i = 1, 2, \ldots, n$, and $x, y \in M$, $M \models T_i x y$ if and only if $x \in U_0^M$, $y \in U_i^M$ and $x \in y$.

- For all $\alpha$ and $\beta$ in $M$ and equation $E : x^{-1} y \sim_i g$, $M \models R_{E \alpha \beta}$ if and only if $\alpha, \beta \in F(P)/H_i$ and $g \in \alpha^{-1} \cdot \beta$ (in other words, if and only if there exist $a \in \alpha$ and $b \in \beta$ such that (equivalently, for all $a \in \alpha$ and $b \in \beta$) $H_i a^{-1} b H_i = H_i g H_i$).

We first notice that for all $h \in F(P)$ the left multiplication by $h$ is an automorphism of $M$. Call it $h$.

Second, we exploit the fact that $(F)$ has no solution modulo $(H)$. We cannot find elements $c(x, i)$ in $M$, for $x \in X$ and $0 \leq i \leq n$ such that the following set of conditions is satisfied:

\[
\begin{align*}
1. & \text{ for all } x \in X \text{ and } i, 0 \leq i \leq n, \quad M \models U_i c(x, i); \\
2. & \text{ for all } x \in X, \text{ and } i = 1, 2, \ldots, n, \quad M \models T_i c(x, 0) c(x, i); \\
3. & \text{ If } E : x^{-1} \cdot y \sim_i g \text{ belongs to } (F), \quad M \models R_{E c(x, i) c(y, i)}.
\end{align*}
\]

Otherwise, $(c(x, 0); x \in X)$ would be a solution of $(F)$ mod $(H)$.

Write $N$ for the $L$-structure whose base-set is the set $\{c(x, i); x \in X, 1 \leq i \leq n\}$ and where the only relations are those necessary to make the conditions (2) true. Another way to say that $(F)$ has no solution modulo $(H)$ is to say that $N$ cannot be weakly embedded in $M$.

Let now $C_0$ be a finite subset of $F(P)$ containing the parameters occurring in the equations of $(F)$ and for each $i, 1 \leq i \leq n$ a set generating $H_i$. For each element of $c \in C_0$, write it in the form $p_0 \cdot p_1 \cdot \ldots \cdot p_m$ with the $p_i$ in $P \cup P^{-1}$, and let $P_c$ be the set $\{p_1 p_2 \cdot \ldots \cdot p_m, p_2 p_1 \cdot \ldots \cdot p_m, p_m^{-1} p_1 p_2 \cdot \ldots \cdot p_m\}$. Set $C = \bigcup_{c \in C_0} P_c$ and let $M_0$ be the substructure of $M$ whose base set is $C \cup \{cH_i; c \in C, 1 \leq i \leq n\}$. For all $p \in P$, we can define a partial automorphism $\hat{p}$ on $M_0$ as the restriction of $p$, the left multiplication by $p$, to $M_0$.

Applying Theorem 5.2 we deduce that there exists a finite $L$-structure $M_1$, extending $M_0$, inside which $N$ cannot be weakly embedded and for each $p \in P$, an automorphism $\hat{p}$ of $M_1$ extending $p$. Then, there is a natural group homomorphism $h \mapsto \hat{h}$ from $F(P)$ onto $\text{Aut}(M_1)$. The point is that for $i = 1, 2, \ldots, n$, and $c \in C$, $\hat{c}(H_i) = cH_i$. In particular, this is true for a set generating $H_i$. In other words, if we set

\[ K_i = \{h \in F(P); \hat{h}(H_i) = H_i\}, \]

then $K_i$ contains $H_i$. Moreover, $K_i$ is a subgroup of finite index of $F(P)$.

Set $\mathcal{K} = (K_1, K_2, \ldots, K_n)$. We shall conclude by proving that $(F)$ has no solution modulo $\mathcal{K}$.

Assume, for a contradiction, that $(g_x; x \in X)$ is a solution of $(F)$ modulo $\mathcal{K}$. For $x \in X$ and $i, 1 \leq i \leq n$, set $x_i = \hat{g}_x(H_i)$ and $x_0 = \hat{g}_x(e)$. We prove that the $x_i$ satisfy the conditions (2), and thus that $N$ is weakly embedded in $M_1$, which is contradictory.

1. For all $x \in X$ and $i, 1 \leq i \leq n$, we have $M_1 \models U_i (\hat{g}_x(H_i))$ since $M_1 \models U_i H_i$ and $\hat{g}_x$ is an automorphism. Similarly, $M_1 \models U_0 \hat{g}_x(e)$.

2. For $i = 1, 2, \ldots, n$ and $x \in X$, we have $M_1 \models T_i \hat{g}_x(e) \hat{g}_x(H_i)$ since $M_1 \models T_i c H_i$ and $\hat{g}_x$ is an automorphism.

3. If $E : x^{-1} y \sim_i g$ is in $(F)$, then there exist $k, k'$ in $K_i$ such that $(\hat{g}_x)^{-1} \cdot g_y = k \cdot g \cdot k'$, so $\hat{g}_x^{-1} \cdot \hat{g}_y \cdot H_i = k \cdot g \cdot k' \cdot H_i = \tilde{k} \cdot g \cdot H_i$. It is clear that
and consider equations of the form:

\[ P \\\n\]

of being short proof of Theorem 3.2 using Theorem 3.3. One has to translate the condition directly in Section 5. But let us point out that the method of section 2 provides a short proof of Theorem 3.3 using Theorem 3.2. One has to translate the condition of being \( T \)-free into a finite system of equations.

We can give an alternative formulation to Theorem 3.3: fix the finite alphabet \( P \) and consider equations of the form:

\[ x \equiv y \cdot v \mod \langle w_1, w_2, \ldots, w_n \rangle \]

where the \( x \) and \( y \) are unknowns and the \( v \) and the \( w \) are words in the alphabet \( P \cup P^{-1} \). Given a group \( G \) and, for each \( p \in P \), a value \( p \) of \( p \), this equation has an obvious meaning: each of the words \( v \) or \( w \), occurring in these equations, is interpreted by the element of \( G \) obtained by replacing the \( p \) by \( p \), and we must find values in \( G \) for the unknowns satisfying all the equations. Then Theorem 3.3 can be rephrased as follows:

**Theorem 3.5.** Let \( S \) be a finite set of equations of the above form. If for all finite groups \( G \) and interpretations of the \( p \) in \( G \), the system \( S \) has a solution, then for every group \( G' \) and for all interpretations of the \( p \) in \( G' \) the system \( S \) has a solution.

## 4. Extension lemmata

### 4.1. A simple combinatorial proof of the theorem of Hrushovski

In this subsection, we will give a simple combinatorial proof of Theorem 1.4. We begin with two definitions.

**Definition 4.1.** Let \( X \) be a finite set and \( n \) a positive integer; then \( \Gamma(X,n) \) denotes the graph whose base is \( [X]^n \), the set of subsets of \( X \) of cardinality exactly \( n \), and where the binary relation \( R \) is defined by: for \( a, b \in [X]^n \), \( Rab \) if and only if \( a \cap b \neq \emptyset \).

**Definition 4.2.** A subgraph \( \Gamma_0 \) of \( \Gamma(X,n) \) is poor if 1) for all \( x \in X \), \( \text{card}\{a \in \Gamma_0; x \in a\} \leq 2 \) and 2) for all \( a, b \in \Gamma_0 \), if \( a \neq b \), then \( \text{card}(a \cap b) \leq 1 \).

If \( \alpha \) is a permutation of \( X \), we will denote by \( \alpha^* \) the induced automorphism of \( \Gamma(X,n) \).

The theorem is an immediate consequence of the two following lemmata.

**Lemma 4.3.** Every finite graph is poorly represented: if \( \Gamma \) is a finite graph, then there exist a finite set \( X \), a positive integer \( n \) and a poor subgraph \( \Gamma_0 \) of \( \Gamma(X,n) \) isomorphic to \( \Gamma \).

**Proof.** Let \( X_0 \) be the set of edges of \( \Gamma \). For each point \( a \) in \( \Gamma \), let \( f(a) = \{x \in X_0; x \) is adjacent to \( a\} \). If the cardinality of \( f(a) \), for \( a \in \Gamma \), is constant equal to \( n \) bigger than one, then we are done, because \( f \) is an isomorphism from \( \Gamma \) to a poor subgraph of \( \Gamma(X_0,n) \). In the general case, let \( n = \sup(\sup(f(a); a \in \Gamma), 2) \) and let \( X \) be a finite set containing \( X_0 \) and sufficiently large so that it is possible to define a map \( h \) from \( \Gamma \) to \( [X]^n \) such that for all distinct \( a \) and \( b \) in \( \Gamma : f(a) \subseteq h(a) \), \( h(a) - f(a) \subseteq X - X_0 \), \( h(a) - f(a) \cap (h(b) - f(b)) = \emptyset \).

**Lemma 4.4.** Let \( \Gamma_0 \) and \( \Gamma_1 \) be two poor subgraphs of \( \Gamma(X,n) \), and \( f \) an isomorphism from \( \Gamma_0 \) to \( \Gamma_1 \). Then there exist a permutation \( \alpha \) of \( X \) such that \( \alpha^* \) extends \( f \).
Proof. First define $\alpha(x)$ for $x \in X$ belonging to two elements $a$ and $b$ of $\Gamma_0$: there is no choice, it has to be the unique element of $f(a) \cap f(b)$; then, for all $a \in \Gamma_0$, extend $\alpha$ to $a$ by defining a bijection between 

$$\{x \in a; \text{ for all } b \in \Gamma_0 - \{a\}, x \notin b\}$$

and 

$$\{x \in f(a); \text{ for all } b \in \Gamma_1 - \{f(a)\}, x \notin b\}.$$  

This is possible because these two sets have the same cardinality. Then extend to a permutation of $X$.

Remark. In his paper, Hrushovski remarks that the cardinality of the resulting homogeneous graph $Z$ is bounded by something like $2^{2^k}$, if $k$ is the cardinality of $\Gamma$. He asks whether it is possible to find a graph $Z$ of cardinality bounded by $2^{ck^2}$, for some constant $c$.

The above proof show that, in fact the graph $Z$ can be found of cardinality less than $k^{2k}$. We will make the precise computation.

Let $k$ be the cardinality of $\Gamma$ and $n$ the valency of $\Gamma$, that is the maximal number of edges adjacent to a given vertex. The “homogeneous” graph $Z$ is a graph $\Gamma(X,n)$; let us compute precisely the cardinality of $X$. Let $m$ be the number of edges of $\Gamma$, and for every $a \in \Gamma$, $c(a)$ the number of edges adjacent to $a$. We have

$$\sum_{a \in \Gamma} c(a) = 2m.$$  

On the other hand, the set $X$ is the disjoint union of the set of edges and, for all $a \in \Gamma$, of a set of cardinality $n - c(a)$. So the cardinality of $X$ is

$$m + \sum_{a \in \Gamma} (n - c(a)) = nk - m$$

and the cardinality of $Z$ is bounded by $(nk)^n$.

So, for graphs of bounded valency, the cardinality of $Z$ is bounded polynomially in the cardinality of $\Gamma$ (but we should be careful that the graph $Z$ has a much bigger valency).

If we do not want to take the valency into account, a first estimation gives $k^{2k}$ for the bound of the cardinality of $Z$. But we can get a slightly better bound: we may assume that $m$ the number of edges in $\Gamma$ is bigger than or equal to the number of non-edges, so the cardinality of $X$ can be bounded by $3k^2/4$ and the cardinality of $Z$ can be bounded by 

$$\left(\frac{3k^2}{4}\right)^k \times \frac{1}{k!}.$$  

Remarking that $k! \geq (k/e)^k$, we see that the cardinality of $Z$ can be bounded by $(3ek/4)^k$.

4.2. Generalisation to arbitrary relational languages.

Theorem 4.5. Let $r > 1$. Suppose the language $\mathcal{L}$ consists of just one $r$-ary predicate $R$. Let $A$ be an $\mathcal{L}$-structure of cardinality $c$. There exists an $\mathcal{L}$-structure $B$ with $A \subseteq B$ and $\text{card}(B) \leq 2^{\text{trc}^r}$ such that every partial automorphism on $A$ extends to an automorphism of $B$. 

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Before we prove the theorem we give some helpful definitions.

**Definition 4.6.** Let $X$ be a finite set. We define the $\mathcal{L}$-structure $M(X)$: its domain is $(\varphi(X))^r$, so its elements are $r$-tuples of subsets of $X$. For an element $a \in M(X)$ we denote by $a_j$ the $j$-th coordinate of $a$ $(1 \leq j \leq r)$. We define the $r$-ary relation $R$ on $M(X)$. For $a^1, \ldots, a^r \in M(X)$:

$$M(X) = R(a^1, \ldots, a^r) \text{ if and only if } \bigcap_{1 \leq i \leq r} a^i = \emptyset.$$

Note that the group Sym$(X)$ of permutations of $X$ acts as automorphisms on $M(X)$.

**Definition 4.7.** Let $k$ be an integer. We say that a substructure $N$ of $M(X)$ is $k$-regular, if there exists an integer $p_k > 0$ such that, for every $a^1, \ldots, a^k \in N$ and every $i_1, \ldots, i_k \in \{1, \ldots, r\}$, if $i_1, \ldots, i_k$ are pairwise distinct, then $\text{card}(\bigcap_{j=1}^k a^i_{i_j}) = p_k$. We say that $N$ is regular, if

- it is $k$-regular for every $k < r$;
- for $a, b \in N$ distinct and $1 \leq i \leq r$ we have $a_i \cap b_i = \emptyset$;
- for $a^1, \ldots, a^r \in N$, if $Ra^1 \cdots a^r$, then $\text{card}(\bigcap_{i=1}^r a^i) = 1$.

For an integer we denote by $M(X, m)$ the substructure of $M$ consisting of all $r$-tuples of sets of size $m$. Note that if $N$ is a 1-regular substructure of $M(X)$, then there exists an $m$ such that $N$ is a substructure of $M(X, m)$. Furthermore Sym$(X)$ also acts on $M(X, m)$.

The theorem follows immediately from the following two lemmata:

**Lemma 4.8.** Let $A$ be an $\mathcal{L}$-structure of cardinality $c$. There exist a set $X$ with $\text{card}(X) \leq r!c^r$ and a regular substructure of $M(X)$ which is isomorphic to $A$.

