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COHOMOLOGY OF UNIFORMLY POWERFUL p-GROUPS

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Abstract. In this paper we will study the cohomology of a family of p-groups associated to $\mathbb{F}_p$-Lie algebras. More precisely, we study a category **BGrp** of p-groups which will be equivalent to the category of $\mathbb{F}_p$-bracket algebras (Lie algebras minus the Jacobi identity). We then show that for a group G in this category, its $\mathbb{F}_p$-cohomology is that of an elementary abelian p-group if and only if it is associated to a Lie algebra.

We then proceed to study the exponent of $H^*(G;\mathbb{Z})$ in the case that G is associated to a Lie algebra $\mathfrak{L}$. To do this, we use the Bockstein spectral sequence and derive a formula that gives B_2^* in terms of the Lie algebra cohomologies of $\mathfrak{L}$. We then expand some of these results to a wider category of p-groups. In particular, we calculate the cohomology of the p-groups $\Gamma_{n,k}$ which are defined to be the kernel of the mod p reduction $GL_n(\mathbb{Z}/p^{k+1}\mathbb{Z}) \overset{mod}{\longrightarrow} GL_n(\mathbb{F}_p)$.

1. Introduction and Motivation

Throughout this paper, p will be an odd prime. First some definitions:

Definition 1.1. Given a p-group G, $\Omega_1(G) = \langle g \in G : g^p = 1\rangle$ where the brackets mean "smallest subgroup generated by". G is called p-central if $\Omega_1(G)$ is central.

Definition 1.2. For any positive integer k, $G^{p^k} = \langle g^{p^k} : g \in G\rangle$.

Definition 1.3. $Frat(G) = G^p[G,G]$.

We will study a category **BGrp** of p-groups which is naturally equivalent to the category of bracket algebras over $\mathbb{F}_p$. This is exactly the category of p-central, p-groups G which have $\Omega_1(G) = G^p = Frat(G)$. For such a group G, the associated bracket algebra will be called $Log(G)$.

In the case that one of these groups is associated to a Lie algebra, we will show that it has the same $\mathbb{F}_p$-cohomology as an elementary abelian p-group. (The necessary definitions for the theorems quoted in this introductory section can be found in the relevant parts of the paper.) More precisely we will show:

Theorem 1 (2.10)**.** *Let $G \in \mathrm{Obj}(\mathbf{BGrp})$ and $n = \dim(\Omega_1(G))$. Then*

$$H^*(G;\mathbb{F}_p) = \wedge(x_1,\ldots,x_n) \otimes \mathbb{F}_p[s_1,\ldots,s_n]$$

(where the x_i have degree 1 and the s_i have degree 2) if and only if $Log(G)$ is a Lie algebra. When this is the case, the polynomial algebra part restricts isomorphically to that of $H^*(\Omega_1(G);\mathbb{F}_p)$ and the exterior algebra part is induced isomorphically from that of $H^*(G/\Omega_1(G);\mathbb{F}_p)$ via the projection homomorphism.

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(This theorem was proven independently by T. Weigel for the case $p \neq 3$ in [W1], [W2].)

However, the integral cohomology of these groups is complicated and, indeed, we will show that one can recover the Lie algebra associated to the group from knowledge of the Bockstein on its $\mathbb{F}_p$-cohomology. To do this we calculate the full structure of the $\mathbb{F}_p$-cohomology as a Steenrod-module. As an application, in corollary 2.26, we completely determine the comodule algebra structure (see [W2]) of $H^*(G; \mathbb{F}_p)$.

With the mild additional hypothesis that the associated Lie algebra lifts to one over $\mathbb{Z}/p^2\mathbb{Z}$, we compute B_2^* of the Bockstein spectral sequence in terms of the Lie algebra cohomologies of the corresponding Lie algebra:

Theorem 2 (2.36). *Let $G \in \mathrm{Obj}(\mathbf{BGrp})$ with $Log(G) = \mathfrak{L}$, a Lie algebra, and suppose that $\mathfrak{L}$ lifts to a Lie algebra over $\mathbb{Z}/p^2\mathbb{Z}$. Then B_2^* of the Bockstein spectral sequence for G is given by*

$$B_2^* = \bigoplus_{k=0}^{\infty} H^{*-2k}(\mathfrak{L}; S^k).$$

Here S^k is the $\mathfrak{L}$-module of symmetric k-forms on $\mathfrak{L}$ with the usual action. (This action is described in detail before the proof of the theorem.)

The integral cohomology of p-groups is a rich subject and there are many theorems and conjectures about the exponent of $\bar{H}^*(P; \mathbb{Z})$. (Where the bar denotes reduced cohomology and we recall that the exponent of an abelian group G is the smallest positive integer n such that $ng = 0$ for all $g \in G$.) We will obtain some partial results on the exponent of the integral cohomology of the p-groups studied in this paper via the Bockstein spectral sequence.

In the last section of this paper, we extend the results mentioned to a bigger family of groups: the uniform, p-central, p-groups. (Elementary abelian p-groups are uniform and, more generally, one defines inductively that a p-central, p-group G is uniform if and only if $\Omega_1(G) = G^{p^k}$ for some nonnegative integer k and $G/\Omega_1(G)$ is itself uniform. Thus a uniform p-group will give rise to a tower of uniform p-groups called a uniform tower where each group in the tower is the quotient of the previous group G by $\Omega_1(G)$. In this paper such uniform towers will be indexed so that G_1 is always an elementary abelian p-group. When this is done, G_2 will always correspond to a group in the category $\mathbf{BGrp}$.)

Specifically we prove:

Theorem 3 (3.14). *Fix $p \geq 5$. Let*

$$G_{k+1} \to G_k \to \cdots \to G_2 \to G_1 \to 1$$

be a uniform tower with $k \geq 2$. Let $\mathfrak{L} = Log(G_2)$ and let c_{ij}^k be the structure constants of $\mathfrak{L}$ with respect to some basis. Then for suitable choices of degree 1 elements $x_1, \ldots, x_n$ and degree 2 elements $s_1, \ldots, s_n$, one has

$$H^*(G_k; \mathbb{F}_p) \cong \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$$

with

$$\beta(x_t) = -\sum_{i<j}^{n} c_{ij}^t x_i x_j,$$

$$\beta(s_t) = \sum_{i,j=1}^{n} c_{ij}^t s_i x_j + \eta_t$$

for all $t = 1, \dots, n$. *Furthermore, the* η_t *define a cohomology class* $[\eta] \in H^3(\mathfrak{L}; ad)$ *which vanishes if and only if there exists a uniform tower*

$$G'_{k+2} \to G'_{k+1} \to G_k \to \cdots \to G_2 \to G_1 \to 1$$

or in other words if and only if $Log(G_k)$ *(which is a Lie algebra over* $\mathbb{Z}/p^{k-1}\mathbb{Z}$*), has a lift to a Lie algebra over* $\mathbb{Z}/p^k\mathbb{Z}$*. Thus in this case where the Lie algebra lifts, one can drop the* η_t *term in the formula for the Bockstein.*

A version of this theorem for the category of powerful, p-central, p-groups is also stated in theorem 3.16.

Some examples of p-groups covered in this paper are the groups $\Gamma_{n,k}$ for $n, k \geq 1$, where the group $\Gamma_{n,k}$ is defined as the kernel of the map

$$GL_n(\mathbb{Z}/p^{k+1}\mathbb{Z}) \xrightarrow{mod} GL_n(\mathbb{F}_p).$$

2. Bracket groups

2.1. **Some preliminaries.** First we will recall a method of obtaining a bracket algebra from a p-group under suitable conditions.

Let us consider the following situation. Suppose we have a central short exact sequence of finite groups:

$$1 \to V \to G \xrightarrow{\pi} W \to 1$$

where W and V are elementary abelian p-groups, then we define

$$\langle \cdot, \cdot \rangle : W \times W \to V \text{ given by } \langle x, y \rangle = \hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1} \tag{1}$$

where $\hat{x}, \hat{y}$ are lifts of x, y to G, i.e., $\pi(\hat{x}) = x, \pi(\hat{y}) = y$.

We also define a p-power map function

$$\phi : W \to V \text{ given by } \phi(x) = \hat{x}^p. \tag{2}$$

It is routine to verify that $\langle \cdot, \cdot \rangle$ is a well-defined alternating, bilinear map and that ϕ is a well-defined linear map. (The linearity of ϕ uses that p is odd.)

We will now restrict ourselves to the case where the p-power map ϕ is an isomorphism. So V and W are isomorphic. In this case one can define

$$[\cdot, \cdot] : W \otimes W \to W \text{ by } [w_1, w_2] = \frac{1}{2}\phi^{-1}(\langle w_1, w_2 \rangle). \tag{3}$$

Then it is easy to see this is still an alternating, bilinear map on W. Note W is a vector space over $\mathbb{F}_p$, the field on p elements. (The factor of $\frac{1}{2}$ is put in for convenience in order to avoid messy expressions later on.)

Definition 2.1. A bracket algebra over $\mathbb{F}_p$ is a finite dimensional vector space W over $\mathbb{F}_p$ equipped with an alternating, bilinear form $[\cdot, \cdot] : W \otimes W \to W$. The dimension of a bracket algebra is the dimension of the underlying vector space.

Definition 2.2. **Brak** is the category whose objects consist of bracket algebras over $\mathbb{F}_p$ and, with morphisms, the linear maps which preserve the brackets.

Now we can define a category of p-groups which is essentially equivalent to the category **Brak**. These groups will be our primary objects of study in the first part of this paper.

Definition 2.3. A p-power exact extension is a central short exact sequence:

$$1 \to V \to G \xrightarrow{\pi} W \to 1$$

where V, W are elementary abelian and where the p-power map ϕ is an isomorphism.

Note in such an extension, $V = \Omega_1(G)$ and $W = G/\Omega_1(G)$ (see definition 1.1). So any homomorphism between two groups G_1, G_2 in the middle of p-power exact extensions takes $V_1 = \Omega_1(G_1)$ into $V_2 = \Omega_1(G_2)$ and hence induces maps $V_1 \to V_2$ and $W_1 \to W_2$ where $W_i = G_i/\Omega_1(G_i)$. These will be called, respectively, the induced maps on the $V, (W)$ level.

Definition 2.4. **BGrp** is the category whose objects consist of finite groups G which fit in the middle of a p-power exact extension. These will be called bracket groups. The morphisms in **BGrp** are just the usual group morphisms between these groups, except we will identify two morphisms if they induce the same map on the V and W levels.

One can show that the two categories **Brak** and **BGrp** are naturally equivalent (see [Pak]). This follows from a general exponent-log correspondence (see [W1], [W3] or [Laz]), suitably reworded for our purposes. We will give a short description of the functors involved in this equivalence of categories.

Define a covariant functor $Log : \mathbf{BGrp} \to \mathbf{Brak}$ as follows, to a bracket group G we associate the bracket algebra $Log(G) = (W, [\cdot,\cdot])$ which is obtained as explained before. Notice that the underlying vector space of $Log(G)$ is just $G/\Omega_1(G)$; so given $\psi \in \mathrm{Mor}(G_1, G_2)$ we note ψ induces a well-defined linear map $Log(\psi) : Log(G_1) \to Log(G_2)$, and it is easy to check that this is a map of bracket algebras.

A description of the inverse functor $Exp : \mathbf{Brak} \to \mathbf{BGrp}$ is given as follows. Given $(L, [\cdot,\cdot])$ a bracket algebra, let K be a free $\mathbb{Z}/p^2\mathbb{Z}$-module of rank equal to the dimension of L. Let $\phi : L \to K$ and $\pi : K \to L$ be injective/surjective maps such that $\phi \circ \pi = p$ (multiplication by p). Then $Exp(L) = (K, \circ)$ where

$$l \circ m = l + m + \phi[\pi(l), \pi(m)]. \tag{4}$$

(Here $+$ is the addition of K as a $\mathbb{Z}/p^2\mathbb{Z}$-module. Notice in this notation, 0 is the identity of $Exp(L)$, l^{-1} corresponds to $-l$, l^p corresponds to pl, and the p-power map is indeed ϕ.)

For clarity we state the following proposition which summarizes these facts:

Proposition 2.5. *The functors $Log : \mathbf{BGrp} \to \mathbf{Brak}$ and $Exp : \mathbf{Brak} \to \mathbf{BGrp}$ give a natural equivalence between the categories* **BGrp** *and* **Brak**. *Thus to every bracket algebra there naturally corresponds a unique bracket group.*

Example. A direct computation shows that the group $\Gamma_{n,2}$ which is the kernel of the map

$$GL_n(\mathbb{Z}/p^3\mathbb{Z}) \xrightarrow{mod} GL_n(\mathbb{F}_p),$$

is a bracket group, and that $Log(\Gamma_{n,2}) = \mathfrak{gl}_n$, the Lie algebra of $n \times n$ matrices.

Now we will study the groups in **BGrp** from a cohomological viewpoint. Recall that if $G \in \mathrm{Obj}(\mathbf{BGrp})$, then G fits in a central short exact sequence

$$1 \to V \to G \xrightarrow{\pi} W \to 1$$

where we can identify V and W via the p-power isomorphism ϕ. If we do not put the restriction that ϕ is an isomorphism, then such extensions as above are in one-to-one correspondence with $H^2(W;V)$. Now for W an elementary abelian p-group of rank n (recall p is odd), we have that

$$H^*(W;\mathbb{F}_p) \cong \wedge^*(x_1,\ldots,x_n) \otimes \mathbb{F}_p[\beta x_1,\ldots,\beta x_n],$$

the tensor product of an exterior algebra on degree 1 generators and a polynomial algebra on degree 2 generators which are the Bocksteins of the degree 1 generators. (Here $\beta : H^1(W;\mathbb{F}_p) \to H^2(W;\mathbb{F}_p)$ is the Bockstein.) In basis free notation, this can be written:

$$H^*(W;\mathbb{F}_p) \cong \wedge^*(W^*) \otimes \mathbb{F}_p[\beta(W^*)]$$

where $W^* = H^1(W;\mathbb{F}_p)$ is the dual vector space of W and $\beta(W^*)$ is its image under the Bockstein β. Now,

$$\begin{aligned} H^2(W;V) &\cong H^2(W;\mathbb{F}_p) \otimes V \text{ by the Universal Coefficient Theorem} \\ &\cong (H^2(W;\mathbb{F}_p))^n \text{ using a choice of basis of } V. \end{aligned} \tag{5}$$

Note that a bracket on W gives us a map $br : \wedge^2(W) \to W$. If we take the dual of this map, we get a map $br^* : W^* \to \wedge^2(W^*)$ (once we identify the dual of $\wedge^2(W)$ with $\wedge^2(W^*)$ in the usual way). With this notation, it is easy to argue by comparisons (see [BC]) and from the description of the Exp functor in (4) that the extension elements are of the form $\beta(x_i) + br^*(x_i)$ for $i = 1,\ldots,n$. (If we use a suitable basis for V in (5), one chooses a basis E for W, the basis $\phi(E)$, for V and the canonical dual basis for V^* and W^* so, for example, $\{x_1,\ldots,x_n\}$ are dual to E.) In this formulation $br^*(x_i)$ are just the components of the bracket of $Log(G)$, with respect to the basis E. We will use this form of the extension element from now on.

2.2. **Lie algebras and co-Lie algebras.**

Definition 2.6. Given a bracket algebra $(W,[\cdot,\cdot])$, we define the Jacobi form of $(W,[\cdot,\cdot])$ to be

$$J(x,y,z) = ([[x,y],z] + [[y,z],x] + [[z,x],y]).$$

This is easily checked to be an alternating 3-form. Thus $J : \wedge^3(W) \to W$. Bracket algebras with $J = 0$ are called Lie algebras.

Given a bracket algebra, $(W,[\cdot,\cdot])$, the bracket defines a map $br : \wedge^2(W) \to W$. Thus by taking duals and identifying the dual of $\wedge^*(W)$ with $\wedge^*(W^*)$ in the usual way, we obtain a map $br^* : W^* \to \wedge^2(W^*)$, where W^* is the dual vector space of W. This motivates the following definition:

Definition 2.7. A co-bracket algebra is a vector space W^* equipped with a linear map $br^* : W^* \to \wedge^2(W^*)$. We will always extend br^* as a degree 1 map on the whole of the graded algebra $\wedge^*(W^*)$ in the unique way that makes it a derivation on that algebra. We call such a co-bracket algebra a co-Lie algebra if $br^* \circ br^* = 0$.

As mentioned before, the dual of a bracket algebra has a natural co-bracket algebra structure and similarly, the dual of a co-bracket algebra has a natural bracket algebra structure.

Given a co-bracket algebra W^*, it is easy to check, as br^* is a derivation, that $br^* \circ br^* = 0$ if and only if $br^* \circ br^* : W^* = \wedge^1(W^*) \to \wedge^3(W^*)$ is zero. The dual of this map is a map from $\wedge^3(W) \to W$ (where we are calling $W^{**} = W$). It is easy to check that this is just the Jacobi form defined before for the bracket algebra W. Thus we can conclude that the dual of a co-Lie algebra is a Lie algebra and vice versa.

Let $\pi : \wedge^2(W) \to W \otimes W$ be the map defined by $\pi(a \wedge b) = a \otimes b - b \otimes a$. Thus the usual definition of a module over a bracket (Lie) algebra can be worded in the following more categorical way:

Definition 2.8. We say V is a module over the bracket algebra (W, br) if we have a map $\lambda : W \otimes V \to V$ such that the following diagram commutes:

$$\begin{array}{ccc} W \wedge W \otimes V & \xrightarrow[br\otimes 1]{} & W \otimes V \\ {\scriptstyle (1\otimes\lambda)\circ(\pi\otimes 1)}\downarrow & & \downarrow{\scriptstyle \lambda} \\ W \otimes V & \xrightarrow{\lambda} & V \end{array}$$

Taking the dual of this definition, and noting that the dual of π, $\pi^* : W^* \otimes W^* \to \wedge^2(W^*)$ is just the canonical quotient map, gives us the following definition:

Definition 2.9. We say V^* is a comodule over the co-bracket algebra (W^*, br^*) if we have a map $\lambda^* : V^* \to W^* \otimes V^*$ such that the following diagram commutes:

$$\begin{array}{ccc} W^* \wedge W^* \otimes V^* & \xleftarrow[br^*\otimes 1]{} & W^* \otimes V^* \\ {\scriptstyle 1\wedge\lambda^*}\uparrow & & \uparrow{\scriptstyle \lambda^*} \\ W^* \otimes V^* & \xleftarrow{\lambda^*} & V^* \end{array}$$

It is easy to see that the dual of a comodule over the co-bracket algebra W^* is a module over the bracket algebra $W = W^{**}$ and vice versa.

2.3. **The $\mathbb{F}_p$-cohomology of a group in BGrp.** In this section we will show that a group in **BGrp** has the $\mathbb{F}_p$-cohomology of an elementary abelian p-group if and only if it is associated to a Lie algebra. More precisely, we will prove:

Theorem 2.10. *Let $G \in \mathrm{Obj}(\mathbf{BGrp})$ and $n = \dim(\Omega_1(G))$. Then*

$$H^*(G; \mathbb{F}_p) = \wedge(x_1, \dots, x_n) \otimes \mathbb{F}_p[s_1, \dots, s_n]$$

(where the x_i have degree 1 and the s_i have degree 2) if and only if $Log(G)$ is a Lie algebra. When this is the case, the polynomial algebra part restricts isomorphically to that of $H^*(\Omega_1(G); \mathbb{F}_p)$ and the exterior algebra part is induced isomorphically from that of $H^*(G/\Omega_1(G); \mathbb{F}_p)$ via the projection homomorphism.

Fix $G \in \mathrm{Obj}(\mathbf{BGrp})$, we now study the Lyndon-Hochschild-Serre (L.H.S.) spectral sequence (with $\mathbb{F}_p$-coefficients) associated to the extension

$$1 \to \Omega_1(G) = V \xrightarrow{i} G \xrightarrow{\pi} W \to 1.$$

Recall that this is a spectral sequence with

$$E_2^{p,q} = H^p(W, H^q(V, \mathbb{F}_p))$$

and it abuts to $H^*(G;\mathbb{F}_p)$. Since this is a central extension, if we use $\mathbb{F}_p$ coefficients throughout, then

$$E_2^{p,q} = H^p(W, H^q(V,\mathbb{F}_p)) = H^q(V,\mathbb{F}_p)\otimes H^p(W,\mathbb{F}_p).$$

So using that p is odd, one gets explicitly:

$$E_2^{*,*} \cong \wedge^*(V^*)\otimes \mathbb{F}_p[\beta(V^*)]\otimes \wedge^*(W^*)\otimes \mathbb{F}_p[\beta(W^*)]. \tag{6}$$

As before, we are using basis free notation, for example $V^* = H^1(V;\mathbb{F}_p)$ and β is the Bockstein.

The dual of the p-power map ϕ gives us an isomorphism $\phi^* : V^* \to W^*$; so given a basis $E = \{e_1,\dots e_n\}$ for V^*, one can use $\phi^*(E) = \{x_1,\dots,x_n\}$ as a basis for W.

By standard comparisons it is easy to see that $d_2|_{W^*} = d_2|_{\beta(W^*)} = d_2|_{\beta(V^*)} = 0$ while

$$d_2(e_i) = (\text{the } i\text{th component of the extension element}) = \beta(x_i) + br^*(x_i)$$

for all i. Here we are using the form of the extension element as presented in the end of section 2.1. As $\phi^*(e_i) = x_i$, one has

$$d_2|_{V^*} = \beta\circ\phi^* + br^*\circ\phi^*.$$

Let us calculate $E_3^{*,*}$.

Let

$$A = \wedge^*(X_1,\dots,X_n)\otimes\mathbb{F}_p[Y_1,\dots,Y_n]\otimes\wedge^*(T_1,\dots,T_n)\otimes\mathbb{F}_p[S_1,\dots,S_n]$$

be an abstract free graded-commutative algebra where the polynomial generators are degree 2 and the exterior generators are degree 1. Since this is a free object, the assignment $X_i \to x_i, T_i \to e_i, S_i \to \beta(e_i), Y_i \to \beta(x_i) + br^*(x_i)$ defines a map Ψ of graded-algebras from A to $E_2^{*,*}$. As $\beta(x_i)+br^*(x_i)$ differs form $\beta(x_i)$ by a nilpotent element, it is easy to see by induction on the grading that Ψ is an isomorphism. The induced differential D_2 on A of d_2 under this isomorphism has $D_2(X_i) = D_2(Y_i) = D_2(S_i) = 0$ and $D_2(T_i) = Y_i$. Applying Künneth's Theorem, we then see that the cohomology of (A, D_2) is isomorphic to $\wedge^*(X_1,\dots,X_n)\otimes\mathbb{F}_p[S_1,\dots,S_n]$. Using this, we see that $E_3^{*,*}$ is given by

$$E_3^{*,*} \cong \wedge^*(W^*)\otimes\mathbb{F}_p[\beta(V^*)]. \tag{7}$$

We have obviously that $d_3|_{W^*} = 0$ and we also have (see for example page 155 of [Be])

$$\begin{aligned} d_3\circ\beta|_{V^*} &= \{\beta\circ d_2|_{V^*}\} \in E_3^{3,0} = \wedge^3(W^*)\\ &= \{\beta\circ(\beta\circ\phi^* + br^*\circ\phi^*)|_{V^*}\}\\ &= \{\beta\circ br^*\circ\phi^*|_{V^*}\} \text{ since } \beta\circ\beta = 0.\end{aligned} \tag{8}$$

Now, note that β is a derivation and that $\beta(x_i)$ is identified with $-br^*(x_i)$ in $E_3^{*,0} = \wedge^*(x_1,\dots,x_n)$ hence β induces the same map as $-br^*$ on $E_3^{*,0}$. Thus we see that the final expression in (8) becomes $-br^*\circ br^*\circ\phi^*|_{V^*}$. Since ϕ^* is an isomorphism, we can conclude that $d_3|_{\beta(V^*)} = 0$ and hence $d_3 = 0$ if and only if $br^*\circ br^* = 0$ on $W^* = \wedge^1(W^*)$ or in other words if and only if (W^*, br^*) is a co-Lie algebra. Hence we conclude:

Lemma 2.11. *$d_3 = 0$ if and only if $Log(G) = (W, br)$ is a Lie algebra.*

When this happens $E_3 = E_4$ but then for grading reasons, $d_r|_{\beta(V^*)} = d_r|_{W^*} = 0$ for all $r \geq 4$. Hence as the differentials are derivations which vanish on the generators, all further differentials in the spectral sequence must be 0. We have proven

Proposition 2.12. *$E_3 = E_\infty$ if and only if $Log(G)$ is a Lie algebra. In this case,*

$$E_3^{*,*} = E_\infty^{*,*} = \wedge^*(W^*) \otimes \mathbb{F}_p[\beta(V^*)].$$

Due to the free nature of this graded ring, one can then show, in a routine manner, that

$$H^*(G;\mathbb{F}_p) = \wedge^*(W^*) \otimes \mathbb{F}_p[S^*]$$

where we have abused notation a bit and identified W^* with its image under the map $\pi^* : H^*(W;\mathbb{F}_p) \to H^*(G;\mathbb{F}_p)$. S^* is a subspace which restricts isomorphically to the subspace $\beta(V^*)$ of $H^*(V;\mathbb{F}_p)$. Hence we have proved theorem 2.10.

Let us define **LGrp** to be the full subcategory of **BGrp** whose objects are the bracket groups G where the associated $Log(G)$ is a Lie algebra. Let **Lie** be the full subcategory of **Brak** whose objects are the Lie algebras. Then restricting the natural functors we had before, we see **LGrp** and **Lie** are naturally equivalent categories.

2.4. **A formula for the Bockstein on $H^*(G;\mathbb{F}_p)$.** From now on we consider groups $G \in \mathrm{Obj}(\mathbf{LGrp})$. By the previous theorem,

$$H^*(G;\mathbb{F}_p) = \wedge^*(W^*) \otimes \mathbb{F}_p[S^*]$$

where $W = G/\Omega_1(G)$.

We wish to study the Bockstein $\beta : H^*(G;\mathbb{F}_p) \to H^{*+1}(G;\mathbb{F}_p)$. The reason for this is that one can determine the groups $H^*(G;\mathbb{Z})$ from knowledge of β and the "higher Bocksteins" on $H^*(G;\mathbb{F}_p)$.

Recall that β is a derivation, i.e., for homogeneous elements $u, v \in H^*(G;\mathbb{F}_p)$ we have

$$\beta(uv) = \beta(u)v + (-1)^{\deg(u)}u\beta(v).$$

Also recall that $\beta \circ \beta = 0$. Since β is a derivation, we need only describe it on the generating subspaces W^* and S^* of $H^*(G;\mathbb{F}_p)$.

Now recall the central short exact sequence

$$1 \to \Omega_1(G) = V \xrightarrow{i} G \xrightarrow{\pi} W \to 1.$$

Then we have seen that the subalgebra $\wedge^*(W^*)$ of $H^*(G;\mathbb{F}_p)$ is the image of the map $\pi^* : H^*(W;\mathbb{F}_p) \to H^*(G;\mathbb{F}_p)$. In the L.H.S.-spectral sequence of the last section, we can identify this image with $E_\infty^{*,0} = E_3^{*,0}$. We saw there that β agrees with $-br^*$ on this subalgebra.

Thus $\beta|_{W^*} = -br^*$. It is easy to check that the differential complex $(\wedge^*(W^*), \beta)$ is the standard Koszul resolution used to calculate the Lie algebra cohomology $H^*(\mathfrak{L};\mathbb{F}_p)$, where $\mathfrak{L} = (W, br)$. Also, we see that one can obtain the Lie algebra structure of $\mathfrak{L}$ from knowledge of β on the exterior algebra part of $H^*(G;\mathbb{F}_p)$.

Now we are left with finding $\beta|_{S^*}$ which is harder. Let us develop some more notation. Purely, from considering degree restrictions one has:

$$\beta|_{S^*} : S^* \to (W^* \otimes S^*) \oplus \wedge^3(W^*).$$

Thus we can write

$$\beta|_{S^*} = \beta_1 + \beta_2 \tag{9}$$

where $\beta_1 : S^* \to W^* \otimes S^*$ and $\beta_2 : S^* \to \wedge^3(W^*)$. If we let $j = \beta^{-1} \circ i^* : S^* \to V^*$ be the natural isomorphism, we can define $\hat{\beta}_1 = (1 \otimes j) \circ \beta_1 \circ j^{-1} : V^* \to (W^* \otimes V^*)$ and $\hat{\beta}_2 = \beta_2 \circ j^{-1} : V^* \to \wedge^3(W^*)$.