**Lemma 4.9.** If $N$ is a regular substructure of $M(X)$, then every partial automorphism of $N$ extends to an automorphism of $M(X)$, which is induced by the action of Sym$(X)$ on $M(X)$.

**Proof of the first lemma.**

Let $X_0$ be the set of links of $A$ (here, the $r$-tuple $(a^1, a^2, \ldots, a^r)$ of $A$ such that $A \models Ra^1a^2 \cdots a^r$). We first embed $A$ into $M(X_0)$: for each $a \in A$ we let $\alpha(a) = (t_1, \ldots, t_r)$, where $t_i = \{q \in X_0 : q_i = a\}$. We first remark that the last two conditions for regularity are satisfied. We will increase the set $X_0$ step by step and change the embedding $\alpha$ such that $\alpha[A]$ becomes $k$-regular for every $k$ with $r - 1 \geq k \geq 1$. Do not worry that $\alpha$ is not necessarily an embedding to begin with. All the isolated points get mapped to the same $r$-tuple. In our construction we will maintain the condition, that for $a, b \in A$ distinct and $1 \leq j \leq r$ we have $\alpha(a)_j \cap \alpha(b)_j = \emptyset$. So the final mapping $\alpha$ will be injective.

We are first aiming for $(r - 1)$-regularity. Consider all sets of the form $\bigcap_{1 \leq j \leq r-1} \alpha(a^j)_i$ for $a^1, a^2, \ldots, a^{r-1} \in A$ and $i_1, \ldots, i_{r-1} \in \{1, \ldots, r\}$ pairwise distinct. Let $v$ be the maximal cardinality of these sets. You may think of $v$ as the (maximal) valency of the $\mathcal{L}$-structure $A$. We let $p_{r-1} = v$. Easily $p_{r-1} \leq c$. For every $a^1, \ldots, a^{r-1} \in A$ and $i_1, \ldots, i_{r-1} \in \{1, \ldots, r\}$ pairwise distinct, if $\text{card}(\bigcap_{j=1}^{r-1} \alpha(a^j)_i) = p' < p_{r-1}$ we choose $(p_{r-1} - p')$ new elements which we add to each set $\alpha(a^j)_i$ (for $1 \leq j \leq (r - 1)$). As every new element will belong to exactly $(r - 1)$ sets of the form $\alpha(a)_j$ with different $j$ each, the second two conditions for regularity will remain true. If we let $X_1$ be the set $X_0$ together with the
new points, and if we change \( \alpha \) as indicated we will have \( \alpha[A] \) is \( (r - 1) \)-regular in \( M(X_t) \). Note that \( X_t = \bigcup \{ \cap_{j=1}^{r-1} \alpha(a^j)_{i_j} : a^1, \ldots, a^{r-1} \in A, i_1, \ldots, i_{r-1} \in \{1, \ldots, r\} \} \) pairwise distinct.

Now we suppose we already have constructed a set \( X_t \) \( (1 \leq t < (r - 1)) \) and an embedding \( \alpha : A \hookrightarrow M(X_t) \) which is \( j \)-regular for \( (r - 1) \geq j \geq (r - t) \), the constant for \( (r - t) \)-regularity being \( p_{r-t} = t! \cdot c^{t-1} \cdot v \). Furthermore we suppose \( X_t = \bigcup \{ \cap_{j=1}^{r-1} \alpha(a^j)_{i_j} : a^1, \ldots, a^{r-t} \in A, i_1, \ldots, i_{t-1} \in \{1, \ldots, r\} \} \) pairwise distinct.

Now we consider all sets of the form \( \bigcap_{j=1}^{r-1} \alpha(a^j)_{i_j} \) for \( a^1, \ldots, a^{r-(t+1)} \in A \) and \( i_1, \ldots, i_{r-(t+1)} \in \{1, \ldots, r\} \) pairwise distinct. As we have

\[
\bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j} = \bigcup \left\{ \cap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j} \cap \alpha(b)_k : b \in A, k \in \{1, \ldots, r\} - \{i_1, \ldots, i_{r-(t+1)}\} \right\},
\]

we have \( \operatorname{card} \left( \bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j} \right) \leq (t + 1)! \cdot c^{t-1} \cdot v \). This means we can set \( p_{r-(t+1)} = (t + 1)! \cdot c^{t-1} \cdot v \). Now we add \( (p_{r-(t+1)} - q) \) many new points to a set of the form \( \bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j} \) of cardinality \( q \) to define the set \( X_{t+1} \).

At the end we constructed a set \( X = X_{r-1} \) and an embedding \( \alpha : A \hookrightarrow M(X) \) such that \( \alpha[A] \) is regular, the constant for 1-regularity being \( p_1 = (r - 1)! \cdot c^{r-2} \cdot v \). Every point of \( X \) is of the form \( \alpha(a)^i \) with \( a \in A \) and \( i \in \{1, \ldots, r\} \). That means \( \operatorname{card}(X) \leq r \cdot c \cdot p_1 = r! \cdot c^{r-1} \cdot v \leq r! \cdot c^r \). This proves the Lemma. Note that we could get a slightly better bound in the theorem by letting \( B = M(X, p_1) : \operatorname{card}(B) \leq (r \cdot c^r) \cdot (r! \cdot c^{r-2} \cdot v) \leq (ec^r) \cdot c^{r-1} \).

Before we prove Lemma \( \text{[4.10]} \) we do a little preparation:

**Definition 4.10.** Let \( X, Y \) be sets of the same cardinality and let \( q \) be a partial function from \( \mathcal{P}(X) \) to \( \mathcal{P}(Y) \). We say \( q \) is induced by a bijection \( \pi : X \rightarrow Y \), if for every \( a \) in the domain of \( q \) \( q(a) = \pi[a] \).

**Lemma 4.11.** Let \( X, Y \) be finite sets of the same cardinality and let \( q \) be a partial function from \( \mathcal{P}(X) \) to \( \mathcal{P}(Y) \). Then \( q \) is induced by a bijection \( \pi : X \rightarrow Y \) if and only if for every subset \( s \) of \( \operatorname{dom}(q) \), the domain of \( q \), we have: \( \operatorname{card}(\bigcap_{a \in s} q(a)) = \operatorname{card}(\bigcap_{a \in s} \pi[a]) \).

Suppose for every \( s \subseteq \operatorname{dom}(q) \), \( \operatorname{card}(\bigcap_{a \in s} a) = \operatorname{card}(\bigcap_{a \in s} \pi[a]) \). We can suppose \( \operatorname{dom}(q) \) is closed under intersection, as for \( a, b \in \operatorname{dom}(q) \) we can define \( q(a \cap b) \) to be \( q(a) \cap q(b) \) and still maintain the condition on \( q \). We can also suppose \( \operatorname{dom}(q) \) is closed under complements, as for every \( a \in \operatorname{dom}(q) \) we can define \( q(a^c) = q(a)^c \). Here we are using the finiteness of \( X \) and \( Y \). Finally we can suppose that for every \( x \in X \), \( \{x\} \in \operatorname{dom}(q) \); for \( x \in X \) we define \( s_x = \{b \in \operatorname{dom}(q) | x \in b\} \) and by the condition on \( q \), we have \( \bigcap_{b \in s_x} q(b) \neq \emptyset \). We choose \( y \in \bigcap_{b \in s_x} q(b) \) and let \( q(\{x\}) = \{y\} \) and check that we still have the condition on \( q \). This means we can assume \( \operatorname{dom}(q) = \mathcal{P}(X) \) and in this case we can define \( \pi \) by letting \( \pi(x) \) be the unique element of \( q(\{x\}) \). Easily we have for every \( a \in \mathcal{P}(X) \) that \( q(a) = \pi[a] \). 

Now we prove the Lemma \( \text{[4.13]} \). Let \( p \) be a partial automorphism of \( N \) with domain \( D \). Define a partial map \( q \) from \( \mathcal{P}(X) \) with domain \( \{a_i : a \in D, 1 \leq i \leq n\} \)
$i \leq r$) by defining for $a \in D$ and $1 \leq i \leq r$ : $q(a_i) = p(a)_i$. Now let $k \in \omega$ and let $a_1^1, \ldots, a_k^k \in D$ and $j_1, \ldots, j_k \in \{1, \ldots, r\}$. We want to check that $\text{card}(a_1^1 \cap \cdots \cap a_k^k) = \text{card}(p(a_1)_1 \cap p(a_2)_2 \cap \cdots \cap p(a_k)_k)$. We can suppose that $(a_1^1, j_1), \ldots, (a_k^k, j_k)$ are pairwise distinct. Also we can suppose that $j_1, \ldots, j_k$ are pairwise distinct as otherwise both cardinalities are 0 by the second condition for regularity. If $k < r$, then by $k$-regularity both cardinalities are equal to $p_k$. If $k = r$ by changing the enumeration we can suppose that $j_1 = 1, \ldots, j_k = r$. In that case only the cardinalities 0 and 1 appear. We have $\text{card}(a_1^1 \cap \cdots \cap a_k^k) = 1$ if and only if $Ra^1a^2 \cdots a^r$ if and only if $Rp(a^1)p(a^2) \cdots p(a^r)$ if and only if $\text{card}(p(a_1^1 \cap \cdots \cap p(a_k^k)_r) = 1$.

Thus $q$ and therefore also $p$ is induced by a permutation of $X$. \hfill $\lozenge$

The following lemma shows how to compute bounds for bigger languages:

**Lemma 4.12.** Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be a finite relational language. Suppose $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$. Let $A$ be an $\mathcal{L}$-structure. Let $B_1$ be an $\mathcal{L}_1$-structure with $A \subset B_1$ such that every partial automorphism of $A|_{\mathcal{L}_1}$ can be extended to an automorphism of $B_1$ and let $B_2$ be an $\mathcal{L}_2$-structure with $A \subset B_2$ such that every partial automorphism of $A|_{\mathcal{L}_2}$ can be extended to an automorphism of $B_2$. Suppose $\text{card}(B_1) = n_1$ and $\text{card}(B_2) = n_2$.

There exists an $\mathcal{L}$-structure $B$ with $\text{card}(B) = n_1 \cdot n_2$ and $A \subset B$, such that every partial automorphism of $A$ extends to an automorphism of $B$.

**Proof.** The domain of $B$ is $B_1 \times B_2$.

For $R \in \mathcal{L}_1$, $r$-ary and $(a_1^1, a_2^1), \ldots, (a_1^r, a_2^r) \in B$ we define:

$R^B(a_1^1, a_2^1) \cdots (a_1^r, a_2^r) \iff R^{B_1}a_1^1 \cdots a_1^r$

and for $R \in \mathcal{L}_2$, $r$-ary and $(a_1^1, a_2^1), \ldots, (a_1^r, a_2^r) \in B$ we define:

$R^B(a_1^1, a_2^1) \cdots (a_1^r, a_2^r) \iff R^{B_2}a_1^1 \cdots a_1^r$.

Note that for $\alpha_1 \in \text{Aut}(B_1)$ and $\alpha_2 \in \text{Aut}(B_2)$ we have that $(\alpha_1, \alpha_2) \in \text{Aut}(B)$. Let $i_1 : A|_{\mathcal{L}_1} \hookrightarrow B_1$ and $i_2 : A|_{\mathcal{L}_2} \hookrightarrow B_2$ be the inclusion mappings. Then $(i_1, i_2) : A \hookrightarrow B$ is an embedding of $\mathcal{L}$-structures.

Via this embedding we can suppose that $A \subset B$. Let $p$ be a partial automorphism of $A$. As $p$ is a partial automorphism of $A|_{\mathcal{L}_1}$, it can be extended to an automorphism $\alpha_1$ of $B_1$ and for the analogous reason to an automorphism $\alpha_2$ of $B_2$. Now $(\alpha_1, \alpha_2) \in \text{Aut}(B)$ extends $p$. \hfill $\lozenge$

**Corollary 4.13.** Let $\mathcal{L}$ be a finite relational language. Let $r$ be the maximal arity of symbols in $\mathcal{L}$. For $1 \leq j \leq r$ let $l_j$ be the number of $j$-ary symbols in $\mathcal{L}$. Let $A$ be a finite $\mathcal{L}$-structure of cardinality $c$. There exists a structure $B$ such that $A \subset B$ and every partial automorphism of $A$ extends to an automorphism of $B$ and $\text{card}(B) \leq c \cdot 2^{p(c)}$, where $p(c) = \sum_{j=2}^{r} j!j^c$.

The proof consists of putting together Theorem 4.5 and Lemma 4.12 and observing that if $B$ only consists of unary predicates, then $B = A$ will do. \hfill $\lozenge$

4.3. **The case of the cycle-free $n$-partitioned graph.** The language $L$ consists of $n$ unary predicates $U_1, \ldots, U_n$ and one binary predicate $R$. We write $a \rightarrow b$ for $\text{Rab}$. Denote by $K$ the class of cycle-free, $n$-partitioned directed graphs.

If $M$ is in $K$ and $a, b \in M$, then we define $a \rightarrow^M b$ if there exist $j \leq k$ and $a_j \in U_j^M, \ldots, a_k \in U_k^M$ such that $a = a_j \rightarrow a_{j+1} \rightarrow \cdots \rightarrow a_k = b$.

**Lemma 4.14.** $K$ has the EPPA.
Proof. Let \( A \in K \) be finite, and let \( p_1, \ldots, p_t \) be partial automorphisms on \( A \). We assume there exist \( M \in K \) and \( g_1, \ldots, g_t \in \text{Aut}(M) \) such that \( g_i \) extends \( p_i \). We let \( D_i \) be the domain of \( p_i \) and \( D_i' \) be its range.

First step.

Without loss of generality we can suppose that for \( a, b \in A \), if there exists an \( i \) such that \( a, b \in D_i \) or \( a, b \in D_i' \), then \( a \to^*_M b \) if and only if \( a \to^*_A b \). (Just put enough points as witnesses into the structure \( A \) without extending \( p_i \).) Now for every \( a, b \in D_i \) : \( a \to^*_A b \) if and only if \( p_i(a) \to^*_A p_i(b) \) (because \( g_i \) is an automorphism it is clear that \( a \to^*_M b \) if and only if \( g_i(a) \to^*_M g_i(b) \)). This will be the only place where we use the existence of the structure \( M \); later we will only use the fact that now \( p_i \) also respects \( \to^*_A \). Now we can forget the structure \( M \) and use \( \to^* \) for \( \to^*_A \).

Second step.

For \( U \subseteq A \) define \( \text{cl}(U) := \{ b \in A ; \exists a \in U : a \to^* b \} \). We say \( U \) is closed, if \( \text{cl}(U) = U \). The first level \( U_1 \) of our structure will play a special role in the argument, we define \( A_u := A - U_1 \) (the upper part of \( A \)). We order \( \mathbb{N}^{n-1} \) lexicographically and define a dimension function \( \text{dim} : \mathcal{A}_u \to \mathbb{N}^{n-1} \) by:

If \( U \) is closed, then \( \text{dim}_k(U) = \text{card}(U \cap U_k) \) and for \( k \geq 2 \), \( \text{dim}_{k+1}(U) = \text{card}((U - \text{cl}(U \cap U_k)) \cap U_{k+1}) \) and \( \text{dim}(U) := (\text{dim}_1(U), \ldots, \text{dim}_n(U)) \); finally if \( U \) is arbitrary: \( \text{dim}(U) := \text{dim}(\text{cl}(U)) \). Informally \( \text{dim}_k(U) \) counts the points in the \( k \)-th level of \( U \), which are not already in the closure of a point of lower level of \( U \).

If \( U \subseteq A_u \) is arbitrary one can use the following method, to determine \( \text{dim}(U) \): call \( b \in U \) a root of \( U \) if there does not exist an element \( a \in U \) such that \( a \neq b, a \to^* b \). Then \( \text{dim}_k(U) = \text{card}(\{ b \in U \cap U_k ; b \text{ a root of } U \}) \).

It is not hard to check, that

- \( U \subseteq V \) implies \( \text{cl}(U) \subseteq \text{cl}(V) \);
- \( U \subseteq V \) implies \( \text{dim}(U) \leq \text{dim}(V) \);
- \( \text{cl}(U) \subseteq \text{cl}(V) \) implies \( \text{dim}(U) < \text{dim}(V) \);
- if \( U \subseteq D_i \), then \( \text{dim}(U) = \text{dim}(p_i(U)) \).

For the last statement use the fact that \( p_i \) respects the relation \( \to^* \) and use the above-mentioned method to compute the dimension.

Lemma 4.15. There exist a set \( C \) of unary predicates (called colors), containing for every \( a \in U_1 \) a color \( Q_a \) and an expansion \((A,(Q^A)_{Q \in C})\) of the structure \( A \) (a coloring of \( A \)), such that:

- \( Q_a^1 = \text{cl}(a) \);
- For \( Q \subseteq C - \{Q_a : a \in U_1^1\} \), \( Q^A \) is included in \( A_u \) and is closed;
- “For a closed subset \( V \) the number of colors of \( V \) (i.e. the colors \( Q \) such that \( V \subseteq Q^A \)) only depends on the dimension of \( V \).” More formally:

There exists a function \( f : \mathbb{N}^{n-1} \to \mathbb{N} \), such that for every \( V \subseteq A_u \), \( V \) closed, \( \text{card}(Q \subseteq C; V \subseteq Q^A) = f(\text{dim}(V)) \).

Let \( \{d_1, \ldots, d_r\} = \{\text{dim}(V ; V \subseteq A_u) ; d_1 < \cdots < d_r \} \) lexicographically. For every single closed \( V \subseteq A_u \) we will decide how many colors \( Q \) we want to put into \( C \) with \( Q^A = V \). We will do this and define the function \( f \) by downward induction on \( \text{dim}(V) \). Note that the value of \( f \) only has to be defined for \( d_r, \ldots, d_1 \).

Let \( C_r = \{Q_a : a \in U_1^1\} \); we will define \( C_r \subseteq \cdots \subseteq C_0 = C \). Suppose (by induction) that for a given \( i < r \), \( C_{i+1} \) is already defined and for \( d' > d_{i+1} = df(d') \)}
is already defined such that for every \( V \subseteq A_u \): if \( \dim V > d \), then \( \text{card}\{Q \in C_{i+1}; V \subseteq Q^A\} = f(\dim(V)) \).

Define \( f(d) = \max\{\text{card}\{Q \in C_{i+1}; V \subseteq Q^A\}; V \subseteq A_u, V \text{ closed}, \dim V = d\} \) and add for every closed \( V \subseteq A_u \) with \( \dim V = d \) enough colors \( Q \) with \( Q^A = V \) (i.e. \( f(d) - \text{card}\{Q \in C_{i+1}; V \subseteq Q^A\} \) many) to \( C_{i+1} \) to define \( C_i \) such that \( \text{card}\{Q \in C_i; V \subseteq Q^A\} = f(d) \).

The strict monotonicity of the dimension ensures that this works (no different closed subsets of dimension \( d \) are contained in each other and if \( \dim V > d \) the equation \( \text{card}\{Q \in C_i; V \subseteq Q^A\} = f(\dim(V)) \) will be maintained). \( C = C_0 \) fulfills the requirement of the claim.

Note that also for arbitrary \( V \subseteq A_u \), we have

\[
\text{card}\{Q \in C; V \subseteq Q^A\} = f(\dim(V))
\]

because \( \{Q \in C; V \subseteq Q^A\} = \{Q \in C_i; \text{cl}(V) \subseteq Q^A\} = f(\text{dim}(\text{cl}(V))) \) and \( \dim(\text{cl}(V)) = \dim(V) \).

**Definition 4.16.** Let \( D \subseteq A \) (think of \( D = D_i \cap A_u \)) and let \( V \subseteq D \). \( V \) is called relatively closed in \( D \), if \( V = \text{cl}(V) \cap D \) or, equivalently, if

\[
\forall a \in V, \forall b \in D : (a \rightarrow^* b \Rightarrow b \in V).
\]

Note that for every \( Q \in C \) and \( D \subseteq A_u \), \( Q^A \cap D \) is relatively closed in \( D \).

**Lemma 4.17.** Let \( 1 \leq i \leq t \) and let \( V \subseteq D_i \cap A_u \) be relatively closed in \( D_i \cap A_u \). Then \( V' := p_i(V) \) is relatively closed in \( D'_i \cap A_u \) and

\[
\text{card}\{Q \in C; Q^A \cap D_i \cap A_u = V\} = \text{card}\{Q \in C; Q^A \cap D'_i \cap A_u = V'\}.
\]

We will prove the equality by downward induction on \( \dim(V) \). First note that the lattice of relatively closed subsets of \( D_i \cap A_u \) is isomorphic via \( p_i \) to the corresponding lattice for \( D'_i \cap A_u \) and this isomorphism respects dimensions.

Suppose \( V \subseteq D_i \cap A_u \) is relatively closed. By induction we can assume that for \( W \subseteq D_i \cap A_u \) relatively closed with \( V \subsetneq W \)

\[
\text{card}\{Q \in C; Q^A \cap D_i \cap A_u = W\} = \text{card}\{Q \in C; Q^A \cap D'_i \cap A_u = p_i(W)\}.
\]

(The same equation for \( W \) not relatively closed is trivial: then both sides are 0.) Write \( S := \{V; V \subseteq W \subseteq D_i \cap A_u\} \) and \( S' := \{V; V' \subseteq W \subseteq D'_i \cap A_u\} \). Then:

\[
\text{card}\{Q \in C; Q^A \cap D_i \cap A_u = V\}
\]

\[
= \text{card}\{Q \in C; V \subseteq Q^A\} - \sum_{W \in S} \text{card}\{Q \in C; Q^A \cap D_i \cap A_u = W\}
\]

\[
= f(\dim(V)) - \sum_{W \in S} \text{card}\{Q \in C; Q^A \cap D_i \cap A_u = p_i(W)\}
\]

\[
= f(\dim(V')) - \sum_{W \in S'} \text{card}\{Q \in C; Q^A \cap D_i \cap A_u = W'\}
\]

\[
= \text{card}\{Q \in C; Q^A \cap D'_i \cap A_u = V'\}.
\]

Now we want to get the colors into the language by introducing a new binary predicate relating the points to their colors.

**Definition 4.18.** We let \( L' = \{R, U_1, \ldots, U_n\} \cup \{C, D\} \). We define an \( L' \)-structure \( B \) with domain \( A \cup C \). We let \( R^B = R^A, U_1^B = U_1^A, \ldots, U_n^B = U_n^A \) and \( C^B = C \) and we put \( D^B q a \) if and only if \( q \in C \) and \( a \in A \) and \( a \) is of color \( q \).
Lemma 4.19. For every \( i \) there is a permutation \( \chi_i \) of the set \( C \) of colors such that \( p_i \cup \chi_i \) is a partial automorphism of \( B \).

Certainly every color \( Q \in C \) belongs to exactly one set of the form

\[
\{ Q \in C; Q^A \cap D_i \cap A_u = V \}
\]

(for some \( V \subset D_i \)), so we can define a permutation \( \chi_i \), mapping \( \{ Q \in C; Q^A \cap D_i \cap A_u = p_i(V) \} \) bijectively to \( \{ Q \in C; Q^A \cap D'_i \cap A_u = p_i(V) \} \). Furthermore, we can find the \( \chi_i \) such that for \( a \in D_i \cap U^A_i \), \( Q_{\chi_i(a)} = Q_{p_i(a)} \) (because: if \( Q_u \cap D_i \cap A_u = V \), then \( Q_{p_i(a)} \cap D'_i \cap A_u = p_i(V) \) and that is because \( p_i \) respects \((\rightarrow^*)\)).

Now it remains to show, that \( r_i := p_i \cup \chi_i \) is a partial automorphism of \( B \); so let \( a \in D_i \), \( Q \in C \). We have to show that \( a \in Q^A \) if and only if \( a' \in Q'^A \) (where \( a' \) is \( p_i(a) \) and \( Q' \) is \( \chi_i(Q) \)).

If \( a \in U^A_i \), then \( a \in Q^A \) if and only if \( Q = Q_a \), if and only if \( Q' = Q_{a'} \), if and only if \( a' \in Q'^A \).

If \( a \in A_u \), let \( V := Q^A \cap D_i \cap A_u \) so by definition of \( \chi_i \), \( Q'^A \cap D'_i \cap A_u = p_i(V) \). So \( a \in Q^A \) if and only if \( a \in V \), if and only if \( a' \in p_i(V) \), if and only if \( a' \in Q'^A \).

Third step.
The structure \( B \) has the following properties:

- for every element \( a \) of \( U_1 \) there exists exactly one \( q \in C \) (namely \( q = Q_a \)) such that \( Dqa \);
- if \( a \in U_i \) (\( 1 \leq i < n \)) and \( a \to b \) and \( Dqa \), then \( Dqb \) (just because the interpretation of the color \( q \) in \( A \) is closed);
- if \( a \in U^A_i \) and \( a \to b \) and \( Dqa \), then not \( Dqb \).

For the last point suppose \( Dqb \); because \( b \in U^A_i \), it follows that \( q = Q_b \), but \( a \in Q_a \) means that \( b \to ^* a \), and this would mean that there is an \( s \)-cycle in \( A \).

Now we are using the Corollary \( [1.3] \). So we know, there exists a finite \( \mathcal{L}' \)-structure \( E, B \subseteq E, \) and \( g_1, \ldots, g_n \in \text{Aut}(E), \) \( g_i \) extending \( p_i \) for every \( i \). By the remark after Definition \( [2.1] \) we can choose \( B \) to be slim. Note that this automatically ensures that \( C^E = C^B = C \).

Claim. The structure \( E \) has the following properties:

- for every point \( a \in U_1 \) there exists exactly one \( q \in C \) such that \( Dqa \);
- if \( a \in U_i \) (\( 1 \leq i < n \)) and \( a \to b \) and \( Dqa \), then \( Dqb \);
- if \( a \in U^A_i \) and \( a \to b \) and \( Dqa \), then not \( Dqb \).

This follows directly from the fact that \( E \) is slim, and the corresponding property of \( A \). Suppose e.g. that \( a \in U^E_i \) (\( 1 \leq i < n \)) and \( a \to b \) and \( Dqa \). Then we know that there exists \( h \in F(P) \) such that \( \hat{h}(a), \hat{h}(b) \in A \) and of course \( \hat{h}(a) \to \hat{h}(b) \). So we have \( D\hat{h}(q)\hat{h}(a) \), which implies \( D\hat{h}(q)\hat{h}(b) \) which implies \( Dqb \).

Now to complete the proof, we have to check that \( E \) is cycle-free. Suppose there are elements \( a_1 \in U^E_1, \ldots, a_n \in U^E_n \), such that \( a_1 \to a_2 \to \cdots \to a_n \to a_1 \). Choose \( q \in C \) such that \( Dqa_1 \), inductively follows \( Dqa_n \) and not \( Dqa_1 \), a contradiction.

5. Proof of Theorem \( [3.2] \)

This section is entirely devoted to the proof of Theorem \( [3.2] \). The strategy is to reduce the problem, by successive steps, to an easy one.
5.1. Reduction to stretched structures. To begin with, we will assume that all structures we are considering are irreflexive: a structure \( M \) in a language \( \mathcal{L} \) is irreflexive if, for every \( n \)-ary predicate \( R \) in \( \mathcal{L} \) and \( a_1, a_2, \ldots, a_n \in M \), if \( M \models Ra_1a_2\cdots a_n \), then the \( a_i \) are pairwise distinct. We can do that without loss of generality (see the last section of [6]).

The first real reduction states that it is enough to prove it for a certain kind of structures, which we will call the stretched structures and which we define now.

The language \( \mathcal{L} \) of a stretched structure \( M \) should contain unary predicates \( U_0, U_1, \ldots, U_k \). The universe of \( M \) is the disjoint union of the sets \( U_i^M \). Moreover, for each \( n \)-ary relation symbol \( R \) of \( \mathcal{L} \), and for all \( a_1, a_2, \ldots, a_n \) in \( M \), if \( M \models Ra_1a_2\cdots a_n \), then the set \( \{a_1, a_2, \ldots, a_n\} \) intersects each set \( U_i \), for \( 1 \leq i \leq k \) in at most one element (notice that \( U_0 \) has a special status).

A small structure \( M \) is a stretched structure such that for all \( i \), \( 0 \leq i \leq k \), \( U_i^M \) has at most one element.

The first reduction is a very important one (we will consistently call each of these reductions ‘proposition’).