Before we go on, we should note an inherent ambiguity. Note $\hat{\beta}_1$ and $\hat{\beta}_2$ determine β; however, their definition requires a choice for the subspace S^*. We would like that all our expressions involving β be determined from knowledge of the underlying Lie algebra $\mathfrak{L} = Log(G)$. Note $W = G/\Omega_1(G)$ can be identified with the underlying vector space of $\mathfrak{L}$ and hence so can $V = \Omega_1(G)$ via the p-power isomorphism ϕ. Thus the subalgebra $\wedge^*(W^*)$ of $H^*(G;\mathbb{F}_p)$ is unambiguously determined by $\mathfrak{L}$. However, the subspace S^*, which generates the polynomial part of $H^*(G;\mathbb{F}_p)$, is only determined by the fact that it restricts isomorphically to the natural subspace $\beta(V^*)$ of $H^*(V;\mathbb{F}_p)$. We can see that given a homomorphism $\mu : S^* \to \wedge^2(W^*)$, then the image of S^* under $\eta = Id+\mu$ works just as well, and that all possible other choices for this generating subspace arise in this way. Let us see how changing ones choices effects the decomposition of β in (9).

It is easiest to see this by using a basis $\{\bar{s}_1, \dots \bar{s}_n\}$ for the space S^*. If we put this basis in a column vector

$$\mathbf{s} = \begin{pmatrix} \bar{s}_1 \\ \vdots \\ \bar{s}_n \end{pmatrix},$$

then we have that

$$\beta(\mathbf{s}) = \xi\mathbf{s} + \beta_2(\mathbf{s})$$

where ξ is an $n \times n$ matrix with entries in $\wedge^1(W^*)$.

If we change our choice of subspace S^* using the homomorphism μ mentioned above, we get a new subspace $\bar{S}^* = \eta(S^*)$ with basis $\mathbf{s}' = \mathbf{s} + \mu(\mathbf{s})$. We then calculate:

$$\beta(\mathbf{s}') = \beta(\mathbf{s} + \mu(\mathbf{s})) = \xi\mathbf{s} + \beta_2(\mathbf{s}) + \beta \circ \mu(\mathbf{s}) = \xi\mathbf{s}' + (\beta_2(\mathbf{s}) + (-br^* - \xi)(\mu(\mathbf{s}))).$$

(Here we have used again that $\beta = -br^*$ on $\wedge^*(W^*)$.) So if we decompose $\beta = \beta_1' + \beta_2'$ using this new subspace $\bar{S}^*$ and note that $(1 \wedge \mu) \circ \beta_1(\mathbf{s}) = \xi\mu(\mathbf{s})$, we see that

$$(1 \otimes \eta^{-1}) \circ \beta_1' \circ \eta = \beta_1 \text{ and } \beta_2' \circ \eta = \beta_2 + (-br^* \circ \mu - (1 \wedge \mu) \circ \beta_1) \tag{10}$$

or

$$\hat{\beta}_1' = \hat{\beta}_1 \text{ and } \hat{\beta}_2' = \hat{\beta}_2 + (-br^* \circ \mu - (1 \wedge \mu) \circ \hat{\beta}_1). \tag{11}$$

(Here we have made the natural identification of μ as a map $\mu : V^* \to \wedge^2(W^*)$.) Thus $\hat{\beta}_1$ is not ambiguous while $\hat{\beta}_2$ depends on the particular subspace S^* chosen, as indicated in (11). We will return to this later.

Let us now concentrate on $\hat{\beta}_1$, which is well defined, given the Lie algebra $\mathfrak{L}$. Since $\beta \circ \beta = 0$ we have

$$0 = \beta \circ \beta|_{S^*} = \beta \circ \beta_1 + \beta \circ \beta_2. \tag{12}$$

However, since β is a derivation and $\beta = -br^*$ on $\wedge^*(W^*)$, it follows easily from (12) that

$$0 = (-br^* \otimes 1) \circ \beta_1 - (1 \otimes \beta_1) \circ \beta_1 - (1 \otimes \beta_2) \circ \beta_1 - br^* \circ \beta_2. \tag{13}$$

Taking components in the usual way, we get the equations:

$$(-br^* \otimes 1) \circ \beta_1 = (1 \otimes \beta_1) \circ \beta_1 \tag{14}$$

and

$$(1 \otimes \beta_2) \circ \beta_1 = -br^* \circ \beta_2. \tag{15}$$

Equation (14) gives us the following commutative diagram:

$$\begin{array}{ccc} V^* & \xrightarrow{-\hat{\beta}_1} & W^* \otimes V^* \\ {\scriptstyle -\hat{\beta}_1}\downarrow & & \downarrow{\scriptstyle 1\otimes -\hat{\beta}_1} \\ W^* \otimes V^* & \xrightarrow{br^* \otimes 1} & \wedge^2(W^*) \otimes V^* \end{array}$$

Thus, $-\hat{\beta}_1$ equips V^* with the structure of a comodule over the co-Lie algebra (W^*, br^*). Thus, taking duals, we see that $-\hat{\beta}_1^*$ equips V with the structure of a module over the Lie algebra (W, br). However, we can naturally identify V with W using the p-power map, thus for every Lie algebra $\mathfrak{L} = (W, br)$ we obtain a map $\lambda : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ from the module structure above. Furthermore, it follows, that if we give $\mathfrak{gl}(\mathfrak{L})$ the canonical Lie algebra structure, then this map λ will be a map of Lie algebras.

These maps λ obtained above are natural in the following sense:

Claim 2.13. Let $\mathfrak{L}, \mathfrak{L}' \in \mathrm{Obj}(\mathbf{Lie})$ and $\psi \in Mor(\mathfrak{L}, \mathfrak{L}')$, then

$$[\lambda'(\psi(x))](\psi(y)) = \psi([\lambda(x)](y)).$$

(The notation $[\lambda(x)](y)$ means, of course, $\lambda(x) \in \mathfrak{gl}(\mathfrak{L})$ applied to the element y.)

Proof. Suppose $\mathfrak{L} = (W, br)$ and $\mathfrak{L}' = (\bar{W}, \bar{br})$. Then $\psi : W \to \bar{W}$ induces a unique morphism $Exp(\psi) : Exp(\mathfrak{L}) \to Exp(\mathfrak{L}')$ in the category $\mathbf{LGrp}$. This is a class of group homomorphisms where any two representatives of this class induce the same mapping on the Ω_1 level as mentioned earlier. It is easy to check that any representative of this class will induce the dual map ψ^* from

$$\bar{W}^* = H^1(Exp(\mathfrak{L}'); \mathbb{F}_p) \to W^* = H^1(Exp(\mathfrak{L}); \mathbb{F}_p).$$

Since the Bockstein is a natural operation, it will commute with any such map on cohomology and from this, one obtains the following commutative diagram:

$$\begin{array}{ccc} \bar{V}^* & \xrightarrow{-\hat{\bar{\beta}}_1} & \bar{W}^* \otimes \bar{V}^* \\ {\scriptstyle \psi^*}\downarrow & & \downarrow{\scriptstyle \psi^* \otimes \psi^*} \\ V^* & \xrightarrow{-\hat{\beta}_1} & W^* \otimes V^* \end{array}$$

The claim follows easily now by taking the dual of the diagram above and recalling that the map λ was defined via $-\hat{\beta}_1^*$. □

Notice that knowledge of the map λ is equivalent to knowledge of the component $\hat{\beta}_1$ of the Bockstein. We have isolated enough properties of the maps λ and will now set out to determine them explicitly.

2.5. **Self-representations.** In this section, let $\mathbf{k}$ be a finite field with $\mathrm{char}(\mathbf{k}) \neq 2$. Let **Lie** be the category of finite dimensional Lie algebras over $\mathbf{k}$ with morphisms the Lie algebra maps. (We will of course be interested in the case $k = \mathbb{F}_p$.)

Definition 2.14. A natural self-representation on **Lie** is a collection of linear maps $\kappa_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$, which satisfy the naturality condition

$$[\kappa_{\mathfrak{L}'}(\psi(x))](\psi(y)) = \psi([\kappa_{\mathfrak{L}}(x)](y))$$

for all $\mathfrak{L}, \mathfrak{L}' \in \mathrm{Obj}(\mathbf{Lie})$, $\psi \in \mathrm{Mor}(\mathfrak{L}, \mathfrak{L}')$ and $x, y \in \mathfrak{L}$. One says that a self-representation is strong if the maps $\kappa_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ are maps of Lie algebras.

Examples. (a) If we set $\kappa = 0$, we see easily that we get a strong self-representation which we call the zero representation.

(b) By claim 2.13, we see that the maps λ fit together to give a strong natural self-representation which we will call λ.

(c) If we set $\kappa = ad$ where $ad : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ is given by $ad(x)(y) = [x, y]$ for all $x, y \in \mathfrak{L}$, then it is easy to see that each map ad is a map of Lie algebras. Furthermore, if $\mathfrak{L}'$ is another Lie algebra and $\psi : \mathfrak{L} \to \mathfrak{L}'$ is a morphism of Lie algebras, then we have for $x, y \in \mathfrak{L}$:

$$[ad'(\psi(x))](\psi(y)) = [\psi(x), \psi(y)] = \psi([x, y]) = \psi([ad(x)](y)).$$

So we see this assignment defines a strong natural self-representation which we call the adjoint representation.

Now we will study self-representations so that we can show that λ is the adjoint representation. Note that if κ_0, κ_1 are two self-representations, then if they assign the same map $\mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ for some Lie algebra $\mathfrak{L}$ (we will say they agree on $\mathfrak{L}$), then they assign the same map for all Lie algebras isomorphic to $\mathfrak{L}$. So we will implicitly use this from now on without mention.

Lemma 2.15. *If κ_0, κ_1 are two self-representations, then if they agree on a Lie algebra $\mathfrak{L}$, they agree on all Lie subalgebras of $\mathfrak{L}$.*

Proof. Follows from the naturality of self-representations. □

Recall that $\mathfrak{gl}_n$ is the Lie algebra of $n \times n$ matrices and $\mathfrak{sl}_n$ is the Lie algebra of $n \times n$ matrices with trace zero. Define $\mathfrak{N}$ to be the 3-dimensional Lie algebra with basis $\{x, y, z\}$ and bracket given uniquely by $[x, y] = z, [x, z] = [y, z] = 0$. Note that x, z generate a 2-dimensional abelian subalgebra of $\mathfrak{N}$. Define $\mathfrak{S}$ to be the 2-dimensional Lie algebra with basis $\{x, y\}$ and bracket given uniquely by $[x, y] = x$. Note as $\mathrm{char}(\mathbf{k}) \neq 2$ this Lie algebra embeds into $\mathfrak{sl}_2$ by the Lie algebra map Ψ where

$$\Psi(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\Psi(y) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Lemma 2.16. *Let κ_0, κ_1 be two self-representations, then if κ_0 and κ_1 agree on $\mathfrak{sl}_2$ and $\mathfrak{N}$, then $\kappa_0 = \kappa_1$ as self-representations.*

Proof. Assume κ_1, κ_0 are two self-representations satisfying the assumptions above. By the Ado-Iwasawa theorem, all Lie algebras embed as subalgebras of $\mathfrak{gl}_n$ for some n. So by lemma 2.15, to show $\kappa_1 = \kappa_0$, one need only show that they agree on $\mathfrak{gl}_n$ for all n. A basis for $\mathfrak{gl}_n$ is $\{\delta_{i,j} : i, j = 1, \dots n\}$ where $\delta_{i,j}$ is the matrix with entry one at the (i, j)-position and zero elsewhere. One calculates easily that

$$[\delta_{i,j}, \delta_{l,m}] = \begin{cases} 0 & \text{if } j \neq l, m \neq i, \\ \delta_{i,m} & \text{if } j = l, m \neq i, \\ -\delta_{l,j} & \text{if } j \neq l, m = i, \\ \delta_{i,i} - \delta_{j,j} & \text{if } j = l, m = i. \end{cases}$$

Let $\mathfrak{L}^n_{ij,lm}$ be the Lie subalgebra of $\mathfrak{gl}_n$ generated by $\delta_{i,j}$ and $\delta_{l,m}$ for all $i, j, l, m \in \{1, \dots, n\}$. Suppose we have shown that κ_1 and κ_0 agree on $\mathfrak{L}^n_{ij,lm}$ for all $i, j, l, m \in \{1, \dots, n\}$. Then by an easy linearity argument it follows that κ_1 and κ_0 agree on $\mathfrak{gl}_n$ for all n and hence are equal. So we see that if we can show that κ_1 and κ_0 agree on $\mathfrak{L}^n_{ij,lm}$ for all $n \in \mathbb{N}, i, j, l, m \in \{1, \dots, n\}$, we are done. (Since $\mathfrak{gl}_n \subset \mathfrak{gl}_{n+1}$, we see that we can assume $n \geq 5$ say.) Now let us identify the Lie algebras $\mathfrak{L}^n_{ij,lm}$. We have the following cases:

1. $\mathfrak{L}_{ij,lm}$ where $j \neq l, m \neq i$. Here we have $[\delta_{i,j}, \delta_{l,m}] = 0$ so this Lie algebra is an abelian Lie algebra of dimension ≤ 2. Hence it is contained inside $\mathfrak{N}$.
2. $\mathfrak{L}_{ij,jm}$ where $m \neq i, m \neq j, j \neq i$. This Lie algebra has basis $\{\delta_{i,j}, \delta_{j,m}, \delta_{i,m}\}$ and has bracket relations, $[\delta_{i,j}, \delta_{j,m}] = \delta_{i,m}$, $[\delta_{i,j}, \delta_{i,m}] = [\delta_{j,m}, \delta_{i,m}] = 0$. Thus it is easy to see that this Lie algebra is isomorphic to $\mathfrak{N}$.
3. $\mathfrak{L}_{ij,jj}$ where $j \neq i$. Here we have $[\delta_{i,j}, \delta_{j,j}] = \delta_{i,j}$. So we see that this Lie algebra is isomorphic to $\mathfrak{S}$ and so is inside $\mathfrak{sl}_2$.
4. $\mathfrak{L}_{jj,jm}$ where $m \neq j$. We see easily that we get the same case as in case 3.
5. $\mathfrak{L}_{ij,li}$ where $j \neq l$. Here we note that $\mathfrak{L}_{ij,li} = \mathfrak{L}_{li,ij}$ so it occurs as one of the cases 2–4.
6. $\mathfrak{L}_{ij,ji}$ where $i \neq j$. Here we have that $\mathfrak{L}_{ij,ji}$ has basis $\{\delta_{i,j}, \delta_{j,i}, \delta_{i,i} - \delta_{j,j} = \Delta\}$. The bracket is given by $[\delta_{i,j}, \delta_{j,i}] = \Delta$, $[\Delta, \delta_{i,j}] = \delta_{i,j} + \delta_{i,j} = 2\delta_{i,j}$ and $[\Delta, \delta_{j,i}] = -\delta_{j,i} - \delta_{j,i} = -2\delta_{j,i}$ and so we see easily that this Lie algebra is isomorphic to $\mathfrak{sl}_2$.
7. $\mathfrak{L}_{ii,ii}$. Obviously, this Lie algebra is 1-dimensional and hence lies in either $\mathfrak{sl}_2$ or $\mathfrak{N}$.