Proposition 5.1. Let \( T \) be a finite set of small structures. Then the class of stretched \( T \)-free structures has the EPPA.

Proof. We deduce Theorem 3.2 from Proposition 5.1: suppose we are given a language \( \mathcal{L} \), a finite set \( T \) of finite \( \mathcal{L} \)-structures, \( A \) a finite \( \mathcal{L} \)-structure, \( p_1, p_2, \ldots, p_n \in \text{Part}(A, A) \), a \( T \)-free structure \( M \) extending \( A \) and automorphisms \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \text{Aut}(M) \) extending respectively \( p_1, p_2, \ldots, p_n \). Let \( k = \max(\text{card}(T); T \in T) \). From \( A \) we want to define a stretched structure \( \hat{A} \). We first add to the language \( k \) new unary predicates \( U_0, U_1, \ldots, U_{k-1} \). We will write \( \mathcal{L}^+ \) for the language that we obtain this way. The universe of \( \hat{A} \) is \( A \times \{0, 1, \ldots, k-1\} \).

\( \hat{A} \models R(a_1, i_1)(a_2, i_2)\cdots(a_s, i_s) \) if and only if \( A \models Ra_1a_2\cdots a_s \) and for all \( i, 1 \leq i \leq k - 1 \) there is at most one \( m \), \( 1 \leq m \leq s \), such that \( i_m = i \).

It is clear that \( \hat{A} \) is a stretched structure. Moreover, the map \( \pi \) from \( \hat{A}_{|\mathcal{L}} \) onto \( A \) defined by \( \pi((a, i)) = a \) is a weak homomorphism.

For every element \( T \) of \( T \) choose a small \( \mathcal{L}^+ \)-structure \( T^+ \) which expands \( T \) (this is possible because \( k \) has been chosen sufficiently large). Let \( T^+ = \{T^+; T \in T\} \). We can see that \( \hat{A} \) is \( T^+ \)-free: if \( h \) were a weak homomorphism from some \( T^+ \) into \( \hat{A} \), then \( \pi \circ h \) would be a weak homomorphism from \( T^+_{|\mathcal{L}} \) (which is equal to \( T \)) to \( A \).

Now, for each partial automorphism \( p_i \) of \( A \), we may define a partial map \( \hat{p}_i \) of \( \hat{A} \) by: if \( a \in \text{Dom}(p) \) and \( 0 \leq i \leq k - 1 \), \( \hat{p}_i((a, j)) = (p_i(a), j) \). It is straightforward to check that these maps are partial automorphisms. We may also define analogously a stretched \( \mathcal{L}^+ \)-structure \( \hat{M} \) from \( M \), and automorphisms \( \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n \) of \( \hat{M} \). As above, \( \hat{M} \) is \( T^+ \)-free, and it is clear that \( \hat{M} \) extends \( \hat{A} \) and that the \( \hat{\alpha}_i \) extend the \( \hat{p}_i \).

Thus we may apply Proposition 5.1 we get a finite \( T^+ \)-free \( \mathcal{L}^+ \)-structure \( C \) extending \( \hat{A} \) and automorphisms \( \gamma_1, \gamma_2, \ldots, \gamma_n \) of \( C \) extending the \( \hat{p}_i \). Moreover, if we translate the problem into a problem in the free group, as has been done in subsection 2.2, we see that to be \( T^+ \)-free can be forced by negative conditions, so, we may apply Proposition 5.1 and assume that \( C \) is a special extension of \( \hat{A} \).

(Another way to see this is as follows: By a remark after Proposition 2.2 we find a
special solution $D$ and $\rho : D \rightarrow C$. As $C$ is $T^+$-free also $D$ is $T^+$-free and therefore we can suppose $C$ to be special.)

We restrict our attention to the $\mathcal{L}$-structure $B$ that we get in the following way: first we take the substructure of $C$ whose universe is the set of elements of $C$ which belong to $U_0$. Then we take the $\mathcal{L}$-reduct of this structure to obtain $B$. Some fact are immediately clear: Since $C$ is special, the $U_i$ partition the universe of $C$. The $\gamma_i$ leave the set $B$ fixed, so they induce permutations $\beta_i$ of $B$, and these permutations are in fact automorphisms of $B$. We may identify the $U_0$ part of $A$ with $A$ (identifying $(a,0)$ with $a$). Doing this, $B$ will be viewed as an extension of $A$, and the $\beta_i$ as extensions of the $p_i$.

Thus, it will suffice to prove that $B$ is $T$-free.

Let $T \in \mathcal{T}$ and let $\{t_1, t_2, \ldots, t_s\}$ be the universe of $T$. Let $T'$ be the expansion of $T$ to $\mathcal{L}^+$, where all the $t_i$ for $1 \leq i \leq s$ belong to $U_0$. We may construct a sequence $T_0, T_1, \ldots, T_s$ of stretched $\mathcal{L}^+$-structures which are all expansions of $T$, beginning with $T_0 = T^+$ and ending with $T_s = T'$ such that the only possible difference between $T_j$ and $T_{j+1}$ is that $t_{j+1}$, which satisfies some $U_r$ in $T_j$, satisfies $U_0$ instead in $T_{j+1}$. We prove by induction on $j$ that $C$ is $T_j$-free. We already know that it is $T_0$-free, and once we will know it is $T_s$-free, we will know that $B$ is $T$-free.

By way of contradiction, assume that $C$ is $T_j$-free, and that $h$ is a weak homomorphism from $T_{j+1}$ into $C$. Let $G$ be the group of automorphisms of $C$ generated by $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Since every element of $C$ is the image by an element of $G$ of an element of $\hat{A}$, we may assume that $h(t_{j+1}) \in \hat{A}$. Since $T_{j+1} \models U_0 t_{j+1}$ and $h$ is a weak homomorphism, $h(t_{j+1}) = (a,0)$ for some $a \in A$. Let $m$ be the positive integer such that $T_j \models U_m(t_{j+1})$ and $h'$ the map from $T_j$ into $C$ equal to $h$ except in $t_{j+1}$ where $h'(t_{j+1}) = (a,m)$. We show that $h'$ is a weak homomorphism from $T_j$ into $C$, and get a contradiction.

By construction, $h'$ preserves the predicates $U_i$. So let $R$ be a predicate symbol of $L$ and assume that $T_j \models R t_{j+1} \bar{t}$ where $\bar{t}$ is a sequence of $t_i$. Thus, we also have $T_{j+1} \models R t_{j+1} \bar{t}$, and since $h$ is a weak homomorphism, $C \models Rh(t_{j+1})k(\bar{t})$. Because $C$ is slim, there exist $b \in A$, a sequence $\bar{c}$ of elements of $\hat{A}$ such that $\hat{A} \models Rb\bar{c}$ and such that $h(t_{j+1})k(\bar{t}) = \gamma(\bar{c})$ for some $\gamma \in G$. In particular $\gamma(b) = h(t_{j+1}) = (a,0)$. Let $b' = \pi(b)$ (that is, $b = (b',0)$), $\bar{c}' = \pi(\bar{c})$ and $b_1 = (b',m)$. By construction of $\hat{A}$ $\hat{A} \models Rb\bar{c}'$ and $\hat{A} \models Rb_1\bar{c}'$.

We now use the real strength of the hypothesis that $C$ is special: there exist a word $w$ of the free group $F(x_1, x_2, \ldots, x_n)$ such that $\gamma = w(\beta_1, \beta_2, \ldots, \beta_n)$ and $w(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)(b) = (a,0)$. Since $w(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)(b_1) = (a,m)$, we have $\gamma(b_1) = w(\beta_1, \beta_2, \ldots, \beta_n)(b_1) = (a,m) = h'(t_{j+1})$. From $\hat{A} \models Rb_1\bar{c}'$ we deduce $C \models R\gamma(b_1)\gamma(\bar{c})$. Because $T$ is irreflexive $h'(\bar{t}) = h(\bar{t}) = \gamma(\bar{c})$. Thus $C \models Rh'(t_{j+1})h'(\bar{t})$.  

5.2. Short extensions. From now on and until the end of the proof of Theorem 6.2 we will only deal with stretched structures in a fixed language $\mathcal{L}$. To fix the notations, we will suppose that $U_0, U_1, \ldots, U_k$ are the unary predicates necessary to make our structures stretched. Before going any further, we need some definitions.

Definition 5.2. Let $M_0$ be a stretched structure, $M_1$ and $M_2$ be two extensions of $M_0$. Assume that $M_1 \cap M_2 = M_0$. The free amalgam of $M_1$ and $M_2$ over $M_0$, denoted $M_1 \ast_{M_0} M_2$, is the structure whose universe is $M_1 \cup M_2$ and whose set of links is the union of the set of links of $M_1$ and of the set of links of $M_2$.  

If \( M_1 \cap M_2 \neq M_0 \), then \( M_1 *_{M_0} M_2 \) is defined up to \( M_0 \)-isomorphism, and is equal to \( M'_1 *_{M_0} M'_2 \) where \( M'_1 \) and \( M'_2 \) are extensions of \( M_0 \) which are respectively \( M_0 \)-isomorphic to \( M_1 \) and \( M_2 \) and such that \( M'_1 \cap M'_2 = M_0 \).

Now come the main devices of the proof.

**Definition 5.3.** 1) Let \( A \) be a structure. A short extension of \( A \) is a structure which can be written as \( A *_{A_0} C \) where \( A_0 \) is a small substructure of \( A \) or the empty set, and \( C \) is an extension of \( A_0 \) which is also small.

2) Let \( D \subseteq A \). A short extension is based on \( D \) if it can be written as \( A *_{A_0} C \) as above, with the requirement \( A_0 \subseteq D \).

3) Let \( A \subseteq B \) be two \( L \)-structures. We say that \( B \) is a strong extension of \( A \) if, for all short extension \( C \) of \( A \), if there exists \( h : C \rightarrow B \), then there exists \( h' : C \rightarrow A \). To denote that \( B \) is a strong extension of \( A \), we will write \( A \subseteq_s B \).

4) Let \( A \) and \( B \) be two \( L \)-structures and \( p \) a partial isomorphism from \( A \) to \( B \) with domain \( D \subseteq A \) and image \( D' \subseteq B \). We say that \( p \) is strong in \( (A, B) \) if for every short extension \( D_1 \) of \( D \) and \( k : D_1 \rightarrow A \), there exists \( p' : D_1 \rightarrow B \) such that \( p' \) extends \( p \), and conversely, for every short extension \( D'_1 \) of \( D' \) and \( k' : D'_1 \rightarrow B \), there exists \( p' : D'_1 \rightarrow A \) such that \( p' \) extends \( p^{-1} \).

Remark for 1): \textit{A priori}, the structures \( A_0 \) and \( C \) are not uniquely determined. But there is a kind of canonical decomposition of a short extension \( B \): set \( C_1 = B - A \) and

\[ A_1 = \{ a \in A; a \text{ is linked with an element of } C_1 \}. \]

From the definition, it should be clear that, if \( B = A *_{A_0} C \), then \( A_1 \subseteq A_0 \) and \( C \subseteq C_1 \subseteq C \) and \( B = A *_{A_1} (A_1 \cup C_1) \). It is this decomposition which will be used implicitly when a decomposition is needed.

It should be remarked that, in the above definition 2), the fact that \( p \) is strong does not depend only on \( p \) itself, but also on the way that \( D \) and \( D' \) sit in \( A \) and \( B \) respectively. That is why we add “in \( (A, B) \)”. If \( p \in \text{Part}(A, A) \), we will say “\( p \) is strong in \( A \)” instead of “\( p \) is strong in \( (A, A) \)”.

The following facts are easy to prove:

**Lemma 5.4.** 1. If \( A \) is \( T \)-free and \( A \subseteq_s B \) is a strong extension of \( A \), then \( B \) is \( T \)-free.

2. If \( B \subseteq B_1 \), then \( B \subseteq_s B_1 \) if and only if for every small structure \( C \subseteq B_1 \), there exists \( h : C \rightarrow B \).

3. If \( A \subseteq_s B \) and \( B \subseteq_s C \), then \( A \subseteq_s C \).

4. If \( A \subseteq_s B \) and \( A \subseteq C \subseteq B \), then \( A \subseteq_s C \).

5. If \( p \in \text{Part}(A, A) \) is strong in \( A \) and \( A \subseteq_s B \), then \( p \) is strong in \( B \).

6. If \( B \) is a short extension of \( A \), then \( A \subseteq_s B \) if and only if there exists \( h : B \rightarrow A \).

7. If \( B \) is a short extension of \( A \) and \( B = A *_{A_0} C \) (where \( A_0 \) and \( C \) are small structures), then \( A \subseteq_s B \) if and only if there exists \( h : C \rightarrow A \).

8. Assume that \( A \subseteq_s B, A \subseteq C \) and \( h : B' \rightarrow B \). Then \( A \subseteq_s B' \).

9. If \( A \subseteq_s B \) and \( A \subseteq C \), then \( C \subseteq_s B *_{A} C \).

10. If \( A \subseteq_s B \) and \( A \subseteq_s C \), then \( A \subseteq_s B *_{A} C \).
We now get to our second reduction step:

**Proposition 5.6.** Let \( B \) take \( j \) \( \frac{}{} \) Furthermore for \( \text{substructure of } M \) \( i \), then for all \( \text{Proposition 5.5.} \) We prove Proposition 5.5 using Proposition 5.6. So we start with the hypothesis of Proposition 5.5. We define a sequence of finite structures \( E \) of every short extension \( E \) advertising \( \text{Part} (A,A) \), and suppose that the \( i \) are strong in \( A \). This allows us to do that. Next we take \( (A) \) such that: for every short extension \( k : E \longrightarrow M \). Thus if \( C \) is any substructure of \( M \) containing \( B \) and \( \alpha_i[B_i] \), we see that the following is true: for every short extension \( k : E \longrightarrow M \). \( p_i \) can be extended to a weak homomorphism from \( E \) to \( C \). Repeating the operation for \( D' \) in place of \( D_i \), and then for all \( i, 1 \leq i \leq n \), we get a finite substructure \( C \) in which all the \( p_i \) are strong. Then we may apply Proposition 5.5 to get the structure \( B \). 

The next reduction tells us that we can work sort by sort.

**Proposition 5.6.** Let \( A \) be a finite structure and \( p_1, p_2, \ldots, p_n \) be elements of \( \text{Part}(A,A) \), and suppose that the \( p_i \) are strong in \( A \). Fix an integer \( j \), \( 1 \leq j \leq k \). There exist a finite strong extension \( B \) of \( A \) and partial automorphisms \( q_1, q_2, \ldots, q_n \in \text{Part}(B,B) \) extending respectively \( p_1, p_2, \ldots, p_n \) such that all the \( q_i \) are strong in \( B \) and induce a permutation on \( U^B_j \).

**Proof.** We prove Proposition 5.6 using Proposition 5.5. We define a sequence of finite structures \( A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k \), and we extend successively the partial automorphisms \( p_i^0 = p_i : p_i^0 \subseteq p_i^1 \subseteq \cdots \subseteq p_i^k, p_i^k \in \text{Part}(A_j,A_{j+1}) \) such that \( p_i^k \) is strong in \( A_j \). Furthermore for \( j \geq 1, p_i^j \) induces a permutation on \( U^B_j \). Clearly, Proposition 5.6 allows us to do that. Next we take \( C = A \cup U^A_{j+1} \cup U^A_{j+2} \cup \cdots \cup U^A_{k} \subseteq A_k \) and take \( r_i \) to be the restriction of \( p_i^k \) to \( C \). It is a partial automorphism on \( C \) and for \( 1 \leq j \leq k, r_i \) induces a permutation on \( U^C_j = U^A_{j+1} \). Since \( A_k \) is a strong extension of \( A \), it is \( T \)-free, and so is \( C \).

It remains to take care of \( U_0 \). By Corollary 4.13 we can find a finite \( L \)-structure \( B \) extending \( C \), and automorphisms \( \alpha_i \) of \( B \) extending the \( r_i \). By Proposition 2.3 we can take \( (B, \alpha_1, \alpha_2, \ldots, \alpha_n) \) to be a slim extension of \( (C, r_1, \ldots, r_n) \), which is automatically stretched.

Write \( G \) for the group of automorphisms of \( B \) generated by \( \{ \alpha_1, \ldots, \alpha_n \} \). For \( 1 \leq j \leq k, \) if \( b \in U^B_j \), there exist \( c \in U^C_j \) and \( \beta \in G \) such that \( b = \beta(c) \). Since each \( \beta \in G \) induces a permutation on \( U^C_j \), we see that \( b \in U^C_j \), so \( U^C_j = U^B_j \).

We have to prove that \( B \) is \( T \)-free. The fact that \( B \) is a slim extension implies: for \( b \in U^B_0 \) there exists \( \beta \in G \) such that \( \beta(b) \in U^C_0 \). Suppose now that there exist \( T \in T \) and \( r \) a weak homomorphism from \( T \) to \( B \). \( T \) is a small stretched structure. If \( U^T_0 = \emptyset \), then \( r \) is a weak homomorphism from \( T \) to \( \bigcup \{ U^B_j; 1 \leq j \leq s \} \), and this is impossible since \( C \) is \( T \)-free. If \( U^T_0 \) has one element, say \( t \), then we choose...
\( \gamma \in G \) such that \( \gamma \circ r(t) \in U_0^C \) and then \( \gamma \circ r \) is a weak homomorphism from \( T \) to \( C \), which is again impossible. \( \diamondsuit \)

5.3. Types. To simplify the notation slightly, we prove Proposition 5.6 for \( j = 1 \)

Given \( a \in U_1^A \) and concentrating on a partial automorphism, say \( p_1 \), the first step is to find an image for \( a \), that is an element \( b \) in some extension \( B \) of \( A \) such that, if we extend \( p_1 \) to \( q_1 \) by setting: \( q_1(a) = b \), then \( q_1 \) is still a strong partial automorphism of \( B \). Intuitively, the conditions to be satisfied by this element \( b \) are: 1) (for \( q_1 \) to be a partial automorphism) the relations between \( b \) and \( D_1^j \) (the image of \( p_1 \)) should be exactly the image of the relations existing between \( a \) and \( D_1 \) (the range of \( p_1 \));

2) (for \( q_1 \) to be a strong partial automorphism) for every small substructure \( C \) of \( A \) containing \( a \), there should be in \( B \) a small substructure \( C' \) of \( B \) containing \( b \), and a weak homomorphism sending \( C \) to \( C' \) extending \( q_1 \); 3) (same as 2), permute the roles of \( a \) and \( b \) and replacing \( q_1 \) by \( q_1^{-1} \). That is where the notion of type comes in: the type of \( a \) over \( D \) is meant to collect the necessary data.

For the rest of the proof, when considering a short extension \( B \) of \( A \), we will implicitly assume that there is a (unique) element in \( U_1^B \) not in \( A \). We will denote this element \( u(B) \).

We can preorder the set of short extensions of \( A \) in the following way: given \( B \) and \( C \) two such extensions, we write \( B \leq C \) if there exists a weak \( A \)-homomorphism \( h \) from \( B \) into \( C \) such that \( h(u(B)) = u(C) \). It is obviously a reflexive and transitive relation (but not antisymmetric in general, so it is just a preordering).

To define the type of an element \( b \) of \( B \), we need to consider the short extensions \( C \) of \( A \) such that there exist \( h : C \rightarrow_{w,A} B \) such that \( h(u(C)) = b \). To deal only with a finite number of them, we will consider a finite set \( S \) of short extensions of \( A \), such that, for all short extensions \( C \) of \( A \), \( C \) is isomorphic to one and only one element of \( S \). Moreover, we will choose an element \( x \) which does not belong to \( A \) and assume that, for all \( C \in S \), \( u(C) = x \), and that for all \( C, C' \in S \), if \( C \neq C' \), then \( C \cap C' = A \cup \{ x \} \).

Definition 5.7. Let \( B \) be a strong extension of \( A \) and \( b \in U_1^B \). Then:

- \( \Gamma_B(b) \) is the extension of \( A \) whose universe is \( A \cup \{ x \} \) and such that, for all \( R \in \mathcal{L} \), \( R \) different from the equality, and for all sequences \( \mathbf{\pi}_1, \mathbf{\pi}_2 \) from \( A \), \( \Gamma_B(b) \models R \mathbf{\pi}_1 = R \mathbf{\pi}_2 \) if and only if \( B \models R \mathbf{\pi}_1 = R \mathbf{\pi}_2 \);

- \( \mathcal{E}_B(b) \) is the set \( \{ C \in S \mid \text{there exists } h : C \rightarrow_{w,A} B \text{ such that } h(x) = b \} \).

More generally, if \( D \subseteq A \), we define:

- \( \Gamma_B(b) \) is the structure whose base set is \( A \cup \{ x \} \) and such that, for every \( R \in \mathcal{L} \) and sequence \( \mathbf{\pi} \) from \( A \cup \{ x \} \), \( \Gamma_B(b) \models R \mathbf{\pi} \) if and only if \( \Gamma_B(b) \models R \mathbf{\pi} \) and either all the elements of \( \mathbf{\pi} \) belong to \( D \cup \{ x \} \) or all elements of \( \mathbf{\pi} \) belong to \( A \);

- \( \mathcal{E}_B(b) \) is the set \( \{ C \in \mathcal{E}_B(b) \mid C \text{ is based on } D \} \).

Lemma 5.8. Let \( D \subseteq A \cup_s B \subseteq_s B' \). Let \( b \in U_1^B \). Then \( \Gamma_B(b) = \Gamma_B'(b) \), \( \Gamma_B(b) / D = \Gamma_B'(b) / D \), \( \mathcal{E}_B(b) = \mathcal{E}_B'(b) \), \( \mathcal{E}_B(b) / D = \mathcal{E}_B'(b) / D \).

Proof. It is clear that \( \Gamma_B(b) = \Gamma_B'(b) \) and that \( \Gamma_B(b) / D = \Gamma_B'(b) / D \). We prove \( \mathcal{E}_B(b) \subseteq \mathcal{E}_B'(b) \) (the fourth equality follows immediately). Again, it is clear that \( \mathcal{E}_B(b) \subseteq \mathcal{E}_B'(b) \). So, let \( C \in \mathcal{E}_B'(b) \), and let us prove \( C \in \mathcal{E}_B(b) \). We may decompose \( C : C = A *_{A_0} C_0 \), where \( A_0 \) and \( C_0 \) are small structures, and we know that there
exists \( h : C_0 \xrightarrow{w,A_0} B' \) with \( h(x) = b \). Let \( C_1 = h[C_0] \) and \( C_2 = C_1 \cap B \). Thus \( b \in C_2 \) and \( A_0 \subseteq C_2 \). Since \( B \subseteq B' \), there exists \( h_1 : C_1 \xrightarrow{w,C_2} B \), and \( h_1 \circ h : C_0 \xrightarrow{w,A_0} B \), and this proves that \( C \in \mathcal{E}_B(b) \). ⊢

Assume that \( B \) is a short extension of \( A \) based on \( D \subseteq A \), let \( p \) be a partial strong automorphism of \( A \) whose domain contains \( D \), and let \( D' = p(D) \). We define an extension \( p(B) \) of \( A \) as follows: say that \( B = A *_{A_0} C_0 \) according to Definition 5.3 (where \( A_0 \subseteq D \)). Let \( A'_0 = p[A_0] \). It is clear that we can find a small structure \( C'_0 \) containing \( A'_0 \) and an isomorphism \( g \) from \( C_0 \) onto \( C'_0 \) extending \( p[A_0] \), and moreover we may assume that \( A \cap C'_0 = A'_0 \). By definition \( p(B) = A *_{A'_0} C'_0 \). One should consult the remark following Definition 5.3 to be sure that this definition is legal (that is, does not depend on the particular \( A_0 \) chosen). Moreover, we may assume that, if \( B \in S \), then \( p(B) \in S \).

**Lemma 5.9.** With the above hypothesis, and supposing moreover that \( B \) is a strong extension of \( A \), \( p(B) \) is a strong extension of \( A \) based on \( D' \).

**Proof.** It suffices to show that there is a weak \( A \)-homomorphism from \( p(B) \) into \( A \). We use the notations of the previous paragraph: \( p(B) = A *_{A'_0} C'_0 \) and \( g \) is the isomorphism from \( C_0 \) to \( C'_0 \). It is enough to show that there is a weak \( A'_0 \)-homomorphism from \( C'_0 \) into \( A \).

Because \( B \) is a strong extension of \( A \), there exists a weak \( A \)-homomorphism from \( B \) to \( A \), so there exists \( h : D *_{A_0} C_0 \xrightarrow{w,D} A \). Since \( p \) is strong, \( p \) can be extended to a weak homomorphism \( k \) from \( D *_{A_0} C_0 \) into \( A \). Then \( k|_{C_0} \circ g^{-1} \) is a weak \( A'_0 \)-homomorphism from \( C'_0 \) into \( A \). ⊢

**Lemma 5.10.** Let \( B \) and \( C \) be two short extensions of \( A \) based on \( D \) and \( p \) a strong partial automorphism of \( A \) of domain \( D \). Then \( B \leq C \) if and only if \( p(B) \leq p(C) \).

**Proof.** Since \( p^{-1}(p(B)) = B \) and \( p^{-1}(p(C)) = C \), it suffices to prove that, if \( B \leq C \), then \( p(B) \leq p(C) \).

We need some notation: let \( D' \) be the image of \( p \), so that \( p \) is an isomorphism from \( D \) to \( D' \). We may write \( B = A *_{B_1} B_1 \), where \( B_2 \subseteq B_1 \) are small structures and \( B_2 \subseteq D \). Similarly \( C = A *_{C_1} C_1 \). Set \( B' = p(B) \) and \( C' = p(C) \). Let \( B'_2 = p[B_2] \) and \( B'_1 = A *_{B'_2} B'_1 \) and \( g : B_1 \to B_1' \) be the isomorphism given by the definition of \( p(B) \). Let \( F = C_1 \cup D \) so that \( C = A *_{D'} F \). We can write \( C' = A *_{D'} F' \) and \( f : F \to F' \) is an isomorphism extending \( p \). We have to show that there exists \( k_1 : B' \xrightarrow{w,A} C' \); it suffices to find \( k : B'_1 \xrightarrow{w,B'_2} C' \) with \( k(u(B')) = u(C') \).

As \( B \leq C \) there exists \( h_1 : B \xrightarrow{w,A} C \) such that \( h(u(B)) = u(C) \). Let \( h = (h_1)|_{B_1} \circ h : B_1 \xrightarrow{w,B_2} C \). Let \( E_1 = h^{-1}(A), E_2 = h^{-1}(F), E_3 = h^{-1}(D) \). As \( C = A *_{D} F \) and \( h \) is a weak homomorphism we have \( B_1 = E_1 *_{E_3} E_2 \). The structure \( G = D *_{h[E_1]} h[E_1] \) is a small extension of \( D \). Because \( p \) is strong there exists \( r : G \to A, r \) extending \( p \). Now we are gluing together \( f|_{h[E_3]} \circ h|_{E_3} \) and \( r|_{h[E_1]} \circ h|_{E_1} \) which coincides on \( E_3 \) with \( p|_{h[E_3]} \circ h|_{E_3} \) to get \( g' : B_1 \to C' \). Finally let \( k = g' \circ g^{-1}, k : B'_1 \xrightarrow{w,B'_2} C' \).

Now, suppose that \( D \subseteq A \subseteq B \), that \( p \) is a strong partial automorphism of \( A \) of domain \( D \) and image \( D' \) and that \( b \in U_1^B \). We may define \( p(\mathcal{E}_B(b/D)) \)
as \( \{ p(C); C \in E_B(b/D) \} \). We may also define \( p(\Gamma_B(b/D)) \): it is the structure whose base set is \( A \cup \{ x \} \) and such that, for every \( R \in L \) and sequence \( \overline{r} \) from \( A \cup \{ x \}, p(\Gamma_B(b/D)) \models R \overline{r} \) if and only if one of the two following cases holds:

1. all the elements of \( \overline{r} \) belong to \( A \) and \( A = R \overline{r} \) or
2. all elements of \( \overline{r} \) belong to \( D' \cup \{ x \} \), say to simplify that \( \overline{r} = \overline{r}'x \), where \( \overline{r}' \) is a sequence from \( D' \), and \( \Gamma_B(b/D) \models Rp^{-1}(\overline{r})x \).

The next lemma shows that, as announced, we have collected the relevant information.