All the Lie algebras $\mathfrak{L}^n_{ij,lm}$ fit into one of these cases and hence embed in either $\mathfrak{sl}_2$ or $\mathfrak{N}$. Since κ_1 and κ_0 agree on these two Lie algebras by assumption, they must agree on all the $\mathfrak{L}^n_{ij,lm}$ and hence must be equal as argued before. □

Now we will study how a self-representation must look in certain particular cases.

Lemma 2.17. *Let κ be a self-representation. If $x, y \in \mathfrak{L}$ have $[x, y] = 0$, then $[\kappa(x)](y) = 0$. So, in particular, $\kappa = 0$ on abelian Lie algebras.*

Proof. By naturality, it is enough to show that $\kappa = 0$ on abelian Lie algebras of dimension ≥ 2. Fix $\mathfrak{L}$ as an abelian Lie algebra of dimension $n \geq 2$. Let $\gamma \in \mathbf{k} - \{0, 1\}$. Then multiplication by γ induces an automorphism Γ of the Lie algebra $\mathfrak{L}$. Given $x, y \in \mathfrak{L}$, then if $[\kappa(x)](y) = z$ it follows from naturality that

$$[\kappa(\Gamma(x))](\Gamma(y)) = \Gamma(z),$$
$$\gamma^2 z = \gamma z.$$

However, $\gamma^2 \neq \gamma$ so $z = [\kappa(x)](y) = 0$. Since x, y were arbitrary, the lemma follows. □

Corollary 2.18. *Let κ be a self-representation. Then for a Lie algebra $\mathfrak{L}$ and $x, y \in \mathfrak{L}$ one has $[\kappa(x)](x) = 0$ and $[\kappa(x)](y) = -[\kappa(y)](x)$.*

Proof. Since $[x, x] = 0$, $[\kappa(x)](x) = 0$ follows from lemma 2.17. The second part follows easily from $[\kappa(x + y)](x + y) = 0$. □

Recall that $\mathfrak{S}$ is the Lie algebra with basis $\{x, y\}$ and $[x, y] = x$. Let $\pi : \mathfrak{S} \to \mathfrak{S}$ be a linear map given by $\pi(x) = 0$ and $\pi(y) = y$. It is easy to check that π is a map of Lie algebras.

Lemma 2.19. *Let κ be a self-representation. Let $\mathfrak{S}$ be the Lie algebra with basis $\{x, y\}$ and bracket given by $[x, y] = x$. Then there is a number $a(\kappa) \in \mathbf{k}$ such that one has $[\kappa(u)](v) = a(\kappa)[u, v]$ for all $u, v \in \mathfrak{S}$.*

Proof. By corollary 2.18, one sees it is enough to show that $[\kappa(x)](y) = a(\kappa)x$ for some $a(\kappa) \in \mathbf{k}$. One knows a priori that

$$[\kappa(x)](y) = a(\kappa)x + b(\kappa)y, \tag{16}$$

so we want to show $b(\kappa) = 0$. Applying the map of Lie algebras π, mentioned in the paragraph preceding the lemma, to (16) and using naturality, we get $0 = b(\kappa)y$ and so it follows that $b(\kappa) = 0$. □

Recall that $\mathfrak{sl}_2$ has basis $\{\delta_{1,2}, \delta_{2,1}, \delta_{1,1} - \delta_{2,2}\}$. Let us relabel as follows: $H = \delta_{1,1} - \delta_{2,2}$, $X_+ = \delta_{1,2}$ and $X_- = \delta_{2,1}$, then it is easy to verify that the bracket on $\mathfrak{sl}_2$ is given by

$$[X_+, X_-] = H, [H, X_+] = 2X_+ \text{ and } [H, X_-] = -2X_-. \tag{17}$$

Define linear maps $\Psi_{mix} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$, $\Psi_{neg} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ as follows:

$$\Psi_{neg}(H) = H, \Psi_{neg}(X_+) = -X_+, \Psi_{neg}(X_-) = -X_-, \tag{18}$$

and

$$\Psi_{mix}(H) = H + 2X_+, \Psi_{mix}(X_+) = -X_+, \Psi_{mix}(X_-) = H + X_+ - X_-. \tag{19}$$

It is easy to see that Ψ_{neg} is a map of Lie algebras, and a routine calculation shows Ψ_{mix} is one also. Now we are ready to prove

Lemma 2.20. *Let κ be a self-representation. Then there is $a(\kappa) \in \mathbf{k}$ which is the same as in lemma 2.19, such that $\kappa = a(\kappa)ad$ on $\mathfrak{sl}_2$, i.e., $[\kappa(x)](y) = a(\kappa)[x, y]$ for all $x, y \in \mathfrak{sl}_2$. Furthermore, if κ is strong, then $a(\kappa) = 0$ or 1.*

Proof. One has $[\kappa(H)](H) = [\kappa(X_+)](X_+) = [\kappa(X_-)](X_-) = 0$ as usual. Now H, X_+ span a Lie subalgebra of $\mathfrak{sl}_2$ which is isomorphic to $\mathfrak{S}$ via $H \to -2y$ and $X_+ \to x$. Thus by naturality and lemma 2.19 we have $[\kappa(H)](X_+) = a(\kappa)[H, X_+]$. A similar argument works for X_- in place of X_+. So we conclude easily that

$$[\kappa(H)](\cdot) = a(\kappa)[H, \cdot]. \tag{20}$$

Thus it follows that $[\kappa(X_\pm)](H) = a(\kappa)[X_\pm, H]$. Now $[\kappa(X_+)](X_-) = \alpha H + \beta X_+ + \gamma X_-$. Applying the map of Lie algebras Ψ_{neg} to this equation, and using naturality one gets

$$\begin{aligned}
[\kappa(\Psi_{neg}(X_+))](\Psi_{neg}(X_-)) =& \alpha H - \beta X_+ - \gamma X_-, \\
\alpha H + \beta X_+ + \gamma X_- =& \alpha H - \beta X_+ - \gamma X_-.
\end{aligned}$$

So $\beta = \gamma = 0$. Thus $[\kappa(X_+)](X_-) = \alpha H = \alpha[X_+, X_-]$. So if we can show that $\alpha = a(\kappa)$, we can conclude that

$$[\kappa(X_\pm)](\cdot) = a(\kappa)[X_\pm, \cdot]$$

and together with (20) this implies $\kappa = a(\kappa)ad$ on $\mathfrak{sl}_2$. So it remains only to show that $\alpha = a(\kappa)$. Applying the Lie algebra map Ψ_{mix} to the equation $[\kappa(X_+)](X_-) = \alpha H$ one gets

$$\begin{aligned}
[\kappa(\Psi_{mix}(X_+))](\Psi_{mix}(X_-)) =& \alpha\Psi_{mix}(H) \\
-[\kappa(X_+)](H) + [\kappa(X_+)](X_-) =& \alpha(H + 2X_+) \\
-a(\kappa)[X_+, H] + \alpha H =& \alpha H + 2\alpha X_+ \\
2a(\kappa)X_+ =& 2\alpha X_+.
\end{aligned}$$

from which one concludes that $a(\kappa) = \alpha$ as desired.

Now if $\kappa = a(\kappa)ad$ is a map of Lie algebras, then we have

$$\begin{aligned}
\kappa([X_+, X_-]) &= \kappa(X_+) \circ \kappa(X_-) - \kappa(X_-) \circ \kappa(X_+), \\
a(\kappa)ad([X_+, X_-]) &= a(\kappa)^2 ad(X_+) \circ ad(X_-) - a(\kappa)^2 ad(X_-) \circ ad(X_+), \\
a(\kappa)ad([X_+, X_-]) &= a(\kappa)^2 ad([X_+, X_-]).
\end{aligned}$$

Since $ad([X_+, X_-])$ is nonzero, it follows that $a(\kappa) = a(\kappa)^2$ or in other words, that $a(\kappa) = 0$ or 1. ☐

Recall that $\mathfrak{N}$ has basis $\{x, y, z\}$ and bracket given by $[x, z] = [y, z] = 0, [x, y] = z$. Define linear maps $\Psi_x, \Psi_y : \mathfrak{N} \to \mathfrak{N}$ by

$$\Psi_x(y) = \Psi_x(z) = 0, \Psi_x(x) = x$$

and

$$\Psi_y(x) = \Psi_y(z) = 0, \Psi_y(y) = y.$$

It is easy to verify that these maps are maps of Lie algebras.

Lemma 2.21. *Let κ be a self-representation. Then there is $\mu(\kappa) \in \mathbf{k}$ such that $\kappa = \mu(\kappa)ad$ on $\mathfrak{N}$.*

Proof. Since z is central, one sees by lemma 2.17, that $\kappa(z) = 0 = ad(z)$. Now $[\kappa(x)](y) = \alpha x + \beta y + \mu z$. Applying the Lie algebra map Ψ_x to this equation and using naturality, one gets $0 = \alpha x$. So one concludes that $\alpha = 0$. Similarly, using Ψ_y instead one concludes that $\beta = 0$. So $[\kappa(x)](y) = \mu z = \mu[x, y]$. So now it is easy to see that $[\kappa(x)](\cdot) = \mu[x, \cdot]$ and $[\kappa(y)](\cdot) = \mu[y, \cdot]$ via lemma 2.17 and corollary 2.18. Thus setting $\mu(\kappa) = \mu$, we are done. ☐

Recall that $\mathfrak{so}_n \subset \mathfrak{gl}_n$, the subset of skew-symmetric $n \times n$ matrices, is a Lie subalgebra. For char$(k) \neq 2$, $\mathfrak{so}_3$ is 3-dimensional with basis $\{X = \delta_{1,2} - \delta_{2,1}, Y = \delta_{1,3} - \delta_{3,1}, Z = \delta_{2,3} - \delta_{3,2}\}$ and bracket given by the relations $[X, Y] = -Z$,$[Y, Z] = -X$,$[Z, X] = -Y$. Since we are over a finite field of odd characteristic, $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2$ via the map which sends $X \to \frac{X_+ - X_-}{2}, Y \to a\frac{X_+ + X_-}{2} + b\frac{H}{2}, Z \to b\frac{X_+ + X_-}{2} - a\frac{H}{2}$. Here $a, b \in \mathbf{k}$ are such that $a^2 + b^2 = -1$.

Lemma 2.22. *Let κ be a self-representation. Then $\kappa = \mu(\kappa)ad = a(\kappa)ad$ on $\mathfrak{so}_3$ where $\mu(\kappa)$ is the same as that of lemma 2.21 and $a(\kappa)$ is the same as lemma 2.20. In particular, $a(\kappa) = \mu(\kappa)$.*

Proof. The fact that $\kappa = a(\kappa)ad$ follows from lemma 2.20 and the fact that $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2$. So it remains to show that $\kappa = \mu(\kappa)ad$ also. Consider the Lie algebra $\mathfrak{gl}_3$. In the proof of lemma 2.16 we se that $\{\delta_{i,j}, \delta_{j,k}, \delta_{i,k}\}$ formed a basis of a Lie subalgebra isomorphic to $\mathfrak{N}$ for i, j, k distinct. Thus

$$[\kappa(\delta_{i,j})](\delta_{j,k}) = \mu(\kappa)[\delta_{i,j}, \delta_{j,k}] = \mu(\kappa)\delta_{i,k}$$

for all i, j, k distinct. Also $[\kappa(\delta_{i,j})](\delta_{l,m}) = 0$ if $j \neq l, i \neq m$ since under these conditions $[\delta_{i,j}, \delta_{l,m}] = 0$. So

$$\begin{aligned}
[\kappa(\delta_{1,2} - \delta_{2,1})](\delta_{1,3} - \delta_{3,1}) =& [\kappa(\delta_{1,2})](\delta_{1,3}) - [\kappa(\delta_{1,2})](\delta_{3,1}) \\
& - [\kappa(\delta_{2,1})](\delta_{1,3}) + [\kappa(\delta_{2,1})](\delta_{3,1}) \\
=& 0 - (-\mu(\kappa)\delta_{3,2}) - (\mu(\kappa)\delta_{2,3}) + 0 \\
=& \mu(\kappa)(\delta_{3,2} - \delta_{2,3}) \\
=& \mu(\kappa)[\delta_{1,2} - \delta_{2,1}, \delta_{1,3} - \delta_{3,1}],
\end{aligned}$$

and similarly,

$$[\kappa(\delta_{1,2} - \delta_{2,1})](\delta_{2,3} - \delta_{3,2}) = \mu(\kappa)[\delta_{1,2} - \delta_{2,1}, \delta_{2,3} - \delta_{3,2}]$$

and

$$[\kappa(\delta_{1,3} - \delta_{3,1})](\delta_{2,3} - \delta_{3,2}) = \mu(\kappa)[\delta_{1,3} - \delta_{3,1}, \delta_{2,3} - \delta_{3,2}].$$

So with these three equations and corollary 2.18, one concludes that $\kappa = \mu(\kappa)ad$ on $\mathfrak{so}_3$. □

We now state:

Theorem 2.23. *Let κ be a strong self-representation. If $\kappa \neq 0$ on $\mathfrak{sl}_2$, then $\kappa = ad$ as self-representations. If $\kappa = 0$ on $\mathfrak{sl}_2$, then $\kappa = 0$ as self-representations.*

Proof. If $\kappa \neq 0$ on $\mathfrak{sl}_2$, then we have $a(\kappa) = \mu(\kappa) = 1$ from lemmas 2.20 and 2.22. Thus $\kappa = ad$ on $\mathfrak{sl}_2$ and on $\mathfrak{N}$. Thus by lemma 2.16 we have that $\kappa = ad$ as self-representations. The case $\kappa = 0$ on $\mathfrak{sl}_2$ proceeds similarly. □

For use later on, we need to state theorem 2.23 in slightly more generality. To do this we need to introduce the categories $\mathbf{Lie}_k(p)$ for $k \geq 1$ and p a prime. The objects of this category will be Lie algebras over $\mathbb{Z}/p^k\mathbb{Z}$, that is free $\mathbb{Z}/p^k\mathbb{Z}$-modules R of finite rank equipped with a bilinear, alternating form $[\cdot,\cdot] : R \times R \to R$ which satisfies the Jacobi identity. The morphisms will be the $\mathbb{Z}/p^k\mathbb{Z}$-module maps which preserve the brackets. Thus $\mathbf{Lie}_1(p)$ is the same category $\mathbf{Lie}$ considered before for $\mathbf{k} = \mathbb{F}_p$.

For $t \leq k$, we have reduction functors $\mathbf{Lie}_k(p) \to \mathbf{Lie}_t(p)$. These reduction functors are obtained in the obvious way from the reduction map of $\mathbb{Z}/p^k\mathbb{Z}$ to $\mathbb{Z}/p^t\mathbb{Z}$. For $\mathfrak{L} \in \mathrm{Obj}(\mathbf{Lie}_k(p))$ we denote $\bar{\mathfrak{L}}$ to be its reduction in $\mathrm{Obj}(\mathbf{Lie}_1(p))$. We will also use the bar notation for the reduction of a morphism. Note that there is the canonical reduction homomorphism which is a map of abelian groups which preserves the bracket from $\mathfrak{L}$ to $\bar{\mathfrak{L}}$; we will refer to maps of abelian groups which preserve brackets as Lie algebra maps for the rest of this section.

Definition 2.24. A natural self-representation on $\mathbf{Lie}_k(p)$ is a collection of maps $\kappa_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{gl}(\bar{\mathfrak{L}})$ of abelian groups, which satisfy the naturality condition

$$[\kappa_{\mathfrak{L}'}(\psi(x))](\bar{\psi}(y)) = \bar{\psi}([\kappa_{\mathfrak{L}}(x)](y))$$

for all $\mathfrak{L}, \mathfrak{L}' \in \mathrm{Obj}(\mathbf{Lie}_k(p))$, $\psi \in \mathrm{Mor}(\mathfrak{L}, \mathfrak{L}')$, $x \in \mathfrak{L}$ and $y \in \bar{\mathfrak{L}}$. We say the natural self-representation is strong if the maps $\kappa_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{gl}(\bar{\mathfrak{L}})$ are maps of Lie algebras.

Note that this definition extends the previous one.

It is easy to check that if we assign $\kappa_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{gl}(\bar{\mathfrak{L}})$ to be the composition of the reduction homomorphism with $ad : \bar{\mathfrak{L}} \to \mathfrak{gl}(\bar{\mathfrak{L}})$, then we obtain a strong self-representation which we will call ad.

The Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_2$,$\mathfrak{N}, \mathfrak{S}$ and $\mathfrak{so}_3$ will denote the obvious Lie algebras in $\mathrm{Obj}(\mathbf{Lie}_k(p))$. For example, $\mathfrak{gl}_n$ is the Lie algebra of $n \times n$ matrices with $\mathbb{Z}/p^k\mathbb{Z}$ entries. Theorem 2.23 is true for these more general self-representations. We state this in the next theorem.

Theorem 2.25. *Let p be an odd prime and let κ be a strong self-representation over $\mathbf{Lie}_k(p)$. If $\kappa \neq 0$ on $\mathfrak{so}_3$ and $\mathfrak{sl}_2$, then $\kappa = ad$ as self-representations.*

Proof. The proof proceeds as before with only some modifications so we leave it to the reader (see [Pak]). The only thing to note is there is a generalized Ado-Iwasawa theorem (see [W4]) which says that any $\mathfrak{L} \in \mathrm{Obj}(\mathbf{Lie}_k(p))$ embeds as a subalgebra of $\mathfrak{gl}_n(\mathbb{Z}/p^k\mathbb{Z})$. □

2.6. **Bockstein spectral sequence implies $\lambda = ad$.** Recall that we had λ a strong self-representation over $\mathbb{F}_p$ which was defined using the Bockstein. In this section we show that $\lambda = ad$ as self-representations.

By theorem 2.23, it is enough to show λ is not identically zero on $\mathfrak{sl}_2$. To do this we will need the Bockstein spectral-sequence.

Let X be a topological space (with finitely generated integral homology). Then the coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

gives a long exact sequence

$$\cdots \longrightarrow H^*(X;\mathbb{Z}) \xrightarrow{p} H^*(X;\mathbb{Z}) \xrightarrow{mod} H^*(X;\mathbb{F}_p) \xrightarrow{\delta} H^{*+1}(X,\mathbb{Z}) \longrightarrow \cdots$$

which gives a spectral sequence in a standard way if we think of it as an exact couple. This spectral sequence has $B_1^* = H^*(X;\mathbb{F}_p)$ with differential given by the Bockstein β (as defined previously) and converges to $\mathbb{F}_p \otimes (H^*(X;\mathbb{Z})/\text{torsion})$. If we apply this to the case X=BG where G is a finite group, then the Bockstein spectral sequence must converge to zero in positive gradings, as the integral cohomology of a finite group is torsion in positive gradings.

Now we will apply the Bockstein spectral sequence to show that λ is not zero on $\mathfrak{sl}_2$.

Let $\{H, X_+, X_-\}$ be the usual basis for $\mathfrak{sl}_2$ and let $G = Exp(\mathfrak{sl}_2)$. Let $\{h, x_+, x_-\}$ be the obvious dual basis for $\mathfrak{sl}_2$. Then as we have seen before,

$$B_1^* = H^*(G;\mathbb{F}_p) = \wedge^*(h, x_+, x_-) \otimes \mathbb{F}_p[s_h, s_+, s_-]$$

where we have chosen the s-basis to correspond in the obvious way. Then we see, using the formulas found before, that on the exterior algebra part the Bockstein is given by

$$\begin{aligned} \beta(h) &= -[\cdot,\cdot]_H = -x_+x_-, \\ \beta(x_+) &= -[\cdot,\cdot]_{X_+} = -2hx_+, \\ \beta(x_-) &= -[\cdot,\cdot]_{X_-} = 2hx_-. \end{aligned}$$

So as $B_1^1 = \wedge^1(h, x_+, x_-)$ one sees that $B_2^1 = 0$. Now note that $B_1^2 = \wedge^2(h, x_+, x_-) \oplus \operatorname{span}(s_h, s_+, s_-)$ and by the above formulas, the exterior part contributes nothing to B_2^2. If λ is zero on $\mathfrak{sl}_2$, then this means β maps $\operatorname{span}(s_h, s_+, s_-)$ into $\wedge^3(h, x_+, x_-)$ which is 1-dimensional generated by hx_+x_-. So we see that β must have at least a 2-dimensional kernel on the span of $\{s_h, s_+, s_-\}$. Thus $\dim(B_2^2) \geq 2$. Now B_1^3 is equal to

$$\wedge^3(h, x_+, x_-) \oplus \operatorname{span}(\{s_h h, s_h x_+, s_h x_-, s_+ h, s_+ x_+, s_+ x_-, s_- h, s_- x_+, s_- x_-\}).$$

We will show that β is injective on the second part so that $\dim(B_2^3) \leq 1$. Note that $\beta(s_*) = a_* hx_+x_-$, for some $a_* \in \mathbb{F}_p$, where * stands for an arbitrary subscript. So $\beta(s_*)x_* = 0$ and one has

$$\beta(s_* x_*) = \beta(s_*)x_* + s_*\beta(x_*) = s_*\beta(x_*).$$

So because $\{\beta(x_h), \beta(x_+), \beta(x_-)\}$ are linearly independent, and $\{s_h, s_+, s_-\}$ are algebraically independent, it is easy to see that β is injective on the second part of B_1^3 as claimed. Thus we have $B_2^1 = 0$, $\dim(B_2^2) \geq 2$ and $\dim(B_2^3) \leq 1$. However the Bockstein spectral sequence converges to zero in positive gradings implying that we need $\dim(B_2^2) \leq \dim(B_2^3)$ which we don't have. Thus we have a contradiction and our assumption that $\lambda = 0$ must be false.

Thus we finally conclude that $\lambda = ad$ for all Lie algebras. Now let us run through the definitions to get formulas for the β_1 part of β. Fix a Lie algebra $\mathfrak{L}$ over $\mathbb{F}_p$. Fix a basis $E = \{e_1, \ldots, e_n\}$ of $\mathfrak{L}$ and let $G = Exp(\mathfrak{L})$. Then $H^*(G; \mathbb{F}_p) = \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$ where $x_1, \ldots, x_n$ is the dual basis of E and $s_1, \ldots, s_n$ is the corresponding basis for the polynomial part of the cohomology. Let $c_{ij}^k = [e_i, e_j]_k$ be the structure constants of $\mathfrak{L}$ with respect to the basis E. We have shown that

$$[\lambda(e_i)](e_j) = \sum_{k=1}^{n} c_{ij}^k e_k.$$

Recalling that λ is the dual of $-\beta_1 : S^* \to W^* \otimes S^*$, we easily see that

$$-\beta_1(s_k) = \sum_{i,j=1}^{n} c_{ij}^k x_i s_j.$$

Using that $c_{ij}^k = -c_{ji}^k$, one can easily deduce the following equation for β:

$$\beta(s_k) = \sum_{i,j=1}^{n} c_{ij}^k s_i x_j + \beta_2(s_k). \tag{21}$$

Thus we can also recover the Lie algebra structure of $\mathfrak{L}$ from knowledge of the β_1 component of $\beta|_{S^*}$. It remains to study β_2 which is the remaining term of the Bockstein. Before doing that, we will digress to give an application of (21).

2.7. **Comodule algebra structure.** Let $G \in \operatorname{Obj}(\mathbf{LGrp})$, then the multiplication in G induces a homomorphism $\mu : \Omega_1(G) \times G \to G$ and

$$\Delta = \mu^* : H^*(G; \mathbb{F}_p) \to H^*(\Omega_1(G); \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p)$$

gives $H^*(G; \mathbb{F}_p)$ the structure of a comodule algebra over the Hopf algebra $H^*(\Omega_1(G); \mathbb{F}_p)$. (See [W2].)

In order to describe Δ, it is enough to describe it on algebra generators of $H^*(G;\mathbb{F}_p)$. To do this let us write

$$H^*(G;\mathbb{F}_p) = \wedge^*(x_1,\dots,x_n)\otimes \mathbb{F}_p[s_1,\dots,s_n]$$

and $H^*(\Omega_1(G);\mathbb{F}_p) = \wedge^*(t_1,\dots,t_n)\otimes\mathbb{F}_p[\bar{s}_1,\dots,\bar{s}_n]$ with $\bar{s}_i = \beta(t_i)$ for all i. Furthermore, we have made our choices so that $j^*(s_i) = \bar{s}_i$ where j is the inclusion map of $\Omega_1(G)$ into G and also so that (21) holds. Recall that the Hopf algebra structure of $\Omega_1(G)$ is one where all the generators $\bar{s}_i$ and t_i are primitive. Now we can state:

Corollary 2.26. *Let $G \in \mathrm{Obj}(\mathbf{LGrp})$ and c_{ij}^k be the structure constants of $Log(G)$ in a suitable basis. Using the notation of the paragraph above, one has the following formulas which determine Δ:*

$$\begin{aligned}\Delta(x_k) =& 1\otimes x_k,\\ \Delta(s_k) =& \bar{s}_k\otimes 1 + 1\otimes s_k + \sum_{i,j} c_{ij}^k t_i\otimes x_j.\end{aligned}$$

Proof. Precomposing μ with the natural inclusions of $\Omega_1(G)$ and G into $\Omega_1(G)\times G$, it is not hard to show that $\Delta = \mu^*$ must take the following form on the generators

$$\begin{aligned}\Delta(x_k) =& 1\otimes x_k,\\ \Delta(s_k) =& \bar{s}_k\otimes 1 + 1\otimes s_k + \sum_{i,j} a_{ij}^k t_i\otimes x_j.\end{aligned}$$

Thus it is enough to show that $a_{ij}^k = c_{ij}^k$ for all i, j, k. Since the Bockstein β is a natural operation, and since Δ is induced from a homomorphism of groups, they must commute, that is $\Delta\circ\beta = (\beta\otimes 1 + 1\otimes\beta)\circ\Delta$. Applying $\beta\otimes 1 + 1\otimes\beta$ to the formula for $\Delta(s_k)$ and using (21), we get

$$(\beta\otimes 1+1\otimes\beta)(\Delta(s_k)) = 1\otimes\beta(s_k) + \sum_{i,j=1}^{n} a_{ij}^k(\bar{s}_i\otimes x_j - t_i\otimes\beta(x_j)) \tag{22}$$

but

$$\begin{aligned}\Delta(\beta(s_k)) &= \sum_{l,m=1}^{n} c_{lm}^k\Delta(s_l x_m) + \Delta(\beta_2(s_i))\\ &= \sum_{l,m=1}^{n} c_{lm}^k(\bar{s}_l\otimes x_m + 1\otimes s_l x_m + \sum_{u,v} a_{uv}^l t_u\otimes x_v x_m) + 1\otimes\beta_2(s_i).\end{aligned}$$

Now we know the expressions in (22) and the last one are equal, so equating their $H^2(\Omega_1(G);\mathbb{F}_p)\otimes H^1(G;\mathbb{F}_p)$ components we get

$$\sum_{i,j=1}^{n} a_{ij}^k\bar{s}_i\otimes x_j = \sum_{l,m=1}^{n} c_{lm}^k\bar{s}_l\otimes x_m$$

which immediately gives $a_{ij}^k = c_{ij}^k$ for all i, j, k which is what we desired. $\square$

Thus we see that the comodule algebra structure of $H^*(G;\mathbb{F}_p)$ also determines the Lie algebra corresponding to G.