**Lemma 5.11.** Let \( p \) be a strong partial automorphism of \( A \) with domain \( D \) and image \( D' \). Let \( B \) be a strong extension of \( A \), and let \( a \) and \( a' \) be elements of \( U_1^D, a \notin D \) and \( a' \notin D' \). Define a map \( q \) with domain \( D \cup \{ a \} \) by: \( q_1D = p \) and \( q(a) = a' \). Then \( q \) is a strong partial automorphism if and only if \( p(\Gamma_B(a/D)) = \Gamma_B(a'/D') \) and \( p(E_B(a/D)) = E_B(a'/D') \).

**Proof.** It follows immediately from the definitions that \( q \) is a partial automorphism if and only if \( p(\Gamma_B(a/D)) = \Gamma_B(a'/D') \).

Suppose first that \( q \) is strong. Let \( C \in E_B(a/D) \). Let \( C = A \ast A_0 C' \), where \( C' \) is a small structure including \( A_0 \), and \( A_0 \subseteq D \). By definition, \( p(C) = A \ast A_0' C'' \), where \( A_0' = p[A_0] \) and \( C' \) is isomorphic to \( C'' \), via an isomorphism \( p^* \) extending \( p_{A_0} \), and such that \( p^*(x) = x \).

Since \( C \in E_B(a/D) \), there exists \( h : C' \rightarrow B \) such that \( h(x) = a \). Since \( q \) is strong, there exists a weak homomorphism \( h' \) from \( C' \) into \( B \), extending \( q_{A_0} \) and such that \( h'(x) = a' \). Thus, if we set \( k = h' \circ p^{-1} \), \( k \) is a weak \( A_0' \)-homomorphism from \( C'' \) into \( B \), and \( k(x) = a' \). This proves that \( p(C) \in E_B(a'/D') \).

So \( p(E_B(a/D)) \subseteq E_B(a'/D') \), and, for the same reason, \( E_B(a'/D') \subseteq p(E_B(a/D)) \).

Conversely, suppose that \( p(E_B(a/D)) = E_B(a'/D') \). Assume that \( C \) is a short extension of \( D \cup \{ a \} \), and \( h : C \rightarrow B \). As usual \( C \) can be written as \((D \cup \{ a \}) \ast A_0 C' \). If \( a \notin A_0 \), we may assume that \( a \notin C' \). Because \( p \) is strong, there exists a weak homomorphism \( k \) from \( C' \) into \( B \) extending \( p_{A_0} \). Then the map from \((D \cup \{ a \}) \ast A_0 C' \) into \( B \) extending both \( k \) and \( q \) is a weak homomorphism, so there exists a weak homomorphism from \( C \) to \( B \) extending \( q \).

If \( a \in A_0 \), let \( A_1 = A_0 - \{ a \} \) and \( A_1' = p[A_1] \). Obviously, \( h \) is also a weak \( D \)-homomorphism from \( C \) to \( B \). Intuitively, this implies that \( A \ast A_1 C' \) belongs to \( E_B(a/D) \). More precisely, there exist \( C_1 = A \ast A_1 C_1' \in E_B(a/D) \) and an \( A_1 \)-isomorphism \( k \) from \( C' \) onto \( C_1' \) satisfying \( k(a) = x \). Let \( p(C_1) = A \ast A_1' C_2' \) and let \( p^* \) denote the isomorphism from \( C_1' \) onto \( C_2' \) extending \( p_{A_1} \), satisfying and \( p^*(x) = x \). Since \( p(C_1) \in p(E_B(a/D)) = E_B(a'/D') \), there exists a weak homomorphism \( k' : C_2' \rightarrow B \) such that \( k'(x) = a' \). Then \( p_1 = k' \circ p^* \circ k : C' \rightarrow B \) extends \( p_{A_1} \) and satisfies \( p_1(a) = a' \). Thus, the map from \( C \) into \( B \) which extends both \( p_1 \) and \( q \) is a weak homomorphism and it is exactly what we had to find.

**Definition 5.12.** Let \( A \subseteq B \) be two structures. We say that \( B \) is irreducible over \( A \) if, whenever \( B \) is isomorphic to \( B_1 \ast A B_2 \), one of the structures \( B_1 \) or \( B_2 \) is equal to \( B \).

So, \( B \) irreducible over \( A \) exactly means that \( B - A \) is not the disjoint union of two non empty subsets \( B_1 \) and \( B_2 \) such that there is no link containing an element of \( B_1 \) and an element of \( B_2 \).
Definition 5.13. Let $A$ be a finite structure. A tiny extension of $A$ is a short and strong extension $B$ of $A$ which is irreducible over $A$.

Lemma 5.14. Suppose $B$ is a tiny extension of $A$ based on $D$ and that $p$ is a strong partial automorphism of $A$ of domain $D$ and image $D’$. Then $p(B)$ is a tiny extension of $A$ based on $D’$.

Proof. To prove that $p(B)$ is irreducible use the remark after Definition 5.12.

Definition 5.15. Let $D \subseteq A \subseteq B$ and $b \in U_1^B$. Then the type of $b$ in $B$ over $D$ is the pair $(\Gamma_B(b/D), \mathcal{E}_B^{irr}(b/D))$ where $\mathcal{E}_B^{irr}(b/D)$ is the set of maximal elements (for the ordering $>$) of the set

\[ \{ C; C \in \mathcal{E}_B(b/D) \text{ and } C \text{ is irreducible} \}. \]

We will denote $t_B(b/D)$ this type. From Lemma 5.8 it follows that if $B’$ is a strong extension of $B$, then $t_B(b/D) = t_{B’}(b/D)$.

Lemma 5.16. $D \subseteq A \subseteq B$ and $b \in U_1^B$, and let $C$ be a short extension of $A$. Then $C \in \mathcal{E}_B(b/D)$ if and only if there exists $C’ \in \mathcal{E}_B^{irr}(b/D)$ such that $C \subseteq C’$.

Proof. One direction is clear: if $C’ \in \mathcal{E}_B^{irr}(b/D)$ and $C \subseteq C’$, then $C \in \mathcal{E}_B(b/D)$. Conversely, let $C \in \mathcal{E}_B(b/D)$. We can write $C$ as $C_1 \ast C_2$ where $C_1$ is a tiny extension of $A$ and $C_2$ is a short extension of $A$ such that $C_2 - A$ contains no point in $U_1$. Clearly $C_1 \in \mathcal{E}_B(b/D)$, so there exists $C’ \in \mathcal{E}_B^{irr}(b/D)$ such that $C_1 \subseteq C’$. As $A \subseteq B$ there exists $h: C_2 \longrightarrow A$ and thus $C \subseteq C_1$.

Definition 5.17. A type is an object of the form $t_B(b/A)$, where $A \subseteq B$ and $b \in B$. A type based on $D$ (where $D$ is a subset of $A$) is an object of the form $t_B(b/D)$.

These definitions are justified by the following lemma:

Lemma 5.18. Let $D \subseteq A \subseteq B$ and $b \in U_1^B$; then $t_B(b/D)$ is a type.

Proof. Let $B^*$ be an isomorphic copy of $B$: its universe is $B^* = \{ a^*; a \in B \}$, $B \cap B^* = \emptyset$ and the map $\alpha$ from $B$ onto $B^*$ defined by $\alpha(a) = a^*$ is an isomorphism. Let $B_1 = A \cup B^*$ and let $h$ be the map from $B_1$ on $B$ defined by: for all $a \in A$, $h(a) = a$ and for all $a \in B : h(a^*) = a$. We endow $B_1$ with an $\mathcal{L}$-structure by setting: for $R$ an $n$-ary predicate symbol and $a_1, a_2, \ldots, a_n$ elements in $B_1$, $B_1 \models R(a_1, a_2, \ldots, a_n)$ if and only if $B \models R(h(a_1), h(a_2), \ldots, h(a_n))$ and either all the $a_i$ belong to $A$ or all the $a_i$ belong to $B^* \cup D$. With this definition, we see that $B_1$ is an extension of both $A$ and $B^*$, and that $h$ is a weak $A$-homomorphism from $B_1$ to $B$, so by Lemma 5.4 $B_1$ is a strong extension of $A$.

We want to check that $t_{B_1}(b^*/A) = t_B(b/D)$. It is quite clear that $\Gamma_{B_1}(b^*/A) = \Gamma_B(b/D)$.

Let $t_{B_1}(b^*/A) = (\Gamma, \mathcal{E})$ and $C \in \mathcal{E}$ (so $C$ is a maximal irreducible element of $\mathcal{E}_{B_1}(b^*/A)$). In order to show that $C$ is based on $D$, we need the two following general lemmata:

Lemma 5.19. Let $C$ be a tiny extension of $A$, and assume that $h: C \longrightarrow C$.

Then $h$ is the identity or $h[C] = A$. 


Proof. Write \( C = A \ast_{A_0} C' \) where \( C' \) is a small structure, and split the set \( C - A \) in two sets: \( C_1 = \{ a \in C - A; h(a) = a \} \) and \( C_2 = \{ a \in C - A; h(a) \in A \} \). We have to prove that one of the sets \( C_1 \) or \( C_2 \) is empty.

If not, since \( C \) is irreducible, there is a link containing an element \( a \in C_1 \) and an element \( b \in C_2 \). Thus, there is also a link containing \( a \) and \( c = h(b) \). We see that \( b \) and \( c \) are distinct elements, both linked to \( a \), so both belong to \( C' \), and belong to the same level \( U \); this contradicts the fact that \( C' \) is small. \( \square \)

As a corollary, we see that the preordering \( < \) when restricted to the set of tiny extensions in \( \mathcal{S} \) is an ordering.

**Lemma 5.20.** Let \( A \subseteq s B, b \in B - A \) and \( t_B(b/A) = (\Gamma, \mathcal{E}) \). Let \( C \in \mathcal{E} \). So \( C \) is a short extension of \( A \) and there exists \( k : C \to B \) such that \( k(x) = b \). Then, for every \( c \in C - A, k(c) \notin A \).

Proof. Write \( C = A \ast_{A_0} C' \) where \( C' \) is a small structure. Assume for a contradiction that there exists \( k : C' \to B \) with \( k(x) = b \) and \( c \in C' - A \) with \( k(c) \in A \). Choose \( k \) such that the set \( X = \{ c \in C' - A; k(c) \in A \} \) is maximal. Let \( C'_1 = k[C'] \) and \( A'_1 = C_1 \cap A \). Intuitively this implies that \( C'_1 = A \ast_{A'_1} C'_1 \) belongs to \( \mathcal{E}_B(b/A) \). More precisely there exists a small structure \( C'_2 \) containing \( A'_1 \) and \( x \), and an \( A'_1 \)-isomorphism \( k' \) from \( C'_1 \) to \( C'_2 \) satisfying \( k'(b) = x \) such that \( A \ast_{A'_1} C'_2 \in \mathcal{E}_B(b/A) \).

Let \( C_2 = A \ast_{A'_1} C'_2 \). Let us first check that \( C'_1 \) is irreducible over \( A'_1 \); suppose not, say \( C'_1 = C_3 \ast_{A_1} C_4 \) with \( A'_1 \subset C_3 \) and \( A'_1 \subset C_4 \). Let us say \( b \in C_3 \). As \( A \subseteq s B \) there exists \( h : C_4 \to A \). Let \( h' : C'_1 \to B \) be the homomorphism which coincides with \( h \) on \( C_4 \) and is the identity on \( C_3 \). Then \( h' \circ k \) contradicts the choice of \( k \) as \( \{ c \in C - A; k(c) \in A \} \subset \{ c \in C - A; h' \circ k(c) \in A \} \). Thus \( C'_1 \) is irreducible over \( A'_1 \), i.e. \( C_1 \) is irreducible over \( A \). Thus \( C_2 \) is irreducible over \( A \) and in \( \mathcal{E}_B(b/A) \).

Let \( l : C \to C_1 \) and \( l' : C_1 \to C_2 \) be the extension of \( k \) and \( k' \) respectively by the identity on \( A \). As \( l' \circ l : C \to C_2 \) and \( l' \circ l(x) = x \), we have \( C \leq C_2 \). Now, since \( C \) is maximal among the irreducible elements of \( \mathcal{E}_B(b/A) \), we see that there exists \( k_1 : C_2 \to C \) with \( k_1(x) = x \), so, by the previous lemma, \( l' \circ l \) is injective, a contradiction. \( \square \)

Now, we go back to \( C \in \mathcal{E} \). We know that there exists \( k : C \to B_1 \), with \( k(x) = b' \). From the preceding lemma, we see that the elements of \( C - A \) are mapped by \( k \) to elements of \( B^* \), so there is no link containing an element of \( C - A \) and an element of \( A - D \). This implies that \( C \) is based on \( D \) (see the remark following Definition [5.3]).

To conclude, we see that \( \mathcal{E}_B(b/D) \subset \mathcal{E}_{B_1}(b^*/A) \subset \mathcal{E}_B(b/A) \).

The first inclusion comes from the fact that if \( C \in \mathcal{E}_B(b/D) \) and \( k : C \to B \) with \( k(x) = b \), then the map \( k' \) from \( C \) to \( B \) defined by: \( k'(y) = y \) for \( y \in A \) and \( k'(y) = (k(y))^* \) for \( y \in C - A \) is a weak \( A \)-homomorphism and \( k'(x) = b^* \). The second inclusion is true because there exists \( h : B_1 \to B \) with \( h(b^*) = b \). Now it is easy to check that \( t_B(b/D) = t_{B_1}(b^*/D) \). \( \square \)
Let \( t = t_B(b/D) = (\Gamma, \mathcal{E}) \) be a type based on \( D \subseteq A \), and let \( p \) be a strong partial automorphism of \( A \) whose domain contains \( D \). Then we define \( p(t) \) to be the pair \((p(\Gamma), p(\mathcal{E}))\), where \( p(\mathcal{E}) \) is the set \( \{ p(B); B \in \mathcal{E} \} \). It is easy to prove that if \( b \in D \), then \( p(t_B(b/D)) = t_B(p(b)/D') \). It is also important to notice:

Lemma 5.21. With these notations, \( p(t) \) is a type.

Proof. Let \( D' = p[D] \). First we may assume that \( b \notin D \) (if not, \( p(t) = t_B(p(b)/D') \)). Let \( B^* \) be, as above, an isomorphic disjoint copy of \( B \). Let \( B_1 = B \cup B^* \) and \( h \) be the map from \( B^* \cup D' \) to \( B \) such that: for \( a \in B, h(a^*) = a \) and for \( a \in D' \), \( h(a) = p^{-1}(a) \). Now, we define another structure on \( B_1 \): if \( R \) is an \( n \)-ary predicate symbol and \( a_1, a_2, \ldots, a_n \) are elements of \( B_1 \), \( B_1 \models R(a_1 \cdots a_n) \) if and only if either all the \( a_i \) belong to \( B \) and \( B \models R(a_1 \cdots a_n) \); or all the \( a_i \) belong to \( B^* \cup D' \) and \( B \models Rh(a_1)h(a_2) \cdots h(a_n) \). Our aim is to prove \( t_{B_1}(b^*/D') = p(t) \).

To prove \( A \subseteq B_1 \), by Lemma 5.14 it suffices to prove that \( B \subseteq B_1 \), and, again by Lemma 5.4, that, if \( C \) is a small structure of \( B_1 \), then there exists \( k : C \to B \) such that \( k(x) = b \). Set \( C' = p(C) \). Since \( C' \) is based on \( D' \), there is no link containing an element of \( C' \) and for \( a \in A \) and \( k_1(y) = y \) if \( y \in A \) and \( k_1(y) = (k(y^{-1}(y)))^* \) if \( y \in C' \). Thus the map \( k \) defined by \( k_1(y) = y \) if \( y \in A \) and \( k_1(y) = (k(y^{-1}(y)))^* \) if \( y \in C' \) is a weak \( A \)-homomorphism from \( C' \) to \( B_1 \) and \( k_1(x) = b^* \). Consequently, \( p(\mathcal{E}_{B_1}(b^*/D')) \subseteq \mathcal{E}_{B_1}(b^*/D') \). Lemma 5.10 and Lemma 5.11 now imply \( p(t_B(b/D)) = t_{B_1}(b^*/D') \). \( \square \)

As in Lemma 5.11 we see that \( t_B(b/D) \) carries enough information:

Lemma 5.22. Let \( p \) be a strong partial automorphism of \( A \) with domain \( D \) and image \( D' \). Let \( B \) be a strong extension of \( A \) and let \( a \) and \( a' \) be elements of \( U_B \), \( a \notin D \) and \( a' \notin D' \). Define a map \( q \) with domain \( D \cup \{ a \} \) by: \( q|_D = p \) and \( q(a) = a' \). Then \( q \) is a strong isomorphism if and only if \( t_B(a'/D') = p(t_B(a/D)) \).

Proof. One direction is clear: if \( q \) is strong, then, by Lemma 5.11, \( \mathcal{E}_B(a'/D') = \mathcal{E}_B(a/D) \), and \( t_B(a'/D') = t_B(a/D) \). Conversely, we have to prove that \( \mathcal{E}_B(a'/D') = \mathcal{E}_B(a/D) \), assuming that \( \mathcal{E}_B^{irr}(a'/D') = \mathcal{E}_B^{irr}(a/D) \). But this is a direct consequence of Lemma 5.11. \( \square \)

We are now ready to state the next reduction.

Proposition 5.23. Let \( A \) be a finite structure and let \( p_1, p_2, \ldots, p_n \in \text{Part}(A, A) \), and suppose that the \( p_i \) are strong in \( A \). Write \( D_i \) for the domain of \( p_i \) and \( D_i' \) for its image. There exists a finite strong extension \( B \) of \( A \) such that, for every \( i \), \( 1 \leq i \leq n \), and for every type \( t \) based on \( D_i \), the sets \( \{ b \in U_B^i; t_B(b/D_i) = t \} \) and \( \{ b \in U_B^i; t_B(b/D_i) = p_i(t) \} \) have the same cardinality.
1) Let \( w \) be a tiny extension \( A \) of \( D \) such that \( h(1) \) is strong partial automorphism of \( A \). Then \( h(1) \) is a strong partial automorphism of \( A \). Thus, it is possible to extend \( p_i \) to an injective map \( g_i \) from \( D \cup U_i^P \) onto \( D_i^P \cup U_i^B \) such that, for all \( b \in U_i^B \), \( I_B(g_i(b)/D_i^P) = p_i(t_B(b/D_i^P)) \). By Lemma 5.22, for every \( b \in U_i^B \) we have that \( (h_i);D_i^P \cup U_i^B \) is a strong partial automorphism of \( B \). But this easily implies that \( h_i \) is a strong partial automorphism of \( B \) as every small structure contains at most one point in \( U_i \).

5.4. The weight of a type. Now, we must face the following problem. We want to find a strong extension \( B \) of \( A \) satisfying the conclusion of Proposition 5.23. Suppose, for example, that the cardinality of the set \( \{ b \in U_1^A : t_A(b/D_1) = t \} \), where \( t \) is a type based on \( D_i \), is smaller than the cardinality of \( \{ b \in U_1^A : t_A(b/D_1) = p_i(t) \} \).

It is fairly easy to increase the cardinality of the set \( \{ b \in U_1^A : t_A(b/D_1) = t \} \) by one, and eventually to get a strong extension where the cardinality of these two sets are the same. But, we cannot perform such an operation simultaneously for all types and for all partial automorphisms, because when taking care of another partial automorphism \( p_j \), we may destroy what we have done for \( p_i \). It is to control this phenomenon that we need the notion of weight.

Let \( C \) be a tiny extension \( A \). We define:

- \( n(C) \) the number of links of \( C \) which are not links of \( A \).
- If \( a \in C - A \), we define the height of \( a \) inductively: \( h(a) = 0 \) if and only if \( a \in U_1^C \); \( h(a) = n + 1 \) if there is a link containing \( a \) and an element \( b \in C - A \) such that \( h(b) = n \) (and the height of \( a \) has not been already defined).

Because \( C \) is irreducible over \( A \), we know that every element of \( C - A \) has a height. Now we define

- the height of \( C \), \( h(C) = \max(h(a); a \in C - A) \).
- \( h_0 = \max(h(C); C \) is a tiny extension of \( A \).
- For \( i, 0 \leq i \leq h_0, w_i(C) \) is the number of elements \( a \in A \) such that there exist \( b \in C - A \) with \( h(b) \leq i \) and a link containing \( a \) and \( b \).
- The weight of \( C \) is the sequence \( w(C) = ((w_0(C), w_1(C), \ldots, w_{h_0}(C), \text{card}(C - A), n(C))) \) ordered lexicographically.

**Lemma 5.24.** 1) Assume that \( C \subseteq B \). Then \( w(C) \subseteq w(B) \) and if \( w(C) = w(B) \), then \( C \) and \( B \) are isomorphic.

2) If \( B \) is based on \( D \subseteq A \) and \( p \) is a partial automorphism of \( A \) of domain \( D \), then \( w(B) = w(p(B)) \).

**Proof.** 1) Let \( k : C \xrightarrow{w,A} B \) such that \( k(w(C)) = u(B) \). If \( a \in A \) is linked to \( u(C) \) (in \( C \)), it is linked (in \( B \)) to \( u(B) \). Thus \( w_0(C) \leq w_0(B) \). Moreover, suppose that \( c \in C - A \) has height 1 and \( k(c) \in A \). Then \( w_0(C) < w_0(B) \). Indeed, there is a link between \( u(C) \) and \( c \) (because \( c \) has height 1), thus there is a link between \( u(B) \) and \( k(c) \). But since \( c \) and \( k(c) \) belong to the same \( U_i \), there is no link between \( u(C) \) and \( k(c) \). Thus there is at least one element of \( A \) which is linked to \( u(B) \) (in \( B \)), but not to \( u(C) \).

Thus, if \( w_0(C) = w_0(B) \) and \( c \in C - A \) has height 1, \( k(c) \in B - A \), and of course, has height 1. We can continue and prove in the same way that \( w_1(B) \leq w_1(C) \) and if \( w_1(B) = w_1(C) \) and \( c \in C - A \) has height 2, then \( k(c) \in B - A \) and \( h(k(c)) \leq 2 \).
After $h_0$ steps, we get that either
\[ (w_0(C), w_1(C), \ldots, w_{h_0}(C)) < (w_0(B), w_1(B), \ldots, w_{h_0}(B)) \]
or
\[ (w_0(C), w_1(C), \ldots, w_{h_0}(C)) = (w_0(B), w_1(B), \ldots, w_{h_0}(B)) \]
and that $k$ is injective. The first part follows easily.

2) is clear from the definitions. \hfill \Box

**Definition 5.25.** Suppose $t = (\Gamma, \mathcal{E})$ and $t' = (\Gamma', \mathcal{E}')$ are two types. Then:

- $t \leq t'$ if the identity map from $\Gamma$ to $\Gamma'$ is a weak homomorphism and, for every $C \in \mathcal{E}$, there exists $C' \in \mathcal{E}'$ such that $C \leq C'$.

We will need the following easy facts.

**Lemma 5.26.** Let $D \subseteq A \subseteq B, t$ and $t'$ be types based on $D$.

1) $t \leq t'$ and $t' \leq t$ implies $t = t'$.
2) For $b \in B$ we have: $t_B(b/A) > t$ if and only if $t_B(b/D) > t$.

The next definition will give an order homomorphism from the partial order of all types over $A$ into a total order.

**Definition 5.27.** Let $t = (\Gamma, \mathcal{E})$ be a type. The weight of $t$ is the sequence $w(t) = (n_0, n_1, \ldots, n_m)$ where: $n_0$ is the number of links of the structure $\Gamma$, and $(n_1, n_2, \ldots, n_m)$ is the weakly decreasing sequence (that is such that $n_1 \geq n_2 \geq \cdots \geq n_m$) of the form $(w(C_1), w(C_2), \ldots, w(C_m))$, where $\mathcal{E} = \{C_1, C_2, \ldots, C_m\}$. These sequences are ordered lexicographically by $\leq$.

**Lemma 5.28.** Let $t$ and $t'$ be two types.

1) Assume that $t \leq t'$. Then $w(t) \leq w(t')$ and if $w(t) = w(t')$, then $t = t'$.
2) If $t$ is based on $D \subseteq A$ and $p$ is a partial automorphism with domain $C$, then $w(t) = w(p(t))$.

**Proof.** Again, 2) follows immediately from the definitions. Set $t = (\Gamma, \mathcal{E}), t' = (\Gamma', \mathcal{E}')$. The identity map from $\Gamma$ to $\Gamma'$ is a bijective weak $A$-isomorphism, and, if it is not an isomorphism, there are strictly more links in $\Gamma$ than in $\Gamma'$, and $w(t) < w(t')$.

So, suppose that $\Gamma = \Gamma'$. Let $\mathcal{E} = \{C_1, C_2, \ldots, C_m\}, \mathcal{E}' = \{C'_1, C'_2, \ldots, C'_m\}$, and suppose that $(w(C_1), w(C_2), \ldots, w(C_m))$ and $(w(C'_1), w(C'_2), \ldots, w(C'_m))$ are both weakly decreasing. We know that, for all $i$, $1 \leq i \leq m$, there exist an integer, say $k(i)$ such that $1 \leq k(i) \leq m'$ and $C_i \leq C'_{k(i)}$. So, $w(C_1) \leq w(C'_{k(1)}) \leq w(C'_1)$ and either $w(t) < w(t')$ or $w(t) = w(C'_1)$. $C_1$ is equal to $C'_{k(1)}$. By reordering $\{C'_1, \ldots, C'_m\}$ we can suppose $k(1) = 1$. In this case, $k(2) \neq 1$ (because, otherwise, $C_2 \leq C'_1 = C_1$, and two elements of $\mathcal{E}$ cannot be comparable) and $k(2) \geq 2$. Again $w(C_2) \leq w(C'_{k(2)}) \leq w(C'_2)$, and either $w(t) < w(t')$ or $C_2 = C'_{k(2)}$. Going on, we reach the results that, either $w(t) < w(t')$ or $k$ is injective. If $k$ is surjective, this means that $t = t'$, and if not, this implies that $w(t) < w(t')$. \hfill \Box

We can now state our last reduction.

**Proposition 5.29.** Let $A$ be a finite structure. Then there exists a strong extension $B$ of $A$ such that, for every type $t$ and $t'$, if $w(t) = w(t')$, then the set $\{b \in U^B_1; t_B(b/A) \geq t\}$ and $\{b \in U^B_1; t_B(b/A) \geq t'\}$ have the same number of elements.
Proof. Let $B$ be as in the conclusion of Proposition \[5.29\] By Lemma \[5.26\]

\[
\{ b \in U^B_1; t_B(b/D) = t \} = \{ b \in U^B_1; t_B(b/A) \geq t \} - \bigcup_{t' > t, \ t' \text{ is based on } D} \{ b \in U^B_1; t_B(b/D) = t' \}.
\]

We argue by downward induction: assume that we have proved that the sets \{ $b \in U^B_1; t_B(b/D_i) = t$ \} and \{ $b \in U^B_1; t_B(b/D'_i) = p_i(t)$ \} have the same cardinality for every type $t$ based on $D_i$ of weight strictly bigger than a given value $l$, and let $u$ be a type based on $D_i$ of weight $l$. We have

\[
\{ b \in U^B_1; t_B(b/D_i) = u \} = \{ b \in U^B_1; t_B(b/A) \geq u \} - \bigcup_{u' > u, \ u' \text{ is based on } D_i} \{ b \in U^B_1; t_B(b/D) = u' \}
\]

and, analogously,

\[
\{ b \in U^B_1; t_B(b/D'_i) = p_i(u) \} = \{ b \in U^B_1; t_B(b/A) \geq p_i(u) \} - \bigcup_{u' > p_i(u), \ u' \text{ is based on } D'_i} \{ b \in U^B_1; t_B(b/D') = u' \}
\]

by our previous remark. The map $p_i$ induces a one to one correspondence between the set \{ $u' > u, \ u' \text{ is based on } D_i$ \} and the set \{ $u' > p_i(u), \ u' \text{ is based on } D'_i$ \}. Moreover, by induction hypothesis, for each type $u' > u$, $u'$ based on $D_i$,

\[
\text{card}(\{ b \in U^B_1; t_B(b/D_i) = u' \}) = \text{card}(\{ b \in U^B_1; t_B(b/D'_i) = p_i(u') \}).
\]

From these facts, we deduce that

\[
\text{card}(\{ b \in U^B_1; t_B(b/D_i) = u \}) = \text{card}(\{ b \in U^B_1; t_B(b/D'_i) = p_i(u) \}).
\]

We have now finished the reductions, and it remains to prove Proposition \[5.29\]

Let $n_0 = \max(w(t); \ t \text{ is a type})$ (here, by abuse of language, we identify the weight of a type with an integer). We construct by downward induction a chain of strong extensions $(B_n, \ 0 \leq n \leq n_0 + 1)$ such that, for every types $t$ and $t'$, if $w(t) = w(t') \geq n$, then the set \{ $b \in U^B_1; t_{B_n}(b/A) \geq t$ \} and \{ $b \in U^B_1; t_{B_n}(b/A) \geq t'$ \} have the same number of elements. We start with $B_{n_0 + 1} = A$, and we end with $B_0$, the structure that we need.