2.8. **The class $[\eta]$ and lifting uniform towers.** In this section we will show that the β_2 component for the Bockstein, defines a cohomology class in a suitable cohomology group.

Fix $\mathfrak{L} = (W, br) \in \mathrm{Obj}(\mathbf{Lie})$. Recall the existence of a Koszul complex:

$$0 \to \wedge^0(\mathfrak{L}, ad) \xrightarrow{d} \cdots \xrightarrow{d} \wedge^n(\mathfrak{L}, ad) \to 0$$

whose cohomology is $H^*(\mathfrak{L}; ad)$. Here $\wedge^i(\mathfrak{L}, ad)$ denotes the $\mathfrak{L}$-valued alternating i-forms on $\mathfrak{L}$, and d is given by

$$\begin{aligned}(d\omega)(u_0, \ldots, u_l) = &\sum_{i<j}(-1)^{i+j}\omega([u_i, u_j], u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_l)\\ &+ \sum_{i=0}^{l}(-1)^i[u_i, \omega(u_0, \ldots, \hat{u}_i, \ldots, u_l)]\end{aligned}$$

on l-forms ω. If we view $\omega : \wedge^l(W) \to W$ and set $ad : W \otimes W \to W$ to be the adjoint action map, one can show easily that

$$d\omega = -\omega \circ br + ad \circ (1 \wedge \omega).$$

Using $\hat{\beta}_2$ instead of β_2 in (15) and taking duals one obtains

$$0 = -(-\hat{\beta}_2^*) \circ br + (-\hat{\beta}_1^*) \circ (1 \wedge -\hat{\beta}_2^*).$$

However, we have shown that $-\hat{\beta}_1^* = ad$ and so we can conclude that $-\hat{\beta}_2^* : \wedge^3(W) \to V$ is a closed 3-form in the Koszul resolution for $H^*(\mathfrak{L}; ad)$. In a similar fashion, when one looks at the effect of changing the choice of the subspace S^* via a map $\mu : V^* \to \wedge^2(W^*)$ as in (11), one sees that the net result is to add $d(-\mu^*)$ to $-\hat{\beta}_2^*$. So although the 3-form $-\hat{\beta}_2^*$ is not uniquely determined from $\mathfrak{L}$, it is uniquely determined up to the addition of a boundary in the Koszul resolution above and hence defines a unique cohomology class $[\eta] \in H^3(\mathfrak{L}; ad)$. In fact, it is easy to see that one can get any representative of this cohomology class by a suitable choice of μ.

So now it remains to study $[\eta] \in H^3(\mathfrak{L}; ad)$. It is true for $p \neq 3$ that $[\eta] = 0$ if and only if $\mathfrak{L}$ lifts to a $\mathbb{Z}/p^2\mathbb{Z}$ Lie algebra. This requires a bit of work using results of lifting Lie algebras as mentioned in [Pak] and T. Weigel's Exp-Log correspondence. We will first show that $[\eta] = 0$ if and only if $G = Exp(\mathfrak{L})$ has an Ω_1-extension with the ΩEP (these terms will be defined later). This, on the other hand, is quite elementary and does not need knowledge of the Exp-Log correspondence.

We now discuss various concepts necessary to see this fact. First recall definitions 1.1 and 1.2. Furthermore, we define:

Definition 2.27. G is powerful if $[G, G] \subseteq G^p$.

Definition 2.28. A tower of groups will be a sequence of p-central groups

$$\{G_1, \ldots, G_N\}$$

(where $N = 1, 2, \ldots, \infty$) equipped with surjective homomorphisms $\pi_i : G_i \to G_{i-1}$ with kernel $\Omega_1(G_i)$ for $i = 1, \ldots, N$. ($G_0 = 1$ by convention.)

Note that the definition implies $G_1 = \Omega_1(G_1)$ and hence is elementary abelian. Also note that if we have a tower, we have central short exact sequences of groups

$$1 \to \Omega_1(G_i) \to G_i \xrightarrow{\pi_i} G_{i-1} \to 1$$

and these pull back to central short exact sequences

$$1 \to \Omega_1(G_i) \to E_i \xrightarrow{\pi_i} \Omega_1(G_{i-1}) \to 1$$

as considered in the beginning of this paper. Thus these have a p-power map ϕ where $\phi(x) = \hat{x}^p$ for $x \in \Omega_1(G_{i-1})$. Note the ϕ map is automatically injective by definition of $\Omega_1(G_i)$.

Definition 2.29. A uniform tower of groups is a tower of groups where all the p-power maps ϕ of the tower are isomorphisms.

Thus a uniform tower is where the p-power maps ϕ are also surjective. Note, it is well known that if G is p-central, then $G/\Omega_1(G)$ is also p-central, so all p-central groups fit into a tower. By induction, it is also easy to see that G_n, a group in the nth stage of a uniform tower, has exponent p^n and is powerful.

In general, if you have a tower, then G_2 fits in a short exact sequence

$$1 \to \Omega_1(G_2) \to G_2 \to G_1 \to 1$$

and as G_1 is elementary abelian; one sees immediately that G_2 groups in uniform towers are exactly the groups associated to bracket algebras considered before. Now given a tower of length N with end group G_N one can ask when it extends to a tower of length $N+1$, i.e., when does there exist a p-central group G_{N+1} such that $G_{N+1}/\Omega_1(G_{N+1})$ is isomorphic to G_N. We will be primarily interested in uniform towers and extending to uniform towers. This question has been considered independently in [W1]. In order to coincide with the notation there we introduce the concept of ΩEP used in [W1] and [W2]. We will provide theorems which were proved independently by us for the uniform case and refer the reader to [W1] and [W2] for lifting in the more general situation.

Definition 2.30. A p-central group G is said to have the ΩEP if there exists a p-central group $\hat{G}$ such that $\hat{G}/\Omega_1(\hat{G}) = G$.

Note that if we have such an extension with $\hat{G} \xrightarrow{\pi} G$, then $\pi^{-1}(\Omega_1(G)) \subset \hat{G}$ is an extension of $\Omega_1(G)$ which has ϕ map injective. We call such an extension an Ω_1 extension. Let $n = \dim(\Omega_1(G))$ and $l = \dim(\Omega_1(\hat{G}))$. In

$$H^2(\Omega_1(G); \Omega_1(\hat{G})) = [\wedge^*(x_1, \dots, x_n) \otimes \mathbb{F}_p[s_1, \dots, s_n]]^l$$

this extension for $\pi^{-1}(\Omega_1(G))$ is represented by an element of the form

$$(s_1 + \mu_1, \dots, s_n + \mu_n, \mu_{n+1}, \dots, \mu_l)$$

with $\mu_i \in \wedge^*(x_1, \dots, x_n)$. Note that $s_1 + \mu_1, \dots, s_n + \mu_n$ are in the image of the restriction $i^* : H^*(G; \mathbb{F}_p) \to H^*(\Omega_1(G); \mathbb{F}_p)$. Thus one has n algebraically independent elements in the second grading in the image of the restriction i^*. On the other hand, if we had a p-central group such that $\dim(\Omega_1(G)) = n$ and there are n algebraically independent elements of degree two in image $i^* : H^*(G; \mathbb{F}_p) \to H^*(\Omega_1(G); \mathbb{F}_p)$. Then by choosing the right basis for

$$H^2(\Omega_1(G); \mathbb{F}_p) = \wedge^*(x_1, \dots, x_n) \otimes \mathbb{F}_p[s_1, \dots, s_n]$$

one has these elements of the form $s_1 + \mu_1, \dots, s_n + \mu_n$ with $\mu_i \in \wedge^2(x_1, \dots, x_n)$. Thus taking an element in $H^2(G; \Omega_1(G))$ which restricts to $(s_1 + \mu_1, \dots, s_n + \mu_n) \in H^2(\Omega_1(G); \Omega_1(G))$, we see easily that the group this element represents, say $\hat{G}$, has $\hat{G}/\Omega_1(\hat{G}) = G$. Thus G has the ΩEP. So we conclude

Lemma 2.31. *Let G be a p-central group with $n = \dim(\Omega_1(G))$. Then G has the ΩEP if and only if there are n algebraically independent elements of degree 2 in image $i^* : H^*(G;\mathbb{F}_p) \to H^*(\Omega_1(G);\mathbb{F}_p)$.*

Now note in a tower $\{G_1,\dots,G_N\}$, G_i has the ΩEP for $1 \le i < N$ by definition. This lets us prove:

Theorem 2.32. *Let $\{G_1,\dots,G_N\}$ be a uniform tower with $\dim(G_1) = n$. Then for $1 \le i < N$ we have*

$$H^*(G_i;\mathbb{F}_p) = \wedge^*(x_1,\dots,x_n) \otimes \mathbb{F}_p[s_1,\dots,s_n]$$

where $\deg(x_i) = 1$ (and for $i > 1$, x_i is induced from G_{i-1}) and $\deg(s_i) = 2$ (and for $i > 1$, s_i "comes" from $\Omega_1(G_i)$), and $H^*(G_N;\mathbb{F}_p)$ is given by the same formula if and only if the tower extends to a uniform tower $\{G_1,\dots,G_N,G_{N+1}\}$.

Proof. By the comments in the preceding paragraph and lemma 2.31, one obtains the first part of this theorem by induction on i where $1 \le i < N$. More explicitly, note that one has the result for G_1. So fix $i > 1$ and assume that we have shown the first part for all $j < i$. Then we have the central short exact sequence

$$1 \to \Omega_1(G_i) \to G_i \xrightarrow{\pi_i} G_{i-1} \to 1.$$

So the $E_2^{*,*}$ term of the L.H.S.-spectral sequence of this extension is given by

$$\wedge^*(t_1,\dots,t_n) \otimes \mathbb{F}_p[s_1,\dots,s_n] \otimes \wedge^*(x_1,\dots,x_n) \otimes \mathbb{F}_p[y_1,\dots,y_n]$$

where the first two terms are the cohomology of $\Omega_1(G_i)$ and the last two terms are the cohomology of G_{i-1} by induction. One has as usual $d_2(s_i) = d_2(x_i) = d_2(y_i) = 0$ and $d_2(t_i)$ are the elements representing the extension. By the proof of lemma 2.31 and the fact that G_i is an Ω_1 extension of G_{i-1}, one sees (after changing basis for y's) that these elements are of the form $d_2(t_i) = y_i + \mu_i$ for all i where $\mu_i \in \wedge^*(x_1,\dots,x_n)$. Thus as in the proof of theorem 2.10, one gets $E_3^{*,*} = \mathbb{F}_p[s_1,\dots,s_n] \otimes \wedge^*(x_1,\dots,x_n)$ with $d_3(x_i) = 0$. Now $E_\infty^{0,*}$ is the image of $i^* : H^*(G_i;\mathbb{F}_p) \to H^*(\Omega_1(G_i);\mathbb{F}_p)$ and by lemma 2.31 this contains n algebraically independent elements in degree 2. Since $E_2^{0,2}$ has dim n, we see by necessity that $d_3 = 0$ on $E_2^{0,2}$, which implies $d_3 = 0$. Then as in theorem 2.10 one sees that $H^*(G_i;\mathbb{F}_p)$ is as stated. So we are done with the first part. If the tower extends to a uniform tower of length one more, then one can perform the same argument on G_N to conclude its cohomology is as stated. On the other hand, if its cohomology is as stated, image $i^* : H^*(G_N;\mathbb{F}_p) \to H^*(\Omega_1(G_N);\mathbb{F}_p)$ has n algebraically independent elements in degree 2 and so G_N has the ΩEP. Using these elements as in the proof of lemma 2.31, one constructs an extension to a uniform tower of length $N+1$. ☐

Examples.

Definition 2.33. Fix p an odd prime number, then $\Gamma_{n,k}(p)$ is defined as the kernel of the reduction homomorphism

$$GL_n(\mathbb{Z}/p^{k+1}\mathbb{Z}) \to GL_n(\mathbb{Z}/p\mathbb{Z})$$

for all $n, k \ge 1$. Similarly, $\hat{\Gamma}_{n,k}(p)$ is the group obtained by replacing GL_n with SL_n in the definition above.

One shows by induction on k that the $\Gamma_{n,k}(p)$ are p-groups and it is then easy to show that for fixed n they fit together to give an infinite uniform tower:

$$\to \Gamma_{n,k}(p) \to \Gamma_{n,k-1}(p) \to \cdots \to \Gamma_{n,1}(p) \to 1.$$

A similar statement holds for the $\hat{\Gamma}_{n,k}(p)$.

Theorem 2.32 has the following immediate corollary:

Corollary 2.34. *For $n, k \geq 1$ and p an odd prime we have:*

$$H^*(\Gamma_{n,k}(p); \mathbb{F}_p) \cong \wedge^*(x_1, \ldots, x_{n^2}) \otimes \mathbb{F}_p[s_1, \ldots, s_{n^2}]$$

and

$$H^*(\hat{\Gamma}_{n,k}(p); \mathbb{F}_p) \cong \wedge^*(x_1, \ldots, x_{n^2-1}) \otimes \mathbb{F}_p[s_1, \ldots, s_{n^2-1}]$$

where $\deg(x_i) = 1$ *and* $\deg(s_i) = 2$ *for all* i.

Recall that if $G \in \mathrm{Obj}(\mathbf{BGrp})$, then $G = G_2$ of a uniform tower $\{G_1, G_2\}$. This tower extends to a uniform tower of length 3 if and only if $Log(G)$ is a Lie algebra. This follows from theorem 2.32 and theorem 2.10. Next, we will show that this uniform tower $\{G_1, G_2\}$ extends to a tower of length 4 if and only if $[\eta] = 0$ where $[\eta] \in H^3(\mathfrak{L}; ad)$ is the cohomology class defined earlier.