We show how to get $B_n$ from $B_{n+1}$. For every type $t$, let

\[
s(t) = \text{card}(\{ b \in B_{n+1}; t_{B_{n+1}}(b/A) \geq t \})
\]

and $r = \max(s(t); \ t \text{ is a type of weight } n)$. Assume that for a given type $t$ of weight $n$, $s(t) < r$. We are going to construct a strong extension $B'$ of $B_{n+1}$ such that $U^B_1 = U^{B_{n+1}}_1 \cup \{ a \}$, where $t_{B'}(a/A) = t$. Thus for no $t'$ with $w(t') \geq w(t)$ and $t' \neq t$ we have $t_{B'}(b/A) \geq t'$. Repeating this process $r - s(t)$ times, and then doing this for all types of weight $n$, we will get the structure $B_n$ with the desired properties.

By the proof of Lemma \[5.18\] we know that there exists a strong extension $A_1$ of $A$ and an element $a \in A_1 - A$ such that $t_{A_1}(a/A) = t$. Let $A_2$ be the substructure of $A_1$ whose universe is

\[
A_1 - \{ b; b \in U^{A_1}_1 - U^A_1 \text{ and } b \neq a \}.
\]
By Lemma 5.4, $A_2$ is a strong extension of $A$, and it is immediate to check that $t_{A_2}(a/A) = t$. Now let $B' = B_{n+1} \ast_A A_2$. By Lemma 5.4 $B_{n+1} \subseteq B'$ and $A_2 \subseteq B'$ and therefore also $t_{B'}(a/A) = t$. Thus $B'$ has the desired properties.

6. Final Comments

We wish to conclude this article by some comments and one open question.

Just before the difficult writing of this paper was over, we became aware (thanks to J. E. Pin and P. Weil) of the work of Almeida [1] and of Almeida and Delgado [2]. It seems that starting from a theorem of Ash [3] they have proved a theorem which can be seen to be equivalent to our Theorem 3.3. See [2] for more details.

The result in this paper improves the results in [6]. Let us recall some notation from [6]. Suppose $\mathcal{R}$ is a set of link structures and $\mathcal{F}$ is a finite set of structures. We denote by $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ the class of all $\mathcal{F}$-free structures of link type $\mathcal{R}$. If $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ has the amalgamation property (AP) (where we allow the common part to be empty), then there exists a countable homogeneous universal structure in $\mathcal{K}_{\mathcal{R},\mathcal{F}}$, which we call $M_{\mathcal{R},\mathcal{F}}$ and which is uniquely determined up to isomorphism. In [6] the property EPPA was called (WEP). The property (EP) is the following property for a class $\mathcal{C}$: For all finite $M_0$ and $P \subseteq \text{Part}(M_0, M_0)$ the $(M_0, P, \mathcal{C})$-extension problem has a finite solution.

**Theorem 6.1.** Let $\mathcal{L}$ be a finite relational language. Let $\mathcal{R}$ be a set of link structures and $\mathcal{F}$ be a finite set of $\mathcal{L}$-structures.

a) $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ has EPPA.

b) If $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ has (AP), then $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ has (EP) and $M_{\mathcal{R},\mathcal{F}}$ satisfies the small index property.


**Proof.**

a) follows from Theorem 3.2. Use the remark after Definition 2.1 to get a slim solution and observe that a slim solution of the $(M_0, P, \mathcal{C})$-extension problem has the same link type as the structure $M_0$ you started with.

b) If $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ has (AP), then $M_{\mathcal{R},\mathcal{F}}$ provides an infinite solution for every $(M_0, P, \mathcal{K}_{\mathcal{R},\mathcal{F}})$-extension problem, where $M_0$ is finite: Embed $M_0$ into $M_{\mathcal{R},\mathcal{F}}$, which you can do by the universality of $M_{\mathcal{R},\mathcal{F}}$ and by the homogeneity you can extend every partial isomorphism from $P$ to an automorphism of $M_{\mathcal{R},\mathcal{F}}$. So in this case EPPA implies (EP). Now use the method of [7] to prove the small index property: Like in the proof of Lemma 14 in [6] deduce that $\mathcal{K}_{\mathcal{R},\mathcal{F}}$ actually satisfies the free amalgamation property. This implies that also the classes $\mathcal{K}_{\mathcal{R},\mathcal{F}}^n$ (for $n \in \omega$) satisfy (AP). Now Theorem 11 in [6] states that $M_{\mathcal{R},\mathcal{F}}$ has the small index property. ⊗

The case of the class of tournaments raises an interesting problem. Recall the definition of a tournament: it is a directed irreflexive graph $\Gamma$ such that, for every two distinct points $a$ and $b$ in $\Gamma$, either there is an arrow from $a$ to $b$, or there is an arrow from $b$ to $a$, but not both. There is a countable tournament $\Gamma_0$ which is universal (every finite tournament can be embedded in it) homogeneous (every isomorphism between two finite tournaments included in $\Gamma_0$ can be extended to an automorphism of $\Gamma_0$), which in fact is determined up to isomorphism by these properties and which we shall call the generic tournament. The following question is open:

(1) Has the generic tournament the small index property?

But this question would be settled affirmatively if we could prove:
(2) The class of all tournaments has the EPPA.

This last question turns out to be equivalent to a rather natural question about free groups. Let $F(P)$ be the free group generated by the finite set $P$. Consider the topology on $F(P)$ for which a basis of open sets is

$$\{f \cdot H; f \in F(P)H \text{ is a normal subgroup of } F(P) \text{ of finite odd index}\}.$$

We shall call this topology the odd-adic topology. Then, (2) is equivalent to the following assertion:

(3) Let $H$ be a finitely generated subgroup of $F(P)$. Then the two following properties are equivalent:

(i) $H$ is closed for the odd-adic topology.

(ii) For all $a \in F(P)$, if $a^2 \in H$, then $a \in H$.

We sketch the proof that (2) and (3) are equivalent. We first remark that, in any case, (3 i) implies (3 ii). Assume, toward a contradiction, that $H$ is a subgroup of $F(P)$ which is closed for the odd-adic topology, that $a \notin H$ and $a^2 \in H$. Because $H$ is closed for the odd-adic topology, there exists a homomorphism $\varphi$ from $F(P)$ onto a finite group $G$ of odd order, such that, if we set $H' = \varphi[H]$ and $a' = \varphi(a)$, then $a'^2 \in H'$ but $a' \notin H'$. Now, if we consider $G_1 = \{g \in G; gh' = H' \text{ and } ga'\H' = a'\H'\}$ and $G_2 = \{g \in G; gh', ga'\H' = \{H', a'\H'\}\}$, then we see that $G_1$ is a subgroup of order 2 of $G_2$, and this is impossible since $G_1$ has odd order, as subgroup of a group of odd order.

Now assume (2). We show that (3 ii) implies (3 i). So, let $H$ be a finitely generated subgroup of $F(P)$ satisfying (3 ii) and $\alpha$ an element of $F(P)$ not in $H$.

We will first define a tournament $T$, whose base set is $F(P)/H$ in such a way that, for all $f \in F(P)$, the left multiplication by $f$ is an automorphism of $T$. To do that, consider, for each pair $(aH, bH)$ in $F(P)/H$ with $aH \neq bH$, the orbit of $(aH, bH)$ under the action of $F(P)$, that is $X_{a,b} = \{(fa, fb); f \in F(P)\}$. Such an orbit cannot contain both a pair $(cH, dH)$ and $(dH, cH)$. Otherwise, for some $f \in F(P)$, $fcH = dH$ and $fdH = cH$, so $f^2dH = dH$, and $(d^{-1}f)dH = H$. By condition (3 ii), this implies that $(d^{-1}f)$ is in $H$, so $cH = dH$, which is impossible.

Consider now the set $\{(aH, bH); aH \neq bH\}$ and partition it into orbits for the action of $F(P)$. Each orbit $O$ is equal to the set $\{(faH, fbH); f \in F(P)\}$, where $a$ and $b$ are two elements of $F(P)$ and $aH \neq bH$. The subset $X(O) = \{(cH, dH), \{cH, dH\} \in O\}$ of $(F(P)/H)^2$ is the disjoint union of exactly two orbits of $(F(P)/H)^2$ under the action of $F(P)$. Choose one of them, say $Y(O)$, and define the tournament on $F(P)/H$ by deciding that there is an arrow from $aH$ to $bH$, if and only if $(aH, bH) \in Y(O)$ for some orbit $O$.

We are now ready to use our machinery. Let $S_0$ be a finite subset of $F(P)$ containing $\alpha$, the generators of $H$ and closed under initial segments, as in section 2. Let $T_0$ be the cosets modulo this subset, considered as a subtournament of $T$. To each element $p \in P$ corresponds a partial automorphism $\tilde{p}$ of $T_0$. By (2), there exists a tournament $T_1$ containing $T_0$ and automorphisms $\tilde{p}$ of $T_1$ extending $\tilde{p}$. This allows us to define an action of $F(P)$ on $T_1$ that is a homomorphism $f \mapsto \tilde{f}$ from $F(P)$ into $\text{Aut}(T_1)$, the automorphism group of $T_1$. Because we have put enough elements in $T_0$, the stabiliser $H'$ of $H$ for this action includes $H$, does not contain $\alpha$ and, of course, is of finite index. We want to show that $H'$ is open.

The kernel of this action, that is $K = \{f \in F(P); f = 1\}$, is a normal subgroup of $F(P)$ of finite index contained in $H'$, and we will be done if we prove that its
index is odd. But $F(P)/K$ is isomorphic to a subgroup of $\text{Aut}(T_1)$, and $\text{Aut}(T_1)$ has odd order, because it cannot contain an involution.

Let us now prove that (3) implies (2). We start from a finite tournament $T$ and a set $P$ of partial automorphisms of $T$. We consider $T$ as a directed graph, and we choose an element $x_0$ in $T$. We may assume that for all $x \in T$, there exists $p \in P$ such that $x = p(x_0)$. So, there is a correspondence between the subgroups $H$ of $F(P)$ containing the set

$$X_0 = \{p^{-1} \cdot q; p(x_0) = q(x_0)\} \cup \{p^{-1} \cdot p \cdot q; r(x_0) = p \circ q(x_0)\}$$

and disjoint from the set

$$X_1 = \{p^{-1} \cdot q; p(x_0) \neq q(x_0)\}$$

and the tuple $(U, \tilde{p}; p \in A)$ where $T \subset U$ and for all $p \in A$, $\tilde{p}$ is a permutation of $U$ extending $p$. Let $H_0$ be the subgroup of $F(P)$ generated by $X_0$. In fact we have $H_0 = \{p_1 p_2 \cdots p_n; p_1 \circ p_2 \circ \cdots \circ p_n(x_0) = x_0\}$. We show that $H_0$ is closed for the odd-adic topology, by showing, by (3), that if $g^2 \in H_0$, then $g \in H_0$.

Say that $g = p_1 \circ p_2 \cdots \circ p_n$, where $n$ is an integer and the $p_i$ are elements of $P \cup P^{-1}$ and this representation is reduced. It is easy to construct a finite tournament $T_1$ extending $T$ and, for each $p \in P$ a partial automorphism $p'$ of $T_1$ with the following properties: 1) $p'$ extends $p$; 2) $p_1' \circ p_2' \circ \cdots \circ p_n'(x_0)$ is defined, and if $p_1 \circ p_2 \circ \cdots \circ p_n(x_0)$ is not defined, then $p_1' \circ p_2' \circ \cdots \circ p_n'(x_0) \neq x_0$. We may now embed $T_1$ into the generic tournament $T_0$ and extend the maps $p'$ (for $p \in P$) into automorphisms $\tilde{p}$ of $T_0$. As usual, this will provide us with a homomorphism $f \mapsto \tilde{f}$ from $F(P)$ into $\text{Aut}(T_0)$. We see that $\tilde{g}(x_0) = p_1' \circ p_2' \circ \cdots \circ p_n'(x_0)$, and thanks to our hypothesis on $T_1$, if $g \notin H_0$, then $\tilde{g}(x_0) \neq x_0$. Since $g^2 \in H_0$, we see that $\tilde{g}^2(x_0) = \tilde{g}^2(x_0) = x_0$, and since $\tilde{g}$ cannot switch 2 different points, $\tilde{g}(x_0) = x_0$, and $g \in H_0$.

Now, we apply our technique: we find a finite set $\Gamma_1$ extending $T$ and a homomorphism $f \mapsto \tilde{f}$ from $F(P)$ into $\text{Sym}(\Gamma_1)$ such that, for all $p \in A$, $\tilde{p}$ extends $p$. From the fact that $H_0$ is closed for the odd-adic topology, we may demand that the kernel of this homomorphism is of odd index. Thus, the subgroup $G$ of $\text{Sym}(\Gamma_1)$ generated by $\{p; p \in P\}$ is of odd order. We can suppose that $G$ acts transitively on $\Gamma_1$ and as usual we define for $\alpha, \beta \in \Gamma_1$: there is an arrow from $\alpha$ to $\beta$ if there exists $a, b \in \Gamma$ and $g \in F(P)$ such that there is an arrow from $a$ to $b$ and $\tilde{g}(a) = \alpha$ and $\tilde{g}(b) = \beta$. As $G$ is odd this defines the structure of a directed graph on $\Gamma_1$ and $\Gamma$ is a substructure of $\Gamma_1$. It just remains to add arrows in $\Gamma_1$ to turn it into a tournament, in such a way that the $\tilde{p}$ for $p \in P$ remain automorphisms. We do that exactly as above, when we have defined a tournament structure on the set $F(P)/H$.

\section*{References}

PARTIAL AUTOMORPHISMS


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