Now suppose we have a uniform tower $\{G_1, G_2, G_3'\}$ so $G_2 \in \mathrm{Obj}(\mathbf{LGrp})$ and has $\mathbb{F}_p$-cohomology given by theorem 2.10. We will now show that there is a uniform tower $\{G_1, G_2, G_3, G_4\}$ if and only if $[\eta] = 0$ where we say G_3 instead of G_3' since we might have to choose a different Ω_1 extension of G_2 to do this. Note that the tower $\{G_1, G_2\}$ extends to one of length 4 as desired if and only if G_2 has a uniform Ω_1 extension G_3 which itself has the ΩEP. By theorem 2.32, this happens if and only if G_3 has $\mathbb{F}_p$-cohomology $\wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$ where $\deg(x_i) = 1, \deg(s_i) = 2$ and $n = \dim(G_1)$.

Let us study the cohomology of an Ω_1 extension G_3 of G_2. $H^*(G_2; \mathbb{F}_p) = \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$ where $\deg(s_i) = 2, \deg(x_i) = 1$ as usual. Then G_3 is represented in $H^2(G_2; \Omega_1(G_3)) = [H^2(G_2; \mathbb{F}_p)]^n$ by $(s_1 + \mu_1, \ldots, s_n + \mu_n)$ where $\mu_i \in \wedge^2(x_1, \ldots, x_n)$ under a suitable choice of basis of $\Omega_1(G_3)$. However, now note that due to the ambiguous nature of s_i, we can shift them by elements of $\wedge^2(x_1, \ldots, x_n)$; so, without loss of generality, the extension element of G_3 is $(s_1, \ldots, s_n)$. Note that this implies a particular choice of s-basis. Let η be the Bockstein term for this choice of s-basis. Then as is seen in the first sections, when we look at the L.H.S.-spectral sequence for

$$1 \to \Omega_1(G_3) \to G_3 \xrightarrow{\pi_3} G_2 \to 1,$$

then

$$E_2^{*,*} = \wedge^*(t_1, \ldots, t_n) \otimes \mathbb{F}_p[\beta(t_1), \ldots, \beta(t_n)] \otimes \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$$

where the first two terms are the cohomology of $\Omega_1(G_3)$ and the last two are that of G_2. Again, $d_2(\beta(t_i)) = d_2(x_i) = d_2(s_i) = 0$ and $d_2(t_i) = s_i$. Thus,

$$E_3^{*,*} = \mathbb{F}_p[\beta(t_1), \ldots, \beta(t_n)] \otimes \wedge^*(x_1, \ldots, x_n)$$

with $d_3(x_i) = 0$ and $d_3(\beta(t_i)) = \{\beta(d_2(t_i))\} = \{\beta(s_i)\}$ in E_3. So by what we said before, G_3 has the ΩEP if and only if $\{\beta(s_i)\} = 0$ for all i. Now let us calculate this explicitly. There are elements $\xi_{i,l} \in \wedge^1(x_1, \ldots, x_n)$ such that

$$\beta(s_i) = \sum_{l=1}^{n} \xi_{i,l} s_l + \beta_2(s_i) = \sum_{l=1}^{n} \xi_{i,l}(0) + \beta_2(s_i)$$

where in the last step we used that $s_i = 0$ in E_3 since $d_2(t_i) = s_i$. Thus we get $d_3(\beta(t_i)) = \beta_2(s_i)$ for all i. So G_3 defined by the extension element $(s_1, \ldots, s_n)$ has the ΩEP if and only if $\beta_2 = 0$. Since we have argued that the Ω_1 extensions of G_2 are exactly those defined by such extension elements, we see that there is an Ω_1 extension G_3 of G_2 such that G_3 has the ΩEP if and only if there is a choice of $s_1, \ldots, s_n$ by adding elements in $\wedge^2(x_1, \ldots, x_n)$ so that the corresponding $\beta_2 = 0$, i.e., if and only if $[\eta] = 0$. Thus we have shown

Theorem 2.35. *Let $\{G_1, G_2\}$ be a uniform tower. Then it extends to a uniform tower of length 3 if and only if $\mathfrak{L} = Log(G_2)$ is a Lie algebra. In this case it is possible to extend $\{G_1, G_2\}$ to a uniform tower of length 4 if and only if $[\eta] \in H^3(\mathfrak{L}; ad)$ is zero.*

2.9. **Formulas for B_2^*.** In this section we will derive explicit formulas for B_2^* of the Bockstein spectral sequence of a group in **LGrp**. These formulas will express B_2^* in terms of various Lie algebra cohomologies of the Lie algebra corresponding to the group.

Let $G \in \mathrm{Obj}(\mathbf{LGrp})$ and $Log(G) = \mathfrak{L}$. Let $n = \dim(\mathfrak{L})$ and assume $[\eta] = 0$. Then we have that

$$H^*(G; \mathbb{F}_p) = \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n].$$

The Bockstein is given by

$$\beta(x_i) = -\sum_{l<m} c^i_{lm} x_l x_m$$

and

$$\beta(s_i) = \sum_{l,m=1}^{n} c^i_{lm} s_l x_m$$

where c^i_{lm} are the structure constants of $\mathfrak{L}$ in a basis $\{e_1, \ldots, e_n\}$ implicit in all of this. (Note that the other Steenrod operations are axiomatically determined so, indeed, we have determined $H^*(G; \mathbb{F}_p)$ as a Steenrod module. We see that the Steenrod module $H^*(G; \mathbb{F}_p)$ determines the group in the class of objects **LGrp**.) Thus we can view $H^*(G; \mathbb{F}_p)$ as being bigraded in the same way as $E_\infty^{*,*}$ of the LHS-spectral sequence which we used to compute it. We see that when $[\eta] = 0$, the Bockstein β preserves the polynomial degree in this bigrading. Thus the complex $(B_1^*, \beta = \beta_1)$ of the Bockstein spectral sequence splits up as a direct sum of differential complexes, one for each polynomial degree. More precisely, we have

$$\mathbb{F}_p[s_1, \ldots, s_n] = \bigoplus_{k=0}^{\infty} S^k$$

where S^k stands for the vector space of homogeneous polynomials of degree k. Thus we have

$$H^*(G; \mathbb{F}_p) = \bigoplus_{k=0}^{\infty} (\wedge^*(x_1, \ldots, x_n) \otimes S^k)$$

where each $\wedge^*(x_1, \ldots, x_n) \otimes S^k$ becomes a differential complex under β. Thus B_2^* is given as the direct sum of the cohomology of each of these complexes. We can identify these complexes with the Koszul complexes used to calculate $H^*(\mathfrak{L}; S^k)$ where S^k is the vector space of symmetric multilinear k-forms on $\mathfrak{L}$ given the Lie algebra action:

$$(u.f)(u_1,\dots,u_k)=\sum_{i=1}^{k} f(u_1,\dots,[u_i,u],\dots,u_k)$$

for all $f \in S^k$ and $u \in \mathfrak{L}$. To see this, note that the direct sum of these Koszul complexes has a natural identification with

$$\bigoplus_{k=0}^{\infty}(\wedge^*(x_1,\dots,x_n)\otimes S^k)$$

and hence $H^*(G;\mathbb{F}_p)$. In [Pak], it is shown that the sum of the Koszul differentials is a derivation with respect to the algebra structure given, i.e., that of $H^*(G;\mathbb{F}_p)$. So to show that these are the same as the Bockstein differential it is enough to check that they agree on the generators $x_1,\dots,x_n,s_1,\dots,s_n$ of $H^*(G;\mathbb{F}_p)$. This is easy and will be left to the reader (for details see [Pak]). Thus, once grading is taken into consideration one gets the equation:

$$B_2^* = \bigoplus_{k=0}^{\infty} H^{*-2k}(\mathfrak{L},S^k) \tag{23}$$

which holds when $[\eta]=0$. Thus we see that knowing the second term of the Bockstein spectral sequence is equivalent to knowing all the Lie algebra cohomology groups $H^*(\mathfrak{L};S^k)$. For clarity we state the results of this section in the following theorem.

Theorem 2.36. *Let $G \in \mathrm{Obj}(\mathbf{LGrp})$ with $Log(G)=\mathfrak{L}$ and with $[\eta]=0$. Then B_2^* of the Bockstein spectral sequence for G is given by*

$$B_2^* = \bigoplus_{k=0}^{\infty} H^{*-2k}(\mathfrak{L};S^k).$$

We will calculate B_2^* (at least partially) in some cases later on in this paper.

3. The general case

3.1. **Introduction.** Our results so far concern groups G which fit in as G_2 in some uniform tower

$$\cdots \to G_n \to \cdots \to G_2 \to G_1 \to 1.$$

We now study the higher groups G_n in such a tower. The $\mathbb{F}_p$-cohomology of such groups is determined by theorem 2.32. We will now extend our formulas for the Bockstein to the higher groups G_n , $n \geq 3$ in such uniform towers.

To do this we will need to use Weigel's Exp-Log correspondence. Let $\mathbb{Z}_p$ denote the p-adic integers.

Definition 3.1. A $\mathbb{Z}_p$-Lie algebra is a $\mathbb{Z}_p$-module $\mathfrak{L}$ equipped with a bilinear, alternating product $[\cdot,\cdot]:\mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}$ which satisfies the Jacobi Identity.

Definition 3.2. A $\mathbb{Z}_p$-Lie algebra $\mathfrak{L}$ is powerful if $[\mathfrak{L},\mathfrak{L}]\subseteq p\cdot\mathfrak{L}$.

Definition 3.3. $\Omega_1(\mathfrak{L})=\{v\in\mathfrak{L}: p\cdot v=0\}$.

Definition 3.4. A $\mathbb{Z}_p$-Lie algebra $\mathfrak{L}$ is p-central if $[\mathfrak{L},\Omega_1(\mathfrak{L})]=0$.

Definition 3.5. **CLie** is the category of powerful, p-central finite $\mathbb{Z}_p$-Lie algebras, with morphisms the maps of Lie algebras.

Definition 3.6. **CGrp** is the category of powerful, p-central p-groups with morphisms the homomorphisms of groups. (See section 2.8 for the relevant definitions.)

Definition 3.7. In analogy to the case with groups we say that $\mathfrak{L} \in \mathrm{Obj}(\mathbf{CLie})$ has the ΩEP if there exists $\bar{\mathfrak{L}} \in \mathrm{Obj}(\mathbf{CLie})$ such that $\bar{\mathfrak{L}}/\Omega_1(\bar{\mathfrak{L}}) \cong \mathfrak{L}$.

The following theorem was proved in [W3] and stated in [W1]. It generalizes the correspondence between p-adic Lie algebras and uniform pro-p groups (see [DS]) and is based on a careful study of the Baker-Campbell-Hausdorff identity. ($e^A \cdot e^B = e^C$ where C lies in the Lie algebra generated by A and B.)

Theorem 3.8 ([W3]: Exp-Log correspondence)**.** *Let $p \geq 5$. There exist functors* $\mathbf{Exp} : \mathbf{CLie} \to \mathbf{CGrp}$ *and* $\mathbf{Log} : \mathbf{CGrp} \to \mathbf{CLie}$ *which give a natural equivalence of categories. Furthermore, if $\mathfrak{L} \in \mathrm{Obj}(\mathbf{CLie})$, then $\mathfrak{L}$ has the ΩEP if and only if* $\mathbf{Exp}(\mathfrak{L})$ *does.*

Note that this theorem does not cover the case $p = 3$.

Definition 3.9. $\mathbf{CLie}_k$ is the full subcategory of **CLie** whose objects are the powerful, p-central Lie algebras which are free $\mathbb{Z}/p^k\mathbb{Z}$-modules.

Definition 3.10. $\mathbf{CGrp}_k$ is the full subcategory of **CGrp** consisting of groups which fit into the kth stage of some uniform tower.

The "Exp-Log" equivalence above, can easily be shown to restrict to a natural equivalence of $\mathbf{CGrp}_k$ with $\mathbf{CLie}_k$ for all $k \geq 1$.

Definition 3.11. $\mathbf{Brak}_k$ is the category whose objects consist of bracket algebras over $\mathbb{Z}/p^k\mathbb{Z}$. That is, free $\mathbb{Z}/p^k\mathbb{Z}$-modules of finite rank B, equipped with a bilinear, alternating map $[\cdot,\cdot] : B \times B \to B$. The morphisms are maps of the underlying modules which preserve the bracket structure.

In [Pak], we construct a shifting function $\mathbf{S} : \mathbf{CLie}_k \to \mathbf{Brak}_{k-1}$ which is a bijection on the object level and induces certain identifications on the morphisms. We also show that for $\mathfrak{L} \in \mathrm{Obj}(\mathbf{CLie}_k)$, $\mathfrak{L}$ has the ΩEP if and only if $\mathbf{S}(\mathfrak{L})$ is a Lie algebra. Let $Log : \mathbf{CGrp}_k \to \mathbf{Brak}_{k-1}$ be the composition of the functors **Log** and **S**. Note that this extends the definition of $Log : \mathbf{CGrp}_2 \to \mathbf{Brak}_1$ studied earlier. The facts mentioned above, together with theorem 3.8 allow us to prove the following theorem on uniform towers:

Theorem 3.12. *Fix $p \geq 5$. Let*

$$G_k \to \cdots \to G_2 \to G_1 \to 1$$

be a uniform tower of groups of length k. Then this tower extends to a uniform tower of length $k+1$ if and only if $Log(G_k)$ is a Lie algebra.

Proof. The tower extends to a uniform tower of length $k+1$ if and only if G_k has the ΩEP. By the comments made before, this happens if and only if $Log(G_k)$ is a Lie algebra. □

Note that this extends the $k = 2$ case proven in theorem 2.10. Let $k \geq 2$. Since $Log : \mathbf{CGrp}_k \to \mathbf{Brak}_{k-1}$ is a bijection on the object level, we will refer to $Exp(B)$ as the unique group $G \in \mathrm{Obj}(\mathbf{CGrp}_k)$ such that $Log(G) = B$. If one has a morphism $\psi \in \mathrm{Mor}_{\mathbf{Brak}_{k-1}}(B_1, B_2)$, then one gets a map of groups $\Psi : Exp(B_1) \to Exp(B_2)$ which is unique up to multiplication by a map $\eta : Exp(B_1) \to \Omega_1(Exp(B_2))$ (see [Pak]). Although this map is not unique, it induces a unique group homomorphism on the lower levels of the tower. ($Exp(B_i)$ fits into the kth stage of some uniform tower.)

Definition 3.13. For this section, we fix $p \geq 5$, a prime. Let $\mathbf{LGrp}_k$ denote the full subcategory of $\mathbf{CGrp}_k$ consisting of the objects $G \in \mathrm{Obj}(\mathbf{CGrp}_k)$ which have $Log(G)$, a Lie algebra. Let $\mathfrak{L} = (W, br)$ be the reduction of $Log(G)$ to a $\mathbb{F}_p$-Lie algebra.

Then by theorems 2.32 and 3.12, one has that

$$H^*(G; \mathbb{F}_p) \cong \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$$

where $n = \dim(\mathfrak{L})$. Recall that the exterior algebra comes isomorphically from the lower level in the tower so it can be identified with the exterior algebra at the first stage of the tower. Thus by naturality of the Bockstein, we see that

$$\beta(x_i) = -br^*(x_i)$$

which is the same formula as in the second stage of the tower. The polynomial algebra part maps isomorphically to that of $\Omega_1(G)$ which is isomorphic to the first level of the tower as a group via an iterate of the pth power map. We will use this isomorphism to choose basis from now on without mention; thus maps on the group level will induce the same maps on the x_i as they do on the s_i (up to addition of exterior elements). As before, we reason that

$$\beta(\mathbf{s}) = \xi\mathbf{s} + \eta$$

where η is a column vector of elements in $\wedge^3(x_1, \ldots, x_n)$, ξ is a $n \times n$ matrix of elements in $\wedge^1(x_1, \ldots, x_n)$ and $\mathbf{s}$ is the column vector of the polynomial generators. As before, because $\beta \circ \beta = 0$, one can show ξ determines a map of Lie algebras $\lambda : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$.

Let $G_1, G_2 \in \mathrm{Obj}(\mathbf{LGrp}_k)$ and $\psi \in \mathrm{Mor}(Log(G_1), Log(G_2))$. Let $\mathfrak{L}_i$ be the $\mathbb{F}_p$-Lie algebras obtained by reducing $Log(G_i)$ for $i = 1, 2$ and let $\bar{\psi} : \mathfrak{L}_1 \to \mathfrak{L}_2$ be the map of Lie algebras induced by ψ. Then $Exp(\psi) : G_1 \to G_2$ induces a map on the cohomology level which is well defined and induced by pullback via $\bar{\psi}$ on the exterior generators and which is given by the same formula on the polynomial generators, except it is ambiguous up to addition of exterior elements. As before, this enables us to prove naturality of the λ maps; that is, the various λ maps fit together to form a strong natural self-representation λ on $\mathbf{Lie}_{k-1}(p)$.

By theorem 2.25, to show that $\lambda = ad$ it is enough to show that it is nontrivial on $\mathfrak{sl}_2$ and $\mathfrak{so}_3$. This follows exactly as it did before by considering the Bockstein spectral sequence and noting that it must converge to zero in positive gradings. Thus if we fix $G \in \mathrm{Obj}(\mathbf{Lie}_k)$ and let c_{ij}^k be the structure constants for $Log(G)$ with respect to some basis, then we obtain the following formula for the Bockstein as before:

$$\beta(x_k) = -\sum_{i<j} c_{ij}^k x_i x_j,$$

$$\beta(s_k) = \sum_{i,j=1}^{n} c_{ij}^k s_i x_j + \eta_k.$$

Again, one shows that η determines a well defined cohomology class $[\eta] \in H^3(\mathfrak{L}; ad)$ which vanishes if and only if the uniform tower that G is in can be extended two steps above G. We summarize in the following theorem:

Theorem 3.14. *Fix $p \geq 5$. Let*

$$G_{k+1} \to G_k \to \cdots \to G_2 \to G_1 \to 1$$

be a uniform tower with $k \geq 2$. Let $\mathfrak{L} = Log(G_2)$ and let c_{ij}^k be the structure constants of $\mathfrak{L}$ with respect to some basis. Then for suitable choices of degree 1 elements $x_1, \ldots, x_n$ and degree 2 elements $s_1, \ldots, s_n$, one has

$$H^*(G_k; \mathbb{F}_p) \cong \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n]$$

with

$$\beta(x_t) = -\sum_{i<j}^{n} c_{ij}^t x_i x_j,$$

$$\beta(s_t) = \sum_{i,j=1}^{n} c_{ij}^t s_i x_j + \eta_t$$

for all $t = 1, \ldots n$. Furthermore, η defines a cohomology class $[\eta] \in H^3(\mathfrak{L}; ad)$ which vanishes if and only if there exists a uniform tower

$$G'_{k+2} \to G'_{k+1} \to G_k \to \cdots \to G_2 \to G_1 \to 1$$

or in other words if and only if $Log(G_k)$ (which is a Lie algebra over $\mathbb{Z}/p^{k-1}\mathbb{Z}$), has a lift to a Lie algebra over $\mathbb{Z}/p^k\mathbb{Z}$. In this case, one can drop the η terms in the formula for the Bockstein.

Note, for an example of a Lie algebra over $\mathbb{Z}/p^{k-1}\mathbb{Z}$ which does not lift to a Lie algebra over $\mathbb{Z}/p^k\mathbb{Z}$ see [BrP].

Note that the obstruction classes $[\eta]$ are functionally equivalent to the obstruction classes $J_{[\cdot,\cdot]}$ discussed in [BrP]. Also note that if

$$H^*(G; \mathbb{F}_p) \cong \wedge^*(x_1, \ldots, x_n) \otimes \mathbb{F}_p[s_1, \ldots, s_n],$$

then all the other (non-Bockstein) Steenrod operations are axiomatically determined. So the only difference in the $\mathbb{F}_p$-cohomologies as Steenrod algebras of the different groups in a uniform tower (except at an end) is in the classes $[\eta]$. Thus we conclude the following corollary:

Corollary 3.15. *Fix $p \geq 5$. Let*

$$G_{k+2} \to G_{k+1} \to G_k \to \cdots \to G_2 \to G_1 \to 1$$

be a uniform tower with $k \geq 2$. Then $H^*(G_i; \mathbb{F}_p)$ are isomorphic Steenrod modules for $2 \leq i \leq k$.

Finally, one should mention that theorem 3.14 extends to the category of powerful, p-central p-groups with the ΩEP. As mentioned before, there is a natural equivalence **Log** from this category to the category of powerful, p-central, finite $\mathbb{Z}_p$-Lie algebras with the ΩEP. The shifting functor **S**, mentioned before, (see also [Pak]) maps this category into the category of finite $\mathbb{Z}_p$-Lie algebras. (Again, the Jacobi identity holds exactly because of the ΩEP as mentioned earlier in this paper.) Unfortunately, the shifting functor does lose a bit of information on the object level which corresponds to an elementary abelian summand in the group/Lie algebra.

Let $Log = \mathbf{S} \circ \mathbf{Log}$, then for a powerful, p-central, p-group G with the ΩEP, one has a corresponding Lie algebra $\mathfrak{L} = Log(G)$ and this induces a Lie algebra over $\mathbb{F}_p$, $\bar{\mathfrak{L}} = \mathfrak{L}/p\mathfrak{L}$. Due to the loss of the elementary abelian summand, one can have $\dim(\bar{\mathfrak{L}}) < \dim(\Omega_1(G))$. This is unavoidable, as the Bockstein on the generators of the elementary abelian summand are of a different nature (Bockstein of degree 1 generators is no longer nilpotent for example) but easy to describe as we know what the Bockstein is on elementary abelian p-groups. Taking this into account, one can obtain the following extension of theorem 3.14.

Theorem 3.16. *Fix $p \geq 5$. Let G be a powerful, p-central p-group with the ΩEP. Let $\mathfrak{L} = Log(G)$, $\bar{\mathfrak{L}} = \mathfrak{L}/p\mathfrak{L}$ and let c_{ij}^k be the structure constants of $\bar{\mathfrak{L}}$ with respect to some basis. Suppose $\dim(\Omega_1(G)) = n$ and $\dim(\bar{\mathfrak{L}}) = k$. Then for suitable choices of degree 1 elements $x_1, \dots, x_n$ and degree 2 elements $s_1, \dots, s_n$, one has*

$$H^*(G; \mathbb{F}_p) \cong \wedge^*(x_1, \dots, x_n) \otimes \mathbb{F}_p[s_1, \dots, s_n]$$

with

$$\begin{aligned}\beta(x_t) &= -\sum_{i<j}^{k} c_{ij}^t x_i x_j, \\ \beta(s_t) &= \sum_{i,j=1}^{k} c_{ij}^t s_i x_j + \eta_t \text{ for } t = 1, \dots k, \\ \beta(x_t) &= s_t \text{ and } \beta(s_t) = 0 \text{ for } t > k.\end{aligned}$$

Furthermore, the η_t define a cohomology class $[\eta] \in H^3(\bar{\mathfrak{L}}; ad)$ which vanishes if and only if the Lie algebra $\mathfrak{L}$ has the ΩEP. When this is the case, one can drop the η terms in the formula above.

Here the definition of ΩEP has been extended to finite $\mathbb{Z}_p$-Lie algebras in general by saying that $\mathfrak{L}$ has the ΩEP if there is a $\mathbb{Z}_p$-Lie algebra $\bar{\mathfrak{L}}$ such that $\bar{\mathfrak{L}}/\Omega_1(\bar{\mathfrak{L}}) = \mathfrak{L}$.

3.2. **Calculating B_2^* in various cases.** Recall that there is a natural equivalence between the categories **Lie** and **LGrp**. Under this equivalence, abelian Lie algebras correspond to abelian groups and are not so interesting. Let us work out B_2^* for the nonabelian Lie algebra $\mathfrak{S}$ which has basis $\{x, y\}$ and bracket given by the relation $[y, x] = y$. Let the corresponding group in **LGrp** be denoted by $G(\mathfrak{S})$. This is a group of order p^4 and exponent p^2. Then we have the structure constants $c_{y,x}^y = 1$ and $c_{y,x}^x = 0$. Thus using theorem 3.14, we see that

$$H^*(G(\mathfrak{S}); \mathbb{F}_p) \cong \wedge^*(x, y) \otimes \mathbb{F}_p[X, Y]$$

with

$$\begin{aligned}\beta(x) = 0 \ &, \ \beta(y) = xy, \\ \beta(X) = 0 \ &, \ \beta(Y) = Yx - Xy\end{aligned}$$

where we used that the class $[\eta]$ vanishes as $\mathfrak{S}$ lifts to a Lie algebra over the p-adic integers. Let A be the subalgebra of $H^*(G(\mathfrak{S}); \mathbb{F}_p)$ generated by x, yY^{p-1}, X, Y^p. This is isomorphic as graded algebras, to the graded algebra

$$\wedge^*(x, z) \otimes \mathbb{F}_p[X, Z]$$

where $\deg(x) = 1, \deg(z) = 2p - 1, \deg(X) = 2, \deg(Z) = 2p$. It is easy to check that β vanishes on A. Thus there is a well defined map of graded-algebras from

A to B_2^*. A direct calculation shows that this is an isomorphism. By comparisons (restricting to suitable subgroups of $G(\mathfrak{S})$), it is easy to show then that we can choose z, Z so that β_2 is given by $\beta_2(x) = X, \beta_2(z) = Z$. Thus $B_3^* = 0$ for $* > 0$. Thus we can conclude that

$$\exp(\bar{H}^*(G(\mathfrak{S}); \mathbb{Z})) = p^2.$$

Now let us consider the Lie algebra $\mathfrak{sl}_2$ with basis $\{h, x_+, x_-\}$ with bracket given by $[h, x_+] = 2x_+, [h, x_-] = -2x_-$ and $[x_+, x_-] = h$. Let $G(\mathfrak{sl}_2) = \hat{\Gamma}_{2,2}(p)$ be the corresponding group. Again we note that this Lie algebra lifts to the p-adics so by theorem 3.14 we conclude:

$$H^*(G(\mathfrak{sl}_2); \mathbb{F}_p) \cong \wedge^*(h, x_+, x_-) \otimes \mathbb{F}_p[H, X_+, X_-]$$

with Bockstein given by

$$\begin{aligned} \beta(h) &= -x_+x_-, \quad \beta(H) = X_+x_- - X_-x_+, \\ \beta(x_+) &= -2hx_+, \quad \beta(X_+) = 2(Hx_+ - X_+h), \\ \beta(x_-) &= 2hx_-, \quad \beta(X_-) = -2(Hx_- - X_-h). \end{aligned}$$

Using this explicit formula and viewing B_2^* as the sum of Lie algebra cohomologies as was shown before, one computes that $H^*(\mathfrak{sl}_2; \mathbb{F}_p)$ is zero except in dimensions 0 and 3 where it is one dimensional. Also $H^*(\mathfrak{sl}_2; S^1) = 0$ and $H^0(\mathfrak{sl}_2; S^2)$ is one dimensional generated by the Killing form of $\mathfrak{sl}_2$. This implies the following facts: $\dim(B_2^1) = \dim(B_2^2) = 0$, $\dim(B_2^3) = 1$ generated by hx_+x_- and $\dim(B_2^4) = 1$ generated by the Killing form $8(H^2 + X_+X_-)$. By comparisons, $\beta_2(hx_+x_-) = 0$ so that $B_2^3 = B_3^3$ and thus by necessity (since the Bockstein spectral sequence must eventually converge to zero), $B_2^4 = B_3^4$. Thus in particular, $B_3^* \neq 0$ in positive dimensions. Thus we conclude that $\exp(\bar{H}^*(G(\mathfrak{sl}_2); \mathbb{Z})) > p^2$. On the other hand, $G(\mathfrak{sl}_2)$ contains $G(\mathfrak{S}) \times \mathbb{Z}/p\mathbb{Z}$ as a subgroup of index p. Thus by transfer arguments,

$$\exp(\bar{H}^*(G(\mathfrak{sl}_2); \mathbb{Z})) \leq p \exp(\bar{H}^*(G(\mathfrak{S}); \mathbb{Z})) = p^3.$$

Thus

$$\exp(\bar{H}^*(G(\mathfrak{sl}_2); \mathbb{Z})) = p^3.$$

Note that this is neither the order nor the exponent of $G(\mathfrak{sl}_2)$ as $G(\mathfrak{sl}_2)$ has order p^6 and exponent p^2.

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